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by

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EXPERIMENTS ON METHODS FOR THE NUMERICAL SOLUTION OF A CERTAIN NON-LINEAR BIHARMONIC EQUATION

by

M. R. Abbott

SUMMARY

An account is given of two methods of numerical solution which have been tried on a non-linear biharmonic partial differential equation of a type which arises in various fluid flow problems. Neither method is wholly successful on a rather demanding test case, and in fact it is not clear how a fully satisfactory automatic method could be devised. However, at least one method should work in other applications where the non-linear terms are of lesser importance.

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ILLUSTRATIONS

1 INTRODUCTION

Steady axially symmetric incompressible viscous flows are described by linked equations of the form[1]

$$\nu \nabla_*^4\psi = \frac{1}{r} \frac{\partial(\psi,\nabla_*^2\psi)}{\partial(r,Z)} + \frac{2}{r^2} \frac{\partial\psi}{\partial Z} \nabla_*^2\psi + \frac{2\Omega}{r^2} \frac{\partial\Omega}{\partial Z} \tag{1}$$

and

$$\nu \nabla_*^2\Omega = \frac{1}{r} \frac{\partial(\psi,\Omega)}{\partial(r,Z)} , \tag{2}$$

with

$$\nabla_*^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial Z^2} \quad ; \tag{3}$$

in terms of cylindrical polar coordinates (Z,r,λ) with the Z-axis taken along the axis of symmetry and the r-axis at right-angles to it. Conditions are independent of λ due to the symmetry. ψ is a stream function such that, if V_Z is the axial velocity and V_r the radial velocity,

$$V_Z = \frac{1}{r} \frac{\partial\psi}{\partial r} , \qquad V_r = -\frac{1}{r}\frac{\partial\psi}{\partial Z} \quad ; \tag{4}$$

and the circumferential velocity is given by

$$V_\lambda = \frac{1}{r}\,\Omega \quad . \tag{5}$$

ν is the kinematic coefficient of viscosity. Similar equations apply if spherical polar coordinates are used instead.

Similar equations also occur in steady two-dimensional natural convection problems. For example, for the problem of natural convection between two concentric circular cylinders with a horizontal axis, we have equations in plane polar coordinates of the form[2,3]

$$\nabla^4\psi = \frac{1}{2} \frac{\partial(\nabla^2\psi,\psi)}{\partial(r,\theta)} + \frac{1}{4} G \left(\cos\theta \frac{\partial\Omega}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\Omega}{\partial\theta}\right) \tag{6}$$

and

$$\nabla^2\Omega = \frac{1}{2r} P \frac{\partial(\Omega,\psi)}{\partial(r,\theta)} , \tag{7}$$

with

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} . \tag{8}$$

These equations are in non-dimensional form. G is the Grashof number and P the Prandtl number. $\Omega(r,\theta)$ now represents the temperature of the fluid, and the stream function ψ is such that

$$V_r = \frac{1}{r} \frac{\partial\psi}{\partial\theta} , \qquad V_\theta = -\frac{\partial\psi}{\partial r} . \tag{9}$$

In fact, Crawford and Lemlich[2] obtain the solution of these equations by a Gauss-Seidel iterative method for general G. The same method could in theory be applied to all the problems given here, but due to the difficulties of obtaining satisfactory values for the over- or under-relaxation factors, and the amount of experiment needed to do this, it appeared to be worth exploring the alternative methods given in this Note.

Further, for natural convection in vertical two-dimensional closed rectangular cavities we have equations of the form

$$\nabla^4\psi = \frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)} + G \frac{\partial\Omega}{\partial x} \tag{10}$$

and

$$\nabla^2\Omega = P \frac{\partial(\Omega,\psi)}{\partial(x,y)} , \tag{11}$$

with

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \tag{12}$$

and

$$V_x = \frac{\partial\psi}{\partial y} , \qquad V_y = -\frac{\partial\psi}{\partial x} . \tag{13}$$

Batchelor[4] investigated the heat loss across double glazing and similar insulation cavities, and the optimum thickness thereof, from equations equivalent to these (with the Rayleigh number in place of the Grashof number and employing a slightly different scaling to make the equations non-dimensional). The method employed was based on boundary-layer type arguments and analogies rather than on a direct solution of (10) and (11), though an expansion method, useful only for very small G, was tried and will be mentioned further below.

In the above problems ψ, its normal derivative, and one condition on Ω are known round a closed boundary in the (Z,r), (r,θ) or (x,y) plane respectively.

Further, the above sets of equations are each of the form

$$\nabla_*^4\psi \quad \text{or} \quad \nabla^4\psi = f(\psi,\Omega)$$

and

$$\nabla_*^2\Omega \quad \text{or} \quad \nabla^2\Omega = g(\psi,\Omega) \quad ,$$

where f and g contain up to quadratic terms in ψ and Ω, and their derivatives. f contains third order derivatives of ψ and first order derivatives of Ω, while g contains first order derivatives of ψ and Ω.

In this Note we examine methods for the solution of the case $\Omega \equiv 0$. This gives a non-trivial problem only in the first case, of steady axially symmetric incompressible viscous flows, and corresponds to having no circumferential velocity (swirl). In the natural convection problems $\psi \equiv 0$ if $\Omega \equiv 0$.

Further, as a specific example, we consider a flow in a pipe of constant radius $(r = a)$. The velocity distribution at entry $(Z = 0)$ is taken as arbitrary, for instance $V_r = 0$ with V_Z constant across the pipe except for a small boundary layer round the perimeter through which we must have $V_Z \to 0$. For large Z the flow is asymptotic to the Hagen-Poiseuille flow with V_Z parabolic and V_r zero. We can find from the calculation the length in the axial direction (inlet length) which the flow takes to become indistinguishable from the Hagen-Poiseuille flow.

Equation (1) with $\Omega = 0$ is thus the equation we wish to solve. We first introduce dimensionless variables as follows:

$$r' = r/a \,, \qquad Z' = Z/a \,, \qquad \psi' = \psi/\tfrac{1}{2}a^2\bar{V}_Z \,, \qquad R = 2a\bar{V}_Z/\nu \quad ;$$

where $\bar{V}_Z$ is the mean axial velocity over a cross-section of the pipe and R is the Reynolds number. This normalises the ψ boundary condition along r = a as follows:

$$\text{rate of volume flow along pipe} = \pi a^2 \bar{V}_Z , \qquad \text{also} = 2\pi\Delta\psi ,$$

where $\Delta\psi = \psi$ on the boundary $- \psi$ on the axis. Thus

$$\pi a^2 \bar{V}_Z = 2\pi(\tfrac{1}{2} a^2 \bar{V}_Z)\, \Delta\psi' , \qquad \text{or} \qquad \Delta\psi' = 1 ;$$

we can therefore take $\psi' = 1$ on the boundary $r' = 1$ and $\psi' = 0$ on the axis $r' = 0$. Making the above substitutions in (1) and then dropping the dashes gives as our equation

$$\nabla_*^4\psi = \frac{R}{4r} \frac{\partial(\psi,\nabla_*^2\psi)}{\partial(r,Z)} + \frac{R}{2r^2} \frac{\partial\psi}{\partial Z} \nabla_*^2\psi , \tag{14}$$

with the boundary conditions ψ and $\partial\psi/\partial Z$ given on Z = 0 and on some downstream section ($Z = \ell$ say); and $\psi = 1$, $\partial\psi/\partial r = 0$ on $r = 1$, $\psi = 0$, $\partial\psi/\partial r = 0$ on $r = 0$. These conditions ensure specified values of V_Z and V_r at Z = 0 and $Z = \ell$, zero values of V_Z and V_r on the boundary and a zero value of V_r together with a finite value of V_Z on the axis.

If a satisfactory method of solution can be found for the case of $\Omega = 0$ then the more general case with a non-zero swirl (or temperature) should not present further difficulty, since the decay of swirl in a pipe has been shown[5] to be well described by linearised equations, and so likely to be amenable to an extension of the method used below.

2 METHOD OF SOLUTION AND NUMERICAL DETAILS

We attempt an iterative solution of (14) from the following linearised form

$$\nabla_*^4\psi_{K+1} = \frac{R}{8r}\left\{\frac{\partial(\psi_{K+1},\nabla_*^2\psi_K)}{\partial(r,Z)} + \frac{\partial(\psi_K,\nabla_*^2\psi_{K+1})}{\partial(r,Z)}\right\} + \frac{R}{4r^2}\left\{\frac{\partial\psi_{K+1}}{\partial Z}\nabla_*^2\psi_K + \frac{\partial\psi_K}{\partial Z}\nabla_*^2\psi_{K+1}\right\} . \tag{15}$$

We take the starting value ψ_0 to be some guess satisfying boundary and physical conditions as well as possible. ψ_1 is then determined from the <u>linear</u> biharmonic equation (15), and so on until (if the process converges) $\psi_{K+1} = \psi_K$ to a given accuracy and we then have a solution of the original equation.

Previously[3], on the analogous natural convection problem, we used an iterative scheme corresponding to

$$\nabla_*^4 \psi_{K+1} = \frac{R}{4r} \frac{\partial(\psi_K, \nabla_*^2 \psi_K)}{\partial(r, Z)} + \frac{R}{2r^2} \frac{\partial \psi_K}{\partial Z} \nabla_*^2 \psi_K \quad , \tag{16}$$

but this only converged if the non-linear terms were of limited importance (G relatively small). However, if it worked, the method had the advantage that the algebraic operator approximating to ∇_*^4 had only to be inverted once in the course of a calculation, since any ψ_{K+1} is formed by pre-multiplying the new right-hand side (depending only on the known ψ_K) by this inverse matrix. (15) does not have this advantage since the ψ_{K+1} have coefficients depending on ψ_K. Batchelor[4] uses an expansion method for small Ra (equivalent to small G), which leads to equations of the form (16). However, as he relied on obtaining analytical solutions the method is only of use if the sequence ψ_K converges so rapidly that a very few terms will suffice. Employing a computer enables any number of iterates to be obtained quickly by matrix multiplication, so we can make use of a more slowly converging sequence ψ_K. However, there is still a limit to the use of the method as convergence is not obtained if the non-linear terms are too large. An iterative scheme such as (15) should improve the chance of convergence in these cases.

For example, Arnason[6] employed a similar iterative method with success on an elliptic equation of the form

$$2 \left\{ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \, \partial y} \right)^2 \right\} + f \nabla^2 \psi + \nabla f . \nabla \psi = F \quad ,$$

by linearising it as

$$\frac{\partial^2 \psi_{K+1}}{\partial x^2} \frac{\partial^2 \psi_K}{\partial y^2} + \frac{\partial^2 \psi_K}{\partial x^2} \frac{\partial^2 \psi_{K+1}}{\partial y^2} - 2 \frac{\partial^2 \psi_{K+1}}{\partial x \, \partial y} \frac{\partial^2 \psi_K}{\partial x \, \partial y} + f \nabla^2 \psi_{K+1} + \nabla f . \nabla \psi_{K+1} = F \quad ;$$

whereas

$$\nabla^2\psi_{K+1} = \frac{1}{f}\left[F - \nabla f.\nabla\psi_K - 2\left\{\frac{\partial^2\psi_K}{\partial x^2}\frac{\partial^2\psi_K}{\partial y^2} - \left(\frac{\partial^2\psi_K}{\partial x\,\partial y}\right)^2\right\}\right]$$

was not always convergent.

A simple example of the method applied to a boundary value problem for a non-linear <u>ordinary</u> differential equation is the following:

To solve

$$\frac{d^2y}{dx^2} = y\frac{dy}{dx} , \tag{17}$$

with $y = 0$ at $x = 0$ and $y = 2$ at $x = \pi/4$, we write (17) as

$$\frac{d^2y_{K+1}}{dx^2} - \frac{1}{2}y_K\frac{dy_{K+1}}{dx} - \frac{1}{2}\frac{dy_K}{dx}y_{K+1} = 0 . \tag{18}$$

Then, given $y_K(x)$, we replace the derivatives of y_{K+1} by the simplest central difference approximations and obtain a system of linear algebraic equations, with a tri-diagonal matrix, which are easily solved in the usual way to give y_{K+1}. It is found that, even with the function y_o chosen 50% in error at the mid point of the range (though smooth and satisfying $y_o(0) = 1$, $y_o(\pi/4) = 2$), convergence is obtained in not more than 5 iterations. The answers have a maximum error of one unit in the fifth place, using 20 internal pivotal points. The exact solution of the equation is of course $y = 2 \tan x$.

Returning now to the solution of (15), we have to specify the remaining boundary conditions at $Z = 0$, $Z = \ell$. At $Z = 0$ we take V_Z proportional to $1 - e^{-\alpha(1-r)}$, hence

$$\psi = \frac{\alpha^2r^2 - 2(\alpha r - 1)\,e^{-\alpha(1-r)} - 2e^{-\alpha}}{\alpha^2 - 2\alpha + 2 - 2e^{-\alpha}} \tag{19}$$

for some constant value of α; so that $\psi = 1$ on $r = 1$, $\psi = 0$ on $r = 0$ and $\partial\psi/\partial Z = 0$. At $Z = \ell$ we assume the flow is the Hagen-Poiseuille flow; this can easily be shown to be

$$\psi = 2r^2 - r^4 \quad \text{with} \quad \frac{\partial\psi}{\partial Z} = 0 \tag{20}$$

in the present non-dimensional variables. The other boundary conditions are given above.

Equation (15) is very lengthy if written out in full, so we give here only the values of the following expressions

$$\nabla_*^4\psi_{K+1} = \frac{\partial^4\psi_{K+1}}{\partial r^4} - \frac{2}{r}\frac{\partial^3\psi_{K+1}}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2\psi_{K+1}}{\partial r^2} - \frac{3}{r^3}\frac{\partial\psi_{K+1}}{\partial r} - \frac{2}{r}\frac{\partial^3\psi_{K+1}}{\partial r\,\partial Z^2} + 2\frac{\partial^4\psi_{K+1}}{\partial r^2\,\partial Z^2} + \frac{\partial^4\psi_{K+1}}{\partial Z^4}, \quad \ldots (21)$$

and, for example,

$$\frac{\partial(\psi_{K+1},\nabla_*^2\psi_K)}{\partial(r,Z)} = \left(\frac{\partial^3\psi_K}{\partial r^3} + \frac{1}{r^2}\frac{\partial\psi_K}{\partial r} - \frac{1}{r}\frac{\partial^2\psi_K}{\partial r^2} + \frac{\partial^3\psi_K}{\partial r\,\partial Z^2}\right)\frac{\partial\psi_{K+1}}{\partial Z} - \left(\frac{\partial^3\psi_K}{\partial Z\,\partial r^2} - \frac{1}{r}\frac{\partial^2\psi_K}{\partial r\,\partial Z} + \frac{\partial^3\psi_K}{\partial Z^3}\right)\frac{\partial\psi_{K+1}}{\partial r}. \quad \ldots (22)$$

When these substitutions are made in (15), and the various derivatives are approximated by the usual finite difference formulae referred to a uniform rectangular mesh in the r,Z plane (with mesh spacings Δr and ΔZ respectively), we obtain a system of linear algebraic equations for the values of ψ_{K+1} at the internal pivotal points; with coefficients functions of r,Z and pivotal values of ψ_K. In substituting for the higher derivatives near the boundaries, values at "fictitious" points just outside the boundaries are brought in, these are eliminated through using the normal derivative boundary condition in finite difference form. In general each algebraic equation connects 13 adjacent mesh points and the system is of the form

$$A\psi_{K+1} = b \tag{23}$$

say, where ψ_{K+1} is the column vector of the solution at the internal pivotal points. The column vector b contains those terms independent of ψ_{K+1}, but depending on ψ_K and the non-homogeneous boundary conditions. The matrix A when partitioned is of the form

$$A = \begin{bmatrix} M_1 & X_1 & Y_1 & & & & & \\ B_2 & M_2 & X_2 & Y_2 & & & & \\ A_3 & B_3 & M_3 & X_3 & Y_3 & & & \\ & A_4 & B_4 & M_4 & X_4 & Y_4 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & A_{n-1} & B_{n-1} & M_{n-1} & X_{n-1} \\ & & & & & A_n & B_n & M_n \end{bmatrix}, \qquad (24)$$

where n is the number of internal pivotal points in the r-direction, and each submatrix element of A is of order $m \times m$, where m is the number of internal points in the Z-direction. The numbering of the points is first in the direction of increasing Z and then of increasing r. The submatrices A_i and Y_i are diagonal, B_i and X_i are tri-diagonal and M_i is quin-diagonal. For brevity, the actual elements of these submatrices are not given here, as the expressions for them are very lengthy. Contributions from homogeneous boundary conditions are included in A.

A direct method is used for solving (23) which depends on factorising A into a lower triangular and an upper triangular matrix, each with submatrix elements. The store of the Mercury computer imposes a limit of about 160 mesh points. This is not sufficient for high accuracy but should be sufficient to examine the feasibility of any method. An outline of the solution of (23), with A given by (24), is given for reference in an Appendix. It is of interest to note that, using the factorisation method and thus taking advantage of the sparseness of the matrix, rather than using the general Gaussian elimination method, reduces the time required to solve (23) by a factor of about 10 or more for the larger matrices.

3 APPLICATION TO AN EXAMPLE

The calculation described above has been carried out on the Mercury computer for an entry flow given by $\alpha = 10$ in (19). The velocity profile corresponding to this, together with that of the eventual Hagen-Poiseuille flow, is shown Fig.1(a); while the behaviour of the stream function is shown in Fig.1(b).

A guide to the range of Z over which it is necessary to perform the calculations is given by an approximate formula for the inlet length[1]:

$$\text{inlet length} = 0.0575\, a\, R \quad ,$$

so that the inlet length, in terms of our non-dimensional variables, is 0.0575R. This formula strictly applies to the case of constant velocity all over the inlet section, apart from a very thin boundary layer. Our example (Fig.1(a)) is less extreme (chosen thus in order to avoid difficulties with our rather coarse mesh), but the formula can be taken as giving an approximation to the minimum range of Z that it is necessary to use in the calculation. In fact, it is desirable to calculate over a length of pipe equal to at least two or three times the inlet length, since the trend towards Hagen-Poiseuille flow is asymptotic and the formula above refers merely to the distance after which changes in the flow are not experimentally obvious.

The Reynolds number $R = 200$ was used in the test calculations presented here, and for this value it was necessary to calculate ψ in the range $0 < Z < 40$ approximately, with $0 < r < 1$ as usual.

In these calculations difficulty was experienced with the convergence if ΔZ was too small, irrespective of the starting function ψ_0. The answers rapidly became oscillatory, and began to diverge. However, for larger values of ΔZ convergence was obtained after a few iterations, and the method produced results which were physically reasonable, as those in Fig.2.

It is believed that the difficulty arises in this example from the fact that the effect of viscosity diffuses from the boundary towards the axis, and for any Z there is a region about the axis in which the flow is as yet unaffected by viscosity. In particular, for small Z, very little of the cross-section is being influenced by viscosity, so that in this region the non-linear terms are the dominant ones, rather than $\nu \nabla_*^4 \psi$; and it appears that the linearisation chosen does not represent these terms adequately. Using a larger ΔZ gets over this, at a cost of accuracy, due to the first line of mesh points $Z = \Delta Z$ being situated in a region where viscosity is of more importance, and consequently the $\nu \nabla_*^4 \psi$ term is more dominant.

The value of ΔZ necessary for convergence is too large to give results for the inlet length of high accuracy; but as can be seen from the case shown in Fig.2, where ψ is plotted against Z for fixed values of r, the answers are compatible with the approximate formula for the inlet length given above. In fact, Prandtl and Tietjens[7] record that the formula

$$\text{inlet length} = 0.13\, a\, R \quad ,$$

due to Boussinesq, is in good agreement with experiment. This differs appreciably from the formula above and agrees more closely with the present numerical calculations.

The application of the method to the natural convection problems given in the Introduction may lead to similar difficulties as the fluid in the centre of the cavity away from the closed boundary is now relatively unaffected by viscosity. Also, the function ψ_0 may be difficult to guess realistically.

In conclusion, though the above method may work adequately on some restricted non-linear problems it seems at the moment that the best approach in the general (i.e. R or G not necessarily small) fluid mechanics applications discussed here is the Gauss-Seidel iterative method, with over- or under-relaxation as necessary, as used by Crawford and Lemlich[2]. This method, however, is far from automatic, needing experiment in each case to determine by trial and error acceptable relaxation factors which will lead to convergence at an adequate rate.

4 A POSSIBLE ALTERNATIVE METHOD

If we put

$$\xi = \nabla_*^2\psi \tag{25}$$

(this new dependent variable is in fact the vorticity multiplied by (-r)), (14) becomes

$$\nabla_*^2\xi = \frac{R}{4r}\frac{\partial(\psi,\xi)}{\partial(r,Z)} + \frac{R}{2r^2}\frac{\partial\psi}{\partial Z}\xi \quad . \tag{26}$$

An iterative method with lower order operators than before is therefore

$$\nabla_*^2\xi_{K+1} = \frac{R}{4r}\frac{\partial(\psi_K,\xi_{K+1})}{\partial(r,Z)} + \frac{R}{2r^2}\frac{\partial\psi_K}{\partial Z}\xi_{K+1} \quad , \tag{27}$$

along with

$$\nabla_*^2\psi_{K+1} = \xi_{K+1} \quad . \tag{28}$$

For the solution of (27) we require the values of ξ_{K+1} round the closed boundary in the r,Z plane, and to calculate this from (25) we use the values of ψ_K at internal points, the known value of ψ along the boundary and eliminate the values of ψ at fictitious points just outside the boundary by using the $\partial\psi/\partial n$ boundary condition. The boundary condition therefore depends on the previous solution, for example in calculating ξ_1 the boundary condition depends on the initial guess ψ_0. The boundary condition for (28) is of course known in advance.

From preliminary trials of the simpler scheme

$$\nabla_*^2\xi_{K+1} = \frac{R}{4r}\frac{\partial(\psi_K,\xi_K)}{\partial(r,Z)} + \frac{R}{2r^2}\frac{\partial\psi_K}{\partial Z}\xi_K \quad , \tag{29}$$

together with (28), it is apparent that difficulties arise in practice, which will be common to more exact representations such as (27) and (28) above. When a solution is obtained for (28) satisfying the boundary condition on ψ, it does not necessarily satisfy the required condition on $\partial\psi/\partial n$. This tends to introduce a 'kink' or cusp in the behaviour of ψ across the boundary when we later incorporate the $\partial\psi/\partial n$ condition into the calculation of ξ along the boundary ready for the next iteration. The iterations may then diverge. Changing the boundary conditions around so that the $\partial\psi/\partial n$ condition goes with (28) and the ψ condition with (29) will not get over this; the trouble may be partly similar to that of the previous section. The method did work for the very simple case of Hagen-Poiseuille flow at both entry and exit, but was of no use for the present example of a disturbed inlet flow. It again might be of use in applications where the non-linear terms are of lesser importance.

The algebraic equations in this section are only tri-diagonal in terms of submatrices, and can be solved very quickly by a simplified version of the method given in the Appendix.

SYMBOLS

a	radius of pipe
A	matrix given by (24)
A_i, B_i, M_i, X_i, Y_i	submatrices of A
b, b_i	column vectors
G	Grashof number
ℓ	range of Z used in calculation
L	lower triangular matrix
L_{ij}	submatrices of L
P	Prandtl number
r	coordinate in radial direction
R	Reynolds number
U	upper triangular matrix
U_{ij}	submatrices of U
$V_Z, V_r, V_\lambda, V_\theta, V_x, V_y$	various components of velocity

SYMBOLS (Contd)

x,y	plane coordinates
x_i, y_i	column vectors
Z	axial coordinate
α	constant in (19)
Δr	step length in r-direction
ΔZ	step length in Z-direction
θ	angle in plane polar coordinates
λ	angle in cylindrical polar coordinates
ν	kinematic viscosity
ξ	defined by (25)
ψ	stream function
Ω	defined by (5), or represents temperature
∇_*^2	defined by (3)

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APPENDIX

To solve $Ax = b$ (x is written for ψ_{K+1}) we express A, given by (24), as the product of a lower triangular matrix

$$L = \begin{bmatrix} I & & & & & \\ L_{21} & I & & & & \\ L_{31} & L_{32} & I & & & \\ & L_{42} & L_{43} & I & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & L_{n,n-2} & L_{n,n-1} & I \end{bmatrix},$$

with an upper triangular matrix

$$U = \begin{bmatrix} U_{11} & U_{12} & Y_1 & & & \\ & U_{22} & U_{23} & Y_2 & & \\ & & U_{33} & U_{34} & Y_3 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & U_{n,n} \end{bmatrix};$$

in which we have, from the start, taken advantage of the easily seen results that $U_{i,i+2} = Y_i$ for all relevant i, and that it is only necessary to employ tri-diagonal L and U. We obtain by multiplying out and comparing terms that:

$$U_{11} = M_1, \quad U_{12} = X_1,$$

$$L_{21} = B_2 U_{11}^{-1},$$

$$U_{22} = M_2 - L_{21} U_{12}, \quad U_{23} = X_2 - L_{21} Y_1,$$

$$L_{31} = A_3 U_{11}^{-1}, \quad L_{32} = (B_3 - L_{31} U_{12}) U_{22}^{-1},$$

$$U_{33} = M_3 - L_{31} Y_1 - L_{32} U_{23}, \quad U_{34} = X_3 - L_{32} Y_2,$$

and so on until we reach

$$L_{n,n-2} = A_n U^{-1}_{n-2,n-2}; \quad L_{n,n-1} = (B_n - L_{n,n-2} U_{n-2,n-1}) U^{-1}_{n-1,n-1};$$

$$U_{n,n} = M_n - L_{n,n-2} Y_{n-2} - L_{n,n-1} U_{n-1,n}.$$

We have now obtained the submatrix elements of L and U. It is necessary to work out each $U^{-1}_{i,i}$ in the process above, and these inverses are stored as they are needed again below.

The equation Ax = b is now LUx = b, so we can solve Ly = b and then Ux = y to yield x. Note that b, x and y are column vectors $(1 \times n)$ with vector elements $(1 \times m)$. To solve Ly = b: we have at once that

$$y_1 = b_1$$

$$y_2 = b_2 - L_{21} y_1$$

$$y_3 = b_3 - L_{31} y_1 - L_{32} y_2 ,$$

and so on until

$$y_n = b_n - L_{n,n-2} y_{n-2} - L_{n,n-1} y_{n-1} \quad .$$

The final step is to solve Ux = y, this is easily achieved:

$$x_n = U^{-1}_{n,n} y_n$$

$$x_{n-1} = U^{-1}_{n-1,n-1} (y_{n-1} - U_{n-1,n} x_n)$$

$$x_{n-2} = U^{-1}_{n-2,n-2} (y_{n-2} - U_{n-2,n-1} x_{n-1} - Y_{n-2} x_n) ,$$

until we reach

$$x_1 = U^{-1}_{11} (y_1 - U_{12} x_2 - Y_1 x_3) \quad .$$

The solution of Ax = b has now been obtained.

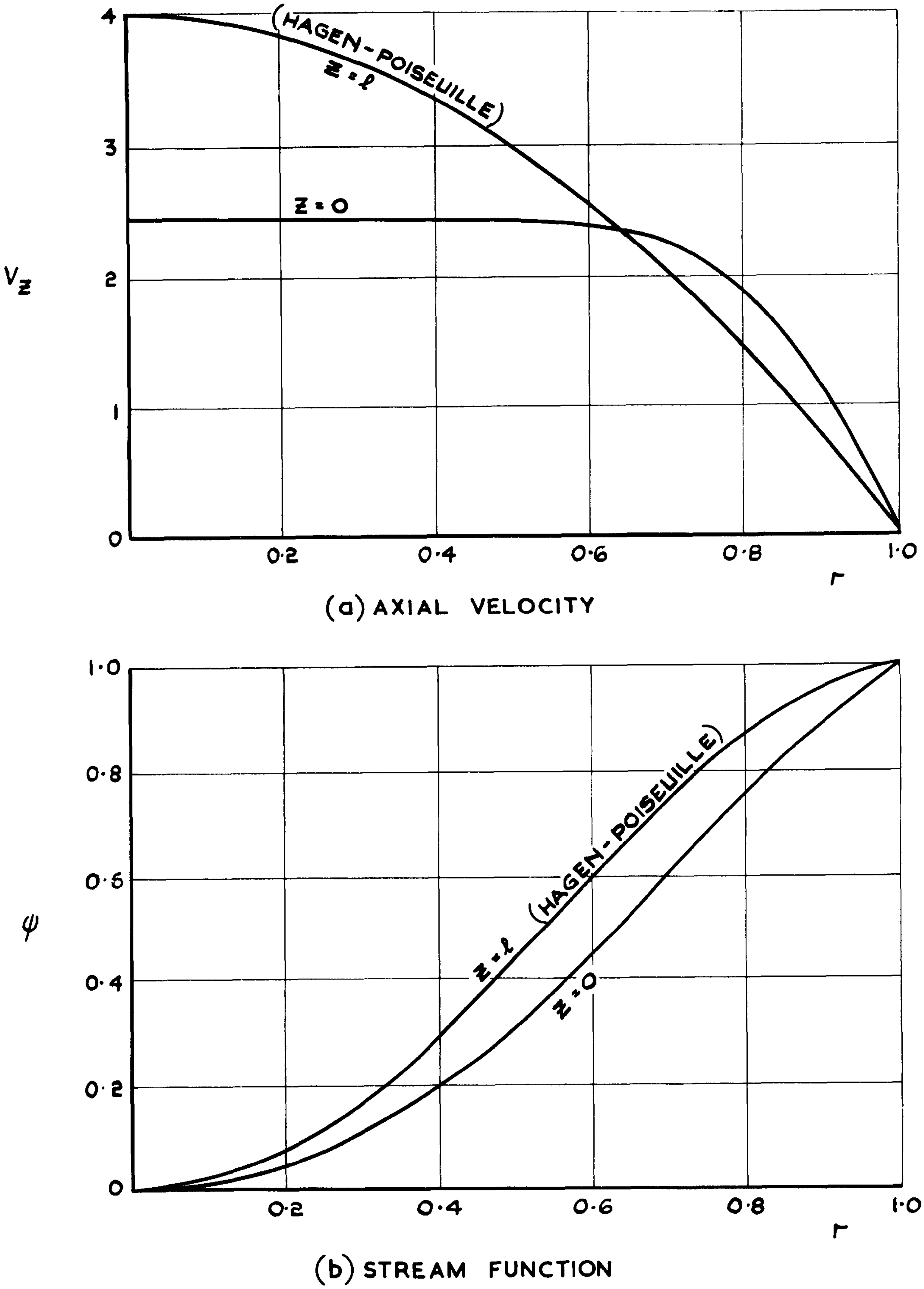

FIG. 1. CONDITIONS AT Z=O & Z=ℓ

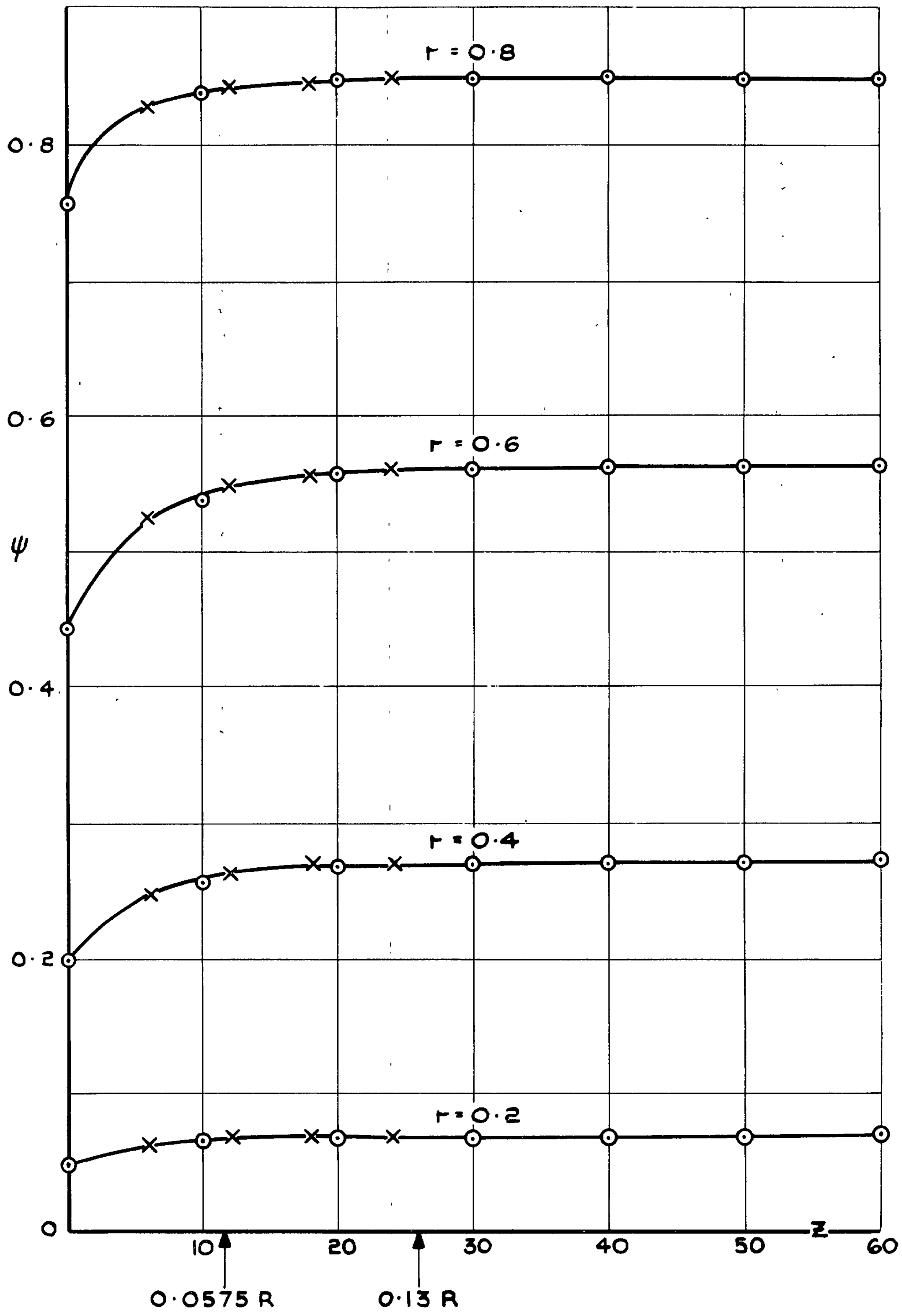

FIG. 2. RESULTS OF CALCULATIONS WITH LARGE Δz

($\times : \Delta z = 6$, $\odot : \Delta z = 10$)

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