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The Supersonic Pressure Drag of a Swept Wing with a Cranked Maximum Thickness Line

By

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SUMMARY

The linearised theory is applied to a particular family of sweptback wings with cranked maximum thickness lines, and the drag of one member is analysed and compared with several other wings whose solutions are well known.

The indications are that one can approximate to the variation of drag with Mach number by combining curves of certain delta and "chevron" wings.

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1 Introduction

A considerable amount of theoretical data is available on the supersonic pressure drag at zero incidence of wings with straight maximum thickness lines but very little is known of the effect of "cranking" the maximum thickness line at a certain station along the span. In this paper the linearised theory is applied to a sweptback wing having a constant chord inboard section and a tapered section outboard, and cranked in such a manner that the whole leading edge is straight, and the outboard trailing edge is perpendicular to the free stream (Fig.1). The problem has been kept as simple as possible by considering a double wedge section with the maximum thickness at 50% chord, and the investigation has been restricted to the case where the Mach cones from the apex lie in front of the leading edge (i.e. a "subsonic" leading edge). A further restriction is that the Mach cones from the disturbances set up at each crank do not cross the opposite half of the wing. The expressions obtained are evaluated for a particular wing with the inboard and outboard maximum thickness lines swept back 60° and approximately 40° respectively, and for this wing the pressure drag has been estimated for the Mach number range M = 1.090 to M = 2.

2 Fundamental Analysis

The drag has been estimated by following the method used by Puckett[1,2]. The wing is replaced by suitable source distributions which satisfy the fundamental linearised perturbation potential equation

$$\phi_{xx}\,(1 - M^2) + \phi_{yy} + \phi_{zz} = 0$$

and also satisfy the boundary conditions for the wing.

The wing in Fig.1 is considered to be replaced by the following source distributions, the strengths being chosen so that the boundary conditions are automatically satisfied (See Refs.1 and 3)

Source distribution	Source strength
AGG'	$+\lambda\,\frac{U}{\pi}$
BHH'	$-2\lambda\,\frac{U}{\pi}$
CFF'	$+\lambda\,\frac{U}{\pi}$
EHG and E'H'G'	$-2\lambda\,\frac{U}{\pi}$

where λ is the semi-angle of the double wedge section, which, for thin sections, is equal to the thickness/chord ratio, τ, and U is the free stream velocity. For the sake of simplicity distributions EHG and E'H'G' have been broken down into distributions EGF, E'G'F' of strength $-2\lambda\,\frac{U}{\pi}$, and EHF, E'H'F' of strength $+2\lambda\,\frac{U}{\pi}$.

Assuming the existence of only small perturbations we can find the pressure coefficient C_p in terms of u the perturbation velocity in the free stream direction; thus to a first approximation $C_p = -\frac{2u}{U}$. If ϕ is known, $u = \phi_x$ may be found, and $C_p = -\frac{2\phi_x}{U}$.

Now the drag increment due to a source distribution acting on an area A is given by

$$\Delta C_D = \frac{1}{S}\int_A C_p \sin\zeta \, dA \simeq \frac{1}{S}\int_A C_p \zeta \, dA$$

$$= \frac{1}{S}\int_A -\frac{2\phi_x}{U} \cdot \zeta \, dA \qquad (1)$$

where S is the wing plan area and ζ is the slope of the elementary area dA in the free stream direction, and is assumed small.

Then $C_D = \Sigma \Delta C_D$ over the whole surface of the wing.

There are two types of source distribution to be considered, namely the symmetrical triangular distributions AGG', BHH' and CFF', and triangular distributions with one side parallel to the free stream direction, such as EGF or EHF. Since we are considering only the case where the wing has a "subsonic" leading edge, the symmetrical distributions will have "subsonic" leading edges, but the "one-sided" distributions may have either "supersonic" or "subsonic" leading edges depending on whether the Mach cone from E lies behind or ahead of G. These cases are considered in detail in Ref.3. If $\xi \frac{U}{\pi}$ is the source strength of the uniform distribution considered, ϕ_x for different zones is given by the equations below.

(a) Symmetrical triangular distribution (Fig.2a)

$$\phi_{x1} = -\frac{2\xi U}{\pi B\sqrt{n^2-1}} \cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}} \qquad (2a)$$

$$\phi_{x2} = -\frac{2\xi U}{\pi B\sqrt{n^2-1}} \cosh^{-1}\sqrt{\frac{n^2-1}{\sigma^2-1}} \qquad (2b)$$

(b) Triangular distribution with a side parallel to the free stream (Fig.2b).

(i) "Subsonic" leading edge

$$\phi_{x1} = - \frac{\xi U}{\pi B \sqrt{n^2-1}} \cosh^{-1} \frac{n^2-\sigma}{n(1-\sigma)} \tag{3a}$$

$$\phi_{x2} = - \frac{\xi U}{\pi B \sqrt{n^2-1}} \cosh^{-1} \frac{n^2-\sigma}{n(\sigma-1)} \tag{3b}$$

$$\phi_{x3} = - \frac{\xi U}{\pi B \sqrt{n^2-1}} \cosh^{-1} \frac{n^2+\sigma}{n(\sigma+1)} \tag{3c}$$

(ii) "Supersonic" leading edge

$$\phi_{x1} = - \frac{\xi U}{B \sqrt{1-n^2}} \left[1 - \frac{1}{\pi} \cos^{-1} \frac{\sigma-n^2}{n(1-\sigma)} \right] \tag{4a}$$

$$\phi_{x2} = - \frac{\xi U}{B \sqrt{1-n^2}} \tag{4b}$$

$$\phi_{x3} = - \frac{\xi U}{\pi B \sqrt{1-n^2}} \cos^{-1} \frac{\sigma+n^2}{n(1+\sigma)} \tag{4c}$$

where

$B = \sqrt{M^2-1}$

$n = k/B$

k = tangent of the sweptback angle of the leading edge of the source distribution*

σ = the ray parameter = $k\,|y/x|$, the modulus being taken in order that σ shall always be a positive quantity

x,y are streamwise and normal cartesian co-ordinates (respectively) in the plane of the wing, measured relative to an origin at the apex of the source distribution concerned.

The subscripts 1, 2, 3 refer to the zones defined in Fig.2.

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* Since the sweepback angle of the leading edges of the source distributions AGG', BHH', CFF' are all equal to the sweepback angle of the leading edge of the wing, the symbol k will be used also for the tangent of this particular angle.

Since we are considering a wing symmetrical in planform about its centreline, and of symmetrical section and at zero incidence, it is necessary to find the drag of only one surface of one half wing and multiply the answer by four to get the total drag. Equation (1) may then be replaced by

$$\Delta C_D = \frac{4}{S} \int_A - \frac{2\phi_x}{U} \zeta \, dA \tag{5}$$

where A is now restricted to areas on one quarter of the wing surface. Accordingly the distributions AGG', BHH', CFF' will be referred to as AGD, BHD, CFD respectively.

3 Evaluation of drag increments

3.1 Drag due to symmetrical source distributions

Consider first the drag increments due to the three symmetrical source distributions AGD, BHD and CFD (Fig.1).

The general expressions for drag increment are given by substituting equations (2) into (5), and since the wing area

$$S = \frac{c^2}{k} \left(\frac{1+a}{1-a}\right)$$

where c is the root chord and $a = \frac{CD}{AD}$ (Fig.1), we get

$$\Delta C_{D1} = \int_A \frac{16 \, \xi \, \zeta \, k}{\pi B \; c^2} \; \frac{(1-a)}{(1+a)} \cdot \frac{1}{\sqrt{n^2-1}} \cdot \cosh^{-1} \sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}} \cdot dA \tag{6a}$$

and

$$\Delta C_{D2} = \int_A \frac{16 \, \xi \, \zeta \, k}{\pi B \; c^2} \cdot \frac{(1-a)}{(1+a)} \cdot \frac{1}{\sqrt{n^2-1}} \cdot \cosh^{-1} \sqrt{\frac{n^2-1}{\sigma^2-1}} \cdot dA \tag{6b}$$

Knowing the values of ξ, ζ we follow the method of conical fields and choose our areas of integration so that they can be expressed in terms of a single variable, the ray parameter, σ. It is then possible to evaluate the drag increments. These will be obtained in the following sections.

3.11 Source distribution AGD ($\xi = +\lambda = +\tau$)

The source distribution AGD affects the area of wing AGFC (Fig.1), in which the slope of area AGEB is $+\lambda$ and the slope of area BEGFC is $-\lambda$ to the free stream. These areas can be suitably divided into

AGEB = AGE + AEB

and BEGFC = AGF - AGE + AFC - AEB

which, it will be seen, all have their apices at A.

Using the notation throughout that $\Delta C_{D_V}(XYZ)$ represents the drag of the area XYZ due to the source distribution with apex at V, we have

$$\Delta C_{D_A}\ (AGEB) = \Delta C_{D_A}\ (AGE) + \Delta C_{D_A}\ (AEB)$$

and

$$\Delta C_{D_A}\ (BEGFC) = \Delta C_{D_A}\ (AGF) - \Delta C_{D_A}\ (AGE) + \Delta C_{D_A}\ (AFC) - \Delta C_{D_A}\ (AEB)$$

(a) Area AGEB ($\zeta = +\lambda = +\tau$)

(i) Area AGE ($\zeta = +\lambda = +\tau$)

The drag increment is given by equation (6a) in which dA has the value

$$dA = \frac{c^2\ (1-\frac{b}{2})^2\ d\sigma}{2k\ (1-a)^2\ (1-\sigma\frac{b}{2})}$$

where dA is the elementary area dA_1 in Fig.3a and b is defined in the same figure. It will be seen that when b = 0, dA refers to a triangular area with one side lying on GF and when b=1 the side lies on GE. Expressing the drag in its general form for the areas AGE and AGF we have

$$\Delta C_D = \frac{8\ \xi\ \zeta}{\pi B\ (1-a^2)} \int_{\sigma=\alpha}^{\sigma=\beta} \frac{(1-\frac{b}{2})^2}{\sqrt{n^2-1}}\ \frac{1}{(1-\sigma\frac{b}{2})^2} \cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}\ d\sigma ,$$

the integration being performed between appropriate limits, and b being given a suitable value for each area.

It will be seen later that a more general expression can be obtained which will cover the incremental drag for the area BHF due to the source distribution B, and this expression is

$$\Delta C_D = \frac{8\ \xi\ \zeta}{\pi B\ (1-a^2)} \int_{\sigma=\alpha}^{\sigma=\beta} \frac{(r-\frac{b}{2})^2}{\sqrt{n^2-1}}\ \frac{1}{(1-\sigma\frac{b}{2})^2} \cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}\ d\sigma$$

where r is defined in Fig.1 as the ratio $\frac{BD}{AD}$.

We write this as

$$\Delta C_D = \frac{8\ \xi_i \zeta}{\pi B\ (1-a^2)}\ E(r,b)\Bigg|_{\sigma=\alpha}^{\sigma=\beta} \tag{7}$$

where

$$E(r,b) = \int \frac{(r-\frac{b}{2})^2}{\sqrt{n^2-1}}\ \frac{1}{(1-\sigma\frac{b}{2})^2}\cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}\,d\sigma.$$

Evaluating* when b = 0 and b = 1 gives

$$E(r,0) = \frac{r^2}{\sqrt{n^2-1}}\left\{\sigma\cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}} + \sqrt{n^2-1}\sin^{-1}\frac{\sigma}{n}\right.$$

$$\left. - \tfrac{1}{2}\log\left|\frac{n(1-\sigma)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(1-\sigma)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right| - \tfrac{1}{2}\log\left|\frac{n(1+\sigma)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(1+\sigma)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right|\right\} \tag{8a}$$

and

$$E(r,1) = \frac{2(r-\frac{1}{2})^2}{\sqrt{n^2-1}}\left\{\frac{\cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}}{(1-\frac{\sigma}{2})} - \log\left|\frac{n(1-\sigma)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(1-\sigma)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right|\right.$$

$$+ \frac{1}{3}\log\left|\frac{n(1+\sigma)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(1+\sigma)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right|$$

$$+ \text{either } \frac{4}{3}\sqrt{\frac{n^2-1}{1-(\frac{n}{2})^2}}\tan^{-1}\frac{n(1-\frac{\sigma}{2})-\sqrt{n^2-\sigma^2}}{\sigma\sqrt{1-(\frac{n}{2})^2}} \quad \text{for } 1 < n < 2$$

$$\left.\text{or } \frac{2}{3}\sqrt{\frac{n^2-1}{(\frac{n}{2})^2-1}}\log\left|\frac{n(1-\frac{\sigma}{2})-\sqrt{n^2-\sigma^2}-\sigma\sqrt{(\frac{n}{2})^2-1}}{n(1-\frac{\sigma}{2})-\sqrt{n^2-\sigma^2}+\sigma\sqrt{(\frac{n}{2})^2-1}}\right| \text{ for } n > 2\right\} \tag{8b}$$

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* If in E(r,b) and all the following functions the substitution $\sigma = n\sin\theta$ is made, the functions after a first integration by parts, all reduce to expressions of the form $\int \frac{d\theta}{A+B\sin\theta}$ which are dealt with in Ref.5.

For the area AGE we integrate with respect to σ between the limits AE ($\sigma = \frac{2a}{1+a}$) and AG ($\sigma = 1$).

Hence (7) will give

$$\Delta C_{D_A}\ (AGE) = \frac{8\ \tau^2}{\pi B\ (1-a^2)}\ E(1,1)\Bigg|_{\frac{2a}{1+a}}^{1} \tag{9}$$

where $E(1,1)\Bigg|_{\frac{2a}{1+a}}^{1}$ can be evaluated from 6(b).

(ii) <u>Area AEB</u> ($\zeta = +\lambda = +\tau$)

Again we use equation (6a) but dA is given by

$$dA = \frac{c^2\ d\sigma}{8k\ (1-\sigma)^2}$$ (dA is the elementary area dA_2 in Fig.3a)

Substituting we get

$$\Delta C_{D_A}\ (AEB) = \frac{2\tau^2\ (1-a)}{\pi B\ (1+a)} \int_0^{\frac{2a}{1+a}} \frac{1}{\sqrt{n^2-1}} \cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}\ \frac{d\sigma}{(1-\sigma)^2}$$

$$= \frac{2\tau^2\ (1-a)}{\pi B\ (1+a)}\ I(n)\Bigg|_0^{\frac{2a}{1+a}} \tag{10}$$

where

$$I(n) = \int \frac{1}{\sqrt{n^2-1}} \cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}\ \frac{d\sigma}{(1-\sigma)^2}$$

and is evaluated to be

$$I(n) = \frac{1}{\sqrt{n^2-1}} \left\{ \frac{\cosh^{-1}\sqrt{\frac{n^2-\sigma^2}{1-\sigma^2}}}{(1-\sigma)} - \frac{\sqrt{n^2-\sigma^2}}{2(1-\sigma)\sqrt{n^2-1}} \right.$$

$$\left. + \frac{n^2+1}{4(n^2-1)} \log\left|\frac{n(1-\sigma)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(1-\sigma)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right| + \frac{1}{4}\log\left|\frac{n(1+\sigma)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(1+\sigma)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right| \right\} \tag{11}$$

(b) Area BEGFC ($\zeta = -\lambda = -\tau$)

(i) Areas AGF - AGE - AEB ($\zeta = -\lambda = -\tau$)

Applying equations (7) and (10), and substituting the appropriate limits for σ, we have

$$\Delta C_{D_A} \text{ (AGF - AGE - AEB)} = - \frac{8\tau^2}{\pi B\,(1-a^2)} E(1,0) \Big|_a^1 + \frac{8\tau^2}{\pi B\,(1-a^2)} E(1,1) \Big|_{\frac{2a}{1+a}}^1$$

$$+ \frac{2\tau^2\,(1-a)}{\pi B\,(1+a)} I(n) \Big|_0^{\frac{2a}{1+a}} \qquad (12)$$

(ii) Area AFC ($\zeta = -\lambda = -\tau$)

Equation (6a) is once again used but dA is given by

$$dA = \frac{c^2}{2k} \frac{d\sigma}{(1-\sigma)^2} \qquad \text{(dA is the elementary area } dA_3 \text{ in Fig.3a)}$$

$$\therefore \quad \Delta C_{D_A} \text{ (AFC)} = - \frac{8\tau^2\,(1-a)}{\pi B\,(1+a)} I(n) \Big|_0^a \qquad (13)$$

Summing up, the drag increment due to source distribution AGD is given by the sum of (9), (10), (12) and (13)

$$\therefore \Delta C_{D_A} = \frac{4\tau^2}{\pi B} \frac{(1-a)}{(1+a)} \left\{ I(n) \Big|_0^{\frac{2a}{1+a}} - 2I(n) \Big|_0^a - \frac{2}{(1-a)^2} \left[E(1,0) \Big|_a^1 + 2E(1,1) \Big|_{\frac{2a}{1+a}}^1 \right] \right\}$$

(14)

3.12 Source distribution BHD ($\xi = -2\lambda = -2\tau$)

There are two cases to be considered here, viz. when the Mach wave from B lies ahead of E and when it cuts EG.

Case (a): Mach wave from B not cutting EG

The area to be considered, BJGFC, can be divided into BHFC and BJGH.

(i) Area BHFC ($\zeta = -\lambda = -\tau$)

This area can be divided into BFC and BHF.

For BFC the drag increment is the same as that for AEB due to the source distribution AGD except for a factor of 2 arising from the increase in the magnitude of ξ.

$$\text{i.e.}\quad \Delta C_{D_B}\ (\text{BFC}) = \frac{4\tau^2}{\pi B}\ \frac{1-a}{1+a}\ I(n)\Bigg|_0^{\frac{2a}{1+a}} \qquad (15a)$$

For the area BHF the equation (6a) is used with dA given by

$$\frac{c^2\ (r - \frac{b}{2})^2\ d\sigma}{2k\ (1-a)^2\ (1 - \sigma\ \frac{b}{2})^2}$$ (dA is the elementary area dA_4 in Fig.3a).

This leads to the general expression for the drag increment given by equation (7), and substituting we obtain

$$\Delta C_{D_B}\ (\text{BHF}) = \frac{16\tau^2}{\pi B\ (1-a^2)}\ E(r,0)\Bigg|_{\frac{2a}{1+a}}^{1} \qquad (15b)$$

∴ for the area BHFC we find the drag increment by adding (15a) and (15b)

$$\Delta C_{D_B}\ (\text{BHFC}) = \frac{4\tau^2}{\pi B}\ \frac{1-a}{1+a}\ I(n)\Bigg|_0^{\frac{2a}{1+a}} + \frac{16\tau^2}{\pi B\ (1-a^2)}\ E(r,0)\Bigg|_{\frac{2a}{1+a}}^{1} \qquad (16)$$

(ii) Area BJGH ($\zeta = +\lambda = +\tau$ over BJGE and $\zeta = -\lambda = -\tau$ over EGH)

Since the area BJGH is outside the boundary of the source distribution, equation (6b) must be used and dA is given by

$$dA = \frac{c^2\ (r - \frac{b}{2})^2\ d\sigma}{2k\ (1-a)^2\ (1 - \sigma\ \frac{b}{2})^2} \qquad (17)$$

(dA is the elementary area dA_4 in Fig.3a).

Substituting equation (17) into (6b) gives for ΔC_D the general form

$$\Delta C_D = \frac{8\ \xi\ \zeta}{\pi B\ (1-a^2)} \int_{\sigma=\alpha}^{\sigma=\beta} \frac{(r - \frac{b}{2})^2}{\sqrt{n^2-1}}\ \frac{\cosh^{-1}\sqrt{\frac{n^2-1}{\sigma^2-1}}}{(1 - \sigma\ \frac{b}{2})^2}\ d\sigma,$$

or if we let

$$F(r,b) = \int \frac{(r - \frac{d}{2})^2}{\sqrt{n^2-1}} \frac{\cosh^{-1}\sqrt{\frac{n^2-1}{\sigma^2-1}}}{(1 - \sigma \frac{b}{2})^2} d\sigma$$

then

$$\Delta C_D = \frac{8 \xi \zeta}{\pi B (1-a^2)} F(r,b) \Bigg|_{\sigma=\alpha}^{\sigma=\beta}$$

where r and b have appropriate values. F(r,b) has been evaluated for b = 0, 1 and 2 and has the following values

$$F(r,0) = \frac{r^2}{\sqrt{n^2-1}} \Bigg\{ \sigma \cosh^{-1}\sqrt{\frac{n^2-1}{\sigma^2-1}} + \sqrt{n^2-1} \sin^{-1}\frac{\sigma}{n}$$

$$+ \frac{1}{2} \log \left| \frac{n(\sigma-1) + \sqrt{n^2-\sigma^2} - \sigma\sqrt{n^2-1}}{n(\sigma-1) + \sqrt{n^2-\sigma^2} + \sigma\sqrt{n^2-1}} \right| - \frac{1}{2} \log \left| \frac{n(\sigma+1) - \sqrt{n^2-\sigma^2} - \sigma\sqrt{n^2-1}}{n(\sigma+1) - \sqrt{n^2-\sigma^2} + \sigma\sqrt{n^2-1}} \right| \Bigg\}$$

(18a)

$$F(r,1) = \frac{(r - \frac{1}{2})^2}{\sqrt{n^2-1}} \Bigg\{ \frac{2 \cosh^{-1}\sqrt{\frac{n^2-1}{\sigma^2-1}}}{(1 - \frac{\sigma}{2})} + \frac{2}{3} \Bigg[\log \left| \frac{n(\sigma+1) - \sqrt{n^2-\sigma^2} - \sigma\sqrt{n^2-1}}{n(\sigma+1) - \sqrt{n^2-\sigma^2} + \sigma\sqrt{n^2-1}} \right|$$

$$- 3 \log \left| \frac{n(\sigma-1) + \sqrt{n^2-\sigma^2} - \sigma\sqrt{n^2-1}}{n(\sigma-1) + \sqrt{n^2-\sigma^2} + \sigma\sqrt{n^2-1}} \right|$$

$$- \text{either } 2\sqrt{\frac{n^2-1}{(\frac{n}{2})^2-1}} \log \left| \frac{n(\frac{\sigma}{2} - 1) + \sqrt{n^2-\sigma^2} - \sigma\sqrt{n^2-1}}{n(\frac{\sigma}{2} - 1) + \sqrt{n^2-\sigma^2} + \sigma\sqrt{n^2-1}} \right| \text{ for } n > 2$$

$$\text{or } \quad 4\sqrt{\frac{n^2-1}{1 - (\frac{n}{2})^2}} \tan^{-1}\left(\frac{n(\frac{\sigma}{2} - 1) + \sqrt{n^2-\sigma^2}}{\sigma\sqrt{1 - (\frac{n}{2})^2}} \right) \text{ for } 1 < n < 2 \Bigg] \Bigg\}$$

(18b)

$$F(r,2) = \frac{(r-1)^2}{\sqrt{n^2-1}}\left\{-\frac{\cosh^{-1}\sqrt{\frac{n^2-1}{\sigma^2-1}}}{(\sigma-1)} - \frac{1}{4}\left[-\frac{2\sqrt{n^2-\sigma^2}}{(\sigma-1)\sqrt{n^2-1}}\right.\right.$$

$$+\frac{(n^2+1)}{(n^2-1)}\log\left|\frac{n(\sigma-1)+\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(\sigma-1)+\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right|$$

$$\left.\left. - \log\left|\frac{n(\sigma+1)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(\sigma+1)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right|\right]\right\} \tag{18c}$$

For the area BJGE, $\zeta = +\lambda = +\tau$

$$\Delta C_{D_B}\ (BJGE) = \Delta C_{D_B}\ (BJG + BGE)$$

$$\therefore\ \Delta C_{D_B}\ (BJGE) = -\frac{16\tau^2}{\pi B\ (1-a^2)}\left[F(r,2)\Big|_{\frac{1}{r}}^{n} + F(r,1)\Big|_{1}^{\frac{1}{r}}\right] \tag{19a}$$

For the area EGH, $\zeta = -\lambda = -\tau$

$$\Delta C_{D_B}\ (EGH) = \Delta C_{D_B}\ (BGH - BGE)$$

$$\therefore\ \Delta C_{D_B}\ (EGH) = \frac{16\tau^2}{\pi B\ (1-a^2)}\left[F(r,0)\Big|_{1}^{\frac{1}{r}} - F(r,1)\Big|_{1}^{\frac{1}{r}}\right] \tag{19b}$$

Summing up, ΔC_{D_B} is given by the sum of equations (16), (19a) and (19b)

$$\therefore\ \Delta C_{D_B} = \frac{4\tau^2}{\pi B}\frac{(1-a)}{(1+a)}\left\{I(n)\Big|_{0}^{\frac{2a}{1+a}} + \frac{4}{(1-a)^2}\left[E(r,0)\Big|_{\frac{2a}{1+a}}^{1} - F(r,2)\Big|_{\frac{1}{r}}^{n} + F(r,0)\Big|_{1}^{\frac{1}{r}}\right.\right.$$

$$\left.\left. - 2F(r,1)\Big|_{1}^{\frac{1}{r}}\right]\right\} \tag{20}$$

Case (b): Mach wave from B cutting EG

In this case ΔC_{D_B} (BJG) = 0. Otherwise the value of ΔC_{D_B} is given by the same expressions as for case (a), except that $\frac{1}{r}$ is replaced by n.

$$\therefore \quad \Delta C_{D_B} = \frac{4\tau^2}{\pi B}\,\frac{(1-a)}{(1+a)}\left\{ I(n)\Big|_0^{\frac{2a}{1+a}} + \frac{4}{(1-a)^2}\left[E(r,0)\Big|_{\frac{2a}{1+a}}^{1} + F(r,0)\Big|_1^{n} - 2F(r,1)\Big|_1^{n}\right]\right\} \tag{21}$$

3.13 Source distribution CFD ($\xi = +\lambda = +\tau$)

Again there are two cases to be considered, namely when the Mach cone form C lies ahead of E and when it lies behind E (see Fig.4).

Case (a) Mach cone from C lying behind E

The drag contribution is given by equation (6b) and the value of dA by (17), with r replaced by a.

We thus have the general formula

$$\Delta C_D = \frac{8\,\xi\,\zeta}{\pi B\,(1-a^2)}\,F(r,b)\Big|_{\sigma=\alpha}^{\sigma=\beta}$$

(i) The Mach cone from C cuts EG

For the area CNGF ($\zeta = -\lambda = -\tau$)

$$\Delta C_{D_C}\,(\text{CNGF}) = -\frac{8\tau^2}{\pi B\,(1-a^2)}\left[F(a,0)\Big|_1^{\frac{1}{a}} + F(a,1)\Big|_{\frac{1}{a}}^{n}\right] \tag{22}$$

For the area NMG ($\zeta = +\lambda = +\tau$)

$$\Delta C_{D_C}\,(\text{NMG}) = \Delta C_{D_C}\,(\text{CMG} - \text{CNG})$$

$$= \frac{8\tau^2}{\pi B\,(1-a^2)}\left[F(a,2)\Big|_{\frac{1}{a}}^{n} - F(a,1)\Big|_{\frac{1}{a}}^{n}\right] \tag{23}$$

Hence for this case ΔC_{D_C} = the sum of equations (22) and (23)

$$\therefore \Delta C_{D_C} = \frac{8\tau^2}{\pi B\ (1-a^2)} \left[F(a,2) \Big|_{\frac{1}{a}}^{n} - 2F(a,1) \Big|_{\frac{1}{a}}^{n} - F(a,0) \Big|_{1}^{\frac{1}{a}} \right] \tag{24}$$

(ii) <u>The Mach cone from C cuts FG</u>

For this case

$$\Delta C_{D_C} = -\frac{8\tau^2}{\pi B\ (1-a^2)}\ F(a,0) \Big|_{1}^{n} \tag{25}$$

Case (b) <u>The Mach cone from C lies ahead of E</u>

The relevant area CLGF may be divided into two areas CKGF and CLK.

(i) <u>Area CKGF</u>

ΔC_{D_C} (CKGF) is the same as that given in equation (24) except that n is replaced by the value of σ for EC, i.e. $2a/(3a-1)$.

(ii) <u>Area CLK</u>

Now CLK = CQE + QLKE.

(iia) <u>Area CQE</u> ($\zeta = -\lambda = -\tau$)

It may be noticed that the areas CQE and CLK are similar and have the same boundary values for σ (viz. $\sigma = n$ along CQL and $\sigma = \frac{2a}{3a-1}$ along CEK), and that the area CQE is one quarter of the area CLK.

$$\therefore \quad \Delta C_{D_C}\ (QCE) = -\frac{2\tau^2}{\pi B\ (1-a^2)}\ F(a,2) \Big|_{\frac{2a}{3a-1}}^{n} \tag{26}$$

(iib) <u>Area QLKE</u> ($\zeta = +\lambda = +\tau$)

$$\Delta C_{D_C}\ (QLKE) = \frac{8\tau^2}{\pi B\ (1-a^2)}\ F(a,2) \Big|_{\frac{2a}{3a-1}}^{n} - \frac{2\tau^2}{\pi B\ (1-a^2)}\ F(a,2) \Big|_{\frac{2a}{3a-1}}^{n}$$

$$= \frac{6\tau^2}{\pi B\ (1-a^2)}\ F(a,2) \Big|_{\frac{2a}{3a-1}}^{n} \tag{27}$$

Hence in the case when the Mach cone from C lies ahead of E, ΔC_{D_C} is given by the equivalent of equation (24) plus equations (26) and (27)

$$\Delta C_{D_C} = \frac{4\tau^2}{\pi B\,(1-a^2)} \left[2F(a,2)\Big|_{\frac{1}{a}}^{\frac{2a}{3a-1}} - 4F(a,1)\Big|_{\frac{1}{a}}^{\frac{2a}{3a-1}} \right.$$

$$\left. - 2F(a,0)\Big|_{1}^{\frac{1}{a}} + F(a,2)\Big|_{\frac{2a}{3a-1}}^{n} \right]$$

$$\therefore \Delta C_{D_C} = \frac{4\tau^2}{\pi B\,(1-a^2)} \left[2F(a,2)\Big|_{\frac{1}{a}}^{n} - F(a,2)\Big|_{\frac{2a}{3a-1}}^{n} - 4F(a,1)\Big|_{\frac{1}{a}}^{\frac{2a}{3a-1}} \right.$$

$$\left. - 2F(a,0)\Big|_{1}^{\frac{1}{a}} \right] \qquad (28)$$

Summing up, if the Mach cone from C lies ahead of E, ΔC_{D_C} is given by (28); if the Mach cone cuts EG, ΔC_{D_C} is given by (24); and if the Mach cone cuts FG C_{D_C} is given by (25).

3.2 Drag due to "one-sided" source distributions

The source distribution EGH can be made up of two source distributions, EGF of strength $\xi = -2\lambda = -2\tau$ and EHF of strength $\xi = +2\lambda = +2\tau$.

Two cases have to be considered, namely when the Mach cone from E lies ahead of EG and when the Mach cone lies behind EG (see Fig.5). The calculations will be restricted in that the Mach cone from E is assumed not to cross the surface of the other half wing. A further limitation already mentioned is that the Mach cone from A is always ahead of the wing leading edge. Thus the source distribution EHF is always of the "subsonic" leading edge type, while EGF may be either "subsonic" or "supersonic".

The general expressions for the drag increment due to this form of triangular source distribution are found by substituting equations (3) and (4) into (5). The substitution leads to the following set of equations, the suffices 1, 2 and 3 representing the zones of influence which are defined in Fig.2.

(a) "Subsonic" leading edge.

$$\Delta C_{D_1} = \frac{8\,\xi\,\zeta\,(1-a)}{\pi B\,(1+a)}\,\frac{k_1}{c^2}\int\frac{1}{\sqrt{n_1^2-1}}\cosh^{-1}\frac{n_1^2-\sigma_1}{n_1(1-\sigma_1)}\,dA \qquad (29a)$$

$$\Delta C_{D_2} = \frac{8\,\xi\,\zeta\,(1-a)}{\pi B\,(1+a)}\,\frac{k_1}{c^2}\int\frac{1}{\sqrt{n_1^2-1}}\cosh^{-1}\frac{n_1^2-\sigma_1}{n_1(\sigma_1-1)}\,dA \qquad (29b)$$

$$\Delta C_{D_3} = \frac{8\,\xi\,\zeta\,(1-a)}{\pi B\,(1+a)}\,\frac{k_1}{c^2}\int\frac{1}{\sqrt{n_1^2-1}}\cosh^{-1}\frac{n_1^2+\sigma_1}{n_1(\sigma_1+1)}\,dA \qquad (29c)$$

(b) "Supersonic" leading edge.

$$\Delta C_{D_1} = \frac{8\,\xi\,\zeta\,(1-a)}{\pi B\,(1+a)}\,\frac{k_1}{c^2}\int\frac{1}{\sqrt{1-n_1^2}}\left\{\pi - \cos^{-1}\frac{\sigma_1-n_1^2}{n_1(1-\sigma_1)}\right\}dA \qquad (30a)$$

$$\Delta C_{D_2} = \frac{8\,\xi\,\zeta\,(1-a)}{B\,(1+a)}\,\frac{k_1}{c^2}\,\frac{A}{\sqrt{1-n_1^2}} \qquad (30b)$$

$$\Delta C_{D_3} = \frac{8\,\xi\,\zeta\,(1-a)}{\pi B\,(1+a)}\,\frac{k_1}{c^2}\int\frac{1}{\sqrt{1-n_1^2}}\cos^{-1}\frac{\sigma_1+n_1^2}{n_1(1+\sigma_1)}\,dA \qquad (30c)$$

In the above equations, k_1 and n_1 refer to the source distribution considered and not the wing leading edge, and σ_1 is always a positive quantity.

The elementary areas (see dA_6 and dA_7 in Fig.3b) are given by the following.

In zones 1 and 2 (Fig.2)

$$dA = dA_6 = g\,\frac{c^2\,(1-b)^2}{8k\cdot(g-\sigma_1\,\frac{b}{2})^2}\,d\sigma_1 \qquad (31a)$$

and in zone 3 (Fig.2)

$$dA = dA_7 = g\,\frac{c^2}{8k\,(g+\sigma_1)^2}\,d\sigma_1 \qquad (31b)$$

where $g = k_1/k$ and k_1 is the tangent of the sweepback angle of the leading edge of the source distribution considered and k is the tangent of the sweepback angle of the leading edge of the wing.

Let E_1 refer to the source distribution EHF, and E_2 refer to EGF.

3.21 Source distribution EHF ($\xi = +2\lambda = +2\tau$)

(a) Area EHF ($\zeta = -\lambda = -\tau$)

Putting $b = 0$, $g = 1$ in (31a) and substituting in (29a) we get for this area the general expression

$$\Delta C_{D_{E_1}} = -\frac{2\tau^2\,(1-a)}{\pi B\,(1+a)} \int_0^1 \frac{1}{\sqrt{n_1^2-1}} \cosh^{-1} \frac{n_1^2-\sigma_1}{n_1(1-\sigma_1)}\, d\sigma_1$$

$$= -\frac{2\tau^2\,(1-a)}{\pi B\,(1+a)} H_1(n_1)\Big|_0^1$$

For EHF $n_1 = n$ and the value of $\Delta C_{D_{E_1}}$ (EHF) is the same as the general value; i.e.

$$\Delta C_{D_{E_1}}\,(\text{EHF}) = -\frac{2\tau^2\,(1-a)}{\pi B\,(1+a)} H_1(n)\Big|_0^1 \qquad (32)$$

where

$$H_1(n) = \int \frac{1}{\sqrt{n^2-1}} \cosh^{-1} \frac{n^2-\sigma}{n(1-\sigma)}\, d\sigma$$

$$= \frac{1}{\sqrt{n^2-1}} \left\{ \sigma \cosh^{-1} \frac{n^2-\sigma}{n(1-\sigma)} + \sqrt{n^2-1}\, \sin^{-1} \frac{\sigma}{n} \right.$$

$$\left. - \log \left| \frac{n(1-\sigma) - \sqrt{n^2-\sigma^2} - \sigma\sqrt{n^2-1}}{n(1-\sigma) - \sqrt{n^2-\sigma^2} + \sigma\sqrt{n^2-1}} \right| \right\} \qquad (33)$$

(b) Area ERGH

This area may be divided into EGH and ERG, the value of dA for each being as in (31a), with $b = 0$ for EGH and $b = 2$ for ERG. The drag is found from equation (29b).

(i) Area EGH ($\zeta = -\lambda = -\tau$)

$$\Delta C_{D_{E_1}} \text{(EGH)} = -\frac{2\tau^2}{\pi B}\frac{(1-a)}{(1+a)} \int_1^2 \frac{1}{\sqrt{n^2-1}} \cosh^{-1} \frac{n^2-\sigma}{n(\sigma-1)}\, d\sigma$$

$$= -\frac{2\tau^2}{\pi B}\frac{(1-a)}{(1+a)} H_2(n)\Big|_1^2 \tag{34}$$

where

$$H_2(n) = \int \frac{1}{\sqrt{n^2-1}} \cosh^{-1} \frac{n^2-\sigma}{n(\sigma-1)}\, d\sigma$$

$$= \frac{1}{\sqrt{n^2-1}} \left\{ \sigma \cosh^{-1} \frac{n^2-\sigma}{n(\sigma-1)} + \sqrt{n^2-1} \sin^{-1} \frac{\sigma}{n} \right.$$

$$\left. + \log \left| \frac{n(\sigma-1)+\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(\sigma-1)+\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}} \right| \right\} \tag{35}$$

(ii) Area ERG ($\zeta = +\lambda = +\tau$)

$$\Delta C_{D_{E_1}} \text{(ERG)} = \frac{2\tau^2}{\pi B}\frac{(1-a)}{(1+a)} \int_2^n \frac{g^2}{\sqrt{n^2-1}\,(g-\sigma)^2} \cosh^{-1} \frac{n^2-\sigma}{(\sigma-1)n}\, d\sigma$$

where $g = 1$.

Let

$$J(n,g) = \int \frac{g^2}{\sqrt{n^2-1}} \frac{1}{(g-\sigma)^2} \cosh^{-1} \frac{n^2-\sigma}{(\sigma-1)n}\, d\sigma$$

Then

$$J(n,1) = -\frac{\cosh^{-1}\frac{n^2-\sigma}{n(\sigma-1)}}{(\sigma-1)\sqrt{n^2-1}} + \frac{\sqrt{n^2-\sigma^2}}{(n^2-1)(\sigma-1)}$$

$$-\frac{1}{(n^2-1)^{3/2}}\log\left|\frac{n(\sigma-1)+\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(\sigma-1)+\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}}\right| \tag{36}$$

$$\therefore\ \Delta C_{D_{E_1}}(ERG) = \frac{2\tau^2\ (1-a)}{\pi B\ (1+a)}\ J(n,1)\Big|_2^n \tag{37}$$

It will be noticed that when EG is "supersonic" the drag increment given by equations (34) and (37) will not hold, since equation (37) will not exist, and equation (34) will become

$$\Delta C_{D_{E_1}}\ (ETH) = -\frac{2\tau^2\ (1-a)}{\pi B\ (1+a)}\ H_2(n)\Big|_1^n \tag{38}$$

(c) Area EFS (or EFU) $(\zeta = -\lambda = -\tau)$.

The drag increment is given by equation (29c) in which dA is given by (31b).

The general expression is

$$\Delta C_{D_{E_1}}\ (EFS) = -\frac{2\tau^2\ (1-a)}{\pi B\ (1+a)}\int_0^n \frac{g^2}{\sqrt{n^2-1}}\ \frac{1}{(g+\sigma)^2}\ \cosh^{-1}\frac{n^2+\sigma}{n(\sigma+1)}\ d\sigma$$

For the case considered, g = 1.

Let

$$K(n,g) = \int \frac{g^2}{\sqrt{n^2-1}}\ \frac{1}{(g+\sigma)^2}\ \cosh^{-1}\frac{n^2+\sigma}{n(\sigma+1)}\ d\sigma .$$

Then

$$K(n,1) = \frac{1}{\sqrt{n^2-1}} \left\{ - \frac{\cosh^{-1} \frac{n^2+\sigma}{n(\sigma+1)}}{(\sigma+1)} + \frac{\sqrt{n^2-\sigma^2}}{(\sigma+1)\sqrt{n^2-1}} \right.$$

$$\left. + \frac{1}{(n^2-1)} \log \left| \frac{n(\sigma+1)-\sqrt{n^2-\sigma^2}-\sigma\sqrt{n^2-1}}{n(\sigma+1)-\sqrt{n^2-\sigma^2}+\sigma\sqrt{n^2-1}} \right| \right\} \quad (39)$$

$$\therefore \Delta C_{D_{E_1}} \text{ (EFS)} = - \frac{2\tau^2}{\pi B} \frac{(1-a)}{(1+a)} K(n,1) \Big|_0^n \quad (40)$$

3.22 <u>Source distribution EGF</u> ($\xi = -2\lambda = -2\tau$)

For source distribution EGF, g has the value $\frac{1}{2}$, and n_1 in equations (29) and (30) equals $\frac{1}{2}n$.

Case (a) <u>EG "subsonic" i.e. n > 2</u>

When EG is "subsonic" equations (29) are used for the drag increment, in conjunction with the elementary areas given by (31)

$$\Delta C_{D_{E_2}} \text{ (EGF)} = \frac{2\tau^2}{\pi B} \frac{(1-a)}{(1+a)} H_1(\tfrac{n}{2}) \Big|_0^1 \quad (41)$$

$$\Delta C_{D_{E_2}} \text{ (ERG)} = - \frac{2\tau^2}{\pi B} \frac{(1-a)}{(1+a)} J(\tfrac{n}{2}, \tfrac{1}{2}) \Big|_1^{\frac{n}{2}} \quad (42)$$

and

$$\Delta C_{D_{E_2}} \text{ (EFS)} = \frac{2\tau^2}{\pi B} \frac{(1-a)}{(1+a)} K(\tfrac{n}{2}, \tfrac{1}{2}) \Big|_0^{\frac{n}{2}} \quad (43)$$

where

$$J\left(\frac{n}{2}, \tfrac{1}{2}\right) = \frac{1}{4\sqrt{\left(\frac{n}{2}\right)^2 - 1}} \left\{ - \frac{\cosh^{-1} \dfrac{\left(\frac{n}{2}\right)^2 - \sigma}{\frac{n}{2}(\sigma - 1)}}{(\sigma - \frac{1}{2})} \right.$$

$$- 2 \log \left| \frac{\frac{n}{2}(\sigma-1) + \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} - \sigma\sqrt{\left(\frac{n}{2}\right)^2 - 1}}{\frac{n}{2}(\sigma-1) + \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} + \sigma\sqrt{\left(\frac{n}{2}\right)^2 - 1}} \right|$$

$$\left. + 4\sqrt{\frac{\left(\frac{n}{2}\right)^2 - 1}{n^2-1}} \log \left| \frac{n(\sigma-\frac{1}{2}) + \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} - \sigma\sqrt{n^2-1}}{n(\sigma-\frac{1}{2}) + \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} + \sigma\sqrt{n^2-1}} \right| \right\} \qquad (44)$$

and

$$K\left(\frac{n}{2}, \tfrac{1}{2}\right) = \frac{1}{4\sqrt{\left(\frac{n}{2}\right)^2 - 1}} \left\{ - \frac{\cosh^{-1} \dfrac{\left(\frac{n}{2}\right)^2 + \sigma}{\frac{n}{2}(\sigma + 1)}}{\sigma + \frac{1}{2}} \right.$$

$$- 4\sqrt{\frac{\left(\frac{n}{2}\right)^2 - 1}{n^2-1}} \log \left| \frac{n(\sigma+\frac{1}{2}) - \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} - \sigma\sqrt{n^2-1}}{n(\sigma+\frac{1}{2}) - \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} + \sigma\sqrt{n^2-1}} \right|$$

$$\left. + 2 \log \left| \frac{\frac{n}{2}(\sigma+1) - \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} - \sigma\sqrt{\left(\frac{n}{2}\right)^2 - 1}}{\frac{n}{2}(\sigma+1) - \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2} + \sigma\sqrt{\left(\frac{n}{2}\right)^2 - 1}} \right| \right\} \qquad (45)$$

Case (b) <u>EG "supersonic" i.e. $n < 2$</u>

Equations (30) are used in this case for the drag increment in conjunction with the elementary areas given by (31)

$$\Delta C_{D_{E_2}} \text{ (EFT)} = \frac{2\tau^2 (1-a)}{\pi B (1+a)} \int_0^{\frac{n}{2}} \frac{1}{\sqrt{1 - (\frac{n}{2})^2}} \left\{ \pi - \cos^{-1} \frac{\sigma - (\frac{n}{2})^2}{\frac{n}{2}(1-\sigma)} \right\} d\sigma$$

$$= \frac{2\tau^2 (1-a)}{\pi B (1+a)} L(\tfrac{n}{2}, \tfrac{1}{2}) \Big|_0^{\frac{n}{2}} \qquad (46)$$

where

$$L(\tfrac{n}{2}, \tfrac{1}{2}) = \int \frac{1}{\sqrt{1 - (\frac{n}{2})^2}} \left\{ \pi - \cos^{-1} \frac{\sigma - (\frac{n}{2})^2}{\frac{n}{2}(1-\sigma)} \right\} d\sigma$$

$$= \frac{1}{\sqrt{1 - (\frac{n}{2})^2}} \left\{ \pi\sigma + \sqrt{1 - (\tfrac{n}{2})^2} \sin^{-1} \frac{2\sigma}{n} - \sigma \cos^{-1} \frac{\sigma - (\frac{n}{2})^2}{\frac{n}{2}(1-\sigma)} \right.$$

$$\left. - 2 \tan^{-1} \frac{\frac{n}{2}(1-\sigma) - \sqrt{(\frac{n}{2})^2 - \sigma^2}}{\sigma \sqrt{1 - (\frac{n}{2})^2}} \right\} \qquad (47)$$

$$\Delta C_{D_{E_2}} \text{ (EGT)} = \frac{8\tau^2 (1-a) k}{B (1+a) c^2} \cdot \frac{\text{area EGT}}{\sqrt{1 - (\frac{n}{2})^2}}$$

and $\quad \text{area EGT} = \frac{c^2}{4k} (1 - \frac{n}{2})$

$$\therefore \quad \Delta C_{D_{E_2}} \text{ (EGT)} = \frac{2\tau^2 (1-a)}{B (1+a)} \sqrt{\frac{1 - \frac{n}{2}}{1 + \frac{n}{2}}}$$

$$= \frac{2\tau^2 (1-a)}{B (1+a)} \cdot M(\tfrac{n}{2}) \qquad (48)$$

where

$$M\left(\frac{n}{2}\right) = \sqrt{\frac{1 - \frac{n}{2}}{1 + \frac{n}{2}}} \qquad (49)$$

$$\Delta C_{D_{E_2}}(EFU) = \frac{2\tau^2\,(1-a)}{\pi B\,(1+a)} \int_0^{\frac{n}{2}} \frac{1}{4\sqrt{1-\left(\frac{n}{2}\right)^2}} \cos^{-1} \frac{\sigma + \left(\frac{n}{2}\right)^2}{\frac{n}{2}\,(\sigma+1)} \frac{d\sigma}{(\sigma+\frac{1}{2})^2}$$

since $g = \frac{1}{2}$.

$$\therefore\ \Delta C_{D_{E_2}}\,(EFU) = \frac{2\tau^2\,(1-a)}{\pi B\,(1+a)}\, N\left(\frac{n}{2}, \frac{1}{2}\right)\Bigg|_0^{\frac{n}{2}} \qquad (50)$$

where

$$N(n,g) = \int \frac{g^2}{\sqrt{1-n^2}} \cos^{-1} \frac{\sigma + n^2}{n(1+\sigma)} \frac{d\sigma}{(\sigma+g)^2}$$

Evaluating for $g = \frac{1}{2}$ gives

$$N\left(\frac{n}{2}, \frac{1}{2}\right) = -\frac{1}{\sqrt{1-\left(\frac{n}{2}\right)^2}} \left\{ \frac{\cos^{-1} \dfrac{\sigma + \left(\frac{n}{2}\right)^2}{\frac{n}{2}\,(1+\sigma)}}{4(\sigma+\frac{1}{2})} + 2\sqrt{\frac{1-\left(\frac{n}{2}\right)^2}{1-n^2}} \tan^{-1} \frac{n(\sigma+\frac{1}{2}) - \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2}}{\sigma\sqrt{1-n^2}} \right.$$

$$\left. - \tan^{-1} \frac{\frac{n}{2}\,(\sigma+1) - \sqrt{\left(\frac{n}{2}\right)^2 - \sigma^2}}{\sigma\sqrt{1-\left(\frac{n}{2}\right)^2}} \right\} \qquad (51)$$

Summing up for the source distribution EGH (or EHF and EGF), ΔC_{D_E} is given by the following:-

If EG is "subsonic", ΔC_{D_E} is made up of the sum of (32), (34), (37), (40), (41), (42) and (43).

If EG is "supersonic", ΔC_{D_E} is given by the sum of (32), (38), (40), (46), (48) and (50).

The total drag coefficient is found by adding the drag increments, thus

$$C_{D_{pressure}} = \Delta C_{D_A} + \Delta C_{D_B} + \Delta C_{D_C} + \Delta C_{D_E}.$$

4 Application to a particular wing

4.1 Evaluation

The formulae derived in para.3 were applied to a particular wing in order to determine numerically the variation of pressure drag coefficient with Mach number. For the calculations a wing was chosen in which the sweepback of the inboard section was 60° and the parameter $a = 0.4$. This gave $k = \sqrt{3}$, $r = 0.7$ and the sweepback angle of the outboard maximum thickness line equal to 40.9°.

The various functions appearing in para.3 are shown tabulated in Table I for a range of Mach numbers between $M = 1.090$ and $M = 2$, the lower limit being that Mach number at which the Mach cones from E and E' passed through the points F' and F respectively (Fig.1). It so happens that at $M = 1.090$, for the selected value of a, the Mach cone from C passes behind E and E', and hence for the range of Mach number considered it was not necessary to calculate any of the functions in which one limit is $\sigma = \frac{2a}{3a-1}$ in section 3.13 case (b). At $M = 2$ the Mach cones from the apex lie along the leading edge. Referring to Table I, it will be seen that the functions are tabulated for n having the value 1.429. This corresponds to the Mach number at which the Mach cone from B passes through G and G' (viz. at $M = 1.572$). The corresponding value of n for the Mach cone from C to pass through G and G' is $n = 2.5$ ($M = 1.217$).

Fig.6 shows the theoretical drag curve for the wing plotted against Mach number. As the Mach number decreases to $M = 1.090$ the drag coefficient begins to increase fairly rapidly. This increase is probably due to the delta-like planform of the outboard portion of the wing, the drag of a delta tending to infinity as M tends to 1, according to the linear theory. Between $M = 1$ and $M = 1.090$ there will be interference between the two halves of the wing, the effect of which has not been calculated. At $M = 1.323$ when the Mach lines are parallel to EG and E'G' a kink occurs. A further kink occurs when the Mach lines from B pass through G and G' at $M = 1.572$, although there are no straight lines of discontinuity in slope between B and the wing tips. One would imagine therefore that there would be a tendency for a kink to occur when the Mach lines from C pass through G and G' (i.e. at $M = 1.217$). However Fig.6 shows that for the particular wing selected the tendency has been entirely suppressed. Since the analysis has been made only for the case of a subsonic leading edge, no calculations were possible above a Mach number of 2. As, however, at this Mach number the Mach lines are parallel to the general sweepback existing over the inboard section of the wing, the drag coefficient is expected to decrease rather more rapidly at higher Mach numbers, causing a slight kink at $M = 2$.

4.2 Comparison with the drag of other wings

In Figs.7A and 7B are plotted the drag curves of several wings of double wedge aerofoil section but different planforms, for comparison with that of the particular wing considered above. The drag of wings No.2 and 4 were derived from graphs given by H.Multhopp and M.Winter in an unpublished paper, using a method essentially equivalent to that of Puckett and Stewart[1,2], and the drag of wings No. 3 and 5 were obtained from Refs.4 and 2 respectively. The derivation of curve No.6 will be discussed later.

Fig.7A is intended to show the difference which exists between the drag of the particular cranked wing evaluated above (wing No.1) and two other well known types of wing. Wing No.2 is an arrowhead formed by replacing the cranked trailing edge and maximum thickness lines by straight lines from the root to the tips. The maximum thickness sweepback angle is 50.5°, corresponding to the lines BG and BG' in Fig.1. The "chevron" planform (wing No.3) has the same plan area as wing No.1, and the same chord and sweepback angle as its inboard section. From Fig.7A it is seen that compared with curves 2 and 3 the drag coefficient of wing 1 varies very little between M = 1.090 and M = 2, the mean value being roughly $2.8\tau^2$. Wings No.2 and 3 both have lower drag than wing No.1 at low supersonic Mach numbers and this is attributed to the fact that wings No.2 and 3 have no lines of discontinuity in slope perpendicular to the free stream. For M > 1.5 the arrowhead wing has values of C_D which are considerably higher than those for the cranked wing, indicating that the regions downstream of CG and CG' on wing 1 have a beneficial effect in reducing drag at these Mach numbers. It will be seen that replacing the wing by a chevron of roughly similar shape and 60° sweepback angle does not give very good agreement with wing No.1 for the case selected, which shows that the tip effect on wing No.1 is fairly large.

It may be concluded that if a cranked wing of the type examined is replaced by a roughly similar chevron or arrowhead wing in order to find a simple approximation to the pressure drag, very poor accuracy will in general be obtained, since insufficient allowance is made for tip effect.

Several other wings and combinations of wings have been examined in an attempt to find a moderately good simple approximation to the pressure drag of the cranked wing, and the best results obtained are shown in Fig.7B. Wing No.4 is an arrowhead with a maximum thickness sweepback angle of 50.5°, and the same span and area (i.e. the same aspect ratio) as Wing No.1. Comparing Wing No.4 with Wing No.2, both of which have the same sweepback of the maximum thickness line, we see that the lower sweepback of the trailing edge of wing No.4 results in better agreement with wing No.1 at low supersonic Mach numbers, than was obtained with wing No.2. However poor agreement is still obtained around M = 1.572, presumably due to the fact that not sufficient allowance has been made for the beneficial effect of the region behind CG and CG' on Wing No.1. In an attempt to allow for this beneficial effect, a delta wing (wing No.5) with the maximum thickness lines swept back 50.5°, was examined, and the drag shows fairly good agreement with that of wing No.1. The agreement should improve as the parameter a (see Fig.1) decreases, exact agreement being obtained when a = 0, since then the two wings are identical. It is to be anticipated, however, that a delta wing such as wing No.5 will give progressively poorer agreement with a cranked wing as the value of the parameter a for the latter increases towards unity. It will over-estimate tip effects at low supersonic Mach numbers, and make too much allowance for the beneficial effects of the regions behind CG and CG' at high Mach numbers.

It was decided, therefore, to seek a method of estimating, for all a, the drag of cranked wings in terms of the abundant data which exist on the drag of "chevron" and arrowhead wings. A method* suggested is to separate the wing into the chevron formed by the inboard sections (i.e. AWFCF'W' in Fig.1), and the delta formed by the outboard sections (i.e. WFG and W'F'G' in Fig.1). The drag coefficients of the chevron and the delta are then evaluated assuming them to be isolated wings, and the drag coefficient of the cranked wing is assumed to be given by a mean, weighted in the ratio of their areas, such that

$$C_{D\,\text{cranked wing}} = \frac{2a}{1+a} C_{D\,\text{chevron}} + \frac{1-a}{1+a} C_{D\,\text{delta}}$$

This method is obviously exact for a = 0 and a = 1, and Fig.7B curve No.6 shows that the agreement for a = 0.4 is moderately good.

5 Conclusions

The theoretical supersonic pressure drag coefficient of a particular wing with a cranked line of maximum thickness and a symmetrical double wedge aerofoil section varies very little between Mach numbers 1.090 and 2, the mean value being roughly $2.8\tau^2$.

Since the computations were long and tedious, an attempt was made to find combinations of wings of known characteristics which would give fairly close approximations to the drag, and the following method appears satisfactory. The cranked wing is separated into a "chevron" wing formed by the inboard sections, and a delta wing formed by the outboard sections. The drag coefficient of the chevron and delta are then evaluated on the assumption that they are isolated wings, and the drag coefficient of the cranked wing is obtained by taking a mean of those for the chevron and delta wings, weighted in the ratio of their plan areas to the plan area of the cranked wing.

If the parameter a defined in Fig.1 is less than 0.4, a rapid approximation to the drag coefficient may be made by replacing the cranked wing by a delta with a double wedge aerofoil section, the sweepback of the maximum thickness lines being equal to that of the lines BG and BG' in Fig.1.

Acknowledgement

The author is much indebted to Miss P.M.Solway for the help she has given in evaluating the integration functions.

* For this suggestion the author is indebted to Mr.C.H.E.Warren.

List of Symbols

a	parameter defined in Fig.1
B	$= \sqrt{M^2-1}$
b	parameter defined in Fig.3
c	root chord
C_D	drag coefficient
$\Delta C_{D_V}(XYZ)$	increment of drag coefficient due to the influence of the source distribution with apex at V on the area XYZ
C_p	pressure coefficient
g	$= k_1/k$
k, k_1	tangent of the sweepback angle of the leading edge, k referring to the main wing, and k_1 being used for those triangular source distributions for which it is different from the value for the main wing
M	Mach number
n	$= \frac{k}{B}$
n_1	$= k_1/B$
r	parameter defined in Fig.1
S	wing plan area
U	free stream velocity
u	perturbation velocity in free stream direction
x,y	streamwise and normal cartesian co-ordinates in the plane of the wing, measured relative to an origin at the apex of a triangular source distribution
ζ	angle between free stream and wing surface at any point
λ	semi-angle of the double wedge section
ξ	strength of source distribution
σ	$= k \left\| \frac{y}{x} \right\|$
τ	thickness/chord ratio of wing
ϕ	perturbation velocity potential

References

No.	Author	Title, etc
1	Puckett, A.E.	Supersonic Wave Drag of Thin Airfoils. Journal of the Aeronautical Sciences, Vol.13, No.9, pp.475-484. September, 1946.
2	Puckett, A.E. and Stewart, H.J.	Aerodynamic Performance of Delta Wings at Supersonic Speeds. Journal of the Aeronautical Sciences, Vol. 14, No.10, pp.567-578. October, 1947
3	Ferri, A.	Elements of Aerodynamics of Supersonic Flows. Chapter 14. Macmillan, New York, 1949.
4	Margolis, K.	Supersonic Wave Drag of Non-lifting Sweptback Tapered Wings with Mach lines behind the Line of Maximum Thickness. NACA Technical Note No. 1672. August, 1948.
5	Dwight, H.B.	Tables of Integrals and Other Mathematical data. Macmillan, New York, 1947.

Wt.2078.CP61.K3. Printed in Great Britain.

TABLE I TABULATION OF INTEGRATION FUNCTIONS

n	1.01	1.05	1.20	1.429	1.50	1.80	2.00	2.10	2.50	2.90	3.30	3.70	4.00
B	1.715	1.650	1.443	1.213	1.155	0.962	0.866	0.825	0.693	0.597	0.525	0.468	0.433
$\dot{M}$	1.985	1.929	1.756	1.572	1.528	1.388	1.323	1.296	1.217	1.165	1.129	1.104	1.090
$\frac{E(r,0)}{r^2}\Big\rvert_a^1$	1.0977	1.0092	0.8750	0.7673	0.7405	0.6565	0.6147	0.5950	0.5332	0.4854	0.4469	0.4153	0.3945
$\frac{E(r,0)}{r^2}\Big\rvert_{\frac{2a}{1+a}}^1$	0.9010	0.8169	0.6937	0.6001	0.5771	0.5066	0.4705	0.4565	0.4068	0.3687	0.3387	0.3136	0.2948
$\frac{E(r,1)}{(r-\frac{1}{2})^2}\Big\rvert_{\frac{2a}{1+a}}^1$	2.7641	2.4872	2.0734	1.7758	1.7028	1.4854	1.3712	1.3172	1.1825	1.0741	0.9818	0.9079	0.8642
$I(n)\Big\rvert_0^a$	0.6926	0.6857	0.6454	0.6010	0.5897	0.5445	0.5176	0.5068	0.4651	0.4318	0.4025	0.3776	0.3608
$I(n)\Big\rvert_0^{\frac{2a}{1+a}}$	1.4381	1.4310	1.3546	1.2547	1.2294	1.1300	1.0740	1.0486	0.9597	0.8868	0.8223	0.7738	0.7380
$\frac{F(r,0)}{r^2}\Big\rvert_1^n$	0.1343	0.3097	0.5857	0.7954	0.8411	0.9818	1.0491	1.0745	1.1592	1.2188	1.2629	1.2972	1.3181
$\frac{F(r,0)}{r^2}\Big\rvert_1^{\frac{1}{r}}$				0.7954	0.8229	0.8397	0.8296	0.8212	0.7786	0.7387	0.7007	0.6669	0.6417
$\frac{F(r,0)}{r^2}\Big\rvert_1^{\frac{1}{a}}$									1.1592	1.1616	1.1286	1.0884	1.0568
$\frac{F(r,1)}{(r-\frac{1}{2})^2}\Big\rvert_1^n$	0.2896	0.6487	1.4463	2.5461									
$\frac{F(r,1)}{(r-\frac{1}{2})^2}\Big\rvert_1^{\frac{1}{r}}$				2.5461	2.7726	2.8287	2.7317	2.7068	2.4703	2.2672	2.1142	1.9779	1.8797
$\frac{F(r,1)}{(r-\frac{1}{2})^2}\Big\rvert_{\frac{1}{a}}^{n'}$										0.5690	1.0037	1.2317	1.3503
$\frac{F(r,2)}{(r-1)^2}\Big\rvert_{\frac{1}{r}}^n$				0	0.0847	0.4705	0.6196	0.6615	0.7676	0.8046	0.8103	0.8016	0.7894
$\frac{F(r,2)}{(r-1)^2}\Big\rvert_{\frac{1}{a}}^n$									0	0.0212	0.0423	0.0584	0.0672
$H_1(n)\Big\rvert_0^1$	2.4267	2.2446	1.9235	1.6628	1.6050	1.3861	1.2811	1.2396	1.0953	0.9864	0.9004	0.8302	0.7855
$H_1(\frac{n}{2})\Big\rvert_0^1$							2.5708	2.2446	1.8514	1.6387	1.4786	1.3585	1.2839
$H_2(n)\Big\rvert_1^n$	0.1407	0.3099	0.5857	0.7938	0.8410	0.9816	1.0483						
$H_2(n)\Big\rvert_1^2$							1.0483	1.0554	1.0059	0.9344	0.8670	0.8079	0.7679
$J(n,1)\Big\rvert_2^n$							0	0.0177	0.1111	0.1712	0.2056	0.2247	0.2352
$J(\frac{n}{2},\frac{1}{2})\Big\rvert_1^{\frac{n}{2}}$							0	0.5166	0.5872	0.5054	0.3953	0.2784	0.1918
$K(n,1)\Big\rvert_0^n$	0.3310	0.3357	0.3435	0.3488	0.3494	0.3493	0.3471	0.3456	0.3379	0.3286	0.3187	0.3089	0.3024
$K(\frac{n}{2},\frac{1}{2})\Big\rvert_0^{\frac{n}{2}}$							0.2397	0.2402	0.2404	0.2383	0.2350	0.2309	0.2267
$L(\frac{n}{2},\frac{1}{2})\Big\rvert_0^{\frac{n}{2}}$	0.9756	1.0137	1.1592	1.3954	1.4758	1.8844	2.5708						
$M(\frac{n}{2})$	0.5735	0.5581	0.5000	0.4082	0.3780	0.2294	0						
$N(\frac{n}{2},\frac{1}{2})\Big\rvert_0^{\frac{n}{2}}$	0.2059	0.2127	0.2208	0.2298	0.2319	0.2375	0.2397						

FIG. 1.

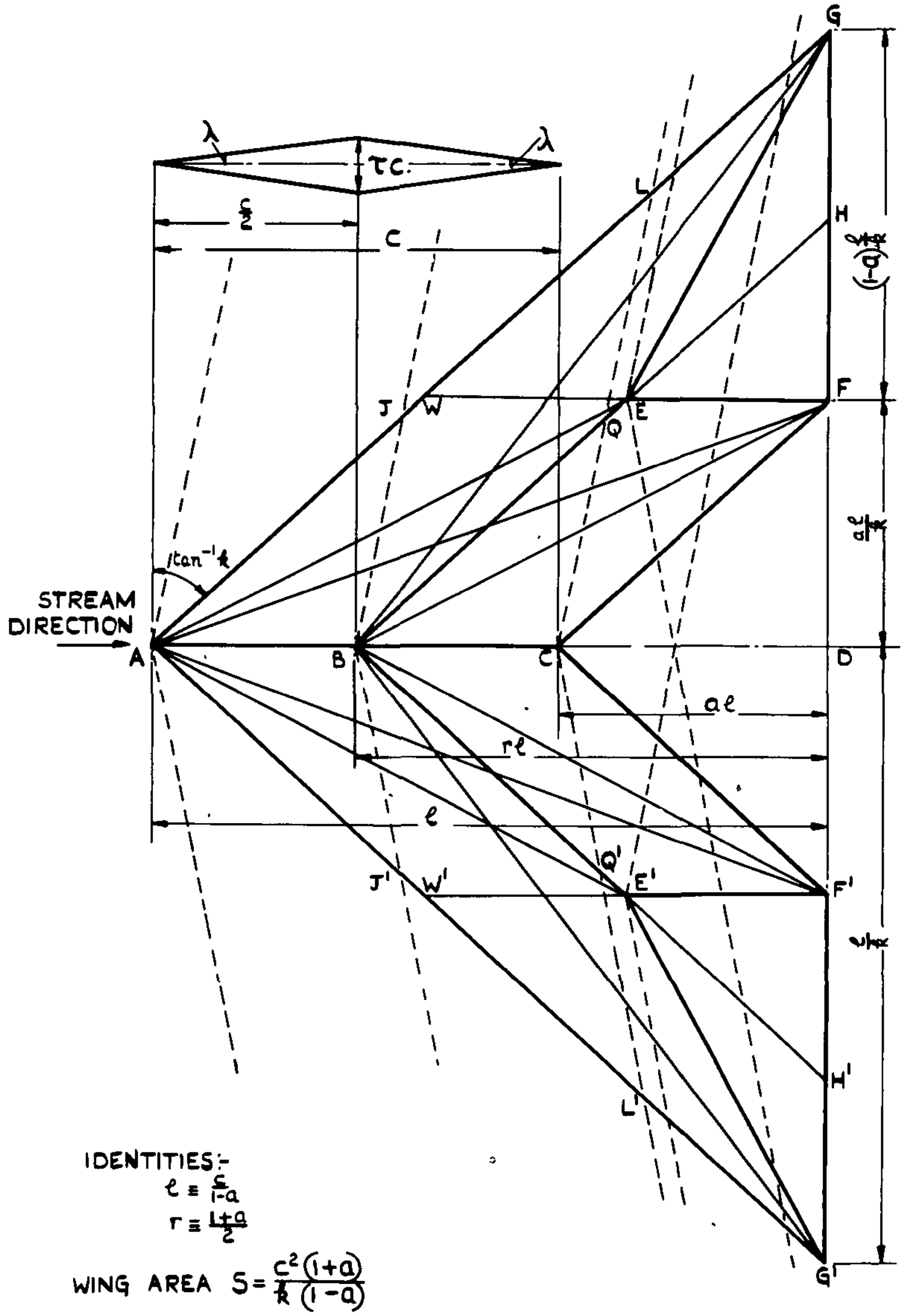

FIG 1. GEOMETRY OF WING WITH CRANKED MAXIMUM THICKNESS LINES.

FIG. 2.

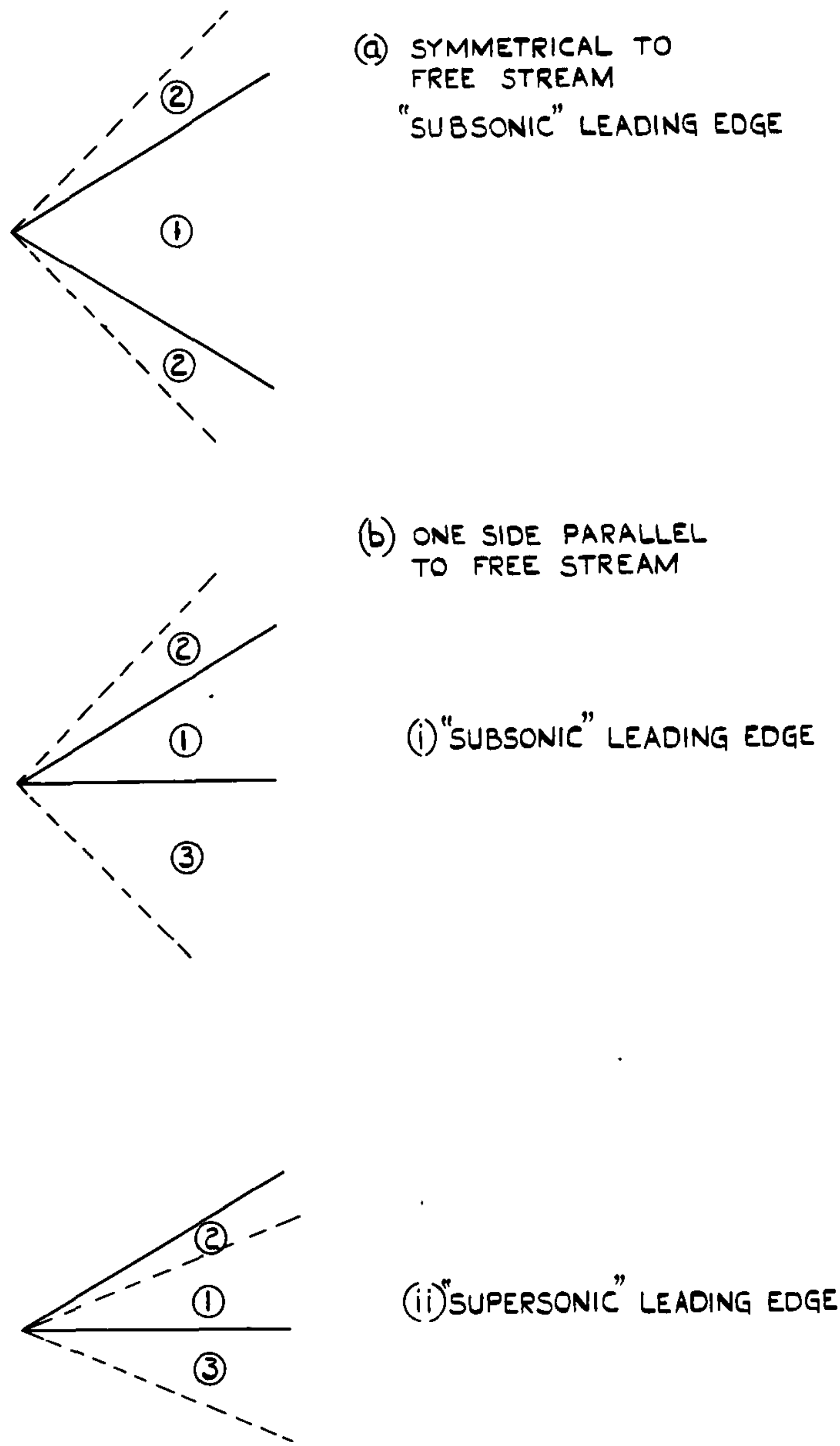

FIG. 2. ZONES OF INFLUENCE FOR TRIANGULAR SOURCE DISTRIBUTIONS.

FIG. 3.

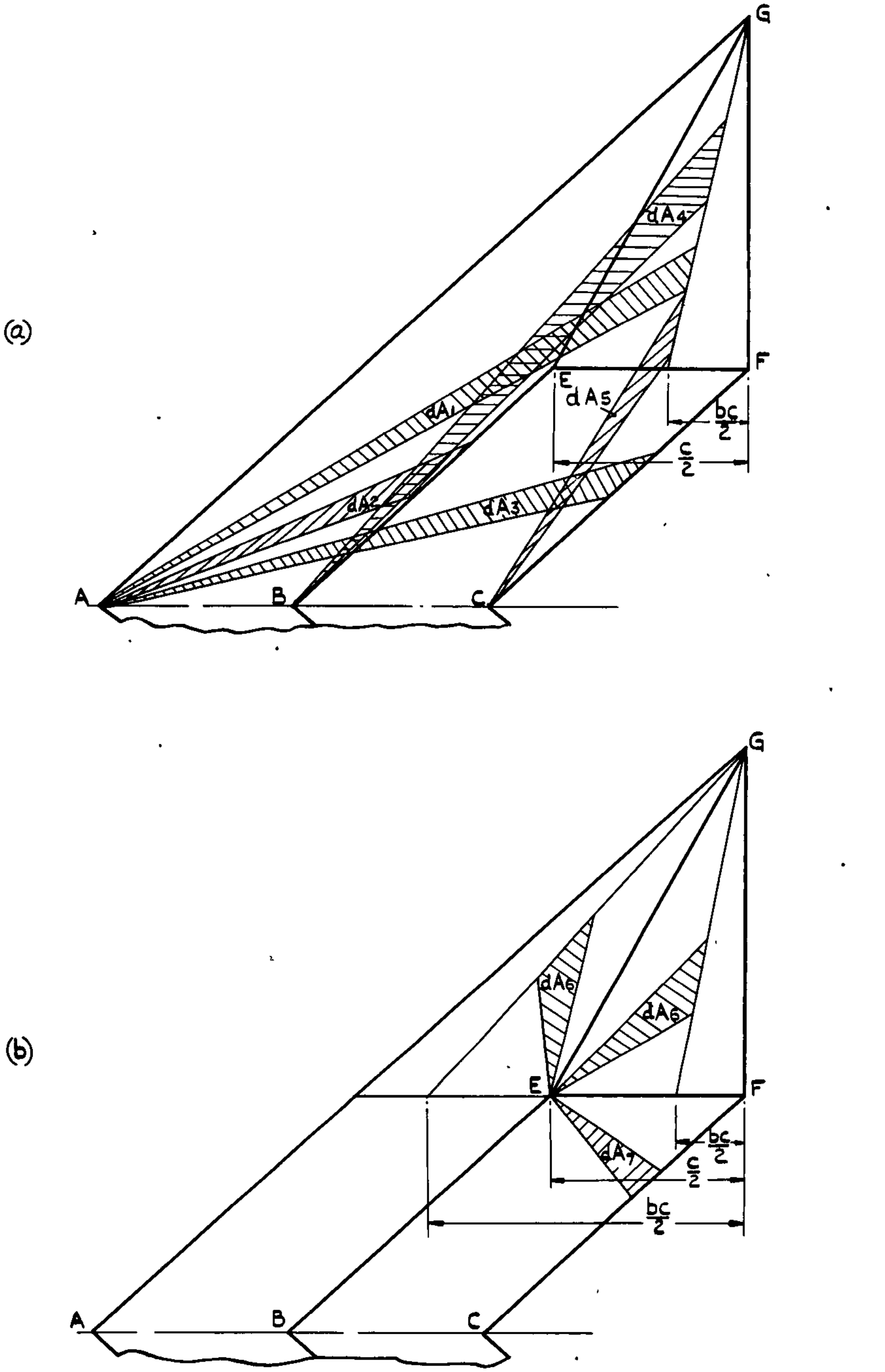

FIG. 3. ELEMENTARY AREAS.

FIG. 4&5.

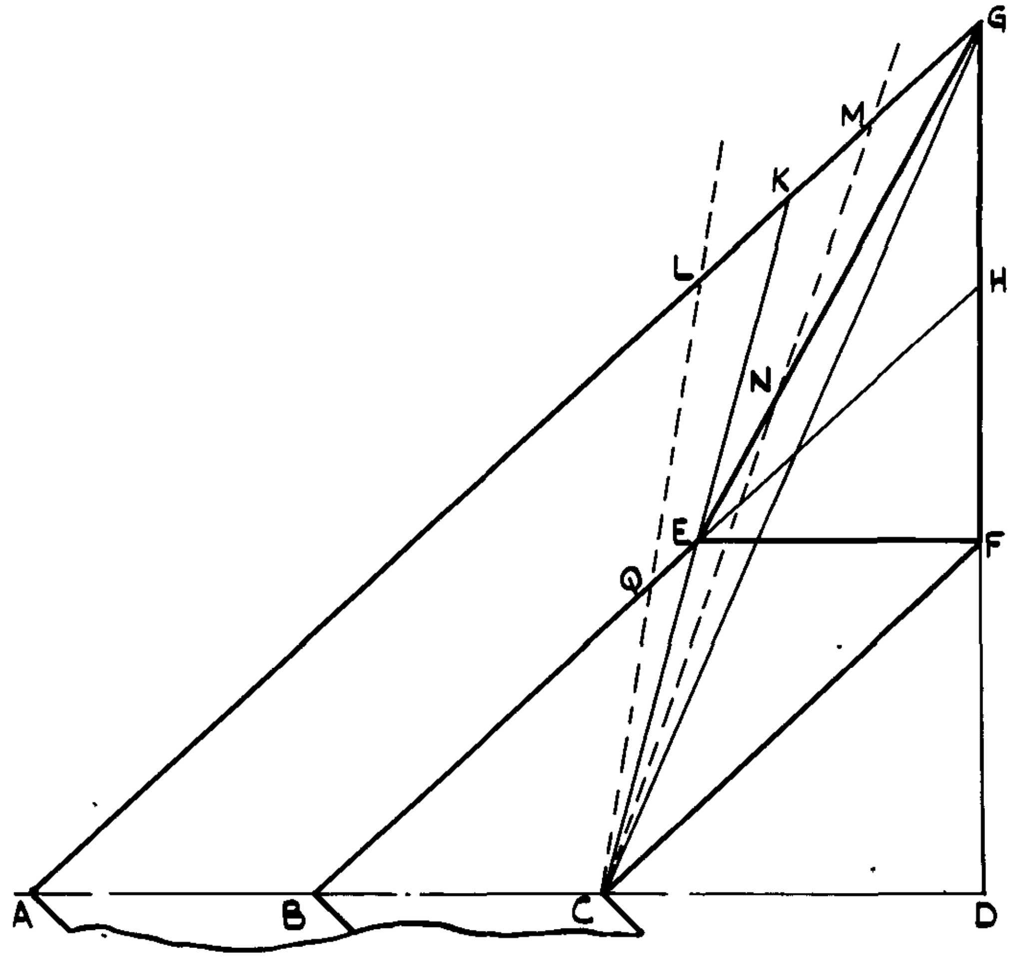

FIG. 4 GEOMETRY FOR CONSIDERATIONS OF SOURCE DISTRIBUTION CFD.

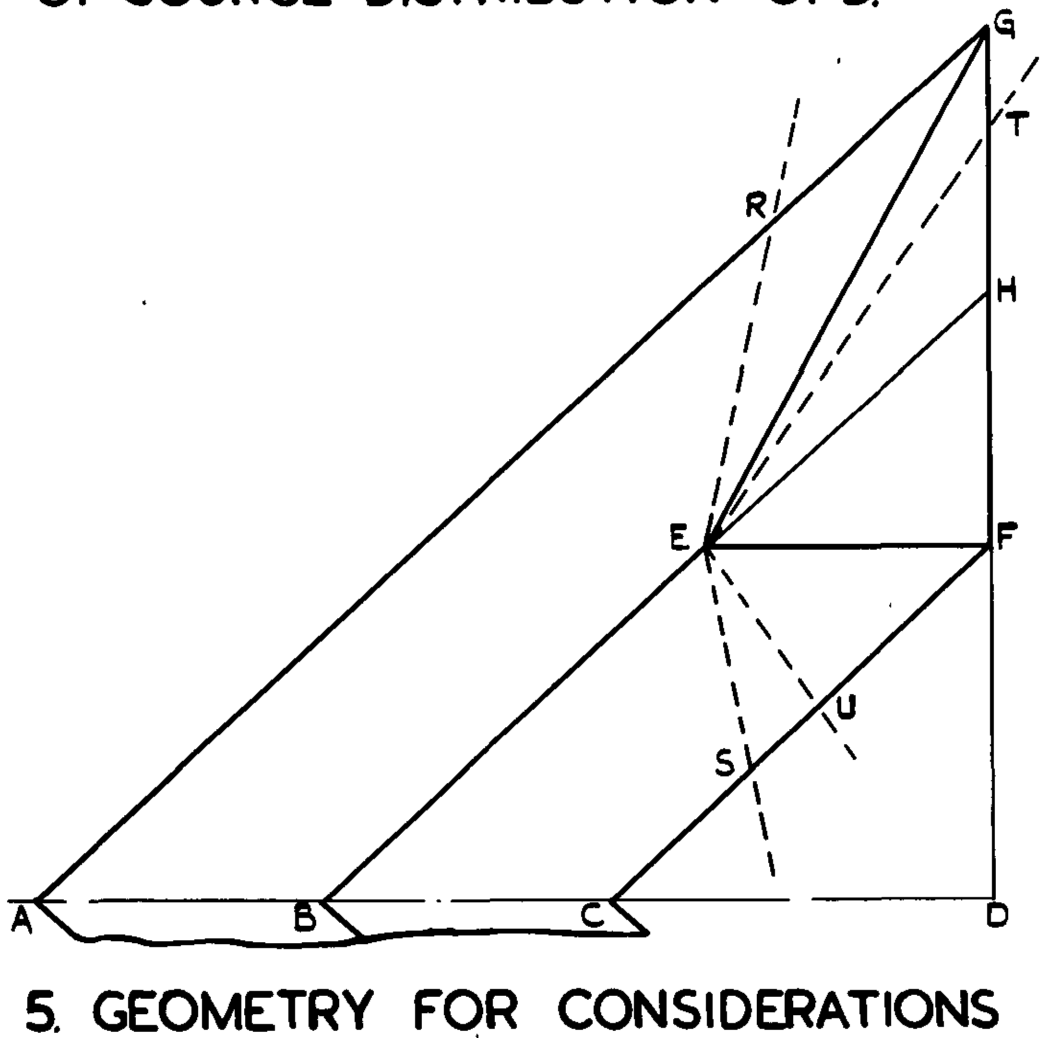

FIG. 5. GEOMETRY FOR CONSIDERATIONS OF SOURCE DISTRIBUTION EGH.

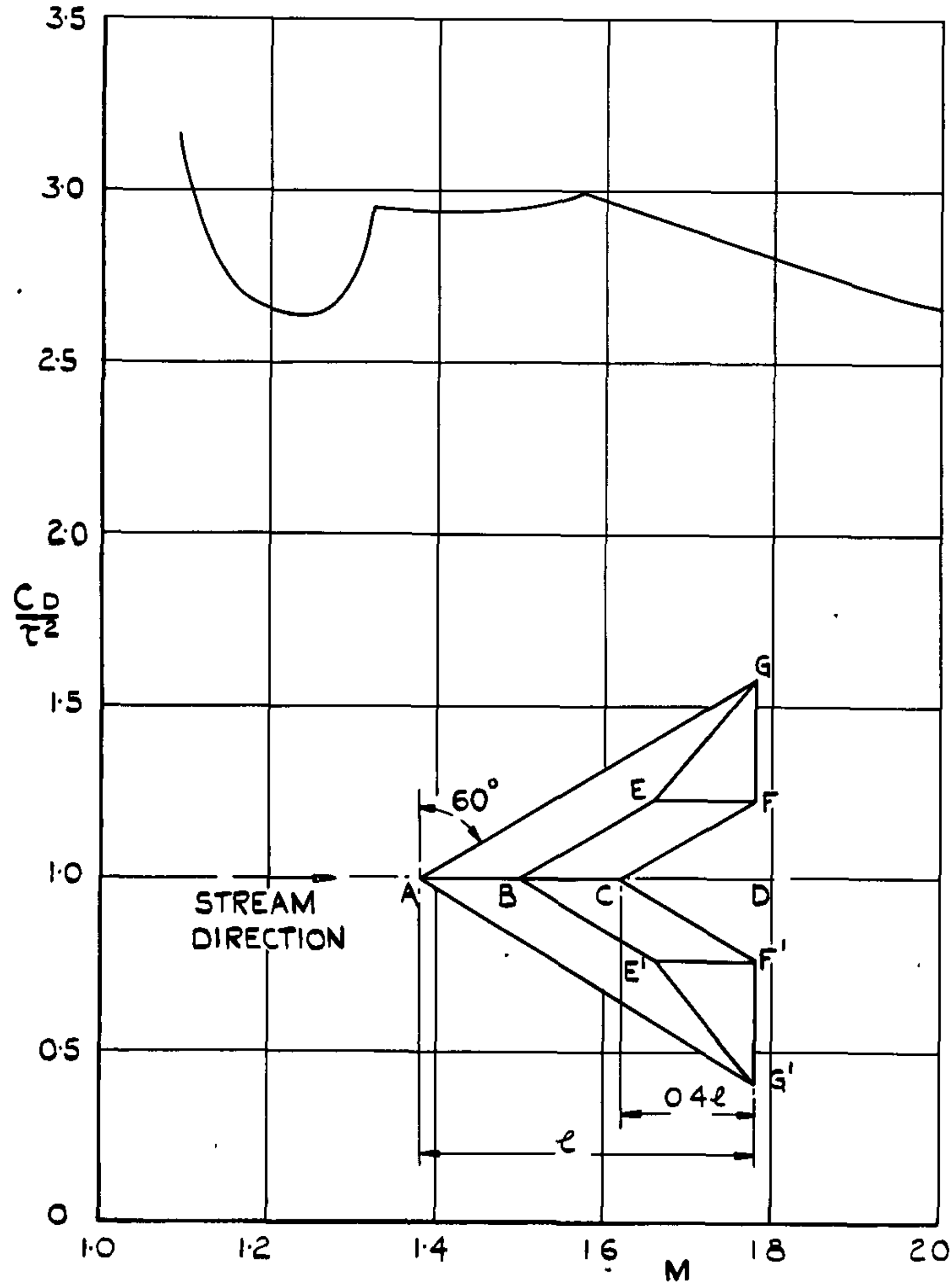

FIG. 6. VARIATION OF PRESSURE DRAG WITH MACH NUMBER FOR A WING WITH CRANKED MAXIMUM THICKNESS LINES.

FIG. 7a.

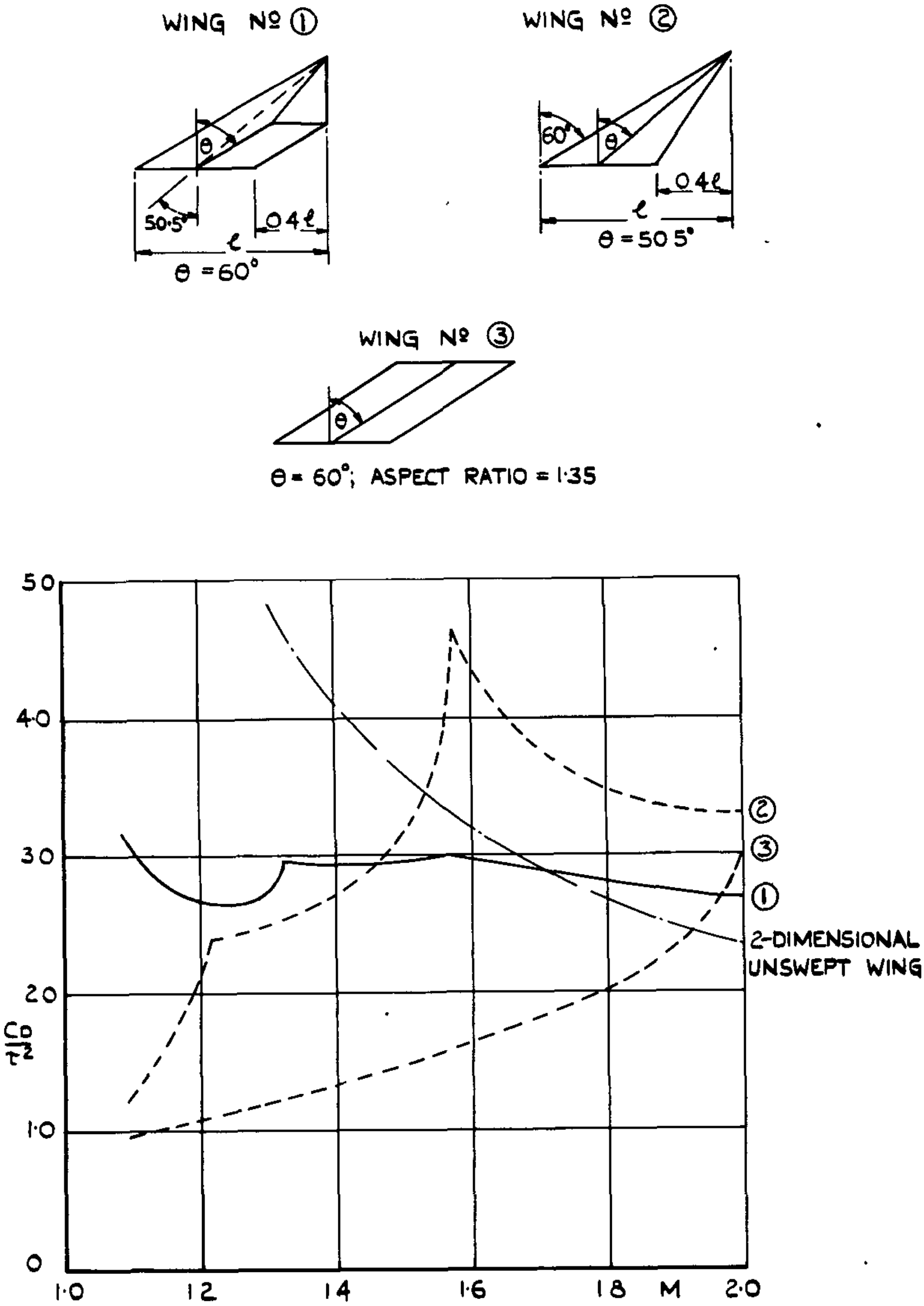

FIG. 7a. COMPARISON OF PRESSURE DRAGS OF WINGS WITH DIFFERENT PLANFORMS.

FIG. 7b.

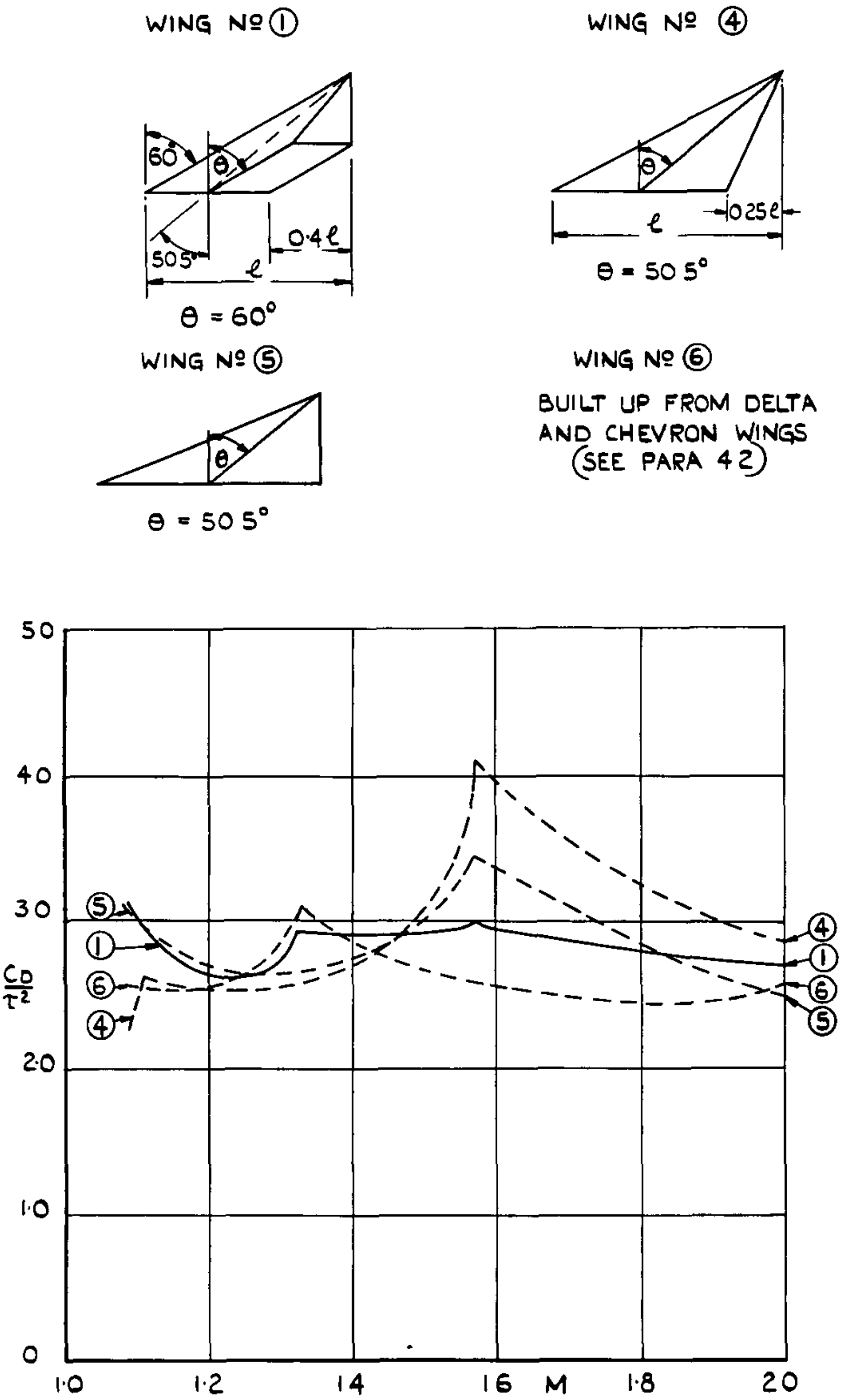

FIG. 7b. COMPARISON OF PRESSURE DRAGS OF WINGS WITH DIFFERENT PLANFORMS.

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