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An Experimental Check on the Theoretical Relationship between the Spectral Density and the Probability Distribution of Crossings for a Stationary Random Process with Gaussian Distribution, using Data Obtained in Measurements of Aircraft Response to Turbulent Air

by

J. Burnham

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AN EXPERIMENTAL CHECK ON THE THEORETICAL RELATIONSHIP BETWEEN THE SPECTRAL DENSITY AND THE PROBABILITY DISTRIBUTION OF CROSSINGS FOR A STATIONARY RANDOM PROCESS WITH GAUSSIAN DISTRIBUTION, USING DATA OBTAINED IN MEASUREMENTS OF AIRCRAFT RESPONSE TO TURBULENT AIR

by

J. Burnham

SUMMARY

The theoretical relationship between the spectral density function and the probability distribution of crossings for a stationary random function with Gaussian distribution has been used to calculate these distributions from spectra measured experimentally. The distributions so obtained are shown to compare well with distributions measured directly from the experimental data, the measured distributions being approximately Gaussian and the experimental data behaving approximately in the manner of a stationary random process.

Replaces R.A.E. Technical Note No. Aero 2917 - A.R.C. 25667.

CONTENTS

ILLUSTRATIONS

1 INTRODUCTION

In studies of the response of physical systems to random disturbances, a knowledge of the probability distribution of crossings of different output levels, and of the effects on this distribution of changing the system dynamics, is often required. The theoretical relationship between this distribution and the spectral density function was derived by Rice[1] for a stationary process with Gaussian distribution. Both of these quantities can be measured experimentally although for finite samples only, and since the theory considers infinite samples of stationary Gaussian processes, it is of interest to discover how well Rice's relationship accords with experiment, in which the process may be only approximately Gaussian, and for which the meaning of the theoretical concept of stationarity is not entirely clear.

One aeronautical problem in which this relation is of interest is that of the fatigue loading of an aircraft flying through turbulent air where the statistical quantity concerned is essentially the number of crossings of different normal acceleration levels. It should be noted however that the distribution of crossings considered is not identical with either the distribution of acceleration peaks[1] or the distribution obtained from readings of a fatigue load meter[2,3]. The results of the present study are likely to be of interest in many engineering applications of random process theory. The data used here are taken from measurements of the behaviour of an aircraft in turbulent air.

2 THEORY

The relation between the spectral density, $\phi(f)$, of a stationary random function, $a(t)$, and the average frequency, $N_{n-\bar{n}}$, with which $a(t)$ crosses the value + or - $(n-\bar{n})$ with positive (or negative) slope, $\bar{n}$ being the mean value of $a(t)$, was obtained by Rice[1] and is given by

$$N_{n-\bar{n}} = N_o \exp\left\{-(n-\bar{n})^2/2\sigma^2\right\} \tag{1}$$

where N_o is the average frequency of crossings of the mean level with positive (or negative) slope, and

$$N_o^2 = \frac{\int_0^\infty f^2\phi(f)df}{\int_0^\infty \phi(f)df} . \tag{2}$$

σ is the r.m.s. value of $a(t)$ relative to its mean level, so that

$$\sigma^2 = \int_0^\infty \phi(f)df \, . \tag{3}$$

The numerator of the expression for N_o^2 is the variance of the rate of change of a(t). N_o is therefore the ratio of the r.m.s. of the rate of change of a(t) to the r.m.s. of a(t) itself.

The effect of high frequencies on the distribution is of interest since some upper limit must exist in any experimental work. To determine the effects of the higher frequency components of the measured spectra on the resulting calculated distributions, these have been calculated neglecting the contribution of frequencies greater than a frequency which will be called the cut-off frequency, f_c, for various values of f_c.

The contribution of all frequencies from zero to f_c to the average frequency of crossings of the mean level, $N_o(f_c)$, is given by

$$N_o(f_c)^2 = \frac{\int_0^{f_c} f^2\phi(f)df}{\int_0^{f_c} \phi(f)df} \, . \tag{4}$$

The denominator of this expression is the contribution of all frequencies between zero and f_c to the variance of a(t), i.e.

$$\sigma(f_c)^2 = \int_0^{f_c} \phi(f)df \, . \tag{5}$$

The contribution of all frequencies between zero and f_c to the distribution of crossings is given by $N_{n-\bar{n}}(f_c)$ where

$$N_{n-\bar{n}}(f_c) = N_o(f_c) \exp\left\{-(n-\bar{n})^2/2\sigma(f_c)^2\right\} . \tag{6}$$

3 THE APPLICATION OF THE CONCEPT OF STATIONARITY TO EXPERIMENTAL WORK

The derivation of Rice's formula (equations (1), (2) and (3)) assumes that the random function is stationary. A function is said to be stationary if the statistical quantities which describe it, and which are based on the behaviour of the function starting from some time, t_o, and continuing for all subsequent time, are independent of the time origin t_o. Alternatively, stationarity can be defined on the basis of an infinite number of samples of finite length, but the following remarks still apply. Stationarity is thus a property of the statistics which describe a random function and not directly a property of the random function itself. This may be of some importance in applying theoretical results derived under the assumption of stationarity to engineering problems, since it cannot be assumed that if for some statistical quantity the agreement between theory and experiment is good, then the agreement will necessarily be equally good for other statistical quantities. Also, whether a random process can be considered, from an engineering point of view, to be stationary, may depend on the frequency range which it is required to cover.

The concept of stationarity is very useful in theoretical work and little progress would have been made without it. It is, however, difficult to apply it to an experimental result, since this must be based on a finite amount of data. The concept does not involve the way in which a statistical quantity based on a sample extending from time t_o to time T converges as T tends to infinity, only the existence of a limit which is independent of t_o. Thus, from a practical point of view, before theoretical relations derived under the assumption of stationarity can be relied on, it is necessary to carry out two kinds of check. The first is that the value of any statistical quantity of interest can be defined within the required accuracy by a finite sample of practicable length, which can be done by considering samples of different lengths and/or samples taken at different times. The second is, that if theoretical relations between different statistical quantities are of interest, these relations can be checked by experimental measurements. The object of the work described here is to carry out the second kind of check. The first was carried out in Ref.4 on the spectral densities used here and these were shown to be reasonably consistent when based on different samples.

It should also be remembered that the finite length of a sample acts, in the calculation of spectral density, like a filter. A knowledge that the process is stationary, or at least a knowledge of how the process behaves during the time that it has not been measured, is necessary to determine the properties of this filter. This question is dealt with fully by Blackman and Tukey[5] and will not be discussed here.

4 THE MEASURED SPECTRAL DENSITIES USED

The measured spectral densities used are taken from Ref.3 and were obtained from acceleration measurements on an aeroplane flying through turbulent air. Three spectra of accelerations at two different points on the aircraft are considered. Two are based on different samples of acceleration at the aircraft's centre of gravity, which give spectra which decay rapidly with frequency over almost the whole frequency range, and one of the acceleration at the wing tip, which gives a spectrum with a number of sharp peaks. The C.G. acceleration data are those for flights 7 run 2 and 18 run 1 of Ref.4 and will be referred to as samples A_1 and A_2, while the wing tip acceleration data are for flight 18 run 1 and will be referred to as sample B.

The spectra given in Ref.4 were obtained as described in Refs.6 and 7, the method being basically that described in Ref.5 except that the effective bandwidth of the filter used (Tukey's No.41 [8], R.A.E. Programme Deuce 156 [7]) is wider than those recommended in Ref.5. This makes no practical difference to the distributions obtained up to about half the Nyquist frequency, i.e. up to a quarter of the sampling frequency. At frequencies near the Nyquist frequency the methods of Refs.5, 6 and 7 all overestimate the spectral density by factors of two or more. The spectra from Ref.4 used here have been corrected, for frequencies between half the Nyquist frequency and the Nyquist frequency, to remove this effect. The spectra used are shown in Fig.1, the highest frequency shown being the Nyquist frequency. The C.G. acceleration spectra, samples A_1 and A_2, are from 80 second samples and the wing tip spectrum, sample B, from a 40 second sample, the former being read from continuous trace records at 0·02 second intervals and the latter at 0·01 second intervals.

5 THE CALCULATION OF THE PROBABILITY DISTRIBUTIONS FROM THE MEASURED SPECTRA

The measured spectra are not accurate at very low frequencies due to the sample length used and the spectral densities at these frequencies make a large contribution to the r.m.s., particularly for samples A. It would therefore be unwise to apply equation (5) directly to the measured spectra. However, the integral of the spectral density from zero to infinity* is the variance σ^2, of the quantity, and this may be obtained directly from the record as the average of the square of the difference between each data point and the mean.

The value of the contribution of frequencies from zero to the cut-off frequency, f_c, may thus be obtained more accurately by using the expression

$$\sigma(f_c)^2 = \sigma^2 - \int_{f_c}^{f_N} \phi(f)df \qquad (7)$$

where σ^2 is the variance, calculated as described above, and f_N is the Nyquist frequency. The contributions of frequencies from zero to f_c to the variance, calculated from the spectra shown in Fig.1 using equation (7), are shown in Fig.2. The validity, in this case, of replacing the infinite upper limit of integration by f_N in deriving equation (7) can be seen by the rapid convergence of $\sigma(f_c)^2$ as f_c increases.

*Care is necessary in choosing the sampling frequency, since if significant energy exists at frequencies greater than the Nyquist frequency (half the sampling frequency) quite large errors can be made, see Ref.5. If significant energy does not exist there, infinity can be taken to mean a very high frequency and, for practical purposes, the integration need only extent to the Nyquist frequency.

The numerator of the expression for $N_o(f_c)^2$, (equation (4)), $\int_0^{f_c} f^2\phi(f)df$, is very much more dependent on the high frequency components and its values, calculated from the three spectra shown in Fig.1, are shown in Fig.3, plotted against the cut-off frequency, f_c.

The values obtained, using equations (4), (5) and (7), of the contributions of frequencies from zero to the cut-off frequency to the zero crossing frequency, $N_o(f_c)$, are shown in Fig.4. If the spectra used had been calculated according to the methods of Refs.5, 6 and 7 over the whole frequency range, the values of $N_o(f_c)$ would not have been noticeably affected up to about half the Nyquist frequency, and increasingly affected as the Nyquist frequency is approached. The values obtained at the Nyquist frequency would be increased by about 10% for samples A_1 and A_2 and 20% for sample B.

The distributions of crossings of acceleration levels obtained from equation (6) are shown in Fig.5 for several cut-off frequencies.

6 THE MEASURED PROBABILITY DISTRIBUTIONS

6.1 Method of measurement

The measured average numbers of crossings of several acceleration levels for the three samples considered are also shown in Fig.5. These were determined from continuous trace records of the measured quantities, the same records which were read at discrete time intervals to give the spectra. A transparent overlay with a number of parallel lines, ruled approximately 4 mm apart (the actual distance being equivalent to 0·125g) for the records of samples A_1 and A_2 and 2 mm apart for the record of sample B, was placed over the records and the number of crossings of each line with positive slope was counted. Sections of the records used are shown in Fig.6.

Although the mean value of the acceleration was known for each record used, (it having been calculated as part of the computing procedure to determine the spectra), this information was not used in determining the distributions since it is frequently desirable to apply the technique of counting crossings to records the mean of which is not accurately known. The method used was to plot the distribution and determine, by eye, the value of the 'mean' which gave the best straight line when the distribution was plotted as increment squared against log frequency, all frequencies corresponding to fewer than 10 crossings in the sample length being neglected. This, in effect, is a simple-minded way of finding the Gaussian distribution which is the best fit to the data.

6.2 A further examination of one measured distribution

The distribution obtained from sample A_1 shows poorest agreement with the distribution calculated from the measured spectrum so, to check the accuracy of the method of counting used and to look for possible reasons for the poorer agreement, this distribution has been examined in more detail. For this purpose the digital readings, taken from the record (at 0·02 second intervals) in order to compute the spectra, were replotted on a much larger scale than the original record and the resulting points joined by straight lines. The numbers of crossings of several levels by the resulting curve were then determined and these are shown in Fig.7a, plotted against the difference between the level and the calculated mean for all the data points. Since the points from the record were plotted from digital readings, some peaks were exactly on the levels at which crossings were counted. The frequencies obtained at some of the levels, when peaks exactly on those levels were counted as crossings, are also shown on Fig.7a. The distribution originally obtained is also shown in this Figure and in this case the mean used is that determined from the symmetry of the distribution, as described in Section 6.1 above. The agreement between the frequencies obtained from the original record and those from the digital readings is good except for the point at $n-\bar{n}$ = 0·275 and this point is farthest from the Gaussian distribution which the others fit fairly well. The data from Fig.7a are replotted in Fig.7b on an increment squared scale and distributions calculated from the measured spectrum at two cut-off frequencies are shown for comparison.

7 DISCUSSION

The slope of a distribution, plotted as in Fig.5, is proportional to the variance (square of r.m.s.) and the agreement between the slopes measured from the records and those calculated from the spectra is quite good. If lines had been drawn through the points of the measured distributions to determine the r.m.s., the values so obtained would have differed little from the calculated values, although the point on the far right in sample A_1 is quite a long way from the line. This point does not fit well with the more detailed count, see Fig.7b, which is in fairly good agreement with the calculated slope.

The zero crossing frequency is strongly dependent on the high frequency content of the time histories and it is necessary to enquire into possible errors in the measured spectral densities at high frequencies. A discussion of the effects of errors arising from inaccurate reading of the records is given in Ref. 9, where it was found that they caused an approximately uniform error over the frequency range (white noise) and that the r.m.s. reading error was 0·003 in.; this error is assumed to include also rounding off errors in the digitising process. Although the magnitude (r.m.s.) of the error may well be different in the present case since different reading equipment was used, it seems likely that the errors in the present case are also white noise. If this is so, and if the r.m.s. reading error of Ref. 9 is assumed, the spectral density at the Nyquist frequencies would, in the present spectra, be overestimated by about 10%. For the whole of the spectral density at the Nyquist frequencies to be due to reading errors, these would need to have an r.m.s. value of about 0·01 in. on the record, which was read via a projector which magnified (linearly) about three times. Although an r.m.s. error of 0·01 in. in the present case would probably be an overestimate, the real r.m.s. error is not likely to be much less than 0·005 in. It is therefore reasonable to suppose that the spectral densities at the Nyquist frequencies are overestimated, in the present spectra, by about 30% due to reading errors.

Errors in the spectral density at high frequencies also arise due to aliasing effects[5], the spectral density from frequencies higher than the Nyquist frequency appearing spuriously in the calculated spectra. The sampling frequency used in reading the records was designed to cut aliasing effects down to a practicable minimum for frequencies up to 15 c/s for samples A_1 and A_2 and 30 c/s for sample B, since these were the highest frequencies of interest for the purpose for which the spectra were intended (described in Ref.4). Whilst aliasing effects are considered to be negligible up to these frequencies, this may well not be the case when the spectra are calculated, as they are here, right up to the Nyquist frequency.

The agreement between the measured values of the zero crossing frequency, N_0, and those calculated from the measured spectra, see Figs.5 and 7, appears to be quite good, and adequate from an engineering point of view. The above argument that the calculated value of N_0 at $f_c = 25$ c/s in samples A is too large due to reading errors seems to be borne out by the values measured from the records. The value for the sample A_1 is overestimated by the calculation from the measured spectrum to a greater extent than the value for sample A_2. This may be simply because the spectral density for sample A_1 is the more inaccurate due to reading errors etc. However, since the measured distributions are obtained from finite samples, they would not be expected to be exactly Gaussian even if they became better approximations to the Gaussian as the length of the sample was increased. The differences in agreement for samples A_1 and A_2 may, in part, be due to the former being a poorer approximation to a Gaussian distribution.

In the case of sample B, the calculated value of N_0 for $f_c = 50$ c/s is slightly less than the measured value. Due to reading errors the value for $f_c = 30$ c/s should be nearer to the value which would be obtained from a correct spectrum, if this was zero at frequencies greater than 50 c/s. However, if some of the spectral density between 30 c/s and 50 c/s in sample B was due to aliasing effects, this could lead to an underestimate of N_0, cancelling out the overestimate due to reading errors. To this extent, the apparently almost exact agreement between the measured value of N_0 for sample B and the calculated value for $f_c = 50$ c/s may be spurious.

From an engineering point of view the above arguments are of only academic interest since in the present case the agreement between the measured and calculated zero crossing frequencies shows that the agreement between the theory of Ref.1 and experimental data is adequate for practical purposes which can be envisaged at the moment. There is no reason to believe that the agreement would not be equally good for other random processes which give approximately Gaussian distributions of crossings and which behave approximately in the manner of stationary random processes. It is, however, clear that in the use of the relations derived in Ref.1 to calculate the zero crossing frequency from measured or theoretically calculated spectra, great care must be taken in ensuring that the high frequency components of the spectra are sufficiently accurate.

8 CONCLUSIONS

The theoretical relation between the probability distribution of crossings and the spectral density of a stationary random process with Gaussian distribution derived by Rice[1] has been applied to spectral densities obtained in measurements of the response of an aeroplane to turbulent air. The resulting probability distributions of crossings of acceleration levels have been compared with distributions obtained directly from records of the measured quantities by counting how often the traces crossed given levels. The distributions obtained by the two methods are in reasonable agreement. If the relation derived in Ref.1 is to be applied to spectral densities either obtained from experimental data or calculated theoretically, great care is necessary to ensure that the high frequency components of the spectra are known with sufficient accuracy.

It is concluded that, for random processes in which the distribution of crossings is approximately Gaussian and which behave approximately in the manner of stationary random processes, the theoretical relation derived in Ref.1 can be considered, for engineering purposes, to be valid. It must, however, be remembered that this relation applies strictly only to the distribution of crossings and not distributions of other quantities, such as peak values and the readings of fatigue load meters, which are frequently used for similar purposes.

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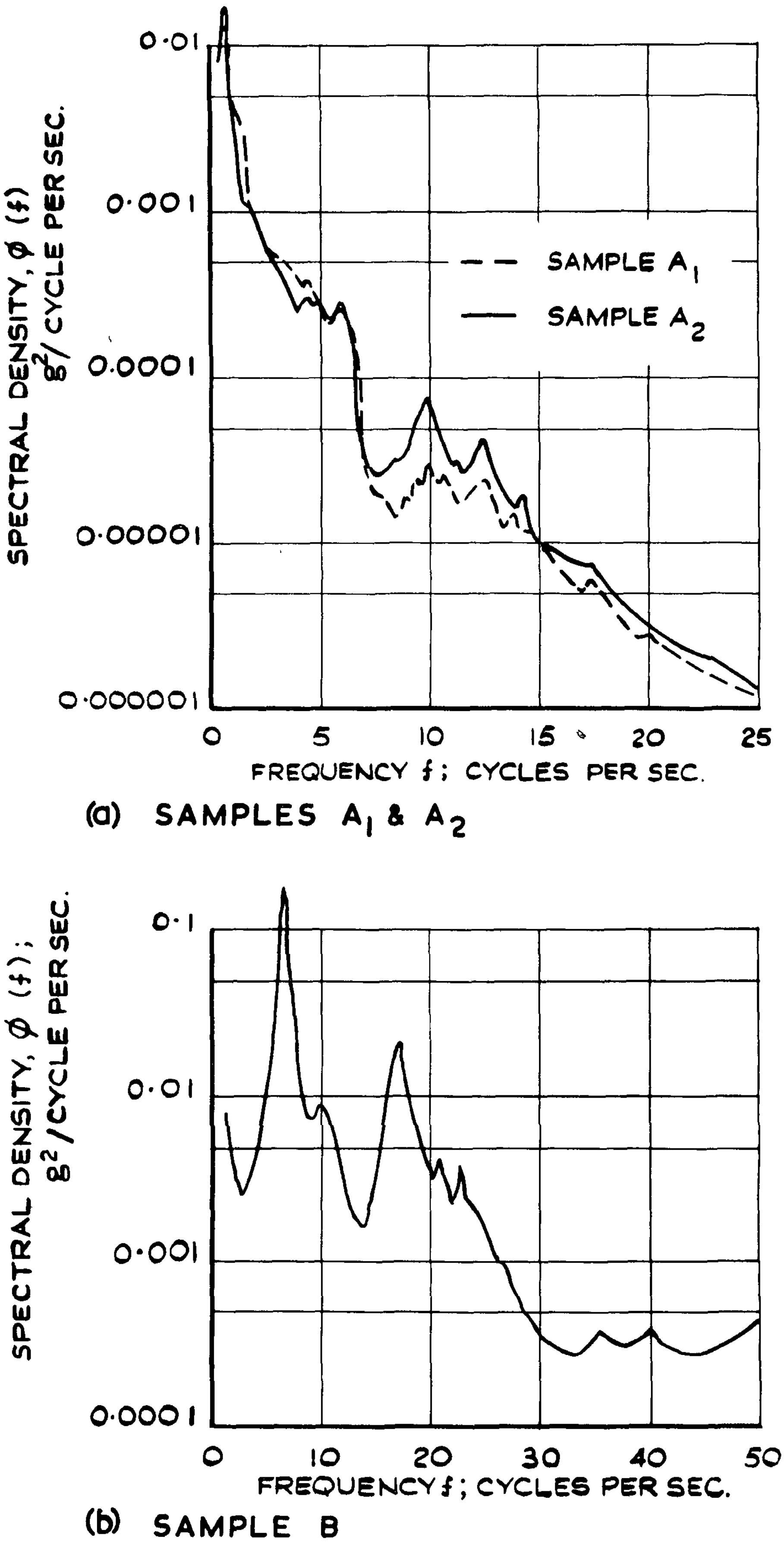
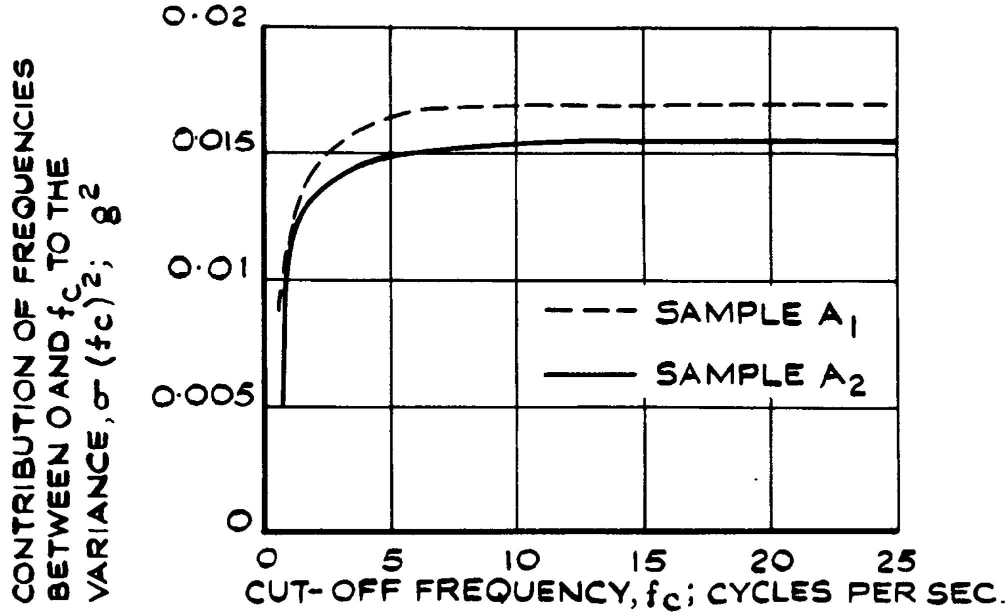
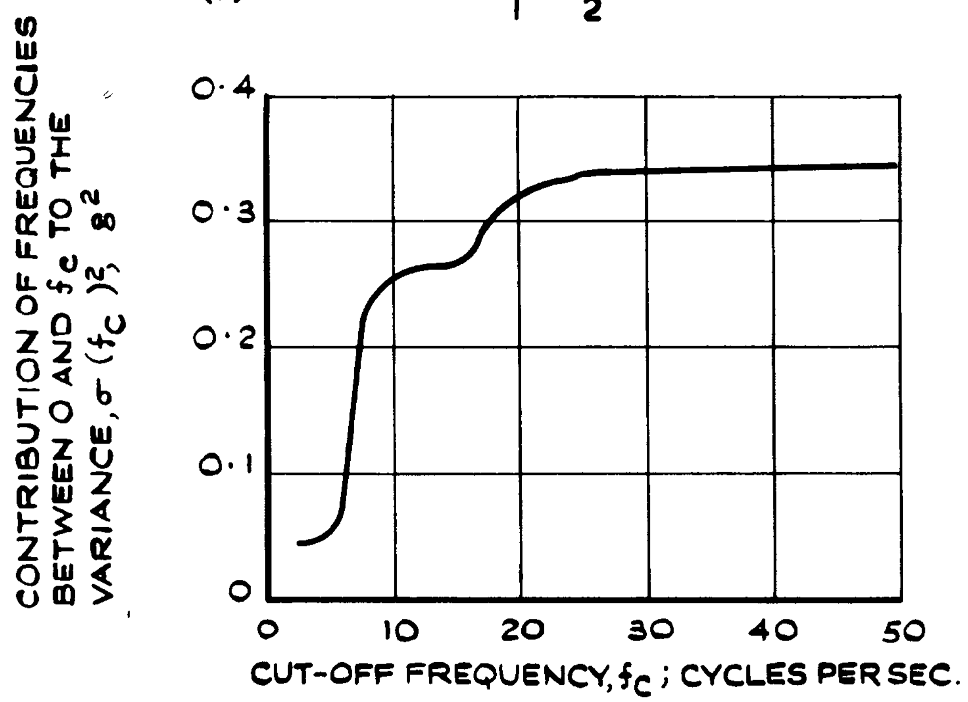


FIG.1. THE MEASURED SPECTRAL DENSITIES.



(a) SAMPLES A_1 A_2



(b) SAMPLE B

FIG. 2. THE CONTRIBUTION OF FREQUENCIES BETWEEN ZERO AND CUT-OFF FREQUENCY TO THE VARIANCE.

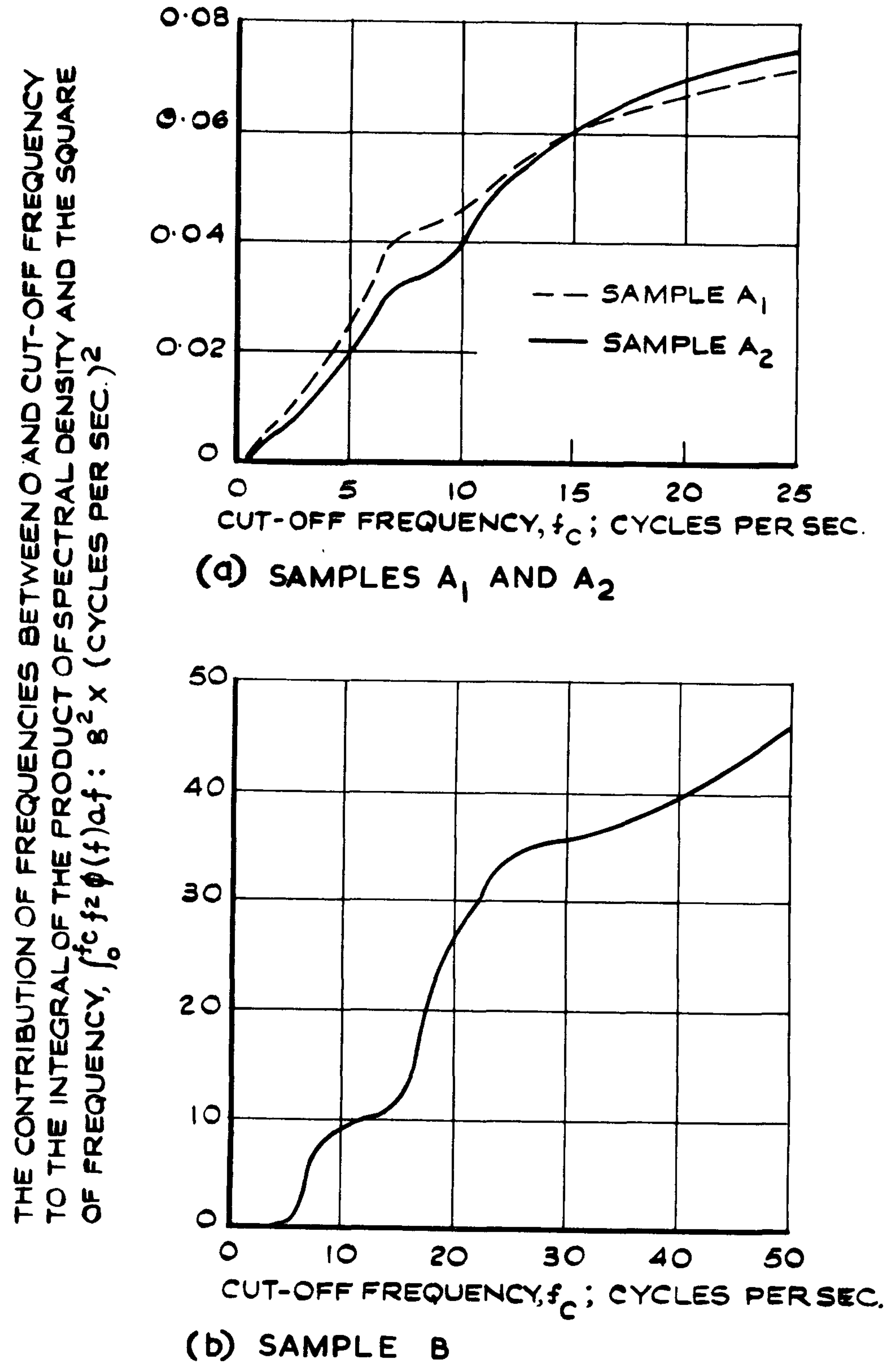
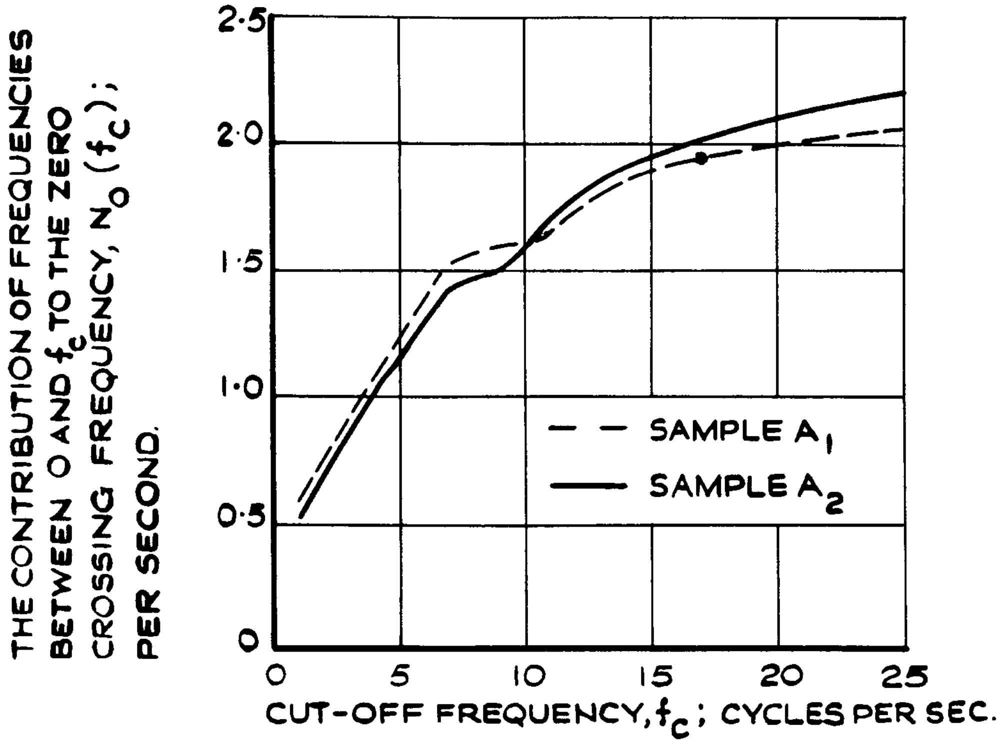
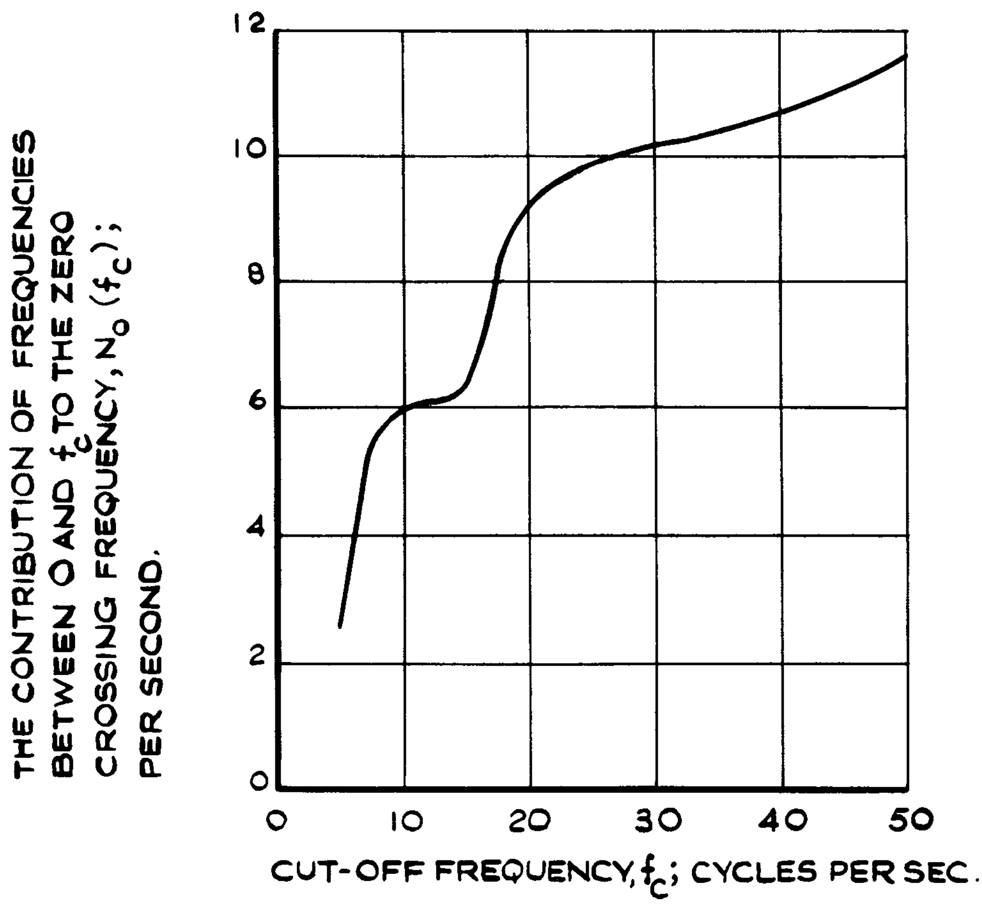


FIG. 3. THE CONTRIBUTION OF FREQUENCIES BETWEEN ZERO & CUT-OFF FREQUENCY TO THE INTEGRAL OF THE PRODUCT OF SPECTRAL DENSITY & THE SQUARE OF FREQUENCY.



(a) SAMPLES A_1 AND A_2



(b) SAMPLE B

FIG. 4. THE CONTRIBUTION OF FREQUENCIES BETWEEN ZERO & CUT-OFF FREQUENCY TO THE ZERO CROSSING FREQUENCY.

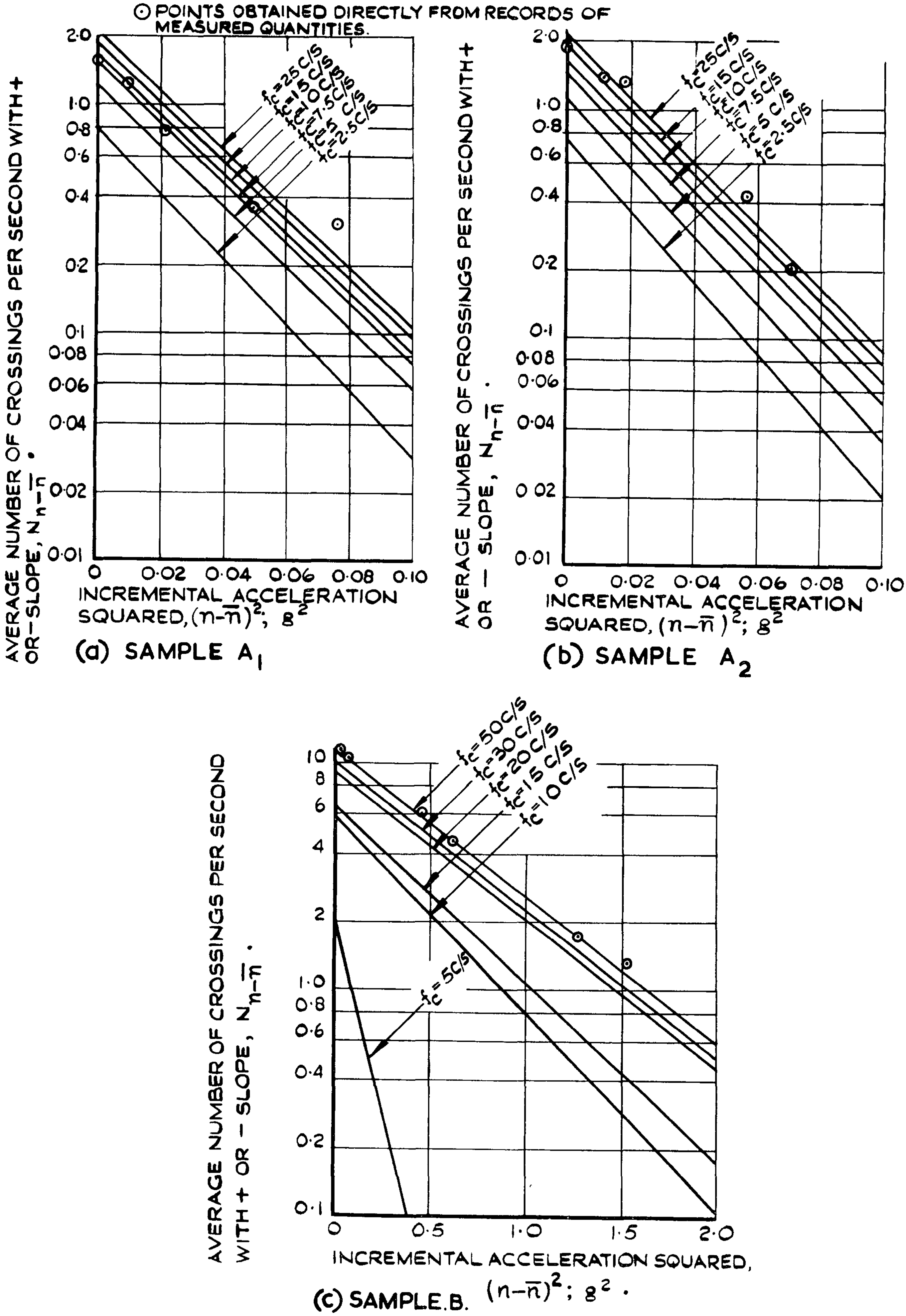
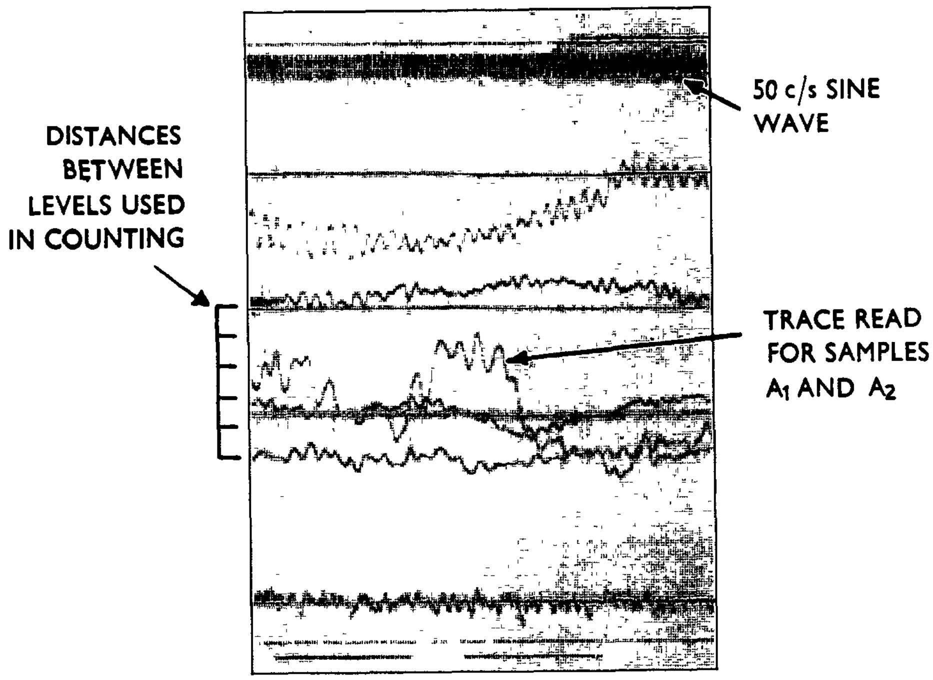


FIG.5. PROBABILITY DISTRIBUTIONS OF CROSSINGS CALCULATED FROM THE MEASURED SPECTRA FOR SEVERAL CUT-OFF FREQUENCIES, f_c, & MEASURED PROBABILITY DISTRIBUTIONS.



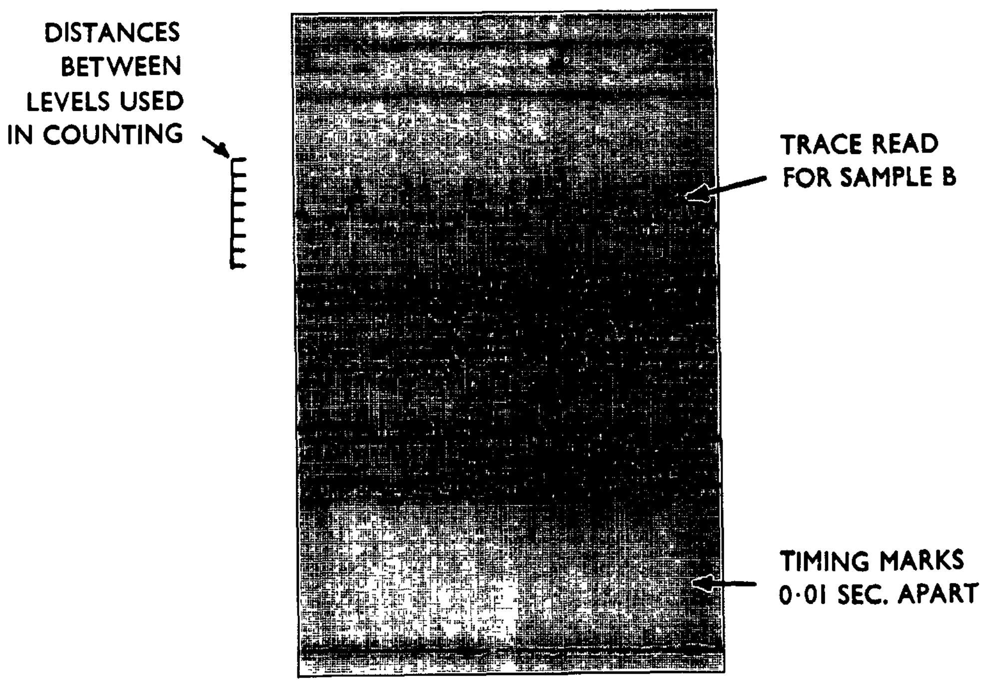


FIG.6. EXAMPLES OF THE TRACE RECORDS FROM WHICH THE SAMPLES WERE OBTAINED (ACTUAL SIZE)

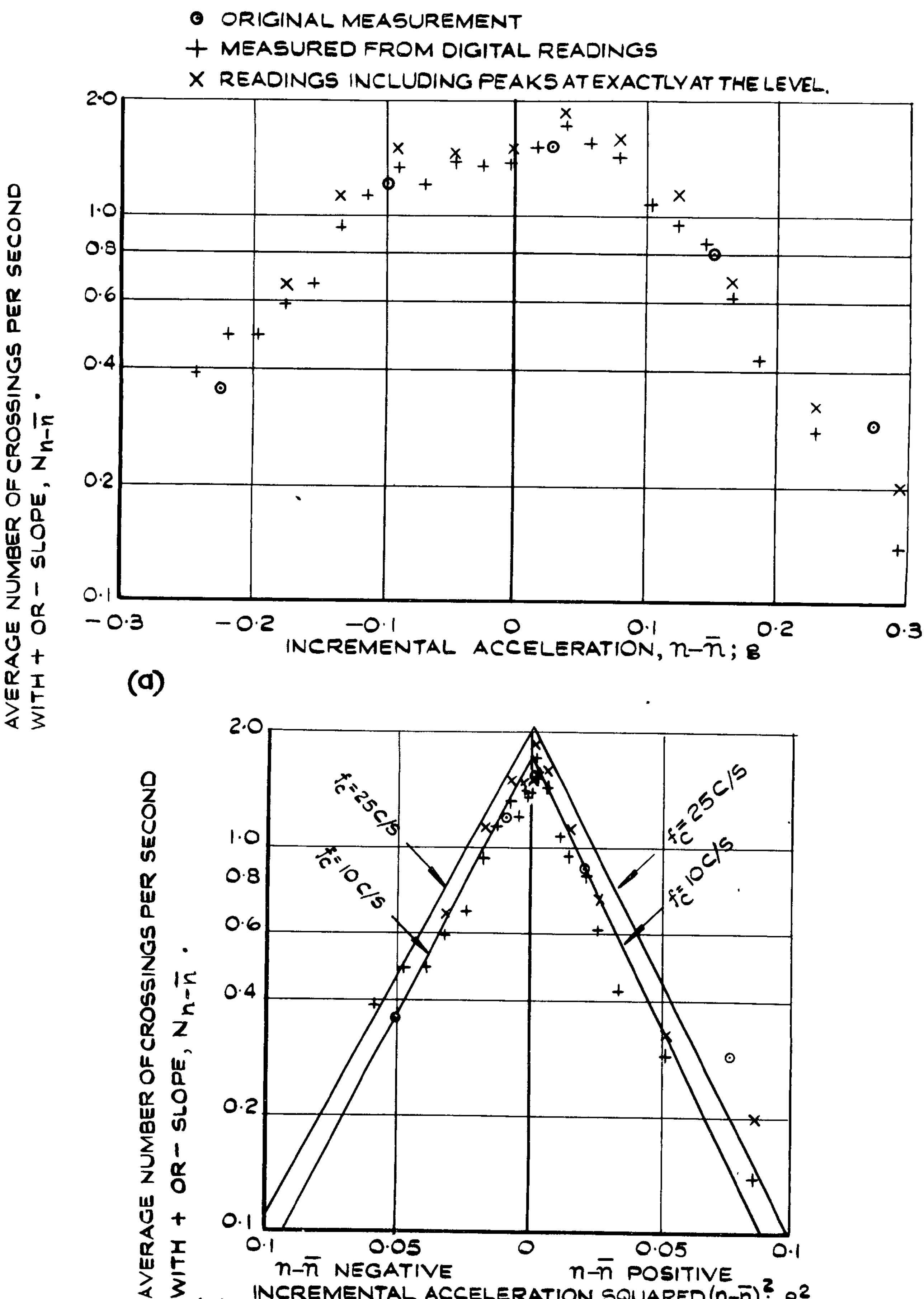


FIG.7. THE MEASURED DISTRIBUTION FOR SAMPLE A_1 IN GREATER DETAIL.

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