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Prandtl-Meyer Flow in a Relaxing Gas

By

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SUMMARY

The structure of a Prandtl-Meyer fan in a vibrationally relaxing gas is examined for the case when the amount of vibrational energy is small. The vibrational energy distribution both through the fan and downstream of the fan is obtained analytically and a criterion governing freezing within the fan is developed. It is shown that there exists two length scales, measured along the wave head, which characterise regions corresponding to near frozen and near equilibrium states respectively.

1. Introduction

In a previous report (Blythe, (1962)) the author considered non-equilibrium quasi-one-dimensional flow through a nozzle when the amount of energy in the lagging mode was small. The analysis presented here is an extension of this approach to a simple two-dimensional flow. The assumption that the energy, σ, in the lagging mode is small compared with the total energy implies that as a first approximation the flow variables are given by the usual isentropic solution. From this basic solution the first approximations to the streamline shapes, velocity distribution, etc., are known. The distribution of the energy in the lagging mode is then found by integrating the appropriate rate equation along the streamlines given by the basic isentropic solution. Since the streamtube shapes are known, this integration is analogous to that carried out in the quasi-one-dimensional case for a specified area distribution.

Of particular interest in the nozzle flow case was the position at which "freezing" would occur in the flow. This problem has interested many workers and various criteria have been deduced from qualitative arguments (see, e.g., Bray, (1959)). Under the assumption of small σ and for a linear rate equation (vibrational relaxation) it was shown (Blythe (1962)) that the type of criterion used by Bray to predict the onset of freezing did indeed emerge from a detailed analysis of the problem. The corresponding criterion for any general steady flow predicts the position along a streamline where freezing becomes important. From this result the locus of freezing points throughout the flow will follow.

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In the nozzle flow case σ tended asymptotically to some constant value downstream of the freezing point. However, for Prandtl-Meyer flow there is a gradual return to equilibrium in the region downstream of the fan, and thus, although along some streamlines freezing will occur within the fan, σ will eventually fall to its equilibrium value at some distance downstream of the fan. A detailed discussion of the various flow regions in non-equilibrium Prandtl-Meyer flow is given in Appleton (1960). Within the fan itself one expects that the flow far from the apex will be in equilibrium, since at large distances from the corner the streamtube area changes relatively slowly and the flow has more time to adjust to its local equilibrium state. Conversely in the region near the apex where the streamtube area changes rapidly near-frozen flow conditions should hold (see Fig. 1). It is interesting to note that the locus of freezing points for Prandtl-Meyer flow indicates that along streamlines meeting the wave head within a certain distance r_{1f} from the corner the energy distribution in the lagging mode will never follow the equilibrium distribution, i.e., it is as though the streamline had passed through a freezing point at some distance upstream of the fan. Some streamlines which pass outside this region encounter a freezing point within the fan (streamline SB in Fig. 1), but far enough from the corner, at distances greater than a certain value r_{1e}, streamlines never encounter a freezing point and the energy distribution in the lagging mode follows closely the equilibrium distribution. Expressions for r_{1f} and r_{1e} are given in the text (Section 3). Downstream of the fan the energy σ decays exponentially with distance, for a linear rate equation, to its equilibrium value, since to a first approximation the temperature, density, etc., are constant in this region and the streamlines are parallel to the wall. In this region one can define a relative relaxation length, L_{rel}, which governs the decay of $\epsilon/\epsilon_{\ell}(r_1)$, where ϵ is the departure from equilibrium and $\epsilon_{\ell}(r_1)$ the value at the tail of the fan corresponding to any streamline r_1. It follows that L_{rel} is the same for all streamlines to the approximation given here. In addition one can also define an absolute relaxation length, L_{abs}, which governs the decay of $\epsilon/\bar{\sigma}_{\ell}$, where $\bar{\sigma}_{\ell}$ is the equilibrium value of σ at the tail of the fan. It is obvious that in the region near the apex L_{abs} will be much larger than in the region far from the apex. By using the solution for the distribution of ϵ derived in Section 4, L_{abs} can be expressed in terms of upstream conditions. The solution obtained in Section 4 for the distribution of ϵ through the fan is specifically for vibrational relaxation under the assumption that the Mach number of the incident stream is large. This latter assumption leads to a simplification in the analytical details. However, the analysis can be formally extended to cover any initial (supersonic) Mach number (see Section 6).

This report is not intended to be a full treatment of non-equilibrium Prandtl-Meyer flow but rather to indicate a simple analytical, as opposed to numerical, approach which predicts some of the main features of this type of flow. Questions such as the decay of the frozen wave head far from the corner, the pressure distribution along the wall, etc., cannot be answered without resort to a higher approximation than that given here. One can regard the solution obtained within as giving the first terms in an expansion of the type

$$K = K_0 + \sigma_r K_1 + \sigma_r^2 K_2 + \dots$$

$$\sigma = \sigma_r \sigma_1 + \sigma_r^2 \sigma_2 + \dots$$

where/

where K is any one of the kinetic variables and $\sigma_r (<< 1)$ is a representative value of σ (suitably normalised). However, in practice it is difficult to obtain the higher order terms for Prandtl-Meyer flow because of the complex nature of the coefficients in the resulting sets of differential equations.

Previous analyses of non-equilibrium Prandtl-Meyer flow have been given, e.g., by Cleaver (1959), Bloom and Steiger (1960), Napolitano (1960) and Appleton (1960).

2. Formal Solution

The problem to be considered is the supersonic inviscid flow of a vibrationally relaxing diatomic gas (system of harmonic oscillators) around a sharp corner. The translational and rotational degrees of freedom of the gas are assumed to be fully excited and in a state of local equilibrium throughout the flow. The basic assumption is that the fraction of excited oscillators is small, or alternatively, if Θ'_{vib} is the characteristic temperature of vibration and T' the translational temperature then

$$\frac{\Theta'_{vib}}{T'} >> 1.$$

Under this assumption the expression for the equilibrium vibrational energy $\bar{\sigma}'$ can be written (see Blythe (1962))

$$\frac{\bar{\sigma}'}{RT'_1} = \bar{\sigma} \approx \Theta_{vib} \exp - \frac{\Theta_{vib}}{T} \qquad \ldots(1)$$

where $$\Theta_{vib} = \frac{\Theta'_{vib}}{T'_1}, \quad T = \frac{T'}{T'_1}$$

and the suffix $_1$ denotes the initial conditions in the incident uniform stream. Furthermore it is reasonable to assume under these conditions that the rate equation governing the approach to equilibrium of the vibrational mode takes the linear form

$$\frac{D\sigma'}{Dt'} = \rho'\Omega'(T')\{\bar{\sigma}'(T') - \sigma'\} \qquad \ldots(2)$$

where ρ' is the density and D/Dt' the usual convective operator. $\rho'\Omega'(T')$ is termed the relaxation frequency (the reciprocal of the relaxation time).

The vibrational energy enters explicitly into the flow equations only through the energy equation and for $\sigma = \sigma'/RT'_1 << 1$ the vibrational energy term can be neglected to a first approximation in this equation (see Blythe (1962)). Consequently the flow equations reduce to those governing the isentropic flow of an ideal gas with constant specific heats. The required solution for Prandtl-Meyer flow is well known and can be written

v/

$$\left.\begin{aligned}
v &= \frac{v'}{q_1'} = \frac{1}{m_1}\frac{\cos\lambda(\theta+\alpha)}{\cos\lambda\alpha} \\
u &= \frac{u'}{q_1'} = \frac{1}{\lambda m_1}\frac{\sin\lambda(\theta+\alpha)}{\cos\lambda\alpha} \\
\rho &= \frac{\rho'}{\rho_1'} = \left[\frac{\cos\lambda(\theta+\alpha)}{\cos\lambda\alpha}\right]^{\frac{1}{\lambda^2}-1} \\
T &= \frac{T'}{T_1'} = \left[\frac{\cos\lambda(\theta+\alpha)}{\cos\lambda\alpha}\right]^{2}
\end{aligned}\right\} \qquad \ldots(3)$$

where θ is the angular co-ordinate measured from the wave head (see Fig. 2). v' is the velocity in the angular direction, u' is the velocity in the radial direction, and q' is the resultant velocity at any point. m is the Mach number based on the frozen speed of sound $\sqrt{\gamma RT'}$, where γ is the ratio of the specific heats neglecting vibration and R is the gas constant. The angle α is defined by

$$\alpha = \frac{1}{\lambda}\tan^{-1}\left(\lambda\sqrt{m_1^2-1}\right)$$

and the quantity λ is defined by

$$\lambda^2 = \frac{\gamma-1}{\gamma+1}.$$

The streamlines are given by

$$\frac{r}{r_1} = \left[\frac{\cos\lambda\alpha}{\cos\lambda(\theta+\alpha)}\right]^{\frac{1}{\lambda^2}} \qquad \ldots(4)$$

where r is the radial distance from the corner and r_1 is the radial position at which a streamline intersects the wave head.

The variation of σ along a streamline can be obtained from equation (2). After suitable non-dimensionalization this equation becomes

$$\frac{\partial\sigma}{\partial\theta} = R_1\frac{r}{r_1}\frac{\rho\Omega}{v}(\bar{\sigma}-\sigma) \qquad \ldots(5)$$

where the differentiation is carried out along a streamline, and $R_1 = \frac{r_1\omega_1'}{q_1'}$, $\Omega = \frac{\Omega'(T')}{\Omega_1'(T_1')}$. To a first approximation the functions $\frac{r}{r_1}\frac{\rho\Omega}{v}$ and $\bar{\sigma}$ are

known/

known functions of θ along a streamline and by defining $\epsilon = \sigma - \bar{\sigma}$ and $F(\theta) = \frac{r}{r_1}\frac{\rho\Omega}{v}$, equation (5) can be rewritten

$$\frac{\partial\epsilon}{\partial\theta} + R_1 F(\theta)\,\epsilon = -\frac{\partial\bar{\sigma}}{\partial\theta}.$$

This equation can be integrated immediately to give

$$\epsilon = \int_0^\theta \left(-\frac{\partial\bar{\sigma}}{\partial z}\right)\left\{\exp\int_\theta^z R_1 F(w)\,dw\right\} dz \qquad \ldots(6)$$

where z and w are dummy variables, and it is assumed that the incident stream is in thermodynamic equilibrium. This result determines the variation of ϵ along any streamline, the streamline being specified by the value of R_1.

3. Freezing Criterion

Equation (6) can be rewritten in the form

$$\epsilon = \left\{\exp - \int^\theta R_1 F(w)\,dw\right\}\int_0^\theta g(z)\,\{\exp - \Theta_{vib} f(z)\}\,dz \qquad \ldots(7)$$

where $g(z) = -\left(\frac{\theta_{vib}}{T}\right)^2 \frac{dT}{dz}$, and $f(z) = \frac{1}{T} - \frac{R_1}{\Theta_{vib}}\int^z F(w)\,dw.$

As was pointed out in Blythe (1962) (7) is a typical steepest descents type integral and it follows that the only important contribution to this integral will come from a region in the neighbourhood of the stationary value of the exponential term. Physically this corresponds to a sudden increase in the departure from equilibrium in this region, i.e., what is termed here freezing of the vibrational energy. The stationary value of the exponential term is given by $f' = 0$ or

$$\frac{1}{\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial\theta} = -R_1 F(\theta). \qquad \ldots(8)$$

Upstream of a certain region near this point (which is dependent on the particular streamline since it depends on R_1) the flow will remain near to equilibrium, while downstream of this region, but within the fan, freezing will occur, (downstream of the fan itself the flow will eventually return to an equilibrium state, see Section 4). In order to evaluate the locus of freezing points given by equation (8) the temperature dependence of Ω must be specified. In general this is a somewhat complicated function of T but for simplicity it will be assumed here that it can be represented by

$$\Omega = T^s$$

where $s > 0$. Under this assumption the function $F(\theta)$ is given by

$F(\theta)$/

$$F(\theta) = m_1 \left[\frac{\cos \lambda(\theta+\alpha)}{\cos \lambda\alpha} \right]^{2(s-1)}$$

For freezing to be a realistic concept $F(\theta)$ should be a monotonically decreasing function of θ. It is certainly true that the relaxation frequency decreases along streamlines but this tendency is offset by the fact that the distance of a streamline from the corner increases with increasing θ, and the above result indicates that $F(\theta)$ decreases only for $s > 1$.

From equation (8) it can be shown that the locus of freezing points, for $\Omega = T^s$, is given by

$$R_f = \frac{\omega_1 r_f}{q_1'} = \frac{2\lambda\, \Theta_{vib}}{m_1} \tan\lambda(\theta+\alpha) \left[\frac{\cos \lambda\alpha}{\cos \lambda(\theta+\alpha)} \right]^{2s+\frac{1}{\lambda^2}} \qquad \ldots(9)$$

It can be seen that R_f grows more rapidly with θ than does the distance $R \left(= \dfrac{\omega_1 r}{q_1'} \right)$ of a streamline from the corner. For $\theta = 0$

$$R_f = R_{1f} = \frac{2\lambda^2\, \Theta_{vib}}{m_1} \sqrt{m_1^2-1} \qquad \ldots(10)$$

(using the definition of α and substituting in (9)). Along streamlines defined by $R_1 < R_{1f}$ the freezing point no longer occurs in the physical flow (mathematically at points upstream of the fan) and in this region the flow variables will never follow the equilibrium distribution, i.e., it is a region of near-frozen flow. If $\theta = \theta_\ell$ defines the tail of the isentropic fan then θ_ℓ is given by

$$\tan \lambda(\theta_\ell+\alpha) = \lambda\sqrt{m_\ell^2-1} = \lambda \cot(\phi+\mu_1-\theta_\ell)$$

where ϕ is the corner angle and μ_1 the Mach angle of the incident stream. Consider the streamline which has its freezing point at $\theta = \theta_\ell$. It can be shown from equations (4) and (9) that the initial radial position along the wave head of this streamline is given by

$$R_1 = R_{1e} = \frac{2\lambda\, \Theta_{vib}}{m_1} \tan \lambda(\theta_\ell+\alpha) \left[\frac{\cos \lambda\alpha}{\cos \lambda(\theta+\alpha)} \right]^{2s}$$

$$\text{i.e.,} \quad R_{1e} = \frac{2\lambda^2\, \Theta_{vib}}{m_1} \sqrt{m_1^2-1} \left[\frac{1 - \lambda^2 + \lambda^2 m_\ell^2}{1 - \lambda^2 + \lambda^2 m_1^2} \right]^{s} \qquad \ldots(11)$$

Along streamlines defined by $R_1 > R_{1e}$ the flow never passes through a freezing point and remains near to equilibrium. In the intermediate region where

$$R_{1f} < R_1 < R_{1e}$$

freezing/

freezing will occur within the fan. The above results are shown schematically in Fig. 1.

4. Particular Solution

For the special case of hypersonic flow ($m_1 >> 1$) the integral in (7) takes a simpler form and can be evaluated in identical fashion to the corresponding integral discussed in Blythe (1962). However, in this case for corner angles of order of magnitude greater than $1/m_1$ the flow downstream of the fan is trivial since the flow will be fully expanded ($\rho = 0$) in that region and the energy in the lagging mode will be frozen along each streamline at the value it has when leaving the fan. For corner angles of $O(1/m_1)$, when the flow is not fully expanded, the solution downstream of the fan can easily be obtained. The streamlines in this region are now, to the approximation given here, parallel to the wall and the density, temperature, etc., are constant ($\neq 0$) in this region. The solution for the vibrational energy distribution downstream of the fan is thus found by integrating the rate equation, with all the kinetic variables remaining constant, along these known streamlines subject to the boundary condition on the vibrational energy distribution along the tail of the fan; this boundary condition is given by the solution within the fan. Details of the solution both within and downstream of the fan are given below in the special case $m_1 >> 1$; it is assumed that the corner angle is $O(1/m_1)$. The generalisation to any $m_1 > 1$ is given in Section 6.

In order to evaluate the integral in equation (7) the various functions are best expressed in terms of the Mach number m which is related to the angle θ by

$$\lambda(\theta+\alpha) = \tan^{-1} \lambda\sqrt{m^2-1}.$$

For $m >> 1$ (strictly $\lambda m >> 1$) the function f is given by

$$f \approx \left(\frac{m}{m_1}\right)^2 + \frac{R_1}{(2s-1)\lambda^2\Theta_{vib}}\left(\frac{m_1}{m}\right)^{2s-1} \qquad (s \neq \tfrac{1}{2}).$$

The stationary point, or freezing point, is given by

$$\frac{m}{m_1} = \left(\frac{R_1}{2\lambda^2\Theta_{vib}}\right)^{\frac{1}{2s+1}} = \Phi_1. \qquad \ldots(12)$$

A new variable ξ is defined by

$$\xi = \frac{m}{m_1\Phi_1}$$

and equation (7) becomes, for $m >> 1$

$$\epsilon = \left\{2\,\Theta_{vib}\,N_1^2 \exp\frac{2N_1^2}{(2s-1)\xi^{2s-1}}\right\}\int_{\Phi_1^{-1}}^{\xi} \psi \exp - N_1^2\left(\psi^2 + \frac{2}{2s-1}\cdot\frac{1}{\psi^{2s-1}}\right) d\psi \qquad \ldots(13)$$

where/

where ψ is a dummy variable and $N_1^2 = \Theta_{vib}\Phi_1^2$. This expression is now in a suitable form for evaluation by the method of steepest descents. The analysis proceeds along exactly the same lines as in the quasi-one-dimensional flow case (see Blythe (1962) for details) and it follows that

$$\epsilon = 2\,\Theta_{vib}\,N_1 \left\{ \exp N_1^2 \left(\frac{2}{2s-1}\,\frac{1}{\xi^{2s-1}} - q(1) \right) \right\}$$
$$\times \left[a_o E(N_1\eta) - \frac{1}{N_1}\,\frac{\left(\psi \frac{d\psi}{d\eta} - a_o \right)}{\eta} \exp - \frac{1}{2} N_1^2\,\eta^2 + O\left(\frac{1}{N_1^2} E(N_1\eta) \right) \right]_{\eta=\eta(\Phi_1^{-1})}^{\eta=\eta(\xi)} \qquad \ldots(14)$$

where

$$\left.\begin{aligned} q(\xi) &= \xi^2 + \frac{2}{2s-1}\cdot\frac{1}{\xi^{2s-1}} \\ \eta(\xi) &= \mathrm{sgn}(\xi-1)\,[2\{q(\xi) - q(1)\}]^{\frac{1}{2}} \\ E(N_1\eta) &= \sqrt{\frac{\pi}{2}}\left[1 + \mathrm{erf}\left(\frac{N_1\eta}{\sqrt{2}} \right) \right] \\ a_o &= \frac{1}{[q''(1)]^{\frac{1}{2}}} \end{aligned}\right\} \qquad \ldots(15)$$

Sufficiently far upstream of the freezing point it can be shown that this expression reduces to the first term in the usual near equilibrium solution (expansion in powers of $1/R_1$, Napolitano (1960)). Conversely it can also be shown that when the freezing point lies sufficiently far upstream of the fan (in a mathematical sense only) the expression reduces to the usual type of near-frozen solution (expansion in powers of R_1).

Upstream of the freezing point the first two terms in the square bracket on the right-hand side of (14) are of the same order of magnitude. In the neighbourhood of the freezing point and downstream of it the first term dominates and represents the rapid departure from equilibrium which occurs there. The behaviour of the above expression indicates that σ approaches some constant value for large ξ. However, the maximum value of ξ permissible in this expression is determined by the tail of the (isentropic) fan, i.e., by

$$\xi = \xi_\ell = \frac{1}{\Phi_1\left[1 - \frac{(\gamma-1)}{2} m_1\phi \right]}. \qquad \ldots(16)$$

Downstream/

Downstream of the fan the streamlines are again parallel to the wall, and if x' is a co-ordinate measured along the wall ($x' = 0$ at the apex) the distribution of the vibrational energy along a streamline in this region is governed by

$$\frac{\partial\sigma}{\partial x} = \frac{\rho_\ell T_\ell^s}{q_\ell}(\bar{\sigma}_\ell - \sigma) \qquad \ldots(17)$$

where $x = \dfrac{x'\omega_1'}{q_1'}$ and the suffix ℓ denotes conditions along the tail of the fan. The x position of a streamline emerging from the fan is given by

$$\frac{x_\ell}{R_1} = \left(\frac{1}{1 - \frac{(\gamma-1)}{2} m_1 \phi}\right)^{\frac{1}{\lambda^2}} = \left(\frac{m_\ell}{m_1}\right)^{\frac{1}{\lambda^2}} = X_\ell. \qquad \ldots(18)$$

Equation (17) can be rewritten

$$\frac{\partial\epsilon}{\partial x} + \frac{\epsilon}{X_\ell^{\{1+(2s-1)\lambda^2\}}} = 0.$$

The solution to this equation using the boundary condition $\epsilon = \epsilon_\ell(\xi_\ell)$ on $x = x_\ell$ is

$$\epsilon = \epsilon_\ell(\xi_\ell) \exp\left\{-\frac{(x-x_\ell)}{X_\ell^{\{1+(2s-1)\lambda^2\}}}\right\} \qquad \ldots(19)$$

where $\epsilon_\ell(\xi_\ell)$ is obtained from equations (14), (15) and (16). The decay of the exponential term in (19) is the same, in terms of $x-x_\ell$, along any streamline since X_ℓ is independent of R_1. Consequently if a relaxation length is defined as the length that governs the decay of ϵ/ϵ_ℓ downstream of the fan then this length will be the same for all streamlines since ϵ/ϵ_ℓ is independent of R_1 in the region downstream of the fan. Such a relaxation length is termed the relative relaxation length since this length governs the decay of ϵ relative to its value when leaving the fan. For convenience this length, denoted by L_{rel}, will be defined as the distance, downstream of the fan, over which ϵ/ϵ_ℓ falls to $1/e$. Hence

$$L_{rel} = X_\ell^{1+(2s-1)\lambda^2} = \left(\frac{R_{1e}}{R_{1f}}\right)^{\frac{1+(2s-1)\lambda^2}{\lambda^2(2s+1)}} \qquad \ldots(20)$$

A relaxation length which governs the decay of $\epsilon/\bar{\sigma}_\ell$ downstream of the fan can also be defined. Such a length is termed an absolute relaxation length

since/

since this length governs the decay of ϵ relative to the equilibrium value $\bar{\sigma}_\ell$ which is the same for all streamlines. Hence this relaxation length, denoted by L_{abs}, depends on the absolute value of ϵ. Correspondingly this relaxation length is defined as the distance over which $\epsilon/\bar{\sigma}_\ell$ falls to $1/e$ and it follows that

$$\frac{L_{abs}}{L_{rel}} = 1 + \log\frac{\epsilon_\ell}{\bar{\sigma}_\ell}. \qquad \ldots(21)$$

In the region far from the corner, where $\epsilon_\ell \ll \bar{\sigma}_\ell$, the expression on the right-hand side of (21) will become negative showing that the flow is effectively in equilibrium when it emerges from the fan. Sufficiently near to the corner, provided $\epsilon_\ell \gg \bar{\sigma}_\ell$, the logarithmic term will dominate and the absolute relaxation length will become much larger than the relative relaxation length. However, it should be noted that this will not always be so in the vicinity of the corner since, e.g., when $\phi \sim 0$, $\epsilon_\ell \ll \bar{\sigma}_\ell$ even in this region and the flow will effectively remain near to equilibrium. Although the general expression for $\log\dfrac{\epsilon_\ell}{\bar{\sigma}_\ell}$ is very complex certain simplifications can be obtained in these limiting regions of the flow.

Sufficiently far from the corner it is expected that a conventional near equilibrium solution (expansion in inverse powers of R_1) will be valid. It can be shown that a necessary condition for such a solution to hold is that $R_1 \gg \text{Max}\,(R_{1e}, 1)$. Conversely it can be shown that in a region near to the corner, given by $R_1 \ll \text{Min}\,(1, R_{1f})$, a conventional near-frozen solution (expansion in powers of R_1) is valid. Utilising these two types of solution it is found that, excluding for the moment the case $R_{1e} \sim R_{1f}$, for $R_1 \gg \text{Max}\,(R_{1e}, 1)$

$$\frac{L_{abs}}{L_{rel}} = 1 + \log\frac{R_{1e}}{R_1} \qquad \ldots(22)$$

for $R_1 \ll \text{Min}\,(1, R_{1f})$ and also $R_{1e} \gg R_{1f}$

$$\frac{L_{abs}}{L_{rel}} = \frac{R_{1f}}{2\lambda^2}\left[\left(\frac{R_{1e}}{R_{1f}}\right)^{\frac{2}{2s+1}} - 1\right] + 1 - \frac{R_1}{\lambda^2(2s-1)}. \qquad \ldots(23)$$

When $R_{1e} \sim R_{1f}$, i.e., the corner angle $\phi \sim 0$, we expect that the flow will everywhere remain near to equilibrium. In fact the relaxation length along streamlines near to the wall becomes, for $R_{1e} - R_{1f} \ll 1$

$$\frac{L_{abs}}{L_{rel}} = 1 + \log\frac{\left(1 - \dfrac{R_1}{R_{1f}}\right)(R_{1e}-R_{1f})}{\lambda^2(2s+1)} \qquad \ldots(24)$$

from/

from which it is apparent that for $R_{1e} - R_{1f}$ sufficiently small the flow is effectively in equilibrium near to the corner. This is not surprising as although a conventional near-frozen type of solution is valid there ($R_1 \ll \mathrm{Min}(1, R_{1f})$), the first term in this solution is $\bar{\sigma}_1$ and when $\phi \sim 0$, $\bar{\sigma} \sim \bar{\sigma}_1$ throughout and hence $\epsilon \sim 0$ throughout.

It is interesting to note the variation of $\epsilon/\bar{\sigma}$ at the freezing point. From equation (14) it is seen that

$$\left(\frac{\epsilon}{\bar{\sigma}}\right)_f = 2a_o\sqrt{\frac{\pi}{2}}\,N_1 - L(0) + \left[L(\eta)\ \exp - \tfrac{1}{2}\,N_1^2\,\eta^2\right]_{\eta=\eta(\Phi_1^{-1})} \qquad \ldots(25)$$

where
$$L(\eta) = \frac{2\left(\xi\dfrac{d\xi}{d\eta} - a_o\right)}{\eta}.$$

For $|N_1\eta(\Phi_1^{-1})| \gg 1$, then

$$(\epsilon/\bar{\sigma})_f \sim 2a_o\sqrt{\frac{\pi}{2}}\,N_1 - L(0) \qquad \ldots(26)$$

and since $N_1 = \Theta_{vib}^{\frac{1}{2}}\Phi_1$ and $\Phi_1 = \dfrac{m_f}{m_1}$, this result implies that $\epsilon/\bar{\sigma}$ varies linearly with the Mach number at the freezing point in the region where the above inequality holds. However, as $\Phi_1 \to 1$, i.e., in the neighbourhood of the wave head, the term neglected in deriving equation (26) from equation (25) must be taken into account.

5. Numerical Example

A specific example was evaluated for the theory of the last section. The Mach number ratio across the fan was taken to be 10, which corresponds for a diatomic gas, with $m_1 \gg 1$, to $m_1\phi = 4.5$. The initial temperature of the gas was taken as $T_1' = \frac{1}{4}\Theta_{vib}'$, i.e., $\Theta_{vib} = 4$, and the parameter s, dependence of the relaxation frequency on temperature, was given the value 2. The variation, within the fan, of $\epsilon/\bar{\sigma}$ along streamlines is plotted in Fig. 3 as a function of the angular ratio θ/θ_ℓ, where θ_ℓ is the value of θ at the tail of the fan. θ/θ_ℓ is related to the Mach number ratio by the expression, valid for $m_1 \gg 1$,

$$\frac{\theta}{\theta_\ell} = \frac{1 - \dfrac{m_1}{m}}{1 - \dfrac{m_1}{m_\ell}}. \qquad \ldots(27)$$

Each/

Each streamline is characterised by a value of Φ_1, since $R_1 = 2\lambda^2\Theta_{vib}\,\Phi_1^{2s+1}$. It is seen that for values of Φ_1 sufficiently greater than unity the flow initially tends to follow closely the equilibrium distribution, but as the freezing point is approached the departure from equilibrium increases rapidly.

However, it should be noted that in the neighbourhood of the wave head ($\theta = 0$) the flow does not strictly follow the equilibrium distribution (Bloom and Ting, (1959)) since in that region on all streamlines

$$\frac{\partial}{\partial\theta}\left(\frac{\epsilon}{\bar{\sigma}}\right) = -\frac{1}{\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial\theta} > 0.$$

For $N_1 \gg 1$ the value of $\epsilon/\bar{\sigma}$ near $\theta = 0$ is too small for this effect to be noticed, but the effect is clearly visible on the streamline defined by $\Phi_1 = 1.5$ ($N_1 = 3$).

In Fig. 3 the exponential decay downstream of the fan is not shown since on the scale of the graph it would be confined to a very narrow region indeed (because of the choice of θ as the independent variable). Also plotted in Fig. 3 are lines of constant r which are characterised by the value of Φ_1 ($= \Phi_1^*$) at $\theta = 0$.

6. General Solution

The above results, which were derived for hypersonic flow using the simple power law temperature dependence for the relaxation frequency, can be formally generalised to include all supersonic Mach numbers and any general temperature dependence for Ω. In this case an explicit expression for the position of the freezing point cannot, in general, be written down. However, it will be given by some Mach number ratio $\dfrac{m}{m_1} = \Phi_1(m_1, \Theta_{vib}, R_1)$, and by scaling the various quantities with respect to their values at the freezing point it can be shown that the generalised equation corresponding to (13) is

$$\epsilon = 2\,\Theta_{vib}\,N_1^2\left\{\exp - 2N_1^2\int^{\xi}\frac{F}{F_f}\,d\psi\right\}\int_{\Phi^{-1}}^{\xi} g(\psi)\ \left\{\exp - N_1^2\,q(\psi)\right\}d\psi \quad \ldots(28)$$

where
$$\left.\begin{aligned} g(\xi) &= \frac{\dfrac{1}{\bar{\sigma}}\dfrac{d\bar{\sigma}}{d\xi}}{\left(\dfrac{1}{\bar{\sigma}}\dfrac{d\bar{\sigma}}{d\xi}\right)_f}, \quad q(\xi) = 2\int^{\xi}\left(\frac{\dfrac{1}{\bar{\sigma}}\dfrac{d\bar{\sigma}}{d\xi}}{\left(\dfrac{1}{\bar{\sigma}}\dfrac{d\bar{\sigma}}{d\xi}\right)_f} - \frac{F}{F_f}\right)d\psi \\ N_1^2 &= \frac{1}{2}R_1\,F_f\,\frac{d\theta}{d\left(\dfrac{m}{m_1}\right)}\,\Phi_1, \quad F = \frac{r}{r_1}\,\frac{\rho\Omega}{v}, \end{aligned}\right\} \quad \ldots(29)$$

and the suffix f denotes conditions at the freezing point. The evaluation of this integral is carried out as before, the result being given by

equations/

equations (14) and (15) provided that in those expressions the above relations (29) are used and the factor $\psi \dfrac{d\psi}{d\eta}$ is replaced by $g(\psi) \dfrac{d\psi}{d\eta}$.

The solution downstream of the fan is again given by (19) provided that X_ℓ is suitably redefined, i.e.,

$$X_\ell = \frac{x_\ell}{R_1} = \sqrt{1 - \frac{1}{m_\ell^2}} \left[\frac{1 - \lambda^2 + \lambda^2 m_\ell^2}{1 - \lambda^2 + \lambda^2 m_1^2} \right]^{\frac{1}{2\lambda^2}} \qquad \ldots(30)$$

and that the modified expression for $\epsilon(\xi)$ is used in computing $\epsilon_\ell(\xi_\ell)$. The value of m_ℓ (and hence ξ_ℓ for a given Φ_1) is obtained from the full isentropic Prandtl-Meyer relations along the tail of the fan.

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References/

References

Appleton, J. P. (1960) The structure of a Prandtl-Meyer expansion fan in an ideal dissociating gas. U.S.A.A. Report No.146.

Bloom, M. H. and Steiger, M. H. (1960) Inviscid flow with non-equilibrium molecular dissociation for pressure distributions encountered in hypersonic flight. I.A.S. Paper No.60-26. Presented at the 28th Annual Meeting, New York.

Bloom, M. H. and Ting, L. (1959) On near-equilibrium and near-frozen behaviour of one-dimensional flow. AEDC-TN-60-156, PIBAL-R-525.

Blythe, P. A. (1962) Non-equilibrium flow through a nozzle. NPL Aero Report 1021. A.R.C.23,871.

Cleaver, J. W. (1960) The two-dimensional flow of an ideal dissociating gas. College of Aeronautics Report 123. A.R.C.21,743.

Napolitano, L. G. (1960) Non-equilibrium centered rarefaction for a reacting mixture. Polytechnic Institute of Brooklyn, AEDC-TN-60-129.

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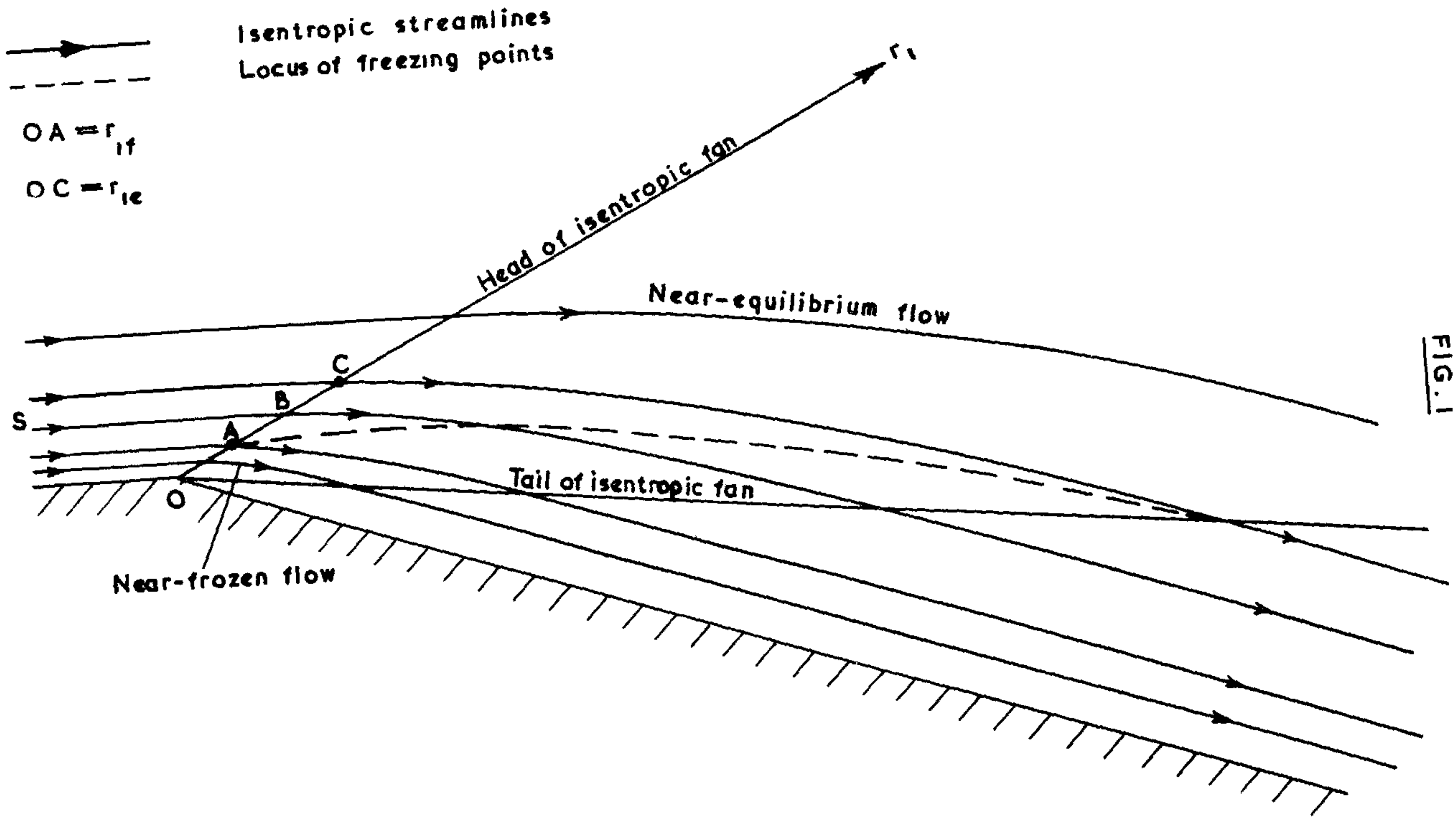


FIG. 1

Isentropic streamlines and locus of freezing points (schematic)

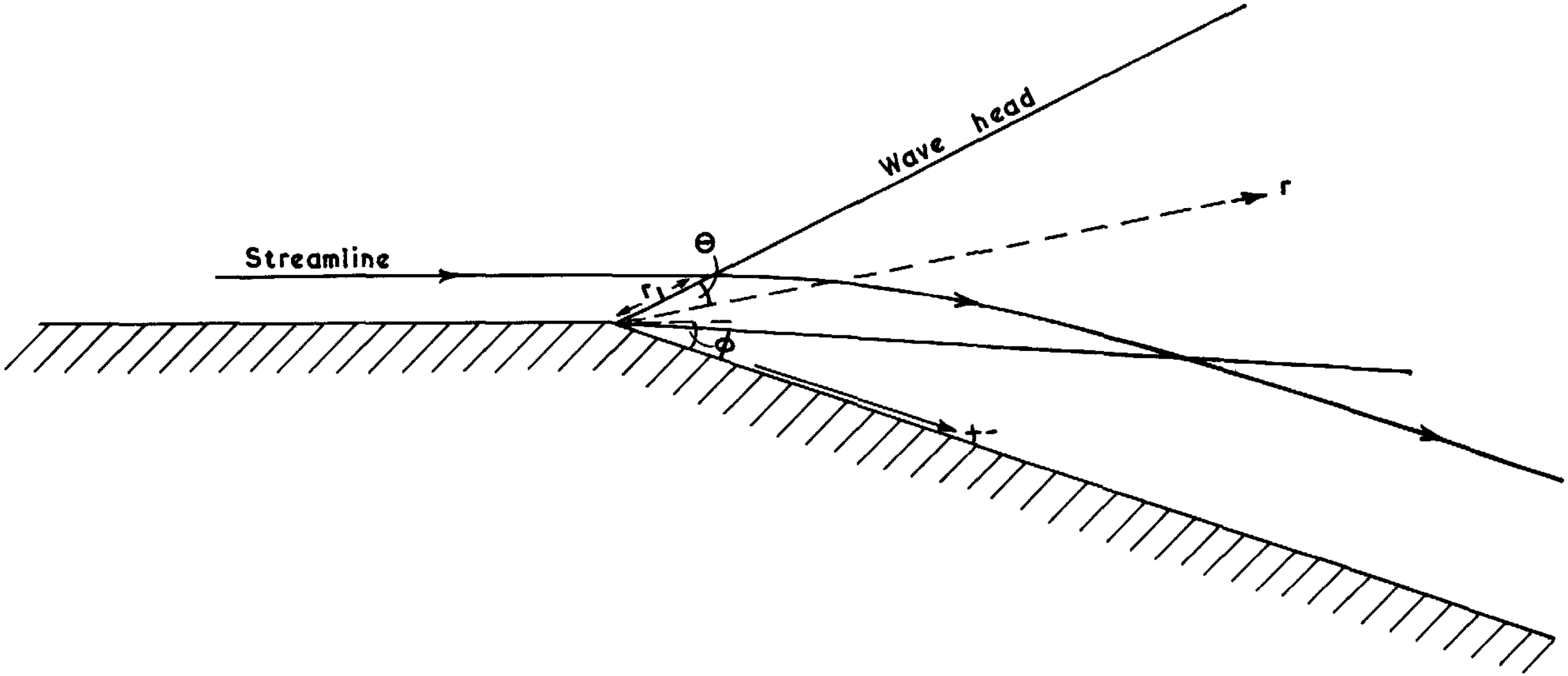
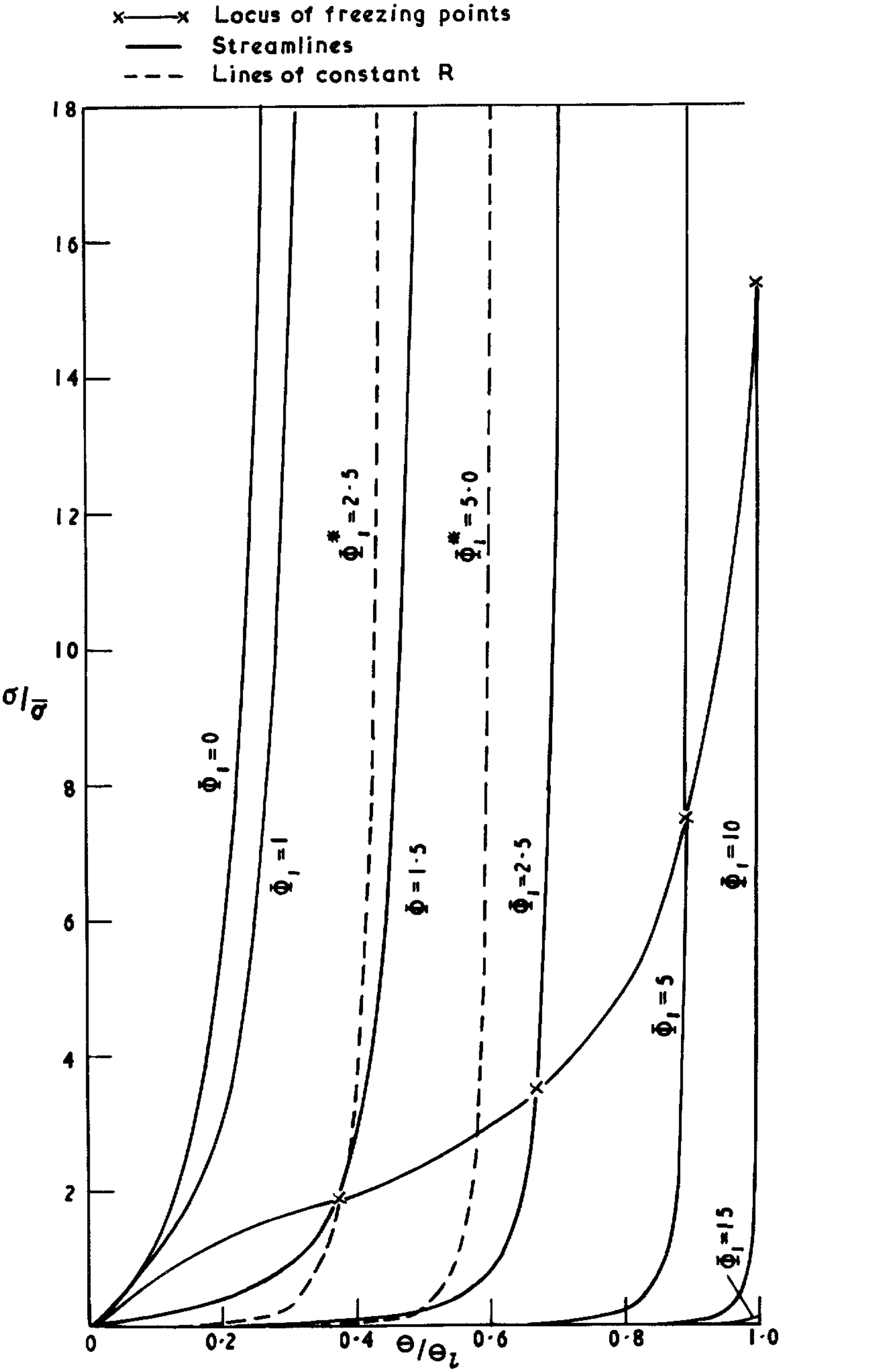


FIG. 2

Schematic diagram of isentropic expansion fan showing co-ordinate systems and notation.

FIG. 3



Distribution of vibrational energy within the fan $\Theta_{vib}=4, S=2, m_1 \gg 1$

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