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Laminar Mixing of a Non-Uniform Stream with a Fluid at Rest

By

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Laminar Mixing of a Non-Uniform Stream with a Fluid at Rest

- By -
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September, 1960

CONTENTS

Notation

x, y	Cartesian co-ordinates
u, v	velocity components
ρ	density
μ	absolute viscosity
ν	kinematic viscosity
T	temperature
L	length of flat plate

ψ/

Previously issued as A.R.C.22,245.

ψ	stream function
θ	momentum thickness
C	Chapman's viscosity constant
β	dummy variable
f(x)	arbitrary function

Subscripts

o	value at separation
1	free-stream value
R	reference value
D	value in dead-air region

Non-Dimensional Quantities

$$x^* = x/L$$

$$\tilde{y} = y\sqrt{\frac{u_R}{\nu x}}$$

$$u^* = u/u_1$$

$$\psi^* = \frac{\psi}{\sqrt{u_1 L \nu}}$$

R_e Reynolds number

SUMMARY

A theoretical analysis is made of the constant pressure laminar mixing process between a stream having an initial boundary layer velocity profile, and a fluid at rest.

The present theory follows the methods of W. Tollmien and S. I. Pai with certain modifications. The results apply to incompressible flow, but can be extended to the compressible case without difficulty.

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1. Reduced Momentum Equation

The momentum differential equation for a thin two-dimensional shear flow at constant pressure can be written

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right). \qquad \ldots(1)$$

The method of Pai[2*], makes the following additional simplifications

T/

---

*The method had previously been employed by Tollmien[1] to solve the incompressible wake problem and also by Oseen for very slow motion.

$$T = \text{constant everywhere},$$

$$v\frac{\partial u}{\partial y} << u\frac{\partial u}{\partial x},$$

and thirdly, that the velocity u appearing as the coefficient of $\partial u/\partial x$ in the first term, can be replaced by some fixed reference value u_R, say.

The momentum equation then reduces to

$$\frac{\partial u}{\partial x} = \frac{\nu}{u_R}\frac{\partial^2 u}{\partial y^2}. \qquad \ldots(2)$$

N.B.

It should be noted that in the present case of the mixing of a stream, with a fluid at rest, the assumption of uniform temperature throughout the flow, precludes the application of the results as they stand, to the compressible case.

The present theory departs from the method of Pai, and its extension in Ref.3, by attaching to u_R a value different from the free-stream value u_1. Since u_R replaces a real velocity component varying between zero and u_1, it seems more reasonable to choose some mean value. Two simple applications of the method, which will be described later, point to a value

$$u_R = \tfrac{1}{2}u_1$$

as being suitable.

2. Solution of the Equation

Equation (2) is in the form of the diffusion equation of classical physics, and its solution is

$$u(x, y) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} u_o\left(y - 2\sqrt{\frac{x\nu}{u_R}}\,.\,\beta\right) e^{-\beta^2} d\beta \qquad \ldots(3)$$

where $u = u_o(y)$, at $x = 0$.

Now, if y_o is the thickness of the initial boundary layer, suitably defined, at the point of separation, the boundary conditions may be formulated

$$\begin{aligned} -\infty < y < 0\,, \quad & u_o = 0 \\ 0 < y < y_o, \quad & u_o = u_o(y) \\ y_o < y < \infty\,, \quad & u_o = u_1. \end{aligned} \qquad \ldots(4)$$

Thus equation (3) can be written

$$u(x, y) = \frac{u_1}{\sqrt{\pi}}\int_{-\infty}^{\left(\frac{\tilde{y}-\tilde{y}_o}{2}\right)} e^{-\beta^2}\,.\,d\beta + \frac{1}{\sqrt{\pi}}\int_{\frac{\tilde{y}-\tilde{y}_o}{2}}^{\frac{\tilde{y}}{2}} u_o\left(y - 2\sqrt{\frac{\nu x}{u_R}}\,.\,\beta\right) e^{-\beta^2} d\beta. \qquad \ldots(5)$$

where/

where
$$\tilde{y} = y\sqrt{\frac{u_R}{\nu x}}$$
$$\tilde{y}_o = y_o\sqrt{\frac{u_R}{\nu x}}.$$

That is

$$u(x, y) = \frac{u_1}{2}\left[1 + \text{erf.}\,\frac{\tilde{y}-\tilde{y}_o}{2}\right] + \frac{1}{\sqrt{\pi}}\int_{\frac{\tilde{y}-\tilde{y}_o}{2}}^{\frac{\tilde{y}}{2}} u_o\left(y - 2\sqrt{\frac{\nu x}{u_R}}\,.\,\beta\right) e^{-\beta^2} d\beta. \qquad \ldots(6)$$

Now for a first approximation, let us assume a linear velocity profile for the initial boundary layer:

$$u_o(y) = u_1\,.\,\frac{y}{y_o}. \qquad \ldots(7)$$

The integral in equation (6) can now be evaluated and we obtain

$$u(x, y) = \frac{u_1}{2}\left[1 + \text{erf.}\,\frac{\tilde{y}-\tilde{y}_o}{2}\right] + \frac{u_1}{y_o\sqrt{\pi}}\int_{\frac{\tilde{y}-\tilde{y}_o}{2}}^{\frac{\tilde{y}}{2}} \left(y - 2\sqrt{\frac{\nu x}{u_R}}\,.\,\beta\right) e^{-\beta^2}\, d\beta$$

$$\begin{aligned} u^* = {} & \tfrac{1}{2}\left[1 + \text{erf.}\,\frac{\tilde{y}-\tilde{y}_o}{2}\right] \\ & + \tfrac{1}{2}\,\frac{\tilde{y}}{\tilde{y}_o}\left[\text{erf.}\,\frac{\tilde{y}}{2} - \text{erf.}\,\frac{\tilde{y}-\tilde{y}_o}{2}\right] \\ & + \frac{1}{\tilde{y}_o\sqrt{\pi}}\left[e^{-\tilde{y}^2/4} - e^{-\frac{(\tilde{y}-\tilde{y}_o)^2}{4}}\right] \end{aligned} \qquad \ldots(8)$$

where $u^* = u/u_1$.

3. Case of $y_o = 0$

For the case of $y_o = 0$, equation (6) reduces to the simple form

$$u^* = \tfrac{1}{2}\left[1 + \text{erf.}\left\{\tfrac{1}{2}\sqrt{\frac{u_R}{u_1}}\,.\,y\sqrt{\frac{u_1}{\nu x}}\right\}\right] \qquad \ldots(9)$$

which describes the mixing process of a uniform stream with a fluid at rest.

This velocity profile is plotted in Fig. 2 for two values of u_R. The curve is compared with the theory of Chapman[4] which is exact within the framework of the boundary-layer approximations. We see that the agreement is improved significantly by taking $u_R = \frac{1}{2}u_1$ rather than $u_R = u_1$. However, the agreement is even better than would appear from Fig. 2. There is an ambiguity in the co-ordinate systems of the two

methods/

methods. The present theory takes rectangular axes aligned with the direction of the initial stream, whereas Chapman measures y from the dividing streamline. There is no reason to suppose "a priori" that the two sets of axes will coincide.

If we determine the velocity on the dividing streamline by the method of Ref.3, we find a value of $0.61\ u_1$. Thus if this point on the profile is chosen as the new y-origin, equation (9) can be correctly compared with the exact curve. Fig.3 shows that the agreement is now rather good.

4. Incompressible Boundary Layer on a Flat Plate

At this point it might be interesting to digress slightly from the subject of laminar mixing, to consider a different solution of equation (2). Since this differential equation is linear, we can add separate solutions to form others.

If we multiply equation (9) by 2 and subtract the solution

$$u^* = 1$$

we obtain an expression for the velocity profiles in the boundary layer on a flat plate:-

$$u^* = \text{erf.} \left\{ \frac{1}{2} \sqrt{\frac{u_R}{u_1}} \cdot y \sqrt{\frac{u_1}{\nu x}} \right\} . \qquad \ldots(10)$$

Equation (10) is plotted in Fig.4 for two values of u_R, and compared with the exact curve of Blasius[5]. It is seen that a value of u_R between $0.4\ u_1$ and $0.5\ u_1$ would give the best fit, though there seems little theoretical justification for such a choice.

5. General Laminar Mixing Problem $y_o \neq 0$.

In the general case of a flow with an initial boundary layer of finite thickness, there is no longer similarity of the velocity profiles.

Suppose that the boundary layer has developed over a flat plate of length L, and also that we wish to define y_o such that the velocity gradient at the wall will be the correct value as given by the Blasius solution. Thus we may write:

$$y_o = \frac{u_1}{\left(\dfrac{\partial u}{\partial y}\right)_{y=0}} = 3.0115 \sqrt{\frac{\nu L}{u_1}} \qquad \ldots(11)$$

and equation (7) becomes

$$u_o(y) = u_1 \cdot 0.332\, y \sqrt{\frac{u_1}{\nu L}} . \qquad \ldots(12)$$

Also, since by definition

$$\tilde{y}_o = y_o \sqrt{\frac{u_R}{\nu x}}$$

we have

$$x/L = x^*, \text{ say}$$

$$= \left(\frac{3.0115}{\tilde{y}_o}\right)^2 \cdot \frac{u_R}{u_1}$$

and/

and taking $u_R = \frac{1}{2}u_1$

$$x^* = \frac{4.5356}{\tilde{y}_o^2}. \qquad \ldots(13)$$

Hence for any value of x^*, we can obtain the corresponding value of $\tilde{y}_o$, and then from equation (8), the velocity profile $u(\tilde{y})$ can be computed. The ordinates can be expressed in the more general form by writing

$$y\sqrt{\frac{u_1}{\nu L}} = 3.0115\left(\frac{\tilde{y}}{\tilde{y}_o}\right). \qquad \ldots(14)$$

6. Method of Locating Streamlines

The stream function ψ is defined by

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}$$

and thus

$$\psi = \int u dy + f(x). \qquad \ldots(15)$$

For convenience we shall assume that the streamline $\psi = 0$ passes through the origin (the point of separation). This streamline is sometimes termed the separating streamline, since it divides the fluid originally in the dead-air region from that in the outside flow. Thus

$$\psi = \int_o^y u_o dy, \quad \text{at} \quad x = 0$$

and for u_o given by equation (12), it follows

$$\psi = 1.5058\sqrt{\nu u_1 L}\,.\left(\frac{u_o}{u_1}\right)^2$$

or in non-dimensional form

$$\psi^* = \frac{\psi}{\sqrt{\nu u_1 L}} = 1.5058\left(\frac{u_o}{u_1}\right)^2. \qquad \ldots(16)$$

For other values of x, the stream function is given by equation (15), where $f(x)$ is an unspecified function. This equation can be written

$$\psi = u_1 y_o \int \left(\frac{u}{u_1}\right) d\left(\frac{y}{y_o}\right) + f(x)$$

$$= 3.0115\sqrt{\nu u_1 L}\int\left(\frac{u}{u_1}\right) d\left(\frac{y}{y_o}\right) + f(x)$$

or in non-dimensional form

$$\psi^* = 3.0115\int u^*\, d\left(\frac{y}{y_o}\right) + f(x). \qquad \ldots(17)$$

We/

We may trace the path of the streamlines in the initial part of the mixing layer by the following argument.

In the immediate vicinity of the separation point, i.e., for very small values of x^*, the velocity profile is modified from its form at $x^* = 0$ only at its extremities. Over a large part of the profile, the velocities are still given effectively by equation (12). Thus neglecting the natural decay of the velocities in the boundary layer as small compared with the velocity changes due to the mixing process, it is fairly safe to assume that particles of fluid on the streamline through the point $(0, \frac{1}{2}y_0)$, for instance, have a velocity $\frac{1}{2}u_1$ for some short distance downstream of separation.

Hence at a given station x^* (where $x^* << 1$) we can fix the value of ψ^* at $u^* = \frac{1}{2}$, as equal to its value at $x^* = 0$ and $u^* = \frac{1}{2}$, viz., from equation (16)

$$(\psi^*)_{u_0^*=\frac{1}{2}} = 0.3765.$$

Then for the remainder of the profile at that station, the local values of the stream function are given by

$$\psi^* = 0.3765 + 3.0115 \int_{\frac{1}{2}}^{y/y_0} u^* \, d\left(\frac{y}{y_0}\right). \qquad \ldots(18)$$

7. Results

The flow field in the neighbourhood of the separation point has been solved using equation (8), and tracing the path of the streamlines by the method outlined above. Fig. 5 shows some typical velocity profiles and streamlines. It is noted that the streamlines closest to the wall in the initial boundary layer, are deflected upwards towards the region of higher velocities after separation.

The results of the present theory have been derived for incompressible flow; however, Chapman[4] showed that assuming a linear viscosity-temperature law, relations expressed as functions of x^* and ψ^*, are independent of Mach number, and can thus be applied to compressible flow. In Fig. 6, we have plotted the velocity profiles in terms of ψ^* instead of the physical ordinate y. The $u(y)$ velocity profiles corresponding to any given Mach number can be obtained from these curves by simple quadrature after the method of Ref. 4.

The velocity along the streamlines is plotted in Fig. 7, and the velocity on the line $y = 0$ is shown for comparison with that on the streamline $\psi = 0$. It is clear that large errors would be introduced by neglecting the migration of such streamlines towards the layers of higher velocity.

It is found that the boundary $u^* = 0$ in the $x^* \sim \psi^*$ plane is a straight line in the part of the flow field, $x^* << 1$, and has a slope

$$\left(\frac{\partial\psi^*}{\partial x^*}\right)_{u^*=0} = -0.66.$$

Now from the definition of the stream function

$$v = -\frac{\partial\psi}{\partial x} = -\sqrt{\frac{\nu u_1}{L}} \cdot \frac{\partial\psi^*}{\partial x^*}. \qquad \ldots(19)$$

Thus/

Thus the vertical velocity component in this part of the dead-air region, $x^* << 1$, is constant and has a value

$$v_D = 0.66 \sqrt{\frac{\nu u_1}{L}} = \frac{0.66}{\sqrt{R_e}} \cdot u_1 . \qquad \ldots(20)$$

By way of comparison, the vertical velocity component at the outer edge of the boundary layer at separation is slightly greater:-

$$v_1 = 0.865 \sqrt{\frac{\nu u_1}{L}}$$

as given by the Blasius solution.

For zero thickness of the initial boundary layer, the value of v_D is infinite at the separation point, and drops as

$$1/\sqrt{x} .$$

In this case of course, the condition $x^* << 1$ cannot be satisfied.

In the compressible laminar mixing process, equation (20) can be replaced by

$$v_D = 0.66 \frac{\rho_1}{\rho_D} \sqrt{\frac{\nu_1 u_1 C}{L}} \qquad \ldots(21)$$

where ρ_D is the density in the dead air region, and C is Chapman's constant. Thus, for a given Reynolds number, the effect of Mach number is to increase the vertical velocity component in the dea-air region.

If θ is the momentum thickness of the initial boundary layer at separation, equation (21) can also be expressed in the form

$$\frac{v_D \theta}{\nu_D} = 0.44 \, C . \qquad \ldots(22)$$

8. True Boundary Conditions

In the above simplified analysis, we have taken a linear function for the velocity profile at separation. The true profile is determined by the past history of the boundary layer; over a flat plate for instance, the velocity distribution would be described by the Blasius solution.

Now we have argued that the effect of separation spreads slowly through the shear layer as it proceeds downstream. Near the separation point, the greater part of the velocity profile still corresponds to a boundary-layer distribution, whilst the lowest velocity region is modified by the mixing process.

Fig. 8 shows this more clearly. At a given station x^* (where $x^* << 1$), a good approximation to the true velocity distribution should be obtained by fairing the mixing profile, equation (8), with the local Blasius profile for a boundary layer which would have developed over a flat plate of length $x + L$.

This procedure of course loses its justification for larger values of x^*.

The/

The next stage of this analysis should be the solution of the complete problem in the $x^* \sim \psi^*$ plane, using the differential equation derived by Chapman[4],

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial}{\partial \psi^*}\left(u^* \frac{\partial u^*}{\partial \psi^*}\right) \qquad \ldots(23)$$

and initial conditions given by the present method at some short distance downstream of separation. The integration of equation (23) cannot be started at the separation point due to the discontinuity in the boundary conditions at the origin.

9. Conclusions

1. Constant-pressure laminar flow problems have been solved using simplified diffusion theory after the small-perturbation methods of Tollmien and Pai. Two such solutions agree well with exact theory.

2. The laminar mixing process between a stream having an initial boundary layer, and a fluid at rest, has been analysed. By assuming a simple form of velocity distribution at separation, a closed solution has been obtained for the velocity profiles in the mixing layer.

3. A method is indicated for tracing the path of the streamlines through the shear layer close to the separation point.

4. Streamlines close to the wall in the initial boundary layer are deflected towards the region of higher velocity after separation.

5. For a short distance downstream of the separation point, the vertical velocity component in the dead-air region is constant, and has a value inversely proportional to the thickness of the boundary layer at separation.

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No.	Author(s)	Title, etc.
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2	S. I. Pai	Two-dimensional jet mixing of a compressible fluid. J.A.S. Vol.16, 1949, pp.463-469.
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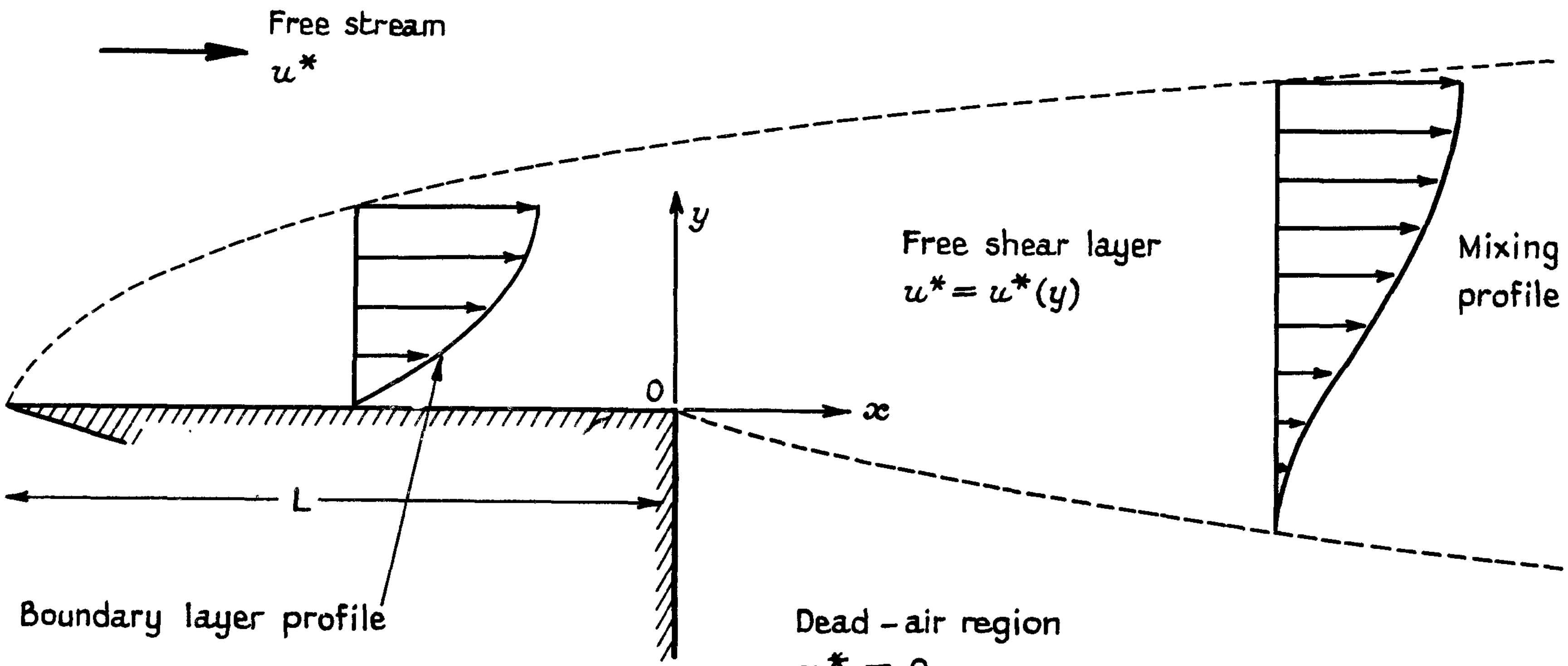
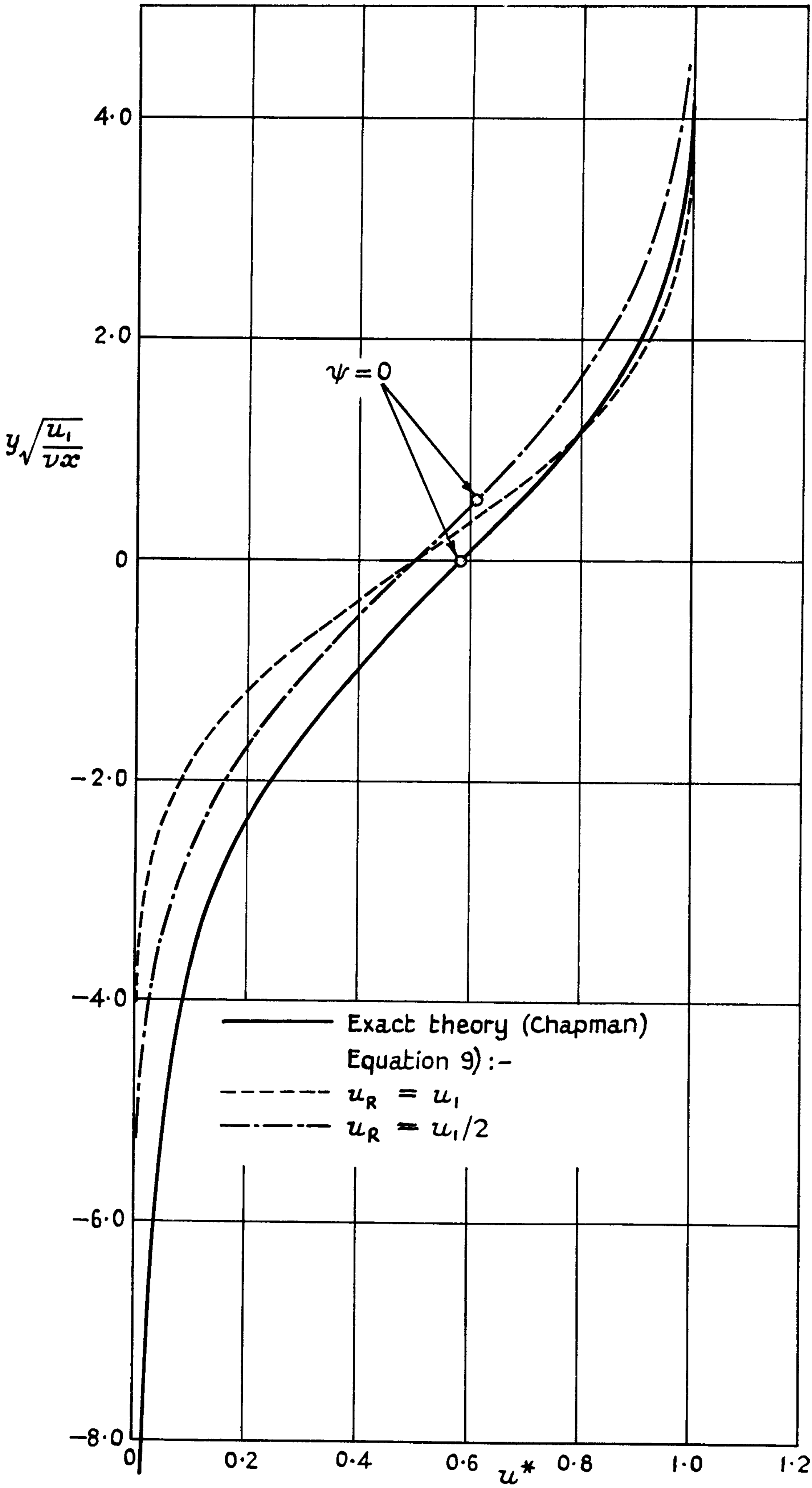
FIG. 1.



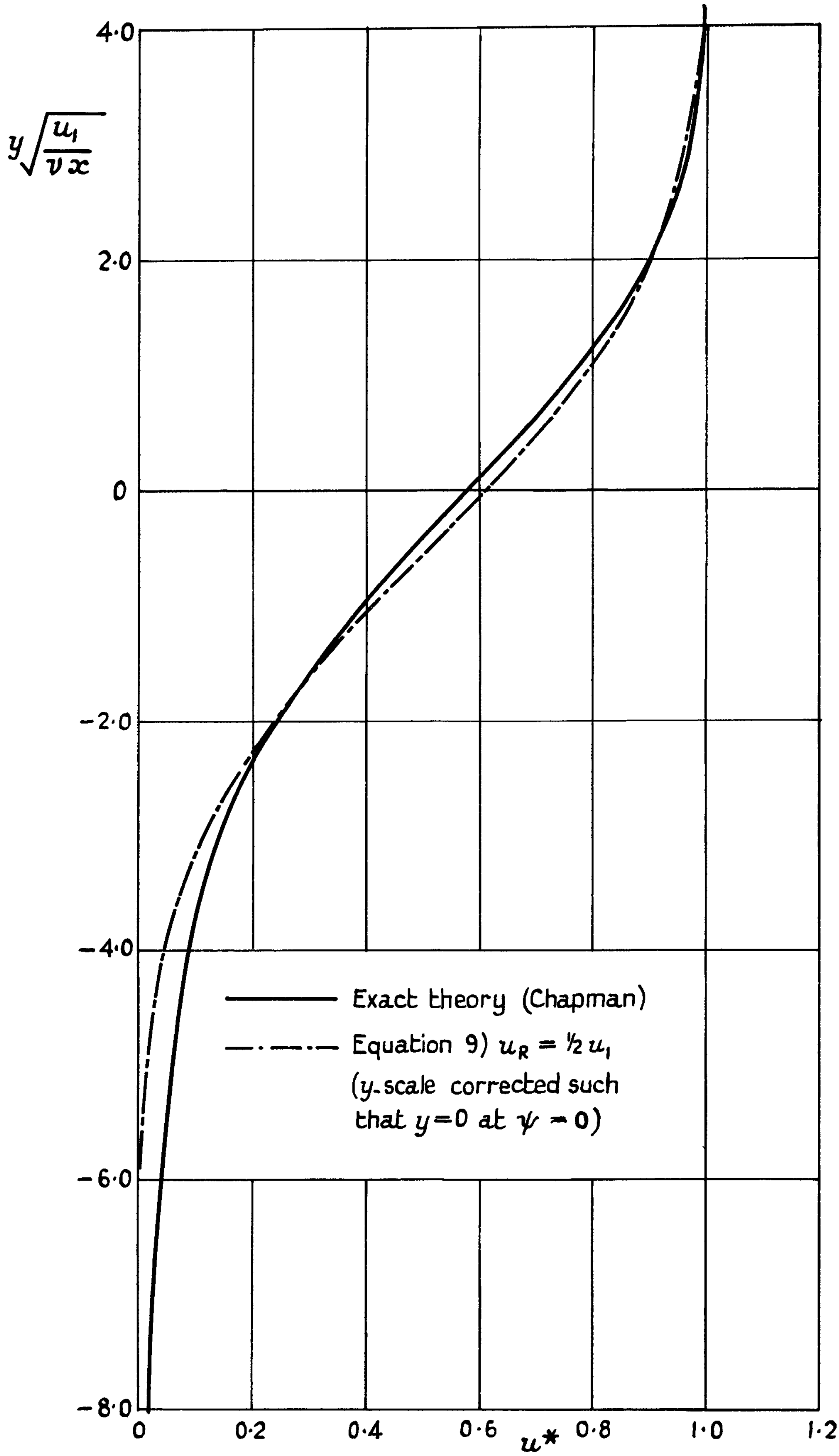
Constant - pressure mixing process

FIG. 2.



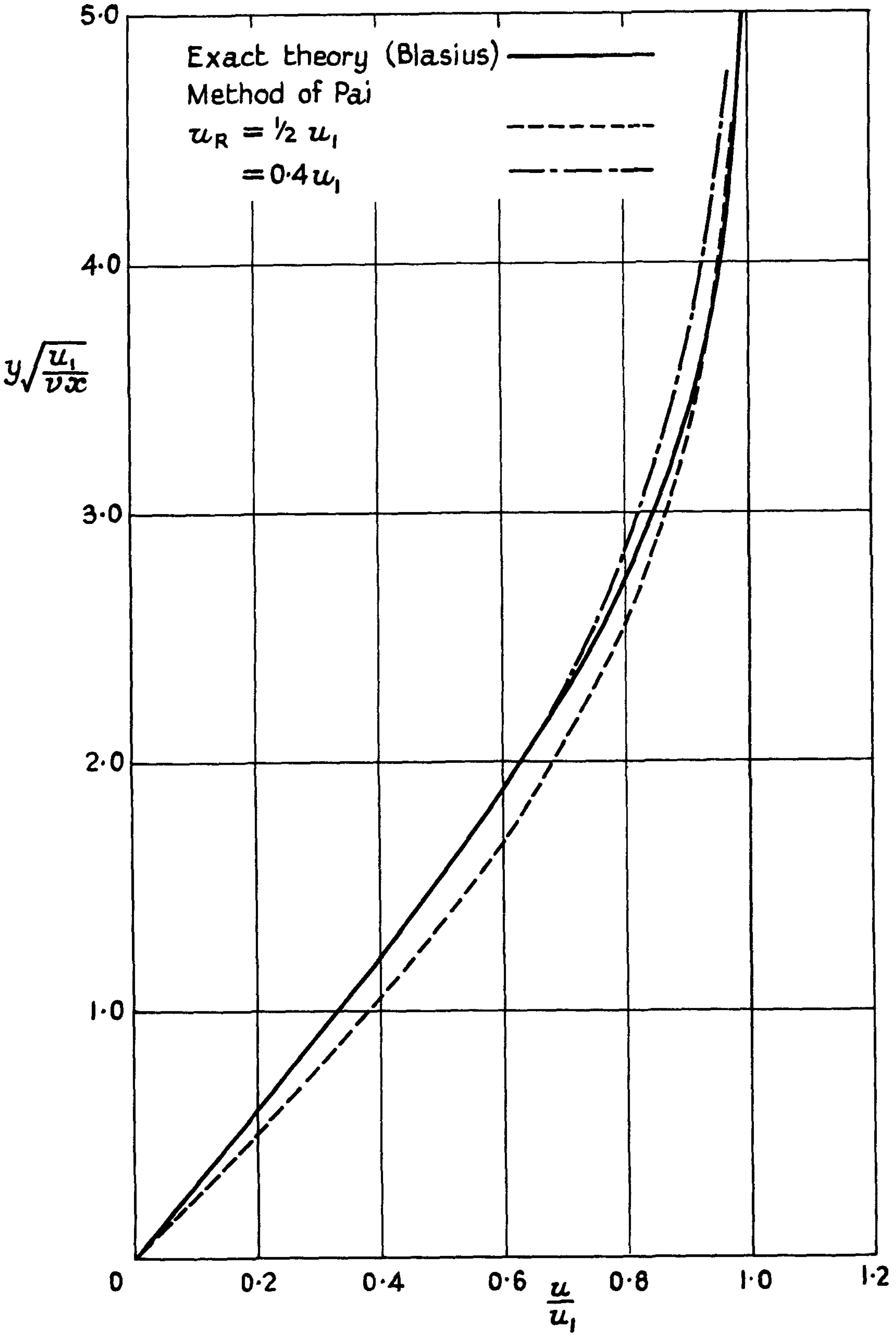
Zero thickness of initial boundary layer

FIG. 3.



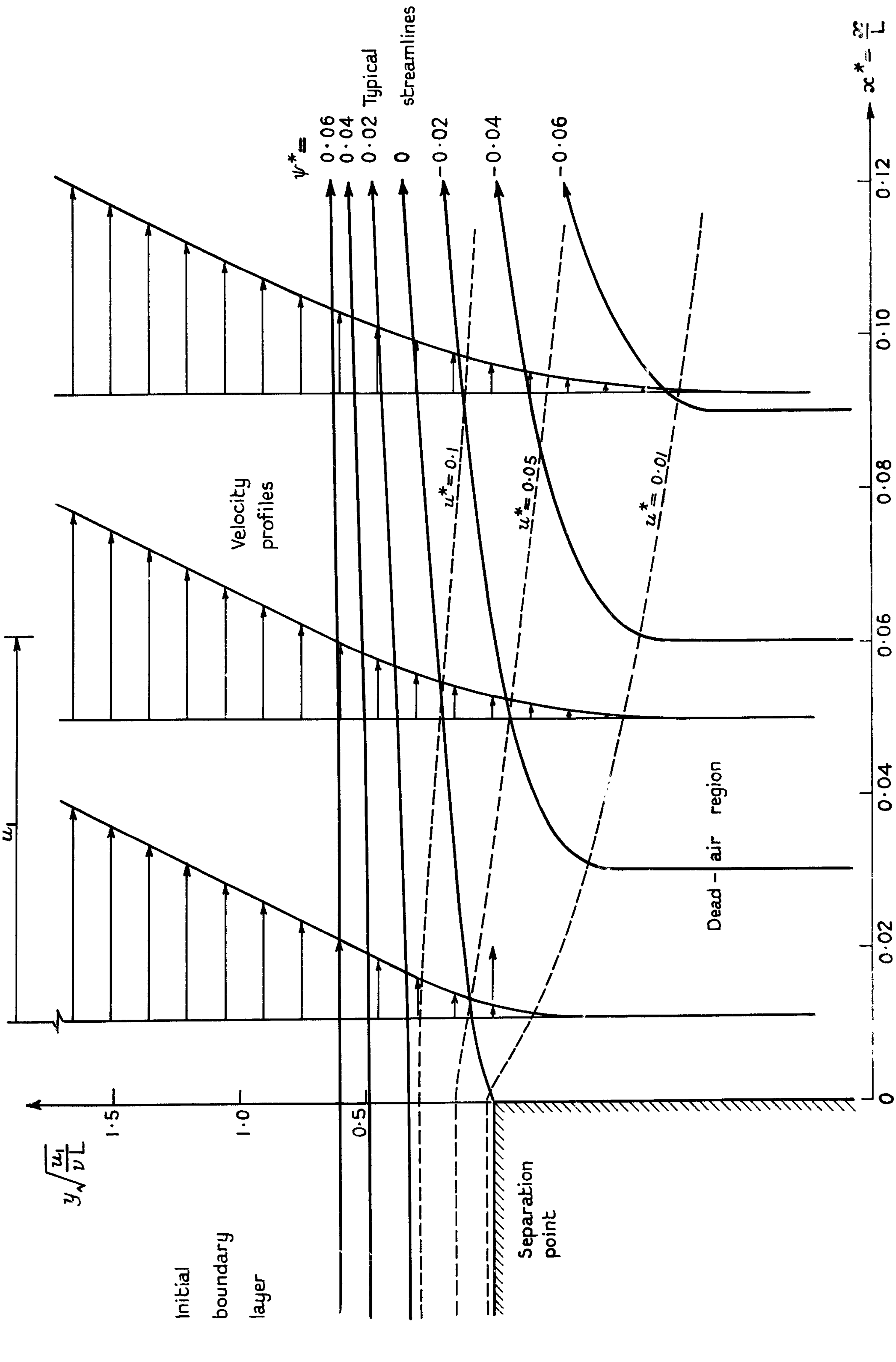
Zero thickness of initial boundary layer

FIG. 4



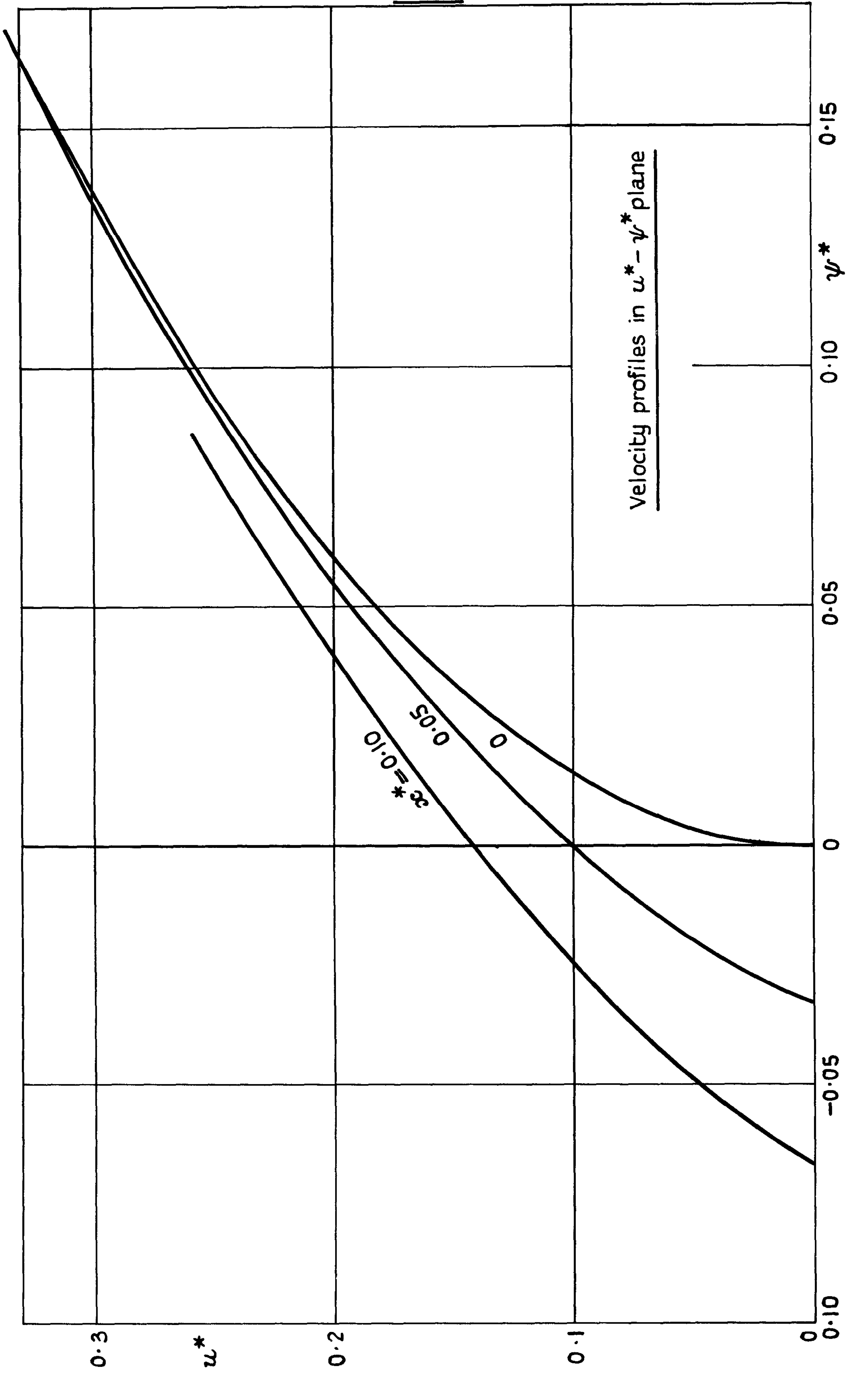
Exact theory (Blasius)
Method of Pai
$u_R = \frac{1}{2} u_1$
$= 0{\cdot}4 u_1$
$y\sqrt{\frac{u_1}{\nu x}}$
5·0
4·0
3·0
2·0
1·0
0
0·2
0·4
0·6
0·8
1·0
1·2
$\frac{u}{u_1}$

FIG. 5.



$y\sqrt{\frac{u_1}{\nu L}}$
1·5
1·0
0·5
u_1
Initial boundary layer
Separation point
Velocity profiles
Dead - air region
$\psi^* =$
0·06
0·04
0·02 Typical streamlines
0
-0·02
-0·04
-0·06
$u^* = 0·1$
$u^* = 0·05$
$u^* = 0·01$
0
0·02
0·04
0·06
0·08
0·10
0·12
$x^* = \frac{x}{L}$

FIG. 6.



Velocity profiles in $u^*-\psi^*$ plane
$x^*=0{\cdot}10$
$0{\cdot}05$
0
u^*
ψ^*
$0{\cdot}3$
$0{\cdot}2$
$0{\cdot}1$
0
$0{\cdot}10$
$-0{\cdot}05$
0
$0{\cdot}05$
$0{\cdot}10$
$0{\cdot}15$

FIG. 7.

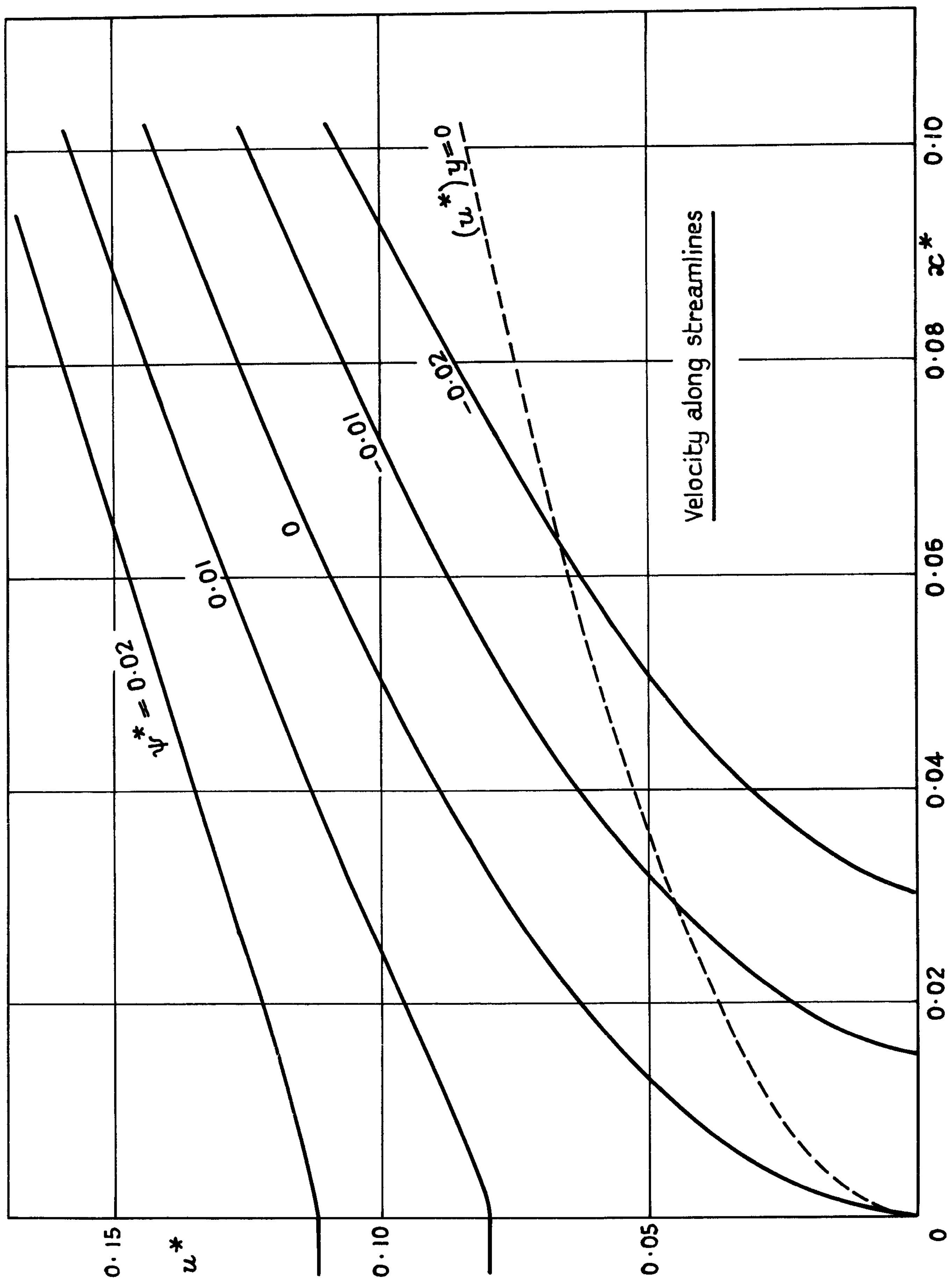
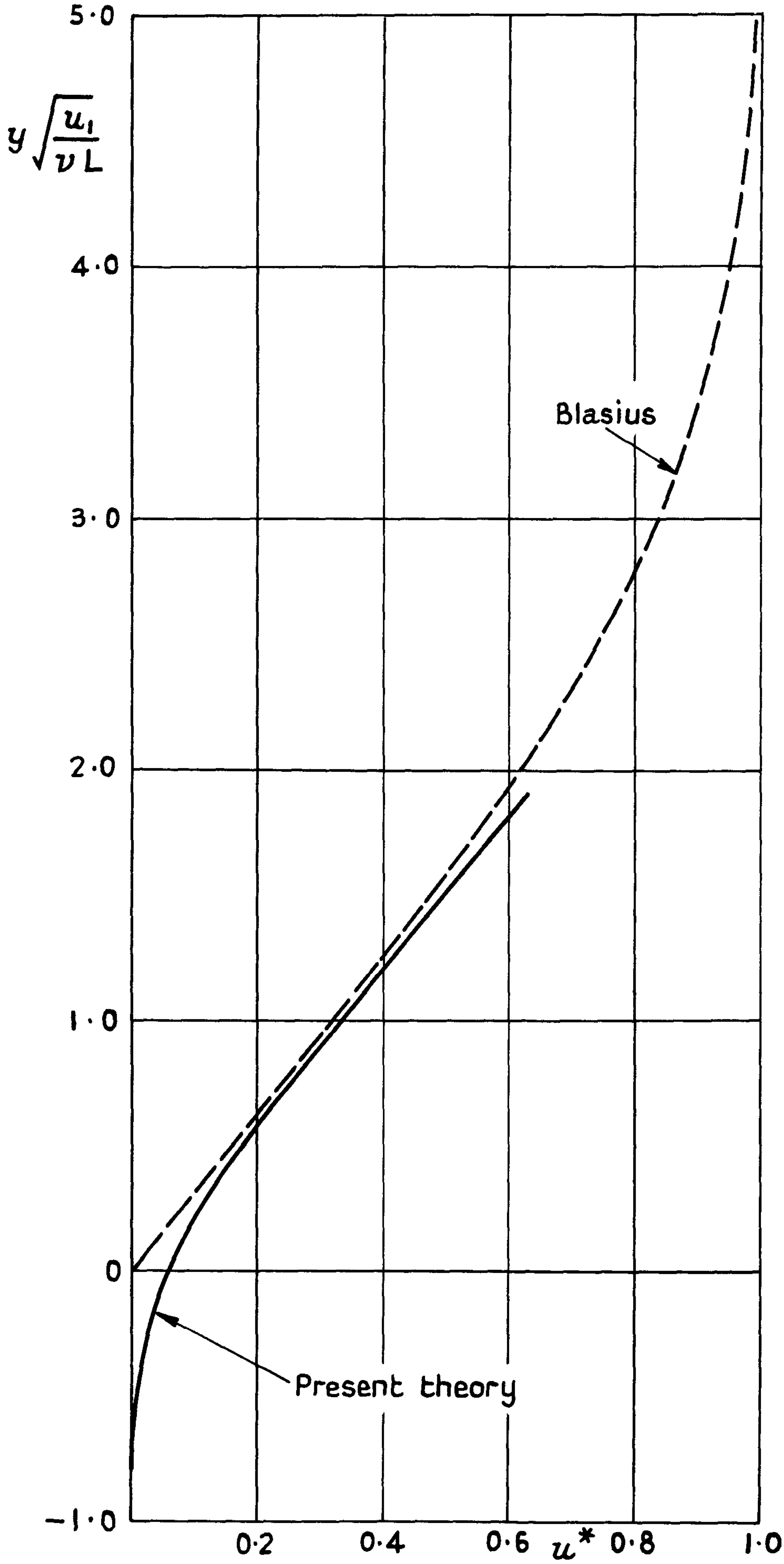

Velocity along streamlines
$\psi^* = 0.02$
0.01
0
−0.01
−0.02
$(u^*)_{y=0}$
u^*
0.15
0.10
0.05
0
x^*
0.02
0.04
0.06
0.08
0.10

FIG. 8.



Velocity profile at $x = 0\cdot05$

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