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Approximate Two-dimensional Aerofoil Theory

Part IV. The Design of Centre Lines

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Approximate Two-Dimensional Aerofoil Theory. Part IV. The Design of Centre Lines

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Summary

Centre lines may be designed from a knowledge of the quantity g_i introduced in Part II[a], eqn.(54). $4g_i$ may be loosely said to give the approximate chordwise distribution of the normal force coefficient $(p_\ell - p_u)/(\frac{1}{2}\rho U^2)$ (but loadings should, in fact, be calculated by Approximation III).

In terms of g_i, the centre line ordinate y_c is given by

$$y_c = \frac{x(1-x)}{\pi}\int_0^1 \frac{G_i(\xi) - \xi G_i(1)}{\xi(1-\xi)(\xi-x)}\,d\xi - \frac{1}{4}A_1\{x\,\ell n\,x + (1-x)\,\ell n\,(1-x)\},^*$$

where the principal value of the integral is to be taken,

$$G_i(x) = \int_0^x g_i\,dx,$$

and

$$A_1 = \frac{4}{\pi}G_i(1).$$

With

$$A_0 = \frac{1}{\pi}\int_0^1 \frac{G_i(\xi) - \xi G_i(1)}{\xi(1-\xi)}\,d\xi,$$

$$\frac{dy_c}{dx} - A_0 = \frac{1}{\pi}\int_0^1 \frac{g_i(\xi)}{\xi - x}\,d\xi,$$

where again the principal value of the integral is to be taken. Also

$$\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{L\,opt} = \pi A = 4\int_0^1 g_i\,dx,$$

α_{opt}/

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*ℓn is used for $\log_e$.

$$\alpha_{opt} = A_0 + \frac{1}{2}\left(\frac{2\pi - a_0}{2\pi + a_0}\right) A_1,$$

$$C_{M_0} = -\int_0^1 (4x - 1)\ g_i(x)\ dx,$$

$$\beta = \tfrac{1}{2} A_1 - A_0,$$

$$\varepsilon_c(\theta) = 2\left[G_i(x) - \frac{1}{4} A_1\theta\right] \operatorname{cosec}\theta - A_0,$$

$$\varepsilon_c(0) = -\tfrac{1}{2} A_1 - A_0,$$

$$\varepsilon_c'(\theta) = g_i - (A_0 + \varepsilon_c) \cot\theta - \tfrac{1}{2} A_1 \operatorname{cosec}\theta.$$

It is shown how explicit formulae may be easily built up when g_i is a polynomial in x in each of any number of segments of the chord. Explicit formulae are set out for the following cases (with occasional remarks on their use and suggestions for experiment): g_i quadratic in each of three segments, or in each of two segments, or over the whole chord; g_i linear in each of three segments (Figs, 1, 2, 3, 4) or in each of two segments (Fig. 5) or over the whole chord (Fig. 6); g_i constant over the whole chord (Fig. 7); g_i constant for $0 \leqslant x \leqslant X$ and decreasing linearly to zero for $X \leqslant x \leqslant 1$ (Fig. 8); g_i constant for $0 \leqslant x \leqslant X$ and parabolic for $X \leqslant x \leqslant 1$, with $g_i(1) = g_i'(1) = 0$ (Fig. 10); g_i discontinuous and constant in each of two segments (Fig. 11).

It is shown how, when the graph of g_i against x is composed of straight lines, the solution may always be built up from certain 'basic' solutions, of which the most important are those of Figs. 7 and 8, and, when g_i is discontinuous, of Fig. 14.

We have occasion to mention a few trite and obvious principles underlying the choice of a particular design of centre line in connection with Figs. 8 and 10 and (for 'suction' aerofoils) Fig. 11.

Extensive tables are given for the centre lines designed according to Figs. 7 and 8.

Introduction/

1. Introduction

In Part III[3] we remarked that if we know Approximation I to the velocity q on both the upper and lower surfaces of an aerofoil in a uniform unlimited stream U, then we know* g_s and $g_c + g_L$ separately, and that we can design the fairing from a knowledge of g_s and the centre line from a knowledge of $g_c + g_L$; the ability to design the aerofoil shape from Approximation I to the velocity is sufficient for many purposes. In Part III we also considered briefly the problem of obtaining values of g_s and $g_c + g_L$ when the exact pressure or velocity distribution is specified, and we then considered in detail the design of the fairing from a knowledge of g_s. In this report we shall be concerned with the design of the centre line; this design is actually worked out from a knowledge of the quantity g_i introduced in Part II, eqn. (54), and related to $g_c + g_L$ by the equation

$$g_c + g_L = g_i + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left(C_L - C_{Lopt}\right) \cot \tfrac{1}{2}\theta - \frac{1}{2}\left(\frac{1}{a_o} - \frac{1}{2\pi}\right) C_L \tan \tfrac{1}{2}\theta, \qquad \ldots(1)$$

where

$$\left.\begin{aligned} C_L &= a_o(\alpha + \beta) \\ \text{and} \quad x &= \tfrac{1}{2}(1 - \cos\theta) = \sin^2 \tfrac{1}{2}\theta^{+}. \end{aligned}\right\} \qquad \ldots(2)$$

It is therefore advisable to consider, by way of preface, the determination of g_i. Since the pressure distribution is specified, C_L is known; we require also C_{Lopt} and a_o. A rounded trailing edge is a stagnation point when $a_o = 2\pi$ (more accurately $2\pi e^{C_o}$),

and/

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*We use the same notation as in Parts I, II and III.[1,2,3]

+More generally, if the (C_L, α) curve is not a straight line, we replace $(C_L - C_{Lopt})/a_o$ in (1) by $\alpha - \alpha_{opt}$, and C_L/a_o by $\alpha + \beta$, where C_{Lopt} and α_{opt} are to be found as the coordinates of the point on the (C_L, α) curve where

$$\alpha_{opt} + \beta + C_{Lopt}/2\pi = A_1,$$

A_1 being the coefficient of $\cos\theta$ in the cosine Fourier series for dy_c/dx (as a function of θ) in the range $0 \leqslant \theta \leqslant \pi$.

and the leading edge is a stagnation point when $C_L = C_{Lopt}$ approximately. When the trailing and leading edges are not stagnation points the values of the velocity at those points determine a_o and C_{Lopt}. (We must suppose either that a_o is specified, or that we are given the correct theoretical pressure for the relevant a_o right up to, and including, the trailing edge.) If C_L is small we may consider the given velocity distribution as Approximation II; for large values of C_L it is necessary, and in any case it is more accurate, to use Approximation III. (Part II, eqn.(67)). The velocities at the leading and trailing edges are briefly considered in Appendix I, where it is shown that they may be expressed in terms of C_L, g_s, a_o and C_{Lopt}, so an estimate of Approximation I carries with it knowledge not only of g_s and of $g_c + g_L$, but also of a_o and C_{Lopt}, and therefore of g_i.

It should be noted that the final value of g_i must be free from singularities at the leading and trailing edges. In fact

$$\left.\begin{aligned} g_i &= \sum_{n=1}^{\infty} A_n \sin n\theta \\ \text{where} \qquad & \\ \frac{dy_c}{dx} &= \sum_{n=0}^{\infty} A_n \cos n\theta, \end{aligned}\right\} \qquad \ldots(3)$$

so g_i is zero at the leading and trailing edges if dy_c/dx is free from singularities.

On the linear theory the non-dimensional normal force distribution is given by

$$\frac{p_\ell - p_u}{\frac{1}{2}\rho U^2} = \left(\frac{q_u}{U}\right)^2 - \left(\frac{q_\ell}{U}\right)^2 = 4(g_c + g_L), \qquad \ldots(4)$$

where the suffixes u and ℓ refer to the upper and lower surfaces, respectively. For $C_L = C_{Lopt}$ and $a_o = 2\pi$, this normal force distribution is therefore given by

$$\frac{p_\ell - p_u}{\frac{1}{2}\rho U^2} = 4g_i, \qquad \ldots(5)$$

so $4g_i$ may be loosely said to give the approximate chordwise loading distribution at the optimum or design C_L. But loadings should, in fact, be calculated by Approximation III.

2. The Quantities to be Calculated

From a knowledge of g_i we now wish to calculate the centre-line ordinate y_c, also the no-lift angle $-\beta$ and the moment coefficient C_{M_0} at zero lift. In terms of the coefficients A_n, β (which is the value of ε_c at $\theta = \pi$) and $\varepsilon_c(0)$ are given by

β/

$$\beta = \tfrac{1}{2} A_1 - A_0, \qquad \varepsilon_c(0) = -\tfrac{1}{2} A_1 - A_0, \qquad \ldots(6)$$

and C_{M_0} by

$$C_{M_0} = \frac{1}{4} \pi(A_2 - A_1), \qquad \ldots(7)$$

so we shall calculate A_0, A_1, A_2. The 'optimum' lift coefficient is given by

$$\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{L\,opt} = \pi A_1, \qquad \ldots(8)$$

and the 'optimum' incidence by

$$\alpha_{opt} = A_0 + \frac{1}{2}\left(\frac{2\pi - a_0}{2\pi + a_0}\right) A_1. \qquad \ldots(9)$$

When we have completed our approximate design of a cambered aerofoil, we shall probably wish to calculate the velocity distribution for several values of C_L on Approximation III. In some cases, for rough guidance, Approximation II or even Approximation I may be useful. We therefore repeat here the relevant formulae, in suitable forms.

Approximation I:

$$\frac{q}{U} = 1 + g_s + g_c + g_L, \qquad \ldots(10)$$

where g_s was settled in designing the fairing and

$$g_c + g_L = g_i + \frac{1}{2}\left[\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_L - A_1\right] \cot \tfrac{1}{2}\theta - \frac{1}{2}\left(\frac{1}{a_0} - \frac{1}{2\pi}\right) C_L \tan \tfrac{1}{2}\theta. \qquad \ldots(11)$$

Approximation II:

$$\frac{q}{U} = \frac{(1 + \frac{1}{2}C_0^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}} (1 + g_s + g_c + g_L)$$

$$= \frac{(1 + \frac{1}{2}C_0^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}} (1 + g_s + g_i)$$

$$\pm \frac{1 + \frac{1}{2} C_0^2}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}} \left[\frac{C_L}{2\pi} + \frac{C_L}{a_0} \cos\theta - \tfrac{1}{2} A_1(1 + \cos\theta)\right]. \qquad \ldots(12)$$

Approximation/

Approximation III:

$$\frac{q}{U} = \frac{e^{C_0}\ (1+\varepsilon')}{(\psi^2+\sin^2\theta)^{\frac{1}{2}}}\left[\left(1-\frac{C_L^2}{a_o^2}\right)^{\frac{1}{2}} \sin(\theta+\varepsilon-\beta) + \frac{C_L}{a_o}\cos(\theta+\varepsilon-\beta) + \frac{C_L\, e^{-C_0}}{2\pi}\right]. \qquad ...(13)$$

If the suffix u refers to the upper surface and ℓ to the lower surface, then

$$\psi_u = \psi_s + \psi_c, \quad \psi_\ell = \psi_s - \psi_c, \qquad ...(14)$$

$$\varepsilon_u = \varepsilon_c + \varepsilon_s, \quad \varepsilon_\ell = \varepsilon_c - \varepsilon_s, \qquad ...(15)$$

$$\varepsilon'_u = \varepsilon'_s + \varepsilon'_c, \quad \varepsilon'_\ell = \varepsilon'_s - \varepsilon'_c, \qquad ...(16)$$

and it must be remembered that θ is positive on the upper and negative on the lower surface. We showed how to calculate C_o, ψ_s, ε_s, ε'_s in Part III; we must here consider the calculation of ψ_c, ε_c, ε'_c. We find ψ_c at once from y_c, since

$$\psi_c = 2y_c \operatorname{cosec}\theta; \qquad ...(17)$$

but it is more convenient to find ε_c, ε'_c from g_i by means of the equation

$$\varepsilon'_c + (\varepsilon_c - \beta)\cot\theta = g_c = g_i - \tfrac{1}{2}A_1 \cot\tfrac{1}{2}\theta \qquad ...(18)$$

3. <u>General Formulae</u>

The analysis is most conveniently carried out, in the main, in terms of x, as in Part I, and not in terms of θ, as in Parts II and III. Consequently it is most convenient to regard g_i as a function of x, and to consider that $g_i(x)$ is given; when we wish to write g_i as a function of θ we must write $g_i(\sin^2\frac{1}{2}\theta)$ and not $g_i(\theta)$. When it is clear what is intended we shall merely write g_i.

From (3) we see that the Fourier cosine series for $(dy_c/dx) - A_0$, considered as a function of θ, is conjugate to the Fourier sine series for $g_i(\sin^2\frac{1}{2}\theta)$ in $(0, \pi)$, so [from Lemma (6) of Part I (Appendix)]

$$\frac{dy_c}{dx} - A_0 = \frac{1}{\pi} P\int_0^{\pi} \frac{g_i(\sin^2\frac{1}{2}t)\sin t}{\cos\theta - \cos t}\,dt$$

$$= \frac{1}{\pi} P\int_0^{1} \frac{g_i(\xi)}{\xi - x}\,d\xi \qquad ...(19)$$

cf. Ref. 4/

[cf. Ref. 4 and Ref. 8], where P denotes, as usual, that the principal value of the integral is to be taken, and we have made the substitution

$$\xi = \tfrac{1}{2}(1 - \cos t) = \sin^2 \tfrac{1}{2}t. \qquad \ldots(20)$$

The Fourier series in (3) terminate if, and only if, g_i is equal to $x^{\frac{1}{2}}(1 - x)^{\frac{1}{2}}$ multiplied by a polynomial in x over the whole range $0 \leqslant x \leqslant 1$. The Fourier series may be used with advantage if they terminate; numerical methods may be used to evaluate the integral in (19), or use may be made of the analytical results of the following sections. The last method is recommended if it is applicable and the Fourier series do not terminate.

When $(dy_c/dx) - A_0$ has been found, the value of $y_c - A_0x$ at x may be obtained by direct integration between the limits 0 and x; and since $y_c = 0$ at $x = 1$, $-A_0$ is the value of $y_c - A_0x$ at $x = 1$.

In actual examples y_c and A_0 may be calculated in this way, but it is desirable to have available explicit formulae for them. We shall find such formulae after considering the calculation of A_1, A_2, ε_c, ε_c'.

We write

$$G_i(x) = G_i(\tfrac{1}{2}\sin^2\theta) = \tfrac{1}{2}\int_0^\theta g_i \sin\theta \, d\theta = \int_0^x g_i \, dx. \qquad \ldots(21)$$

It follows immediately from (3) that

$$A_1 = \frac{2}{\pi}\int_0^\pi g_i \sin\theta \, d\theta = \frac{4}{\pi} G_i(1), \qquad \ldots(22)$$

and

$$A_2 = \frac{2}{\pi}\int_0^\pi g_i \sin 2\theta \, d\theta = \frac{8}{\pi}\int_0^1 g_i(1 - 2x)dx, \qquad \ldots(23)$$

so from (7) and (8)

$$\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{L\,opt} = 4\int_0^1 g_i \, dx,$$

$$C_{M_0} = -\int_0^1 g_i(x)(4x - 1)\, dx. \qquad \ldots(24)$$

From/

From (18) and (6) the equation for ε_c is

$$\frac{d}{d\theta}(\varepsilon_c \sin\theta) = g_i \sin\theta + \beta\cos\theta - \tfrac{1}{2}A_1(1 + \cos\theta)$$

$$= g_i \sin\theta - A_0 \cos\theta - \tfrac{1}{2}A_1. \qquad \ldots(25)$$

Hence

$$\varepsilon_c \sin\theta = 2G_i - \tfrac{1}{2}A_1\,\theta - A_0 \sin\theta, \qquad \ldots(26)$$

and

$$\varepsilon_c = 2\left[G_i(x) - \frac{1}{4}A_1\,\theta\right]\operatorname{cosec}\theta - A_0. \qquad \ldots(27)$$

The limits of this expression for ε_c as $x \to 0$ and $x \to 1$ yield the values of $\varepsilon_c(0)$ and β given by eqn.(6).

From (18), or by differentiation of (27), we find that

$$\varepsilon_c'(\theta) = g_i - (A_0 + \varepsilon_c)\cot\theta - \tfrac{1}{2}A_1 \operatorname{cosec}\theta$$

$$= g_i - 2\operatorname{cosec}^2\theta\left[G_i(x)\cos\theta - \frac{1}{4}A_1\theta\cos\theta + \frac{1}{4}A_1\sin\theta\right] \qquad \ldots(28)$$

The limits of (28) as $\theta \to 0$ and $\theta \to \pi$ are given by

$$\varepsilon_c'(0) = \tfrac{1}{2}g_i(0), \quad \varepsilon_c'(\pi) = \tfrac{1}{2}g_i(1). \qquad \ldots(29)$$

These results refer to $0 \leqslant \theta \leqslant \pi$; ε_c is even and ε_c' is odd; ε_c' is discontinuous at $\theta = 0$ if $g_i(0)$ is not zero and at $\theta = \pi$ if $g_i(1)$ is not zero. In practice it is probably best to fair off $\varepsilon_c'(\theta)$ to 0 at $\theta = 0$ and $\theta = \pi$.

We now return to the calculation of A_0 and y_c. From Part II, eqns. (18) and (19), if

$$\psi_c = \sum_{n=1}^{\infty} D_n \sin n\theta \qquad \ldots(30)$$

then

$$\varepsilon_c = -\sum_{n=1}^{\infty} D_n \cos n\theta. \qquad \ldots(31)$$

Hence, in the first place

$$\int_0^{\pi} \varepsilon_c(\theta)\, d\theta = 0, \qquad \ldots(32)$$

so/

so from (27),

$$A_0 = \frac{2}{\pi}\int_0^\pi \frac{G_1(\sin^2 \tfrac{1}{2}t) - \tfrac{1}{4} A_1 t}{\sin t}\, dt. \qquad \ldots(33)$$

From Lemma 1, Appendix II,

$$\int_0^\pi \frac{\tfrac{1}{2}\pi A_1(1 - \cos t) - A_1 t}{\sin t}\, dt = 0, \qquad \ldots(34)$$

so

$$A_0 = \frac{2}{\pi}\int_0^\pi \frac{G_1(\sin^2 \tfrac{1}{2}t) - \tfrac{1}{8}\pi A_1(1 - \cos t)}{\sin t}\, dt$$

$$= \frac{1}{\pi}\int_0^1 \frac{G_1(\xi) - \xi G_1(1)}{\xi(1 - \xi)}\, d\xi. \qquad \ldots(35)$$

Secondly, it follows from (30) and (31) (by Lemma 6 of Part I) that

$$\psi_c(\theta) = \frac{\sin\theta}{\pi} P\int_0^\pi \frac{\varepsilon_c(t)}{\cos\theta - \cos t}\, dt.$$

Hence, from (27),

$$\psi_c(\theta) = \frac{2\sin\theta}{\pi} P\int_0^\pi \frac{G_1(\sin^2 \tfrac{1}{2}t) - \tfrac{1}{4} A_1 t}{\sin t(\cos\theta - \cos t)}\, dt, \qquad \ldots(36)$$

since

$$P\int_0^\pi \frac{dt}{\cos\theta - \cos t} = 0. \qquad \ldots(37)$$

Therefore

$$y_c = \tfrac{1}{2}\psi_c \sin\theta = \frac{\sin^2\theta}{\pi} P\int_0^\pi \frac{G_1(\sin^2 \tfrac{1}{2}t) - \tfrac{1}{4} A_1 t}{\sin t(\cos\theta - \cos t)}\, dt. \qquad \ldots(38)$$

From Lemma 2, Appendix II,

$$\frac{\sin^2\theta}{\pi} P\int_0^\pi \frac{\tfrac{1}{8}\pi A_1(1 - \cos t) - \tfrac{1}{4} A_1 t}{\sin t(\cos\theta - \cos t)} = -\frac{1}{8} A_1\Big\{(1 - \cos\theta)\,\ell n\tfrac{1}{2}(1 - \cos\theta) + (1 + \cos\theta)\,\ell n\,\tfrac{1}{2}(1 + \cos\theta)\Big\}^* \ldots(39)$$

Hence/

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*ℓn is used for $\log_e$.

Hence

$$y_c = \frac{\sin^2\theta}{\pi} P \int_0^{\pi} \frac{G_i(\sin^2\frac{1}{2}t) - \frac{1}{8}\pi A_1(1 - \cos t)}{\sin t(\cos\theta - \cos t)} dt$$

$$- \frac{1}{8} A_1 \left\{(1 - \cos\theta)\, \ln \tfrac{1}{2}(1 - \cos\theta) + (1 + \cos\theta)\, \ln \tfrac{1}{2}(1 + \cos\theta)\right\}$$

$$= \frac{x(1-x)}{\pi} P \int_0^1 \frac{G_i(\xi) - \xi G_i(1)}{\xi(1 - \xi)(\xi - x)} d\xi - \tfrac{1}{4} A_1 \left\{x\, \ln x + (1-x)\, \ln(1 - x)\right\}. \qquad \ldots(40)$$

Equations (35) and (40) are in a suitable form for the direct computation of A_0 and y_c.

4. Basis for Explicit Formulae when g_i is a Polynomial in x in Each of any Number of Segments.

We have completed our investigation of the general equations, and proceed to show how the various quantities required may be calculated analytically in special examples. It is clear from the form of our general equations that the whole mathematical apparatus of the Appendix to Part I is at our disposal, so results may be obtained analytically for a wide variety of algebraical formulations of g_i; it is probably sufficient for the present, however, to consider that, with the chord divided into any number of segments, in each segment g_i is represented by a polynomial in x. Then we can build up the complete expressions for any such case from appropriate multiples of the contributions corresponding to a term in x^n in the expression for g_i in the interval $x_{r-1} \leqslant x \leqslant x_r$. For such a term the contribution to

$$\left.\begin{aligned} G_i(x) &= 0 \quad \text{for} \quad 0 \leqslant x \leqslant x_{r-1} \\ &= \frac{1}{n+1}\left(x^{n+1} - x_{r-1}^{n+1}\right) \quad \text{for} \quad x_{r-1} \leqslant x \leqslant x_r \\ &= \frac{1}{n+1}\left(x_r^{n+1} - x_{r-1}^{n+1}\right) \quad \text{for} \quad x_r \leqslant x \leqslant 1 \end{aligned}\right\} \ldots(41)$$

The contribution to

$$A_1 = \frac{4}{\pi(n + 1)}\left(x_r^{n+1} - x_{r-1}^{n+1}\right). \qquad \ldots(42)$$

It follows from (41) and Lemma 12 of Part I (Appendix) that the contribution to

A_0/

$$A_0 = \frac{1}{\pi(n+1)} \left\{ \left(1 - x_{r-1}^{n+1}\right) \ell n \left(1 - x_{r-1}\right) - \left(1 - x_r^{n+1}\right) \ell n \left(1 - x_r\right) - x_r^{n+1} \ell n\, x_r + x_{r-1}^{n+1} \ell n\, x_{r-1} - \sum_{m=1}^{n} \frac{x_r^m - x_{r-1}^m}{m} \right\}, \quad \ldots(43)$$

the last term being omitted if $n = 0$. The contribution to β then follows from (6), and to C_{Lopt} and α_{opt} from (8) and (9); the contribution to

$$- C_{M_0} = \frac{4}{n+2} \left(x_r^{n+2} - x_{r-1}^{n+2}\right) - \frac{1}{n+1} \left(x_r^{n+1} - x_{r-1}^{n+1}\right) \ldots(44)$$

From (40) and Lemma (9) Part I (Appendix) we find that the contribution to

$$y_c = \frac{1}{\pi(n+1)} \left\{ \left(x^{n+1} - x_r^{n+1}\right) \ell n \left|x - x_r\right| - \left(x^{n+1} - x_{r-1}^{n+1}\right) \ell n \left|x - x_{r-1}\right| - x\left[\left(1 - x_r^{n+1}\right) \ell n \left(1 - x_r\right) - \left(1 - x_{r-1}^{n+1}\right) \ell n \left(1 - x_{r-1}\right) \right] + (1 - x) \left[x_r^{n+1} \ell n\, x_r - x_{r-1}^{n+1} \ell n\, x_{r-1} \right] - x(1 - x) \sum_{s=0}^{n-2} x^s \sum_{m=1}^{n-s-1} \frac{x_r^m - x_{r-1}^m}{m} \right\} \quad \ldots(45)$$

For $n = 0$ and $n = 1$ the last term inside the {} is to be omitted; for other values of n it may be expressed in the alternative form

$$\sum_{s=2}^{n} x^s \frac{x_r^{n-s+1} - x_{r-1}^{n-s+1}}{n - s + 1} - x \sum_{m=1}^{n-1} \frac{x_r^m - x_{r-1}^m}{m}. \quad \ldots(46)$$

The contribution to $\varepsilon_c(\theta)$ is given by (27) and to $\varepsilon_c(0)$ by (6); the contribution to $\varepsilon_c'(\theta)$ is similarly given by (28) with the contribution to

$$\begin{aligned} g_i &= 0 \quad \text{for} \quad 0 \leqslant x \leqslant x_{r-1} \\ &= x^n \quad \text{for} \quad x_{r-1} \leqslant x \leqslant x_r \\ &= 0 \quad \text{for} \quad x_r \leqslant x \leqslant 1. \end{aligned} \quad \ldots(47)$$

We may note that, from (19) and Lemma 12 of Part I (Appendix), the contribution to

$$\frac{dy_c}{dx} - A_0 = \frac{1}{\pi} \left\{ x^n \ell n \left| \frac{x - x_r}{x - x_{r-1}} \right| + \sum_{m=0}^{n-1} x^m \frac{x_r^{n-m} - x_{r-1}^{n-m}}{n - m} \right\}, \ldots(48)$$

the last term being omitted if $n = 0$, and this result may be verified by direct differentiation of (45).

5./

5. Explicit Results when g_i is Quadratic in x in Each of Three Segments.

We now set out explicit results when, with the chord divided into three segments, g_i is expressible as a quadratic in x in each of them, viz.

$$\left.\begin{aligned} g_i &= a_0 + a_1 x + a_2 x^2 \quad \text{for} \quad 0 \leqslant x \leqslant X_1 \\ &= b_0 + b_1 x + b_2 x^2 \quad \text{for} \quad X_1 \leqslant x \leqslant X_2 \\ &= c_0 + c_1 x + c_2 x^2 \quad \text{for} \quad X_2 \leqslant x \leqslant 1 \end{aligned}\right\} \qquad \ldots(49)$$

Write

$$a_r - b_r = k_r, \quad b_r - c_r = \ell_r \quad (r = 0, 1, 2). \qquad \ldots(50)$$

Then

$$\left.\begin{aligned} G_i(x) &= a_0 x + \tfrac{1}{2}a_1 x^2 + \tfrac{1}{3}a_2 x^3 \quad \text{for} \quad 0 \leqslant x \leqslant X_1 \\ &= b_0 x + \tfrac{1}{2}b_1 x^2 + \tfrac{1}{3}b_2 x^3 + k_0 X_1 + \tfrac{1}{2}k_1 X_1^2 + \tfrac{1}{3}k_2 X_1^3 \quad \text{for} \quad X_1 \leqslant x \leqslant X_2 \\ &= c_0 x + \tfrac{1}{2}c_1 x^2 + \tfrac{1}{3}c_2 x^3 + k_0 X_1 + \tfrac{1}{2}k_1 X_1^2 + \tfrac{1}{3}k_2 X_1^3 + \ell_0 X_2 + \tfrac{1}{2}\ell_1 X_2^2 \\ &\qquad + \tfrac{1}{3}\ell_2 X_2^3 \quad \text{for} \quad X_2 \leqslant x \leqslant 1. \end{aligned}\right\} \qquad \ldots(51)$$

$$A_1 = \frac{4}{\pi} G_i(1) = \frac{4}{\pi}\left\{c_0 + \tfrac{1}{2}c_1 + \tfrac{1}{3}c_2 + k_0 X_1 + \tfrac{1}{2}k_1 X_1^2 + \tfrac{1}{3}k_2 X_1^3 + \ell_0 X_2 + \tfrac{1}{2}\ell_1 X_2^2 + \tfrac{1}{3}\ell_2 X_2^3\right\}; \qquad \ldots(52)$$

$$\begin{aligned} A_2 = -\frac{1}{\pi}\Bigg\{ &\left[k_0(1 - X_1) + \frac{k_1}{2}(1 - X_1^2) + \frac{k_2}{3}(1 - X_1^3)\right] \ell n\,(1 - X_1) \\ &+ \left[\ell_0(1 - X_2) + \frac{\ell_1}{2}(1 - X_2^2) + \frac{\ell_2}{3}(1 - X_2^3)\right] \ell n\,(1 - X_2) \\ &+ \left[k_0 X_1 + \frac{k_1}{2} X_1^2 + \frac{k_2}{3} X_1^3\right] \ell n\, X_1 \\ &+ \left[\ell_0 X_2 + \frac{\ell_1}{2} X_2^2 + \frac{\ell_2}{3} X_2^3\right] \ell n\, X_2 \\ &+ \left(\frac{k_1}{2} + \frac{k_2}{3}\right) X_1 + \frac{k_2}{6} X_1^2 + \left(\frac{\ell_1}{2} + \frac{\ell_2}{3}\right) X_2 + \frac{\ell_2}{6} X_2^2 + \frac{c_1 + c_2}{2} \Bigg\}; \end{aligned} \qquad \ldots(53)$$

$$y_c = \frac{1}{\pi}\left\{\left[k_0(x - X_1) + \tfrac{1}{2}k_1(x^2 - X_1^2) + \tfrac{1}{3}k_2(x^3 - X_1^3)\right] \ell n\,|x - X_1|\right.$$

$$+ \left[\ell_0(x - X_2) + \tfrac{1}{2}\ell_1(x^2 - X_2^2) + \tfrac{1}{3}\ell_2(x^3 - X_2^3)\right] \ell n\,|x - X_2|$$

$$- \left[a_0 x + \tfrac{1}{2}a_1 x^2 + \tfrac{1}{3}a_2 x^3\right] \ell n\, x$$

$$- \left[c_0(1 - x) + \tfrac{1}{2}c_1(1 - x^2) + \tfrac{1}{3}c_2(1 - x^3)\right] \ell n\,(1 - x)$$

$$+ \tfrac{1}{3}\left[k_2 X_1 + \ell_2 X_2 + c_2\right] x^2$$

$$+ \left[\pi A_0 + \frac{k_1}{2} X_1 + \frac{k_2}{6} X_1^2 + \frac{\ell_1}{2} X_2 + \frac{\ell_2}{6} X_2^2 + \frac{c_1}{2} + \frac{c_2}{6}\right] x$$

$$+ (k_0 X_1 + \tfrac{1}{2}k_1 X_1^2 + \tfrac{1}{3}k_2 X_1^3)\, \ell n\, X_1$$

$$\left. + (\ell_0 X_2 + \tfrac{1}{2}\ell_1 X_2^2 + \tfrac{1}{3}\ell_2 X_2^3)\, \ell n\, X_2 \right\}; \qquad \ldots(54)$$

$$\frac{dy_c}{dx} - A_0 = \frac{1}{\pi}\left\{(k_0 + k_1 x + k_2 x^2)\, \ell n\,|x - X_1| + (\ell_0 + \ell_1 x + \ell_2 x^2)\, \ell n\,|x - X_2|\right.$$

$$- (a_0 + a_1 x + a_2 x^2)\, \ell n\, x + (c_0 + c_1 x + c_2 x^2)\, \ell n\,(1 - x)$$

$$+ x(k_2 X_1 + \ell_2 X_2 + c_2) + k_1 X_1 + \tfrac{1}{2}k_2 X_1^2 + \ell_1 X_2$$

$$\left. + \tfrac{1}{2}\ell_2 X_2^2 + c_1 + \tfrac{1}{2}c_2 \right\}; \qquad \ldots(55)$$

$$-C_{M_0} = k_2 X_1^4 + \ell_2 X_2^4 + \tfrac{1}{3}X_1^3(4k_1 - k_2) + \tfrac{1}{3}X_2^3(4\ell_1 - \ell_2)$$

$$+ X_1^2(2k_0 - \tfrac{1}{2}k_1) + X_2^2(2\ell_0 - \tfrac{1}{2}\ell_1) - k_0 X_1 - \ell_0 X_2$$

$$+ c_0 + \tfrac{5}{6}c_1 + \tfrac{2}{3}c_2. \qquad \ldots(56)$$

With the values of g_i, G_i, A_1 and A_0 above, β and $\varepsilon_c(0)$ are given by (6), $C_{L opt}$ and α_{opt} by (8) and (9), $\varepsilon_c(\theta)$ by (27), and $\varepsilon_c'(\theta)$ by (28).

y_c is zero at $x = 0$ and $x = 1$, but dy_c/dx is logarithmically infinite at $x = 0$ if $g(0) \neq 0$, and at $x = 1$ if $g(1) \neq 0$; as we have already remarked, $\varepsilon_c'(\theta)$ is discontinuous at $\theta = 0$ if $g(0) \neq 0$, and at $\theta = \pi$ if $g(1) \neq 0$, though in practice it is probably best to fair off $\varepsilon_c'(\theta)$ to 0 at $\theta = 0$ and $\theta = \pi$.

At/

At X_1 and X_2, G_i, y_c, ε_c are continuous whether g_i is continuous or not; but if g_i is not continuous, dy_c/dx is logarithmically infinite and ε_c' finitely discontinuous (so that the velocity q is finitely discontinuous on Approximation III); similarly, if g_i is continuous but g_i' discontinuous, d^2y_c/dx^2 and the curvature of the aerofoil centre line are logarithmically infinite, with $\varepsilon_c''(\theta)$, and therefore the velocity gradient along the aerofoil surface, finitely discontinuous.

The results for many special cases may be written down at once from the results of this section.

6. g_i Quadratic in Each of Two Segments, or Quadratic over the Whole Chord

If

$$\left.\begin{aligned} g_i &= a_0 + a_1x + a_2x^2 \quad \text{for} \quad 0 \leqslant x \leqslant X_1 \\ &= b_0 + b_1x + b_2x^2 \quad \text{for} \quad X_1 \leqslant x \leqslant 1, \end{aligned}\right\} \quad \ldots(57)$$

we put

$$c_r = b_r, \quad \ell_r = 0 \quad (r = 0, 1, 2)$$

in eqns. (49) - (56) above.

If

$$g_i = a_0 + a_1x + a_2x^2 \quad \text{for} \quad 0 \leqslant x \leqslant 1, \qquad \ldots(58)$$

we put

$$c_r = b_r = a_r, \quad \ell_r = k_r = 0 \quad (r = 0, 1, 2)$$

in eqns. (49) - (56).

7. g_i Linear in Each of Three Segments, or Linear in Each of Two Segments

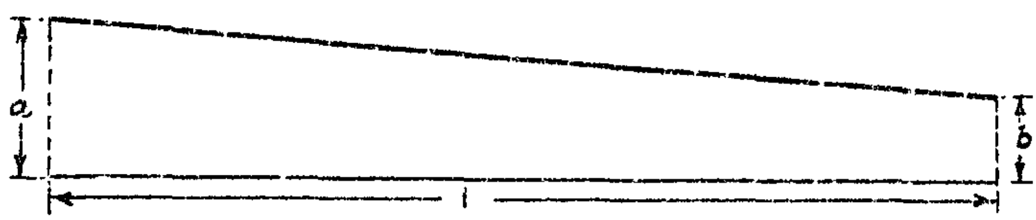
Explicit results have been worked out when g_i is linear in each of three segments (with or without discontinuities: Figs. 1, 2, 3, 4) or in each of two segments (Fig. 5). Since the algebra is complicated, it has been thought worth while to put the results on record; but since they are likely to be used only rarely they have been relegated to Appendix III.

Figures/

8. g_i Linear Over the Whole Chord

If g_i is linear throughout $0 \leqslant x \leqslant 1$, $g_i(0) = a$, $g_i(1) = b$ (Fig. 6),

FIG. 6



then

$$g_i(x) = a(1 - x) + bx \qquad (0 \leqslant x \leqslant 1);$$

$$G_1(x) = a\left(x - \frac{x^2}{2}\right) + \frac{bx^2}{2} \qquad (0 \leqslant x \leqslant 1);$$

$$A_1 = \frac{2}{\pi}(a + b);$$

$$A_0 = \frac{1}{2\pi}(a - b);$$

$$y_c = -\frac{a}{2\pi}\left\{(1 - x)^2 \ln(1 - x) + (2x - x^2) \ln x\right\}$$

$$-\frac{b}{2\pi}\left\{(1 - x^2) \ln (1 - x) + x^2 \ln x\right\};$$

$$\frac{dy_c}{dx} - A_0 = \frac{a}{\pi}\left\{(1 - x) \ln \frac{1 - x}{x} - 1\right\} + \frac{b}{\pi}\left\{x \ln \frac{1 - x}{x} + 1\right\};$$

$$-C_{M_0} = \frac{1}{6}\left\{a + 5b\right\};$$

$$\beta = \frac{1}{2\pi}(a + 3b);$$

$\frac{\pi}{a_0}$ /

8548.
Figs. 1 - 5.

Fig. 1.

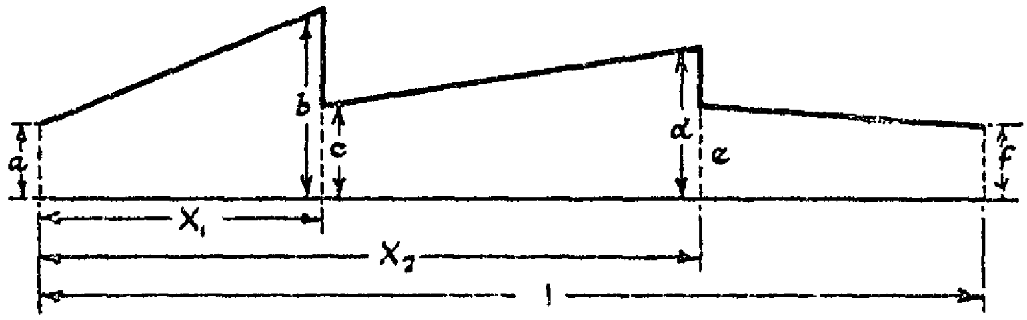


Fig. 2.

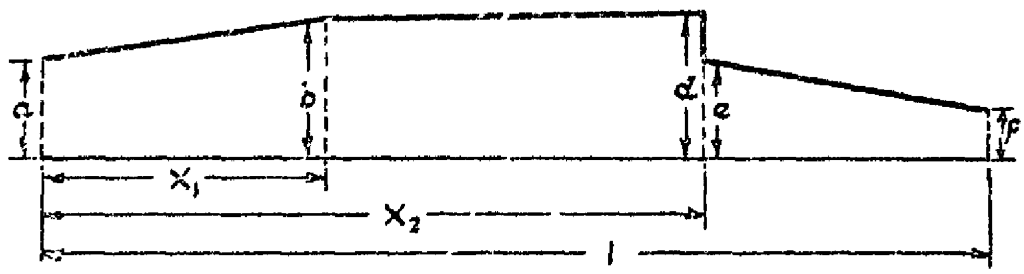


Fig. 3.

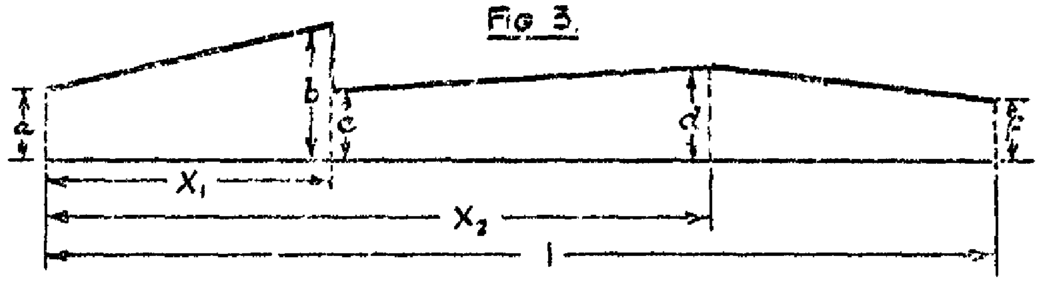


Fig. 4.

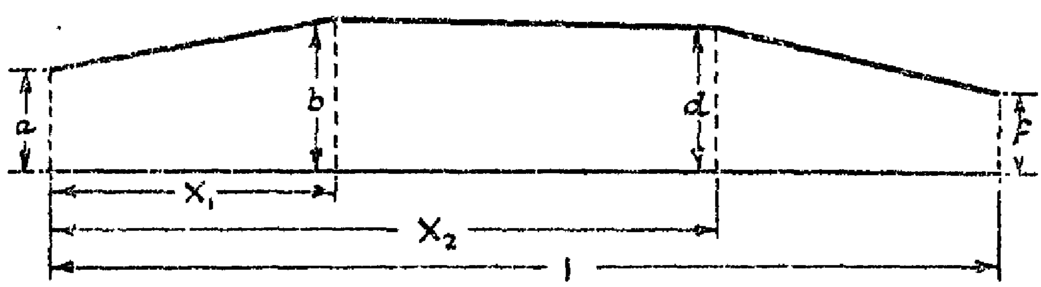
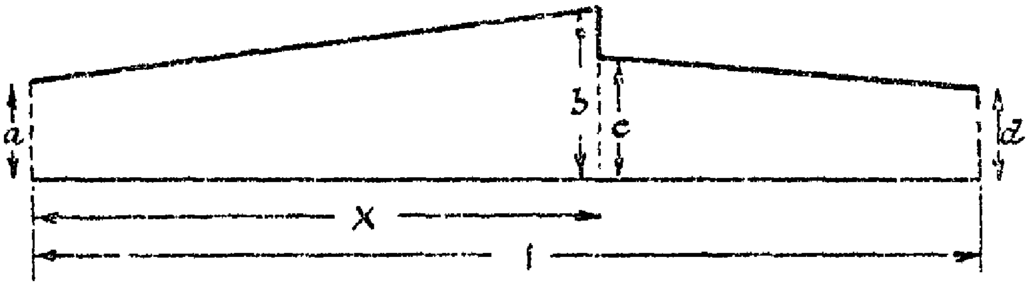


Fig. 5.



$$\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt} = 2(a + b);$$

$$\alpha_{opt} = \frac{a - b}{2\pi} + \left(\frac{2\pi - a_0}{2\pi + a_0}\right)\left(\frac{a + b}{\pi}\right);$$

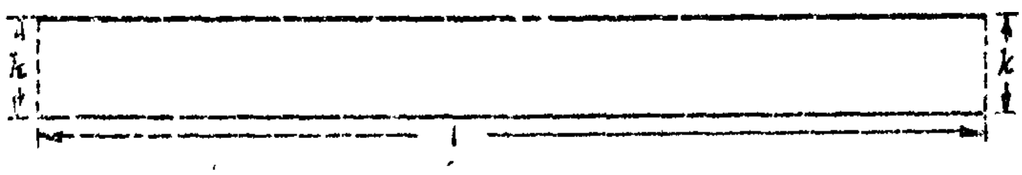
$$\varepsilon_c(0) = -\frac{1}{2\pi}(3a + b).$$

For $\varepsilon_c(\theta)$ and $\varepsilon_c'(\theta)$ see (27) and (28).

9. g_i Constant Over the Whole Chord

If in the last section we put $a = b = k$ (Fig. 7), we obtain

Fig 7



the results for a constant approximate distribution of normal load; these results have been given by a number of authors[5,6,7,8]

$$y_c = -\frac{k}{\pi}\left\{(1 - x)\,\ell n\,(1 - x) + x\,\ell n\,x\right\}$$

$$= -0{\cdot}73293560\,k\left\{(1 - x)\log_{10}(1 - x) + x\log_{10}x\right\};$$

$$\frac{dy_c}{dx} = \frac{k}{\pi}\,\ell n\,\frac{1 - x}{x};$$

$$-C_{M_0} = k = \frac{1}{4}\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt},$$

so if $a_0 = 2\pi$,

$$-\frac{C_{M_0}}{C_{Lopt}} = \frac{1}{4}.$$

Also/

Also

$$\beta = \frac{2k}{\pi}\left(= \frac{C_{L\,opt}}{2\pi} \text{ for } a_o = 2\pi\right),$$

α_{opt} is zero for $a_o = 2\pi$, $\varepsilon_c(0) = -2k/\pi$, and $\varepsilon_c(\theta)$, $\varepsilon_c'(\theta)$ are given by (27) and (28) with $g_i = k$, $G_i = kx$, $A_1 = 4k/\pi$, $A_0 = 0$, i.e.

$$\varepsilon_c(\theta) = 2k\left[x - \frac{\theta}{\pi}\right] \operatorname{cosec} \theta,$$

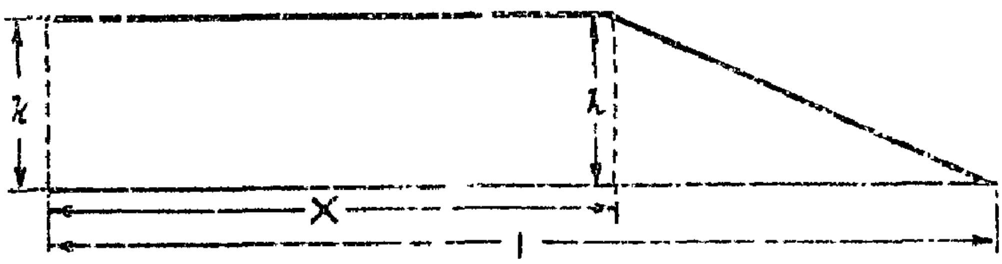
$$\varepsilon_c'(\theta) = k - \varepsilon_c \cot \theta - \frac{2k}{\pi} \operatorname{cosec} \theta.$$

10. g_i Constant for $0 \leqslant x \leqslant X$, and Decreasing Linearly to Zero for $X \leqslant x \leqslant 1$

For low-drag aerofoils it is customary to use centre lines such that g_i is constant from the leading edge to some value X of x, usually chosen to coincide with the design position of maximum suction on the fairing. It may be shown that such centre lines lead to larger C_L-ranges than any others. An important set of such centre lines is obtained[8,9] if g_i decreases linearly to zero between $x = X$ and $x = 1$; but over the rear portion of the aerofoil it is probable that g_i may sometimes be varied with advantage, particularly with regard to the resultant value of C_{M_0}, and other shapes of the graph of g_i between X and 1 will be considered later.

For the shape of graph here to be considered (Fig. 8)

Fig 8.



$$g_i = k \quad (0 \leqslant x \leqslant X),$$

$$= k \frac{1 - x}{1 - X} \quad (X \leqslant x \leqslant 1).$$

We put $a = b = k$, $b - c = 0$, $d = 0$ in the formulae of the second section of Appendix III, and we find that

$$G_i(x) = kx \qquad (0 \leqslant x \leqslant X),$$

$$= -k \frac{(X^2 - 2x + x^2)}{2(1 - X)} \qquad (X \leqslant x \leqslant 1);$$

$$A_1 = \frac{2k}{\pi} (1 + X);$$

$$A_0 = \frac{k}{\pi} \left\{ \frac{1}{2} + \frac{X^2}{2(1 - X)} \ell n\, X - \frac{1 - X}{2} \ell n\, (1 - X) \right\};$$

$$y_c = \frac{k}{2\pi} \left\{ \frac{(x - X)^2}{1 - X} \ell n\, |x - X| - \frac{(1 - x)^2}{1 - X} \ell n\, (1 - x) - 2x\, \ell n\, x \right.$$

$$\left. - x(1 - X)\, \ell n\, (1 - X) - (1 - x) \frac{X^2}{1 - X} \ell n\, X \right\}$$

$$\frac{dy_c}{dx} = \frac{k}{\pi} \left\{ \frac{x - X}{1 - X} \ell n\, |x - X| + \frac{1 - x}{1 - X} \ell n\, (1 - x) - \ell n\, x - \tfrac{1}{2} \right.$$

$$\left. + \frac{X^2}{2(1 - X)} \ell n\, X - \frac{1 - X}{2} \ell n\, (1 - X) \right\}$$

where

$$\left(\frac{\pi}{a_0} + \frac{1}{2} \right) C_{Lopt} = 2k(1 + X).$$

Also

$$-C_{M_0} = \frac{k}{6} (4X^2 + X + 1),$$

so that, for $a_0 = 2\pi$,

$$-\frac{C_{M_0}}{C_{Lopt}} = \frac{4X^2 + X + 1}{12(1 + X)},$$

while

$$\beta = \frac{k}{\pi} \left\{ \tfrac{1}{2} + X - \frac{X^2}{2(1 - X)} \ell n\, X + \frac{1 - X}{2} \ell n\, (1 - X) \right\}.$$

With/

With the values of A_1, A_0, g_i, G_i above, α_{opt} is then given by (9), $\varepsilon_c(0)$ by (6), $\varepsilon_c(\theta)$, $\varepsilon'_c(\theta)$ by (27) and (28) respectively.

For the purposes of computation the equation for y_c is written in the form

$$y_c = K_0 \left\{(x - X)^2 \log_{10}|x - X| - (1 - x)^2 \log_{10}(1 - x)\right\}$$

$$- K_1 \, x \log_{10} x + K_2 x + K_3 .$$

For a fixed X, the quantities k, A_0, A_1, β, C_{M_0}, α_{opt} (if a_0 is fixed), y_c, g_i, G_i, ψ_c, ε_c, ε'_c are all proportional to $\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt}$.

In Table I, the coefficieints K_0, K_1, K_2, K_3 in the expression for y_c, together with k, A_0, β, C_{M_0} and $\varepsilon_c(0)$ are tabulated for $\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt} = 1$ and various values of X; for $\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt} = 1$, $A_1 = \frac{1}{\pi}$.

Table II contains values of the ordinates y_c for X = 0·35(0·05)1·0 and a fairly long list of values of x; Table III contains values of ψ_c, ε_c, ε'_c, required for the calculation of q/U on Approximation III, for the same values of X and a shorter set of values of x.

We shall show later how these Tables may be used to build up values for more complicated forms of g_i.

The Tables were prepared at the National Physical Laboratory, under the supervision of Mrs. Moore, while I was working there.

Some numerical values of the no-lift angle and comparisons with experiment have been given by Jacobs and Abbott[10]. Further comparisons of theory and experiment are to be found in Ref. 11. It should be noted that some disagreement between the measured and calculated values of, say, β and C_{M_0} is to be expected because of the singularity in the theoretical solution at x = 0 (and at x = 1 if X = 1); the exact manner in which the smoothing-away of the singularity in practice will affect the results is difficult to predict, and will probably vary to some extent from model to model. These uncertainties do not seem to be objectionably large in practice.

From the values given in Table I, it appears that, for a given design C_L (i.e. C_{Lopt}), $-C_{M_0}$ may be inconveniently large, especially if X is the position of *maximum* suction on the fairing and this position is required to be fairly far back along the chord. The ratio of $-C_{M_0}$ to C_{Lopt} may be decreased by reflexing the centre line towards the trailing edge; it is always preferable to

re-design/

$$\frac{dy_c}{dx} = \frac{k}{\pi}\left\{\frac{(x-X)(2-X-x)}{(1-X)^2}\,\ell n\,|x-X| - \ell n x + \left(\frac{1-x}{1-X}\right)^2 \ell n\,(1-x) + \frac{x}{1-X}\right.$$

$$\left. - \frac{2}{3}(1-X)\,\ell n\,(1-X) + \frac{(3-2X)X^2}{3(1-X)^2}\,\ell n\,X - \frac{3-X}{3(1-X)}\right\};$$

$$-C_{M_0} = kX^2;$$

$$\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt} = \frac{4k}{3}(1 + 2X).$$

Hence, for $a_0 = 2\pi$,

$$-\frac{C_{M_0}}{C_{Lopt}} = \frac{3X^2}{4(1 + 2X)}$$

and

$$\frac{k}{C_{Lopt}} = \frac{3}{4(1 + 2X)}.$$

For example, for $X = 0{\cdot}5$, $k/C_{Lopt} = 0{\cdot}375$, compared with $0{\cdot}333$ when g_i is given by Fig. 8, while $-C_{M_0}/C_{Lopt} = 0{\cdot}094$ compared with $0{\cdot}139$; for $X = 0{\cdot}6$, $k/C_{Lopt} = 0{\cdot}341$ and $-C_{M_0}/C_{Lopt} = 0{\cdot}123$ compared with $0{\cdot}3125$ and $0{\cdot}158$, respectively. There is therefore a fair reduction in $-C_{M_0}$ at the expense of a quite small increase in k; however, the increased negative pressure gradient just aft of $x = X$ in these designs will certainly lead to a somewhat thicker boundary layer over the rear of the aerofoil, and, in spite of the decrease of the gradient towards the trailing edge, freedom from danger of turbulent boundary-layer separation may require a somewhat thinner aerofoil for the same value of X; it is also possible that separation at the stall may not start at the trailing edge, so the effect on stalling characteristics would have to be investigated. It may, however, be worth while to test an aerofoil with a centre line designed as here described.

12. g_i Discontinuous and Constant in Each of Two Segments

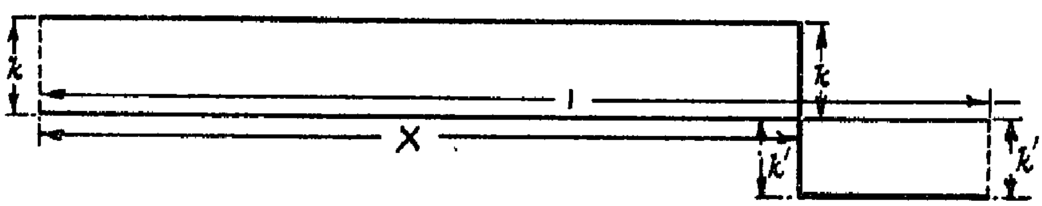
On an aerofoil specially designed for suction, there would appear to be no objection to having g_i discontinuous, so long as the position of the discontinuity is the same as that in the velocity distribution over the fairing (uncambered aerofoil). With such a discontinuity we may, within quite wide limits, obtain any specified value of $-C_{M_0}/C_{Lopt}$ by adjusting the ratio of the loading in front of and behind the discontinuity. Negative loading

over/

over the rear portion of the chord will usually be necessary to obtain values of $-C_{M_0}/C_{Lopt}$, but the danger of turbulent boundary layer separation certainly does not arise, since the whole point of the design of such aerofoils is to concentrate the pressure recovery (fall in velocity) at one chordwise station on each surface, where a slot is cut and boundary-layer suction applied. There will, however, be a rise in the maximum value of g_i, with a consequent rise in the maximum value of the velocity on the surface and decrease in the theoretical critical speed for compressibility effects. Since the most important application of the suction principle is likely, for some time to come at any rate, to be to thick aerofoils for aeroplanes not moving at very high speeds, this effect is not likely to be a serious drawback.

The simplest of such centre lines to design are those in which g_i is constant in each of the two segments $0 \leqslant x \leqslant X$, $X \leqslant x \leqslant 1$, with a discontinuity at $x = X$ (Fig. 11). Moreover, these are the designs which, for various reasons, are most likely to be employed in such cases as we are considering, where it is required to reduce $-C_{M_0}$ for a suction aerofoil to be used at speeds where compressibility effects will not be important. Experimental information should, however, be obtained for aerofoils with such centre lines as soon as convenient.

FIG 11.



With the graph of g_i as in Fig. 11,

$$g_i = k \qquad (0 < x < X),$$

$$= -k' \qquad (X < x < 1);$$

$$G_i = kx \qquad (0 \leqslant x \leqslant X)$$

$$= (k + k')X - k'x \quad (X \leqslant x \leqslant 1);$$

$$A_1 = \frac{4}{\pi}\left\{kX - k'(1 - X)\right\};$$

A_0/

$$A_0 = -\frac{k + k'}{\pi}\left\{X \ln X + (1 - X)\ln(1 - X)\right\};$$

$$y_c = \frac{k + k'}{\pi}\left\{(x-X)\ln|x-X| - [(1-X)\ln(1-X)]\,x + [X \ln X](1-x)\right\}$$

$$-\frac{k}{\pi}x\ln x + \frac{k'}{\pi}(1 - x)\ln(1 - x);$$

$$\frac{dy_c}{dx} - A_0 = \frac{k + k'}{\pi}\ln|x-X| - \frac{k}{\pi}\ln x - \frac{k'}{\pi}\ln(1 - x);$$

$$-C_{M_0} = k(2X^2 - X) - k'(1 + X - 2X^2);$$

$$\left(\frac{\pi}{a_0} + \frac{1}{2}\right)C_{Lopt} = \pi A_1 = 4kX - 4k'(1 - X).$$

β, $\varepsilon_c(0)$, α_{opt}, $\varepsilon_c(\theta)$, $\varepsilon_c'(\theta)$ are, as usual, to be found from eqns. (6), (9), (27) and (28).

When the value of X has been selected, appropriate values of k and k' may be found for the desired values of C_{Lopt} and $-C_{M_0}$. For example, with $X = 0{\cdot}75$, if the desired values of C_{Lopt} and $-C_{M_0}$ are 0·2 and 0·015, respectively, we have, assuming $a_0 = 2\pi$,

$$3k - k' = 0{\cdot}2, \quad 0{\cdot}375\,k - 0{\cdot}625\,k' = 0{\cdot}015,$$

whence $k = 0{\cdot}07\dot{3}$, $k' = 0{\cdot}02$.

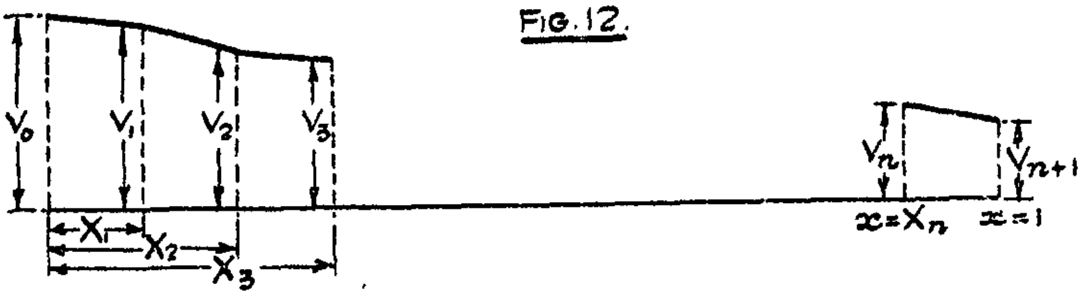
13. The Graph of g_i Composed of Straight Lines. Basic Solutions

If the graph of g_i is composed of straight lines, then g_i may be expressed as the sum of a number of certain 'basic' values, and all those quantities which depend linearly on g_i (G_i, A_1, A_0, C_{Lopt} and α_{opt} (for a given a_0), β, C_{M_0}, y_c, ψ_c, dy_c/dx, $\varepsilon_c(\theta)$ $\varepsilon_c'(\theta)$) may be found as the sum of the corresponding values for the 'basic' g_i. Once the values for the 'basic' g_i are all tabulated, we have a convenient numerical process for finding these values in other cases.

Let the values of g_i be given at $x = 0, X_1, X_2, \ldots X_n, 1$, $(0 < X_1 < X_2 \ldots < X_n < 1)$ and let the graph of g_i against x between any two of these values of x which are consecutive be a straight line. We first assume that g_i is continuous; we shall remove this restriction later. Let the values of g_i at the specified values of x be V_0, V_1, V_2, ... V_n, V_{n+1}, respectively (Fig. 12).

Fig./

FIG. 12.



The simplest case is that in which $V_0 = V_1$ and $V_{n+1} = 0$. The 'basic' values of g_i are then those of §10, Fig. 8, with $X = X_1, X_2, \ldots X_n$ in turn. Denote the value of g_i whose graph is shown in Fig. 8 by $g_i(k;\, X)$. Then if $V_0 = V_1$ and $V_{n+1} = 0$, the value of g_i whose graph is shown in Fig. 12 is given by

$$g_i = \sum_{r=1}^{n} g_i(a_r;\, X_r) \qquad \ldots(59)$$

if we make the values of the two sides of this equation agree at $x = X_r (r = 1, 2, \ldots n)$, since they then necessarily agree at $x = 0$, are both zero at $x = 1$, and both vary linearly with x between 0 and X_1, X_1 and X_2, and so on. We have therefore exactly n linear equations from which to determine the n unknowns a_r. The simplest method of solution is to equate the changes of slope at $x = X_1$, $x = X_2$ and so on. Proceeding in this way we find that

$$\left.\begin{aligned}
a_1 &= (V_1 - V_2)\,\frac{1 - X_1}{X_2 - X_1}, \\
a_r &= \left(V_r - V_{r+1}\right)\frac{1 - X_r}{X_{r+1} - X_r} - \left(V_{r-1} - V_r\right)\frac{1 - X_r}{X_r - X_{r-1}} \\
&\qquad (r = 2, 3, \ldots, n-1;\ n > 2), \\
a_n &= V_n - \left(V_{n-1} - V_n\right)\frac{1 - X_n}{X_n - X_{n-1}} = V_n\,\frac{1-X_{n-1}}{X_n - X_{n-1}} - V_{n-1}\,\frac{1 - X_n}{X_n - X_{n-1}}
\end{aligned}\right\} \qquad \ldots(60)$$

and/

and we may easily verify algebraically that with these values the sum on the right in eqn. (59) has the values V_1 at $x = 0$ and $x = X_1$, V_2 at $x = X_2$, and so on.

The values of β, C_{M_0}, y_c, ψ_c', ε_c, ε_c' for all such cases may therefore be found from Tables I, II and III. In those tables values are given for $\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt} = 1$. Since the value of $\left(\frac{\pi}{a_0} + \frac{1}{2}\right) C_{Lopt}$ corresponding to a_r is $2a_r(1 + X_r)$, the values in these tables must be multiplied by $2a_r(1 + X_r)$, where the a_r have the value found above, and then added together.

The simplest composite case occurs when $n = 2$. (Fig. 9 and Ref. 12). If, following the notation of Ref. 12, we put $V_1 = k$, $V_2 = ks$, then

$$a_1 = k(1 - s)\frac{1 - X_1}{X_2 - X_1}, \quad a_2 = k\frac{s(1-X_1) - (1-X_2)}{X_2 - X_1}.$$

If $V_{n+1} \neq 0$, we must introduce, as another 'basic' g_i, the constant V_{n+1}, which we may also denote by $g_i(V_{n+1};\ 1)$ [Fig. 7 with $k = V_{n+1}$]. Then

$$g_i = \sum_{r=1}^{n} g_i(a_r;\ X_r) + V_{n+1},$$

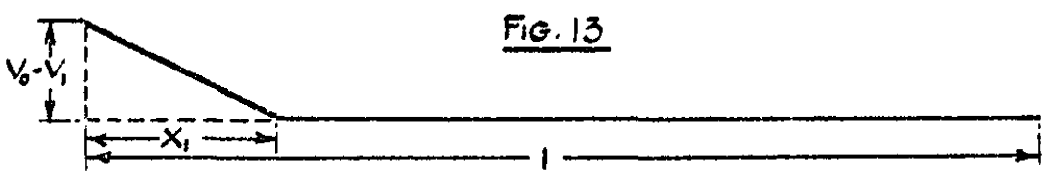
where the values of a_r are given by the same formulae (60) as before with $V_r - V_{n+1}$ in place of V_r, i.e. a_r is given by exactly the same formula for $r = 1, 2, \ldots, n-1$ and

$$a_n = V_n - V_{n+1} - \left(V_{n-1} - V_n\right)\frac{1 - X_n}{X_n - X_{n-1}}.$$

The formulae for g_i constant are contained in §9, and numerical values are shown in Tables I, II, and III under $X = 1$ for $C_{Lopt}\left(\frac{\pi}{a_0} + \frac{1}{2}\right) = 1$; these values must be multiplied by $4V_{n+1}$ and added to the others.

Finally if $V_0 \neq V_1$, we add to the sum on the right in (59) another 'basic' g_i, shown in Fig. 13, for which $g_i = 0$ for $x \geqslant X_1$, $g_i(0) = V_0 - V_1$, and g_i varies linearly with x for $0 \leqslant x \leqslant X_1$.

FIG. 13



The formulae for this 'basic' g_i are contained in the second section of Appendix III if we put $b = c = d = 0$, $a = V_0 - V_1$ (see Fig. 5). No tables have been prepared since it is not clear what useful purpose would be served by designing centre lines with $V_0 \neq V_1$.

We turn next to the case in which g_i has one or more discontinuities. The additional 'basic' g_i required may be taken as that in Fig. 14, and denoted by $g_i(k;\ X;\ 0)$; it is

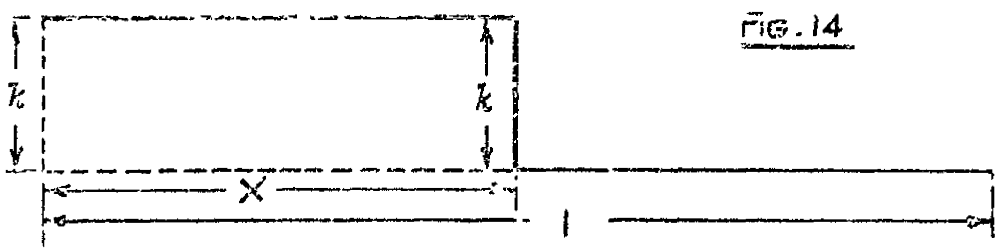


FIG. 14

equal to k for $0 \leqslant x < X$, and to 0 for $X < x \leqslant 1$. Let us suppose first that g_i has only one discontinuity, at $x = X_s$, and that as $x \to X_s - 0$, $g_i \to V_s$, but that as $x \to X_s + 0$, $g_i \to W_s$. It will be sufficient to write out the formulae with $V_0 = V_1$; then

$$g_i = g_i(V_s - W_s;\ X_s;\ 0) + \sum_{r=1}^{n} g_i(a_r;\ X_r) + V_{n+1},$$

and the values of a_r are the same as before, with $V_r - (V_s - W_s)$ substituted for V_r for $r = 1, 2, 3, \dots s$. Hence the a_r are given by the same expression as before, namely

$$a_1 = (V_1 - V_2)\ \frac{1 - X_1}{X_2 - X_1};$$

$$a_r = \left(V_r - V_{r+1}\right)\ \frac{1 - X_r}{X_{r+1} - X_r} - \left(V_{r-1} - V_r\right)\ \frac{1 - X_r}{X_r - X_{r-1}} \quad (r > 1,\ n > 2)$$

for $r \leqslant s-1$ and for $s+2 \leqslant r \leqslant n-1$, while

a_s/

$$a_s = \left(W_s - V_{s+1}\right) \frac{1 - X_s}{X_{s+1} - X_s} - \left(V_{s-1} - V_s\right) \frac{1 - X_s}{X_s - X_{s-1}},$$

$$a_{s+1} = \left(V_{s+1} - V_{s+2}\right) \frac{1 - X_{s+1}}{X_{s+2} - X_{s+1}} - \left(W_s - V_{s+1}\right) \frac{1 - X_s}{X_{s+1} - X_s},$$

and

$$a_n = V_n - \left(V_{n-1} - V_n\right) \frac{1 - X_n}{X_n - X_{n-1}}$$

as before, if $s < n-1$.

If there are two discontinuities, one at $x = X_s$ and one at $x = X_t$, the discontinuity at X_s being as before and that at X_t being given by a change in g_i from V_t to W_t, then

$$g_i - g_i(V_s - W_s;\ X_s;\ 0) - g_i(V_t - W_t;\ X_t;\ 0) - V_{n+1}$$

is continuous, has zero slope for $0 \leqslant x \leqslant X_1$, and is zero at $x = 1$; hence it can be expressed as

$$\sum_{r=1}^{n} g_i(a_r;\ X_r).$$

The values of the expression at $x = X_r$ are

$$V_r - (V_s - W_s) - (V_t - W_t) - V_{n+1} \qquad (r \leqslant s),$$

$$V_r - (V_t - W_t) - V_{n+1} \qquad (s < r \leqslant t),$$

$$V_r - V_{n+1} \qquad (t < r \leqslant n);$$

and the a_r are given by the same expressions (60) as before, if these values be substituted for V_r.

In the simplest composite case, $n = 2$, (Fig. 1), $V_0 = a$, $V_1 = b$, $W_1 = c$, $V_2 = d$, $W_2 = e$, $V_3 = f$,

$$g_i = g_i(a_1;\ X_1) + g_i(a_2;\ X_2) + g_i(a-b;\ 0;\ X_1;\ 0) + g_i(b-c;\ X_1;\ 0)$$
$$+ g_i(d-e;\ X_2;\ 0) + f,$$

where the third term on the right denotes the function whose graph is shown in Fig. 13, and

a_1/

$$a_1 = (c - d)\frac{1 - X_1}{X_2 - X_1}, \quad a_2 = e - f - (c - d)\frac{1 - X_2}{X_2 - X_1}.$$

The solution when g_1 has the form depicted in Fig. 14 is contained in §12, by putting $k' = 0$ in the formulae there given. (See Fig. 11). Systematic tabulation for various values of X has not yet been undertaken, but it is recommended that such tables be prepared as soon as suction aerofoils are likely to come into general use.

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Addendum to above - A.R.C. 8139.

Table 1/

Table 1

Centre Lines with g_1 Constant for $0 \leqslant x \leqslant X$, and Decreasing Linearly to Zero for $X \leqslant x \leqslant 1$

Values for $\left(\frac{\pi}{a_o} + \frac{1}{2}\right) C_{Lopt} = 1$

Some Constants

X_1	k	K_0	K_1	K_2	K_3	A_o	β	$-C_{M_o}$	$-\mathcal{E}_c(0)$
0.25	0.4000 0000	0.1954 4949	0.2931 7424	0.0063 8129	0.0073 5452	0.0700 4326$_5$	0.0891 1	0.1	0.2292 0
0.3	0.3846 1538$_5$	0.2013 5593	0.2818 9831	0.0058 0768	0.0094 7563	0.0670 2112	0.0921 3	0.1064 1	0.2261 8
0.35	0.3703 7037	0.2088 1356	0.2714 5763	0.0048 4290	0.0116 6258$_5$	0.0637 8917	0.0953 7	0.1135 8	0.2229 4
0.4	0.3571 4286	0.2181 3559$_5$	0.2617 6271	0.0035 3274	0.0138 8878	0.0603 7379	0.0987 8	0.1214 3	0.2195 3
0.45	0.3448 2759	0.2297 6038	0.2527 3641	+0.0019 1065	0.0161 3480	0.0567 9166	0.1023 6	0.1298 9	0.2159 5
0.5	0.3333 3333	0.2443 1187	0.2443 1187	0	0.0183 8630	0.0530 5165	0.1061 0	0.1388 9	0.2122 1
0.55	0.3225 8064$_5$	0.2627 0093	0.2364 3084	-0.0021 8457	0.0206 3261	0.0491 5573	0.1100 0	0.1483 9	0.2083 1
0.6	0.3125 0000	0.2863 0297	0.2290 4237$_5$	-0.0046 3672	0.0228 6574	0.0450 9920	0.1140 6	0.1583 3	0.2042 5
0.65	0.3030 3030	0.3172 8814	0.2221 0170	-0.0073 5869	0.0250 7976	0.0408 7008$_5$	0.1182 8	0.1686 9	0.2000 2$_5$
0.7	0.2941 1765	0.3592 8216	0.2155 6929	-0.0103 6273	0.0272 7022	0.0364 4755	0.1227 1	0.1794 1	0.1956 0
0.75	0.2857 1429	0.4188 2034	0.2094 1017	-0.0136 7419	0.0294 3387	0.0317 9865	0.1273 6	0.1904 8	0.1909 5
0.8	0.2777 7778	0.5089 8305$_5$	0.2035 9322	-0.0173 3780	0.0315 6835$_5$	0.0268 7191	0.1322 8	0.2018 5	0.1860 3
0.85	0.2702 7027	0.6603 0234	0.1980 9070	-0.0214 3135	0.0336 7200	0.0215 8350	0.1375 7	0.2135 1	0.1807 4
0.9	0.2631 5789$_5$	0.9643 8895	0.1928 7779	-0.0260 9981	0.0357 4369$_5$	0.0157 8307	0.1433 7	0.2254 4	0.1749 4
0.95	0.2564 1026	1.8793 2205	0.1879 3220$_5$	-0.0316 7009	0.0377 8273	0.0091 3887	0.1500 2	0.2376 1	0.1682 9
1	0.25	← See §9 →				0	0.1591 5	0.25	0.1591 5

Table 2//

Table 2

Centre Lines with g_1 Constant for $0 \leqslant x \leqslant X$, and Decreasing Linearly to Zero for $X \leqslant x \leqslant 1$

Values for $\left(\frac{\pi}{a_o} + \frac{1}{2}\right) C_{Lopt} = 1$ Centre Line Ordinates

x \ X	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.001	0.000976	0.000945	0.000915	0.000887	0.000859	0.000833	0.000808	0.000734	0.000761	0.000738	0.000717	0.000695	0.000674	0.000652	0.000629
0.002	0.001782	0.001726	0.001672	0.001621	0.001572	0.001524	0.001479	0.001435	0.001392	0.001351	0.001310	0.001271	0.001231	0.001192	0.001148
0.003	0.002523	0.002445	0.002370	0.002298	0.002228	0.002161	0.002097	0.002035	0.001974	0.001915	0.001858	0.001801	0.001745	0.001688	0.001625
0.004	0.003223	0.003124	0.003029	0.002937	0.002849	0.002763	0.002718	0.002602	0.002524	0.002449	0.002375	0.002303	0.002230	0.002157	0.002075
0.005	0.003892	0.003773	0.003659	0.003548	0.003442	0.003339	0.003240	0.003144	0.003051	0.002959	0.002870	0.002782	0.002694	0.002605	0.002505
0.006	0.004535	0.004398	0.004265	0.004137	0.004014	0.003895	0.003779	0.003667	0.003558	0.003452	0.003347	0.003244	0.003141	0.003036	0.002919
0.007	0.005158	0.005003	0.004853	0.004708	0.004568	0.004432	0.004301	0.004174	0.004050	0.003928	0.003809	0.003691	0.003574	0.003454	0.003319
0.0075	0.005463	0.005299	0.005140	0.004987	0.004839	0.004696	0.004557	0.004422	0.004290	0.004162	0.004035	0.003910	0.003786	0.003658	0.003515
0.008	0.005764	0.005591	0.005424	0.005263	0.005106	0.004955	0.004809	0.004666	0.004528	0.004392	0.004259	0.004127	0.003994	0.003859	0.003708
0.009	0.006353	0.006164	0.005981	0.005803	0.005632	0.005465	0.005304	0.005147	0.004994	0.004844	0.004697	0.004551	0.004405	0.004255	0.004087
0.010	0.006929	0.006724	0.006525	0.006332	0.006145	0.005964	0.005788	0.005616	0.005449	0.005286	0.005125	0.004965	0.004805	0.004641	0.004456
0.012	0.008045	0.007830	0.007579	0.007356	0.007140	0.006930	0.006726	0.006527	0.006333	0.006143	0.005955	0.005769	0.005582	0.005390	0.005173
0.0125	0.008317	0.008073	0.007836	0.007606	0.007383	0.007166	0.006955	0.006750	0.006549	0.006352	0.006158	0.005965	0.005772	0.005573	0.005347
0.014	0.009118	0.008853	0.008594	0.008343	0.008099	0.007861	0.007631	0.007405	0.007185	0.006969	0.006755	0.006543	0.006330	0.006111	0.005862
0.016	0.010156	0.009862	0.009576	0.009298	0.009027	0.008763	0.008506	0.008255	0.008010	0.007768	0.007530	0.007293	0.007054	0.006808	0.006528
0.018	0.011162	0.010842	0.010529	0.010224	0.009927	0.009638	0.009356	0.009080	0.008810	0.008545	0.008282	0.008020	0.007757	0.007485	0.007174
0.020	0.012140	0.011794	0.011456	0.011126	0.010804	0.010490	0.010183	0.009884	0.009590	0.009300	0.009014	0.008728	0.008440	0.008143	0.007802
0.025	0.014478	0.014073	0.013674	0.013285	0.012903	0.012531	0.012166	0.011809	0.011458	0.011111	0.010768	0.010425	0.010078	0.009718	0.009303
0.030	0.016688	0.016228	0.015774	0.015329	0.014892	0.014465	0.014046	0.013634	0.013229	0.012828	0.012430	0.012032	0.011628	0.011208	0.010722
0.035	0.018790	0.018279	0.017774	0.017277	0.016789	0.016309	0.015839	0.015375	0.014918	0.014466	0.014016	0.013565	0.013106	0.012629	0.012073
0.04	0.020798	0.020240	0.019688	0.019142	0.018605	0.018077	0.017557	0.017044	0.016538	0.016036	0.015536	0.015034	0.014522	0.013988	0.013365
0.05	0.024575	0.023935	0.023296	0.022661	0.022034	0.021415	0.020803	0.020198	0.019599	0.019003	0.018408	0.017808	0.017195	0.016553	0.015797
0.06	0.028083	0.027371	0.026656	0.025942	0.025234	0.024531	0.023836	0.023146	0.022460	0.021776	0.021091	0.020399	0.019691	0.018944	0.018062
0.07	0.031362	0.030589	0.029806	0.029022	0.028240	0.027461	0.026688	0.025918	0.025152	0.024386	0.023616	0.022837	0.022036	0.021190	0.020184
0.075	0.032925	0.032124	0.031312	0.030495	0.029678	0.028864	0.028054	0.027247	0.026442	0.025636	0.024826	0.024004	[illegible]	0.022265	0.021198
0.08	0.034441	0.033615	0.032774	0.031926	0.031077	0.030229	0.029383	0.028540	0.027698	0.026854	0.026004	0.025141	0.024252	0.023311	0.022184
0.09	0.037341	0.036472	0.035580	0.034675	0.033765	0.032852	0.031940	0.031027	0.030114	0.029196	0.028269	0.027327	0.026353	0.025319	0.024075
0.10	0.040080	0.039175	0.038238	0.037283	0.036317	0.035346	0.034371	0.033394	0.032413	0.031425	0.030426	0.029406	0.028351	0.027228	0.025869
0.12	0.045120	0.044166	0.043161	0.042122	0.041061	0.039986	0.038900	0.037805	0.036701	0.035583	0.034447	0.033284	0.032074	0.030779	0.029199

Table Continued/

Table 2 Continued

x \ X	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.0
0.14	0.049640	0.048666	0.047616	0.046515	0.045379	0.044217	0.043036	0.041838	0.040623	0.039388	0.038127	0.036830	0.035476	0.034018	0.032226
0.15	0.051721	0.050748	0.049685	0.048560	0.047393	0.046195	0.044971	0.043726	0.042461	0.041171	0.039852	0.038492	0.037069	0.035534	0.033638
0.16	0.053690	0.052725	0.051654	0.050512	0.049318	0.048086	0.046824	0.045536	0.044223	0.042882	0.041507	0.040086	0.038596	0.036985	0.034988
0.18	0.057309	0.056381	0.055313	0.054149	0.052915	0.051628	0.050299	0.048933	0.047533	0.046096	0.044617	0.043081	0.041464	0.039707	0.037512
0.20	0.060520	0.059659	0.058619	0.057453	0.056195	0.054868	0.053485	0.052054	0.050578	0.049055	0.047480	0.045838	0.044102	0.042206	0.039821
0.22	0.063340	0.062581	0.061592	0.060444	0.059180	0.057828	0.056404	0.054918	0.053377	0.051778	0.050116	0.048375	0.046527	0.044499	0.041930
0.24	0.065778	0.065157	0.064246	0.063137	0.061884	0.060521	0.059069	0.057541	0.055944	0.054278	0.052538	0.050707	0.048753	0.046600	0.043854
0.25	0.066853	0.066319	0.065458	0.064376	0.063136	0.061773	0.060311	0.058766	0.057145	0.055449	0.053672	0.051799	0.049796	0.047583	0.044749
0.26	0.067833	0.067396	0.066593	0.065545	0.064321	0.062962	0.061494	0.059935	0.058293	0.056569	0.054758	0.052844	0.050793	0.048521	0.045602
0.28	0.069494	0.069301	0.068638	0.067675	0.066498	0.065159	0.063689	0.062110	0.060432	0.058660	0.056787	0.054797	0.052655	0.050271	0.047186
0.30	0.070721	0.070867	0.070385	0.069533	0.068423	0.067119	0.065661	0.064074	0.062371	0.060558	0.058631	0.056574	0.054347	0.051856	0.048611
0.32	0.071431	0.072084	0.071833	0.071121	0.070100	0.068848	0.067416	0.065832	0.064115	0.062271	0.060299	0.058180	0.055875	0.053285	0.049885
0.34	0.071733	0.072924	0.072975	0.072440	0.071531	0.070350	0.068958	0.067390	0.065669	0.063804	0.061794	0.059622	0.057246	0.054561	0.051012
0.35	0.071754	0.073181	0.073429	0.072998	0.072155	0.071017	0.069650	0.068095	0.066376	0.064504	0.062478	0.060282	0.057873	0.055144	0.051522
0.36	0.071694	0.073302	0.073802	0.073487	0.072717	0.071627	0.070290	0.068751	0.067037	0.065160	0.063121	0.060903	0.058463	0.055690	0.051997
0.38	0.071352	0.073228	0.074290	0.074254	0.073655	0.072679	0.071414	0.069917	0.068222	0.066343	0.064263	0.062027	0.059530	0.056675	0.052845
0.40	0.070758	0.072807	0.074390	0.074731	0.074344	0.073504	0.072330	0.070891	0.069225	0.067355	0.065283	0.062997	0.060449	0.057520	0.053557
0.42	0.069675	0.072087	0.074004	0.074899	0.074773	0.074099	0.073037	0.071671	0.070049	0.068197	0.066124	0.063815	0.061224	0.058225	0.054136
0.44	0.068786	0.071101	0.073247	0.074723	0.074932	0.074461	0.073533	0.072257	0.070692	0.068871	0.066805	0.064482	0.061855	0.058794	0.054585
0.45	0.068162	0.070516	0.072751	0.074481	0.074906	0.074551	0.073701	0.072477	0.070946	0.069145	0.067086	0.064759	0.062118	0.059028	0.054760
0.46	0.067489	0.069874	0.072184	0.074104	0.074805	0.074580	0.073814	0.072648	0.071155	0.069376	0.067328	0.064999	0.062345	0.059228	0.054904
0.48	0.065999	0.068430	0.070855	0.073051	0.074362	0.074446	0.073874	0.072839	0.071435	0.069712	0.067691	0.065367	0.062693	0.059528	0.055095
0.50	0.064333	0.066786	0.069289	0.071675	0.073545	0.074042	0.073705	0.072826	0.071529	0.069876	0.067896	0.065586	0.062899	0.059693	0.055159
0.52	0.062505	0.064961	0.067510	0.070026	0.072243	0.073346	0.073296	0.072604	0.071434	0.069866	0.067939	0.065653	0.062964	0.059724	0.055095
0.54	0.060529	0.062970	0.065539	0.068138	0.070579	0.072313	0.072632	0.072163	0.071144	0.069680	0.067819	0.065568	0.062886	0.059621	0.054904
0.55	0.059488	0.061916	0.064486	0.067114	0.069636	0.071642	0.072197	0.071857	0.070924	0.069519	0.067696	0.065468	0.062793	0.059518	0.054760
0.56	0.058416	0.060827	0.063392	0.066039	0.068627	0.070831	0.071690	0.071492	0.070653	0.069312	0.067532	0.065329	0.062664	0.059382	0.054585
0.58	0.056178	0.058546	0.061088	0.063750	0.066430	0.068905	0.070437	0.070578	0.069952	0.068756	0.067075	0.064933	0.062295	0.059005	0.054136
0.60	0.053828	0.056140	0.058641	0.061292	0.064019	0.066665	0.068800	0.069398	0.069030	0.068006	0.066442	0.064375	0.061777	0.058490	0.053557
0.62	0.051376	0.053621	0.056066	0.058683	0.061419	0.064165	0.066651	0.067924	0.067872	0.067053	0.065628	0.063654	0.061108	0.057833	0.052845
0.64	0.048834	0.051001	0.053376	0.055938	0.058652	0.061443	0.064133	0.066102	0.066458	0.065884	0.064625	0.062761	0.060281	0.057031	0.051997
0.65	0.047531	0.049657	0.051992	0.054520	0.057212	0.060009	0.062764	0.065027	0.065647	0.065215	0.064049	0.062248	0.059808	0.056575	0.051522

Table Continued/

Table 2 Continued

x \ X	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.0
0.66	0.046210	0.048292	0.050584	0.053073	0.055739	0.058531	0.061332	0.063797	0.064761	0.064486	0.063422	0.061690	0.059294	0.056081	0.051012
0.68	0.043516	0.045505	0.047703	0.050104	0.052697	0.055453	0.058297	0.061015	0.062737	0.062839	0.062009	0.060434	0.058138	0.054977	0.049885
0.70	0.040763	0.042651	0.044745	0.047044	0.049544	0.052232	0.055066	0.057911	0.060298	0.060916	0.060370	0.058982	0.056808	0.053714	0.048611
0.72	0.037959	0.039741	0.041722	0.043906	0.046296	0.048890	0.051668	0.054553	0.057281	0.058678	0.058486	0.057321	0.055294	0.052285	0.047186
0.74	0.035116	0.036785	0.038646	0.040704	0.042968	0.045444	0.048130	0.050987	0.053867	0.056059	0.056330	0.055436	0.053585	0.050683	0.045602
0.75	0.033683	0.035294	0.037091	0.039083	0.041279	0.043689	0.046316	0.049139	0.052049	0.054560	0.055139	0.054404	0.052653	0.049813	0.044749
0.76	0.032244	0.033795	0.035528	0.037452	0.039576	0.041915	0.044477	0.047252	0.050167	0.052878	0.053866	0.053307	0.051668	0.048896	0.043854
0.78	0.029352	0.030782	0.032382	0.034162	0.036136	0.038319	0.040731	0.043380	0.046243	0.049149	0.051034	0.050907	0.049525	0.046914	0.041930
0.80	0.026453	0.027757	0.029219	0.030849	0.032661	0.034675	0.036914	0.039401	0.042145	0.045077	0.047713	0.048198	0.047137	0.044721	0.039821
0.82	0.023556	0.024732	0.026051	0.027525	0.029168	0.030999	0.033047	0.035342	0.037913	0.040756	0.043675	0.045125	0.044473	0.042300	0.037512
0.84	0.020674	0.021719	0.022893	0.024206	0.025671	0.027311	0.029152	0.031229	0.033584	0.036253	0.039186	0.041591	0.041495	0.039625	0.034988
0.85	0.019243	0.020221	0.021321	0.022552	0.023927	0.025468	0.027200	0.029161	0.031395	0.033950	0.036826	0.039580	0.039871	0.038183	0.033638
0.86	0.017819	0.018731	0.019757	0.020905	0.022189	0.023628	0.025249	0.027089	0.029194	0.031622	0.034406	0.037318	0.038146	0.036666	0.032226
0.88	0.015004	0.015782	0.016658	0.017639	0.018737	0.019970	0.021362	0.022949	0.024778	0.026913	0.029434	0.032341	0.034331	0.033379	0.029199
0.90	0.012244	0.012888	0.013613	0.014425	0.015335	0.016358	0.017515	0.018837	0.020369	0.022175	0.024349	0.026997	0.029841	0.029700	0.025869
0.92	0.009555	0.010065	0.010640	0.011283	0.012004	0.012815	0.013733	0.014784	0.016006	0.017456	0.019224	0.021453	0.024272	0.025525	0.022184
0.925	0.008896	0.009373	0.009910	0.010511	0.011185	0.011943	0.012801	0.013784	0.014927	0.016285	0.017946	0.020052	0.022779	0.024384	0.021198
0.94	0.006956	0.007334	0.007759	0.008235	0.008768	0.009367	0.010046	0.010823	0.011729	0.012807	0.014134	0.015842	0.018167	0.020651	0.018062
0.95	0.005698	0.006010	0.006362	0.006755	0.007195	0.007689	0.008249	0.008890	0.009637	0.010528	0.011626	0.013049	0.015031	0.017786	0.015797
0.96	0.004471	0.004719	0.004997	0.005309	0.005657	0.006047	0.006490	0.006996	0.007586	0.008289	0.009157	0.010287	0.011886	0.014406	0.013365
0.975	0.002702	0.002854	0.003025	0.003216	0.003429	0.003667	0.003937	0.004245	0.004604	0.005030	0.005557	0.006244	0.007227	0.008919	0.009303
0.98	0.002133	0.002254	0.002390	0.002541	0.002710	0.002900	0.003113	0.003357	0.003640	0.003977	0.004392	0.004934	0.005709	0.007063	0.007802
0.9875	0.001304	0.001379	0.001463	0.001556	0.001660	0.001776	0.001907	0.002057	0.002230	0.002435	0.002667	0.003016	0.003486	0.004313	0.005347
1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3/

x	$X_1 = 0.7$			$X_1 = 0.75$			$X_1 = 0.8$			$X_1 = 0.85$		
	ψ_c	ε_c	ε'_c	ψ_c	ε_c	ε'_c	ψ_c	ε_c	ε'_c	ψ_c	ε_c	ε'_c
0	0	-0.1956	*	0	-0.1909_5	*	0	-0.1860	*	0	-0.1807	*
0.005	0.04326	-0.1753	0.1402	0.04195	-0.1712	0.1360	0.04069	-0.1669	0.1320	0.03944	-0.1621	0.1283
0.0075	0.04972	-0.1708	0.1389	0.04824	-0.1669	0.1347	0.04677	-0.1627	0.1307	0.04532	-0.1580	0.1269
0.0125	0.05895	-0.1639	0.1369	0.05717	-0.1601_5	0.1326	0.05543	-0.1561	0.1286	0.05369	-0.1517	0.1248
0.025	0.07339	-0.1512	0.1336	0.07117	-0.1479	0.1293	0.06897	-0.1443	0.1252	0.06677	-0.1402	0.1213_5
0.05	0.08993	-0.1337	0.1297	0.08719	-0.1309	0.1253	0.08446	-0.1278	0.1211	0.08171	-0.1243	0.1171_5
0.075	0.1004	-0.1203	0.1273	0.0973	-0.1181	0.1228	0.0943	-0.1154	0.1185	0.0911	-0.1122_5	0.1144
0.10	0.1080	-0.1091	0.1257	0.1047_5	-0.1072_5	0.1211	0.1014	-0.1050	0.1167	0.0980	-0.1022	0.1125
0.15	0.1189	-0.0902	0.1239	0.1153	-0.0890	0.1190	0.1116	-0.0874	0.1143	0.1078	-0.0853	0.1099
0.20	0.1264	-0.0739	0.1232	0.1226	-0.0734	0.1180	0.1187	-0.0725	0.1130	0.1146	-0.0709	0.1083
0.25	0.1320	-0.0591	0.1234	0.1280_5	-0.0593	0.1178	0.1240	-0.0589	0.1125	0.1196	-0.0580	0.1075
0.30	0.1361	-0.0452	0.1243	0.1321_5	-0.0461	0.1183	0.1279	-0.0463	0.1126	0.1234_5	-0.0460	0.1073
0.35	0.1392	-0.0319	0.1258	0.1352	-0.0334	0.1194	0.1310	-0.0343	0.1133	0.1264	-0.0345	0.1075
0.40	0.1413	-0.0187	0.1281	0.1375	-0.0210	0.1211	0.1333	-0.0225	0.1145	0.1286	-0.0234	0.1082
0.45	0.1426	-0.0056	0.1311	0.1390	-0.0086	0.1234	0.1348	-0.0108_5	0.1162	0.1302	-0.0123_5	0.1094
0.50	0.1431	+0.0077	0.1350	0.1397_5	+0.0040	0.1266	0.1358	+0.0009	0.1186	0.1312	-0.0013	0.1111
0.55	0.1426	0.0214	0.1400	0.1397	0.0168	0.1306	0.1361	0.0129	0.1218	0.1316	+0.0099	0.1135
0.60	0.1409	0.0359	0.1464	0.1388	0.0303	0.1359	0.1356	0.0255	0.1260	0.1314	0.0216	0.1166
0.65	0.1376	0.0515	0.1549	0.1367	0.0447	0.1429	0.1343	0.0388	0.1316	0.1305	0.0338	0.1209
0.70	0.1316	0.0686	0.1663	0.1329	0.0604	0.1523	0.1317	0.0532	0.1391	0.1287	0.0470	0.1266
0.75	0.1202	0.0852	0.1316	0.1260	0.0782	0.1654	0.1273	0.0694	0.1496	0.1256	0.0616	0.1345
0.80	0.1054	0.0990	0.0987	0.1127	0.0955	0.1251	0.1193	0.0882	0.1651	0.1205	0.0784	0.1463
0.85	0.0879	0.1099	0.0677	0.0951	0.1095	0.0870	0.1031	0.1066	0.1163	0.1108_5	0.0989	0.1655
0.90	0.0679	0.1179	0.0386	0.0739	0.1198_5	0.0512	0.0812	0.1207	0.0704	0.0900	0.1191	0.1025
0.925	0.0567	0.1207	0.0250	0.0618	0.1236	0.0344	0.0681	0.1259	0.0486	0.0761	0.1268	0.0724
0.95	0.04422	0.1226	0.0123	0.04831	0.1263	0.0185	0.05334	0.1298	0.0279	0.05987	0.1328	0.0436_5
0.975	0.02949	0.1235	0.0013	0.03222	0.1278	+0.0044	0.03559	0.1322	0.0090	0.03999	0.1367	0.0168
0.9875	0.02007	0.1234	-0.0028	0.02192	0.1279	-0.0018	0.02400_5	0.1326	0.0010	0.02715	0.1376	0.0049
1.000	0	0.1227	0	0	0.1274	0	0	0.1323	0	0	0.1376	0

*See eqn.(29) and the remarks following that equation.

Table 3 Continued/

<u>Table 3 Continued</u>

x	$X_1 = 0.9$			$X_1 = 0.95$			$X_1 = 1.0$		
	ψ_c	ε_c	ε'_c	ψ_c	ε_c	ε'_c	ψ_c	ε_c	ε'_c
0	0	-0.1749	*	0	-0.1683	*	0	-0.15915	*
0.005	0.03819_5	-0.1568	0.1247	0.03693	-0.1506_5	0.1213	0.03551_5	-0.1420	0.1181
0.0075	0.04388	-0.1529	0.1233	0.04240	-0.1468	0.1199	0.04074	-0.1382	0.1167
0.0125	0.05195	-0.1467	0.1212	0.05016	-0.1408	0.1178	0.04813	-0.1324	0.1146
0.025	0.06455	-0.1355	0.1177	0.06224_5	-0.1299	0.1142	0.05959	-0.1218	0.1109_5
0.05	0.07890	-0.1201	0.1134	0.07595	-0.1150	0.1098_5	0.07248	-0.1073	0.1065
0.075	0.0879	-0.1085	0.1106	0.0845	-0.1037_5	0.1069	0.0805	-0.0964	0.1035
0.10	0.0945	-0.0988	0.1085	0.0908	-0.0944	0.1048	0.0862	-0.0874	0.1012
0.15	0.1038	-0.0825	0.1057	0.0995	-0.0787	0.1017	0.0942	-0.0722	0.0979_5
0.20	0.1103	-0.0687	0.1039	0.1055	-0.0654	0.0997	0.0995_5	-0.0595	0.0957
0.25	0.1150	-0.0563	0.1028	0.1099	-0.0535_5	0.0983	0.1033_5	-0.0481	0.0940
0.30	0.1186	-0.0448	0.1022	0.1132	-0.0426	0.0974	0.1061	-0.0376_5	0.0928
0.35	0.1213	-0.0339	0.1020	0.1156	-0.0322	0.0968	0.1080	-0.0278	0.0919
0.40	0.1234	-0.0234	0.1023	0.1174	-0.0222	0.0966	0.1093	-0.0183	0.0913
0.45	0.1249	-0.0130	0.1029	0.1186_5	-0.0124	0.0968	0.1101	-0.0091	0.0910
0.50	0.1258	-0.0026	0.1040	0.1194	-0.0027	0.0972_5	0.1103	0	0.0908
0.55	0.1262	+0.0079	0.1056	0.1196	$+0.0070_5$	0.0981	0.1101	+0.0091	0.0910
0.60	0.1261	0.0187	0.1078	0.1194	0.0170	0.0993	0.1093	0.0183	0.0913
0.65	0.1254	0.0299	0.1107	0.1186	0.0274	0.1011	0.1080	0.0278	0.0919
0.70	0.1240	0.0420	0.1147	0.1172	0.0383	0.1035	0.1061	0.0376_5	0.0928
0.75	0.1216	0.0551	0.1203	0.1150	0.0501	0.1068	0.1033_5	0.0481	0.0940
0.80	0.1178	0.0700	0.1286	0.1118	0.0632	0.1117	0.0995_5	0.0595	0.0957
0.85	0.1117	0.0878	0.1418	0.1069	0.0784	0.1193	0.0942	0.0722	0.0979_5
0.90	0.0995	0.1111	0.1670	0.0990	0.0974_5	0.1133	0.0862	0.0874	0.1012
0.925	0.0865	0.1237	0.1204	0.0926	0.1098	0.1462	0.0805	0.0964	0.1035
0.95	0.06897	0.1338	0.0753	0.08161	0.1261	0.1706	0.07248	0.1073	0.1065
0.975	0.04629	0.1408	0.0325	0.05713	0.1425	0.0797	0.05959	0.1218	0.1109_5
0.9875	0.03138	0.1429	0.0127	0.03882	0.1477_5	0.0362_5	0.04813	0.1324	0.1146
1.000	0	0.1434	0	0	0.1500	0	0	0.1591_5	0

*See eqn.(29) and the remarks following that equation.

Appendix I/

Appendix I

The Relation of the Velocities at the Leading and Trailing Edges to a_o and C_{Lopt}

From Part II, eqn.(67), the velocity at the leading edge is given by

$$\frac{q}{U} = \frac{e^{C_0}\,[1 + \varepsilon'(0)]}{\psi_L}\left\{\left(1 - \frac{C_L^2}{a_o^2}\right)^{\frac{1}{2}}[\varepsilon(0) - \beta] + C_L\left[\frac{1}{a_o} + \frac{e^{-C_0}}{2\pi}\right]\right\},$$

and at the trailing edge by

$$\frac{q}{U} = -\frac{e^{C_0}\,[1 + \varepsilon'(\pi)]}{\psi_T}\left[\frac{1}{a_o} - \frac{e^{-C_0}}{2\pi}\right] C_L.$$

C_L is known. ψ_L, ψ_T are equal to $(2\rho_L)^{\frac{1}{2}}$, $(2\rho_T)^{\frac{1}{2}}$, respectively, where ρ_L and ρ_T are the radii of curvature at the leading and trailing edges; hence ψ_L and ψ_T are given in terms of g_s by eqns. (13) and (14) of Part III; C_0 is given in terms of g_s by eqns. (18) and (29) of Part III; $\varepsilon_c'(0) = \varepsilon_c'(\pi) = 0$, so from eqn. (33) of Part III

$$\varepsilon'(0) = \tfrac{1}{2}g_s(0) - \tfrac{1}{2}C_0, \qquad \varepsilon'(\pi) = \tfrac{1}{2}g_s(\pi) - \tfrac{1}{2}C_0,$$

and $\varepsilon'(0)$, $\varepsilon'(\pi)$ are also given in terms of g_s.

Finally $\varepsilon_s(0) = 0$, so

$$\varepsilon(0) - \beta = \varepsilon_c(0) - \beta;$$

from Part II, eqn. (39),

$$\beta = \varepsilon_c(\pi) = \tfrac{1}{2}A_1 - A_0,$$

and similarly it may be shown that

$$\varepsilon_c(0) = -\tfrac{1}{2}A_1 - A_0,$$

so

$$\varepsilon(0) - \beta = -A_1 = -\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{Lopt}$$

(from eqn. 50, Part II). Hence the velocities at the leading and trailing edges may be expressed in terms of C_L, g_s, C_{Lopt} and a_o.

Appendix II/

Appendix II

Lemma 1

$$\int_0^{\pi} \frac{1 - \cos t - \frac{2t}{\pi}}{\sin t}\, dt = 0.$$

This result is obvious, since the integrand is free from singularities at 0 and π and is antisymmetrical about $\frac{1}{2}\pi$.

Lemma 2

$$P\int_0^{\pi} \frac{1 - \cos t - \frac{2t}{\pi}}{\sin t(\cos\theta - \cos t)}\, dt = -\frac{1}{\sin^2\theta}\{(1 - \cos\theta)\ \ell n\ \tfrac{1}{2}(1 - \cos\theta) + (1 + \cos\theta)\ \ell n\ \tfrac{1}{2}(1 + \cos\theta)\}.^{*}$$

To prove this result we note that

$$\int \frac{1 - \cos t}{\sin t(\cos\theta - \cos t)}\, dt = \int \frac{\sin t}{(1 + \cos t)(\cos\theta - \cos t)}\, dt = \frac{1}{1 + \cos\theta}\ \ell n\ \frac{|\cos\theta - \cos t|}{1 + \cos t},$$

and

$$\int \frac{dt}{\sin t(\cos\theta - \cos t)} = \frac{1}{\sin^2\theta}\ \ell n\ |\cos\theta - \cos t| - \frac{1}{2(1 + \cos\theta)}\ \ell n\ (1 + \cos t) - \frac{1}{2(1 - \cos\theta)}\ \ell n\ (1 - \cos t)$$

$$= \frac{1}{\sin^2\theta}\left\{\ell n\ |\cos\theta - \cos t| - \ell n\ \sin t - \cos\theta\ \ell n\ \tan\tfrac{1}{2}t\right\},$$

so

$$\int \frac{t\, dt}{\sin t(\cos\theta - \cos t)} = \frac{t}{\sin^2\theta}\ \ell n\ |\cos\theta - \cos t| - \frac{t}{2(1 + \cos\theta)}\ \ell n\ (1 + \cos t) - \frac{t}{2(1 - \cos\theta)}\ \ell n\ (1 - \cos t)$$

$$- \frac{1}{\sin^2\theta}\int\left\{\ell n|\cos\theta - \cos t| - \ell n\ \sin t - \cos\theta\ \ell n\ \tan\tfrac{1}{2}t\right\} dt.$$

Now/

*ℓn is used to denote $\log_e$.

Now

$$\int_0^\pi \ell n \sin t \, dt = -\pi \ell n 2,$$

$$\int_0^\pi \ell n \tan \tfrac{1}{2}t dt = 0,$$

and

$$\int_0^\pi \ell n |\cos\theta - \cos t| \, dt = \tfrac{1}{2}\int_0^{2\pi} \Big\{\ell n 2 + \ell n |\sin \tfrac{1}{2}(\theta + t)|$$

$$+ \ell n |\sin \tfrac{1}{2}(t - \theta)|\Big\} dt$$

$$= \pi \ell n 2 + \int_0^{2\pi} \ell n \sin \tfrac{1}{2}t \, dt = \pi \ell n 2 + 2\int_0^\pi \ell n \sin t \, dt$$

$$= -\pi \ell n 2,$$

so

$$\int_0^\pi \Big\{\ell n |\cos\theta - \cos t| - \ell n \sin t - \cos\theta \, \ell n \tan \tfrac{1}{2}t\Big\} dt = 0.$$

Hence

$$P\int_0^\pi \frac{1 - \cos t - \frac{2t}{\pi}}{\sin t(\cos\theta - \cos t)} dt = \Bigg[\frac{1}{\sin^2\theta}\left(1 - \cos\theta - \frac{2t}{\pi}\right) \ell n|\cos\theta - \cos t|$$

$$- \frac{1}{1 + \cos\theta}\left(1 - \frac{t}{\pi}\right) \ell n (1 + \cos t) + \frac{t}{\pi(1 - \cos\theta)} \ell n (1 - \cos t)\Bigg]_0^\pi$$

$$= -\frac{1}{1 - \cos\theta} \ell n (1 + \cos\theta) + \frac{1}{1 - \cos\theta} \ell n 2$$

$$- \frac{1}{1 + \cos\theta} \ell n (1 - \cos\theta) + \frac{1}{1 + \cos\theta} \ell n 2$$

$$= -\frac{1}{\sin^2\theta}\left\{(1 - \cos\theta)\, \ell n \frac{1 - \cos\theta}{2} + (1 + \cos\theta)\, \ell n \frac{1 + \cos\theta}{2}\right\}.$$

Appendix III/

Appendix III

g_i Linear in Each of Three Segments

If

$$\left.\begin{aligned} g_i &= a_0 + a_1 x \quad (0 \leqslant x \leqslant X_1), \\ &= b_0 + b_1 x \quad (X_1 \leqslant x \leqslant X_2), \\ &= c_0 + c_1 x \quad (X_2 \leqslant x \leqslant 1) \end{aligned}\right\}$$

we put $a_2 = b_2 = c_2 = k_2 = \ell_2 = 0$ in eqns. (49) – (56) of §5.

If g_i has the values shown in Fig. 1, then

$$a_0 = a, \quad a_1 = (b - a)/X_1,$$

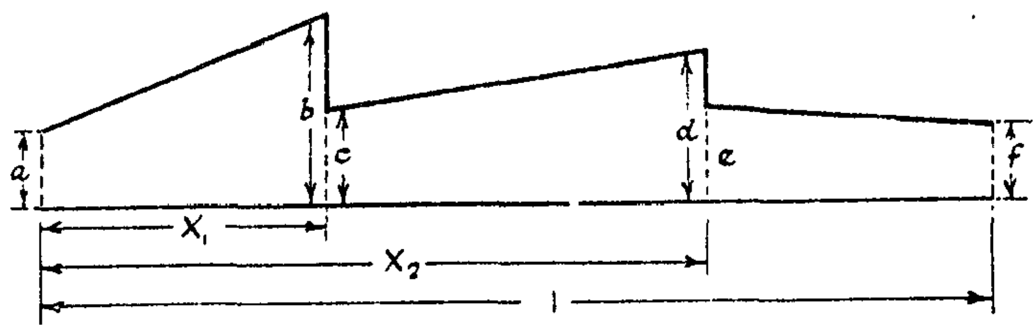
$$b_0 = \frac{b\,X_2 - d\,X_1 - (b-c)X_2}{X_2 - X_1}, \qquad b_1 = \frac{d - b + (b-c)}{X_2 - X_1}$$

$$c_0 = \frac{d - f\,X_2 - (d-e)}{1 - X_2}, \qquad c_1 = \frac{f - d + (d-e)}{1 - X_2}.$$

If $b = c$ (Fig. 2) or $d = e$ (Fig. 3), g_i has only one discontinuity; if $b = c$ and $d = e$ (Fig. 4), g_i is continuous.

For the general case of Fig. 1,

FIG. 1.



$g_i(x)$/

$$g_i(x) = a\left(1 - \frac{x}{X_1}\right) + b\,\frac{x}{X_1} \quad (0 \leqslant x \leqslant X_1),$$

$$= \frac{b}{X_2-X_1}\,(X_2-x) + \frac{d}{X_2-X_1}\,(x-X_1) - \frac{b-c}{X_2-X_1}\,(X_2-x) \quad (X_1 \leqslant x \leqslant X_2),$$

$$= \frac{d}{1-X_2}\,(1-x) + \frac{f}{1-X_2}\,(x-X_2) - \frac{d-e}{1-X_2}\,(1-x) \quad (X_2 \leqslant x \leqslant 1);$$

$$G_i(x) = a\left(x - \frac{x^2}{2X_1}\right) + b\,\frac{x^2}{2X_1} \quad (0 \leqslant x \leqslant X_1),$$

$$= a\,\frac{X_1}{2}\,\frac{b}{2(X_2-X_1)}\left[X_1X_2 - 2X_2x + x^2\right] + \frac{d}{2(X_2-X_1)}\,(x-X_1)^2$$

$$+ \frac{b-c}{2(X_2-X_1)}\,(x-X_1)(x-2X_2+X_1) \quad (X_1 \leqslant x \leqslant X_2),$$

$$= a\,\frac{X_1}{2} + b\,\frac{X_2}{2} - \frac{d}{2(1-X_2)}\left[X_1+X_2-X_1X_2-2x+x^2\right] + \frac{f}{2(1-X_2)}\,(x-X_2)^2$$

$$- (b-c)\,\frac{X_2-X_1}{2} + \frac{d-e}{2(1-X_2)}\,(2X_2-X_2^2-2x+x^2) \quad (X_2 \leqslant x \leqslant 1);$$

$$A_1 = \frac{4}{\pi}\,G_i(1) = \frac{4}{\pi}\left\{a\,\frac{X_1}{2} + b\,\frac{X_2}{2} + d\left(\frac{1-X_1}{2}\right) + f\left(\frac{1-X_2}{2}\right)\right.$$

$$\left. - (b-c)\left(\frac{X_2-X_1}{2}\right) - (d-e)\left(\frac{1-X_2}{2}\right)\right\};$$

A_0/

$$A_0 = -\frac{1}{\pi}\left\{a\left[-\frac{(1-X_1)^2}{2X_1}\,\ell n\,(1-X_1) + \frac{X_1}{2}\,\ell n\,X_1 - \tfrac{1}{2}\right]\right.$$

$$+\, b\left[\frac{X_2(1-X_1)^2}{2X_1(X_2-X_1)}\,\ell n\,(1-X_1) - \frac{(1-X_2)^2}{2(X_2-X_1)}\,\ell n\,(1-X_2) - \frac{X_1\,X_2}{2(X_2-X_1)}\,\ell n\,X_1\right.$$

$$\left. +\, \frac{X_2^2}{2(X_2-X_1)}\,\ell n\,X_2\right] + d\left[-\frac{(1-X_1)^2}{2(X_2-X_1)}\,\ell n\,(1-X_1) + \frac{(1-X_1)(1-X_2)}{2(X_2-X_1)}\,\ell n(1-X_2)\right.$$

$$\left. +\, \frac{X_1^2}{2(X_2-X_1)}\,\ell n\,X_1 - \frac{X_2^2(1-X_1)}{2(1-X_2)(X_2-X_1)}\,\ell n\,X_2\right]$$

$$+\, f\left[-\frac{1-X_2}{2}\,\ell n\,(1-X_2) + \frac{X_2^2}{2(1-X_2)}\,\ell n\,X_2 + \tfrac{1}{2}\right]$$

$$+\, (b-c)\left[\frac{(1-X_1)(2X_2-1-X_1)}{2(X_2-X_1)}\,\ell n\,(1-X_1) + \frac{(1-X_2)^2}{2(X_2-X_1)}\,\ell n\,(1-X_2)\right.$$

$$\left. +\, \frac{X_1(2X_2-X_1)}{2(X_2-X_1)}\,\ell n\,X_1 - \frac{X_2^2}{2(X_2-X_1)}\,\ell n\,X_2 + \tfrac{1}{2}\right]$$

$$\left. +\, (d-e)\left[\tfrac{1}{2}(1-X_2)\,\ell n\,(1-X_2) + \frac{X_2(2-X_2)}{2(1-X_2)}\,\ell n\,X_2 + \tfrac{1}{2}\right]\right\} ;$$

$$y_c = a\eta_1 + b\eta_2 + d\eta_3 + f\eta_4 + (b-c)\,\eta_5 + (d-e)\,\eta_6,$$

where

$$\pi\eta_1 = -\frac{(x-X_1)^2}{2X_1}\,\ell n\,|x-X_1| - \left(x - \frac{x^2}{2X_1}\right)\ell n\,x$$

$$+ \left[\frac{(1-X_1)^2}{2X_1}\,\ell n\,(1-X_1)\right] x + \left[\frac{X_1}{2}\,\ell n\,X_1\right](1-x),$$

$\pi\eta_2$/

$$\pi\eta_2 = \frac{X_2(x-X_1)^2}{2X_1(X_2-X_1)}\,\ell n\,|x-X_1| - \frac{(x-X_2)^2}{2(X_2-X_1)}\,\ell n\,|x-X_2| - \frac{x^2}{2X_1}\,\ell n\,x$$

$$- \left[\frac{X_2(1-X_1)^2}{2X_1(X_2-X_1)}\,\ell n\,(1-X_1) - \frac{(1-X_2)^2}{2(X_2-X_1)}\,\ell n\,(1-X_2)\right] x$$

$$- \left[\frac{X_1X_2}{2(X_2-X_1)}\,\ell n\,X_1 - \frac{X_2^2}{2(X_2-X_1)}\,\ell n\,X_2\right](1-x),$$

$$\pi\eta_3 = -\frac{(x-X_1)^2}{2(X_2-X_1)}\,\ell n\,|x-X_1| + \frac{(1-X_1)(x-X_2)^2}{2(1-X_2)(X_2-X_1)}\,\ell n\,|x-X_2| - \frac{(1-x)^2}{2(1-X_2)}\,\ell n(1-x)$$

$$+ \left[\frac{(1-X_1)^2}{2(X_2-X_1)}\,\ell n\,(1-X_1) - \frac{(1-X_1)(1-X_2)}{2(X_2-X_1)}\,\ell n\,(1-X_2)\right] x$$

$$+ \left[\frac{X_1^2}{2(X_2-X_1)}\,\ell n\,X_1 - \frac{X_2^2(1-X_1)}{2(1-X_2)(X_2-X_1)}\,\ell n\,X_2\right](1-x),$$

$$\pi\eta_4 = -\frac{(x-X_2)^2}{2(1-X_2)}\,\ell n\,|x-X_2| + \frac{(1-x)(2X_2-1-x)}{2(1-X_2)}\,\ell n\,(1-x)$$

$$+ \left[\frac{1-X_2}{2}\,\ell n\,(1-X_2)\right] x + \left[\frac{X_2^2}{2(1-X_2)}\,\ell n\,X_2\right](1-x),$$

$$\pi\eta_5 = \frac{(x-X_1)(2X_2-X_1-x)}{2(X_2-X_1)}\,\ell n\,|x-X_1| + \frac{(x-X_2)^2}{2(X_2-X_1)}\,\ell n\,|x-X_2|$$

$$- \left[\frac{(1-X_1)(2X_2-X_1-1)}{2(X_2-X_1)}\,\ell n\,(1-X_1) + \frac{(1-X_2)^2}{2(X_2-X_1)}\,\ell n\,(1-X_2)\right] x$$

$$+ \left[\frac{X_1(2X_2-X_1)}{2(X_2-X_1)}\,\ell n\,X_1 - \frac{X_2^2}{2(X_2-X_1)}\,\ell n\,X_2\right](1-x),$$

$$\pi\eta_6 = \frac{(x-X_2)(2-X_2-x)}{2(1-X_2)}\,\ell n\,|x-X_2| + \frac{(1-x)^2}{2(1-X_2)}\,\ell n\,(1-x)$$

$$- \left[\tfrac{1}{2}(1-X_2)\,\ell n\,(1-X_2)\right] x + \left[\frac{X_2(2-X_2)}{2(1-X_2)}\,\ell n\,X_2\right](1-x);$$

$\frac{dy_c}{dx}$ /

$$\frac{dy_c}{dx} - A_0 = a\xi_1 + b\zeta_2 + d\xi_3 + f\xi_4 + (b - c)\,\xi_5 + (d - e)\,\xi_6,$$

where

$$\pi\xi_1 = -\frac{x - X_1}{X_1}\,\ell n\,|x-X_1| + \frac{x - X_1}{X_1}\,\ell n\,x - 1,$$

$$\pi\xi_2 = \frac{X_2(x-X_1)}{X_1(X_2-X_1)}\,\ell n\,|x-X_1| - \frac{x-X_2}{X_2-X_1}\,\ell n\,|x-X_2| - \frac{x}{X_1}\,\ell n\,x,$$

$$\pi\xi_3 = -\frac{(x-X_1)}{X_2-X_1}\,\ell n\,|x-X_1| + \frac{(1-X_1)(x-X_2)}{(1-X_2)(X_2-X_1)}\,\ell n\,|x-X_2| + \frac{1-x}{1-X_2}\,\ell n\,(1-x),$$

$$\pi\xi_4 = -\frac{x - X_2}{1 - X_2}\,\ell n\,|x - X_2| + \frac{x - X_2}{1 - X_2}\,\ell n\,(1 - x) + 1,$$

$$\pi\xi_5 = -\frac{x - X_2}{X_2 - X_1}\,\ell n\,|x - X_1| + \frac{x - X_2}{X_2 - X_1}\,\ell n\,|x - X_2| + 1,$$

$$\pi\xi_6 = \frac{1 - x}{1 - X_2}\,\ell n\,|x - X_2| - \frac{1 - x}{1 - X_2}\,\ell n\,(1 - x) + 1;$$

and

$$-C_{M_0} = \frac{1}{6}\Big\{aX_1(4X_1-3) + bX_2(4X_1+4X_2-3) + d(1-X_1)(4X_1+4X_2+1)$$

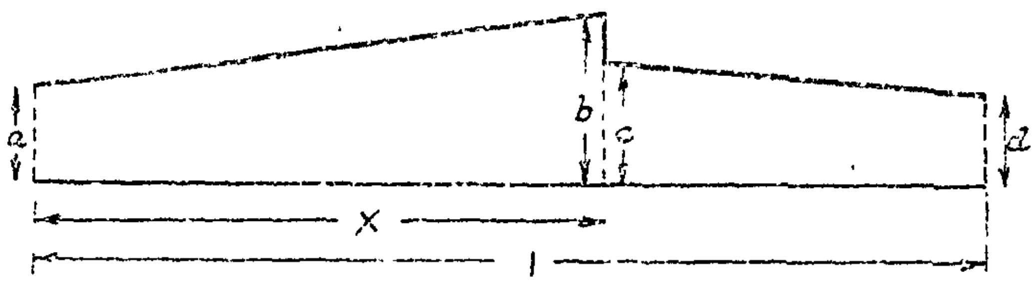
$$+ f(1-X_2)(4X_2+5) - (b-c)(X_2-X_1)(4X_2+8X_1-3) - (d-e)(1-X_2)(1+8X_2)\Big\}$$

For β, $\varepsilon_c(0)$, C_{Lopt}, α_{opt}, $\varepsilon_c(\theta)$, $\varepsilon_c'(\theta)$ see eqns. (6), (8), (9), (27) and (28).

g_i Linear in Each of Two Segments

If g_i is linear in each of two segments, $0 \leqslant x \leqslant X$, $X \leqslant x \leqslant 1$, and has the values shown in Fig. 5, then

Fig. 5



$$g_i(x) = a\left(1 - \frac{x}{X}\right) + b\,\frac{x}{X} \qquad (0 \leqslant x \leqslant X)$$

$$= \frac{b}{1-X}\,(1-x) + \frac{d}{1-X}\,(x-X) - \frac{b-c}{1-X}\,(1-x) \quad (X \leqslant x \leqslant 1).$$

We obtain the required formulae by putting $d = e = f$, and then making $X_2 \rightarrow 1$, in the results immediately preceding.

We thus obtain the following formulae.

$$G_i(x) = a\left(x - \frac{x^2}{2X}\right) + b\,\frac{x^2}{2X} \qquad (0 \leqslant x \leqslant X),$$

$$= a\,\frac{X}{2} - \frac{b}{2(1-X)}\left[X - 2x + x^2\right] + \frac{d}{2(1-X)}\,(x-X)^2 + \frac{b-c}{2(1-X)}(x-X)(x+X-$$

$$(X \leqslant x \leqslant 1);$$

$$A_1 = \frac{4}{\pi}\left\{a\,\frac{X}{2} + \frac{b}{2} + d\,\frac{1-X}{2} - (b-c)\left(\frac{1-X}{2}\right)\right\};$$

A_0/

$$A_o = -\frac{1}{\pi}\left\{a\left[-\frac{(1-X)^2}{2X}\ \ell n\ (1-X) + \frac{X}{2}\ \ell n\ X - \tfrac{1}{2}\right]\right.$$

$$+ b\left[\frac{1-X}{2X}\ \ell n\ (1-X) - \frac{X}{2(1-X)}\ \ell n\ X\right]$$

$$+ d\left[-\frac{1-X}{2}\ \ell n\ (1-X) + \frac{X^2}{2(1-X)}\ \ell n\ X + \tfrac{1}{2}\right]$$

$$\left. + (b-c)\left[\frac{1-X}{2}\ell n\ (1-X) + \frac{X(2-X)}{2(1-X)}\ \ell n\ X + \tfrac{1}{2}\right]\right\};$$

$$y_c = ah_1 + bh_2 + dh_3 + (b-c)\ h_4,$$

where

$h_1 = \eta_1$ (as given in the preceding section of this Appendix),

$$\pi h_2 = \frac{(x-X)^2}{2X(1-X)}\ \ell n\ |x-X| - \frac{(1-x)^2}{2(1-X)}\ \ell n\ (1-x) - \frac{x^2}{2X}\ \ell n\ x$$

$$-\left[\frac{1-X}{2X}\ \ell n\ (1-X)\right] x - \left[\frac{X}{2(1-X)}\ \ell n\ X\right] (1-x),$$

$$\pi h_3 = -\frac{(x-X)^2}{2(1-X)}\ \ell n\ |x-X| - \frac{(1-x)(2X-1-x)}{2(1-X)}\ \ell n\ (1-x)$$

$$+\left[\frac{1-X}{2}\ \ell n\ (1-X)\right] x + \left[\frac{X^2}{2(1-X)}\ \ell n\ X\right] (1-x),$$

$$\pi h_4 = \frac{(x-X)(2-X-x)}{2(1-X)}\ \ell n\ |x-X| + \frac{(1-x)^2}{2(1-X)}\ \ell n\ (1-x) - \left[\frac{1-X}{2}\ \ell n\ (1-X)\right] x$$

$$+\left[\frac{X(2-X)}{2(1-X)}\ \ell n\ X\right] (1-x);$$

$$\frac{dy_c}{dx} - A_o = aH_1 + bH_2 + dH_3 + (b-c)\ H_4,$$

where/

where

$$H_1 = \xi_1 \quad \text{(as given in the preceding section of this appendix)},$$

$$\pi H_2 = \frac{x-X}{X(1-X)} \ell n \, |x-X| + \frac{1-x}{1-X} \ell n \, (1-x) - \frac{x}{X} \ell n \, x,$$

$$\pi H_3 = -\frac{x-X}{1-X} \ell n \, |x-X| + \frac{x-X}{1-X} \ell n \, (1-x) + 1,$$

$$\pi H_4 = \frac{1-x}{1-X} \ell n \, |x-X| - \frac{1-x}{1-X} \ell n \, (1-x) + 1;$$

$$-C_{M_0} = \frac{1}{6} \{aX(4X-3) + b(4X+1) + d(1-X)(4X+5) - (b-c)(1-X)(8X+1)\}.$$

For β, $\varepsilon_c(0)$, C_{Lopt}, α_{opt}, $\varepsilon_c(\theta)$, $\varepsilon_c'(\theta)$ see eqns. (6), (8), (9), (27) and (28).

AH.

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