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A Note on the Use of Time Series in the Analysis of Flight Test Series

By

W. P. Jones, M.A.,
of the Aerodynamics Division, N.P.L.

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30th January, 1950

Summary

The methods suggested in Ref.1 for analysing the behaviour of linear systems are briefly reviewed, the numerical analysis being expressed in terms of matrices. Possible applications of time series representation to the study of aircraft stability characteristics are discussed and a detailed numerical investigation of a simple one degree of freedom undamped system is made. For this system the Tustin method[1] of analysis in terms of Δ units seems satisfactory. An alternative method based on the use of Simpson's integration rule in conjunction with time series representation is also described.

No definite conclusions can be drawn as to the advisability of using the suggested method of analysis at this stage, as it is considered that a detailed numerical study of the stability characteristics of a particular aircraft should be made in order to check fully the accuracy of the method.

1. Introduction

In a note entitled "Flight measurements of aircraft stability and control", it is suggested by Boulton Paul Aircraft Limited that measurements of aircraft response due to known control displacements should be analysed by means of time series. Any function depended on time is in this scheme represented by a series of ordinates corresponding to the values of the function at equal time intervals δ, where δ is assumed to be small enough to ensure accurate representation. Such a procedure was used by Professor Tustin in Ref. 1, but was not applied to aircraft response problems. His methods of dealing with time series are briefly outlined in this note, and it is shown how the analysis can be conveniently expressed in matrix notation. The numerical processes of 'serial multiplication' and 'serial division' as defined by Tustin correspond to matrix multiplication and inversion respectively. Certain operators used in Ref. 1 can also be expressed concisely in matrix form.

Over /

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Over a period of time, any function would be represented in this scheme by a large number of ordinates. Consequently, the numerical work of analysing the behaviour of a linear system might in certain cases be rather laborious with ordinary calculating machines. This disadvantage could perhaps be overcome by using special computing equipment.

An alternative method of approach is suggested in this note which might in certain cases reduce the amount of computation without entailing loss of accuracy. This scheme is based on the use of Simpson's integration rule in conjunction with time series representation. The relative accuracy of the two methods is illustrated by a very simple example of a one variable system with known response characteristics.

By the use of matrices the analysis for either method can be extended to deal with problems involving many degrees of freedom and the results of flight tests. In flight, the response in any degree of freedom due to a known movement of a particular control can usually be measured.. The problem is then to estimate the response in each degree of freedom due to a Δ unit input (or unit impulse) from the control. When these are known accurately, the responses due to any other known input can be estimated. The responses in a number of degrees of freedom due to a combination of inputs, such as from aileron and rudder, can be treated similarly.

It has been suggested that the roots of the stability determinant for the aircraft can be deduced from the numerically equivalent form of the equations of motion as derived by time series representation. This has been done for the simple example considered, but it is difficult to judge whether the method would apply in the case of a system with several degrees of freedom. Since in practice the analysis would be based on data obtained from flight tests, the possibility of small errors in the measured responses would also have to be considered as these would affect the calculated responses due to Δ unit inputs (or unit impulses) which have to be determined by a process of inversion, as presumably such inputs cannot be applied directly. In view of these difficulties it is thought that further test calculations should be done for a particular aircraft, taking into account the appropriate degrees of freedom and assuming control inputs of a form which can be applied in practice. The information obtained from such calculations should give a clear indication as to the advisability, or otherwise, of using the method of time series representation for dealing with stability problems.

2. Time Series Representation

In Tustin's paper[1] any function of time $d(t)$ is represented by a series of ordinates as shown in Fig.1.

Fig.1 /

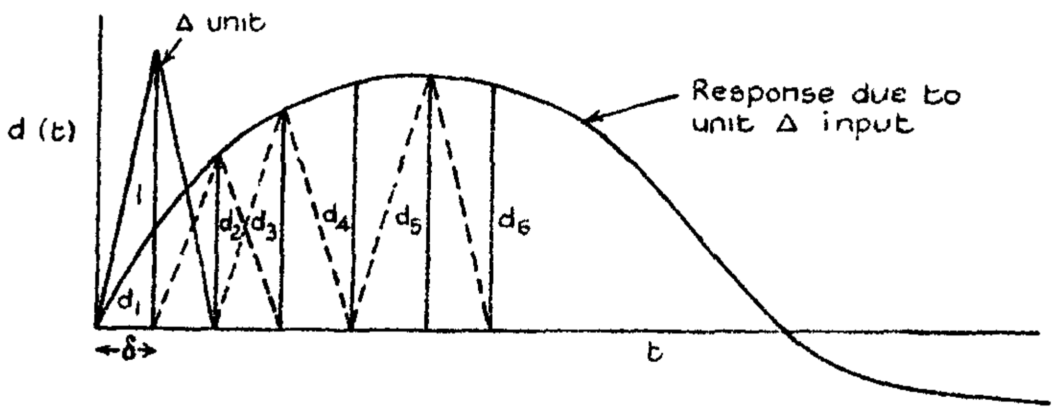


FIG I

The curve $d(t)$ is first replaced by a polygon formed by joining the ordinates at regular time intervals δ , and this in turn is replaced by a system of isosceles triangles of height d_1, d_2, etc. and base 2δ. If the interval δ is sufficiently small the function $d(t)$ will be accurately represented by such a series of ordinates and can be regarded as being composed of superposed Δ units as indicated.

Let us suppose $\{d(t)\} = \{d_1, d_2, d_3 \ldots\ldots\}$

is the response of a particular variable due to a unit Δ input*. Then, since any general input $e(t)$ can be represented by a number of isosceles triangles as above, the response $r(t)$ due to $e(t)$ is expressible in matrix notation in the alternative forms

$$\{r\} = A(d)\,\{e\} = A(e)\,\{d\}, \qquad \ldots(1)$$

where $\{r\}$, $\{e\}$ and $\{d\}$ represent columns of ordinates and the matrix operator

A(d) /

---

* A Δ unit is in the form of an isosceles triangle of unit height and base 2δ.

$$A(d) \equiv \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ d_2 & d_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ d_3 & d_2 & d_1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ d_4 & d_3 & d_2 & d_1 & 0 & \cdot & \cdot & \cdot & \cdot \\ d_5 & d_4 & d_3 & d_2 & d_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \ldots(2)$$

and $A(e)$ is defined similarly. In general A is an infinite triangular matrix, but in practice only a finite number of rows will be needed. For a stable system $d_n \to 0$ when n is large.

Formula (1) above expresses in concise form the table for serial multiplication given in Ref. 1. From (1) it follows by inversion* that

$$\{e\} = \left[A(d)\right]^{-1} \{r\} \qquad \ldots(3)$$

and

$$\{d\} = \left[A(e)\right]^{-1} \{r\}, \qquad \ldots(4)$$

where A^{-1} represents the inverse of A so that $A\,A^{-1} = I$, where I represents the unit matrix.

Suppose

$$[A(d)]^{-1} = \begin{bmatrix} a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ a_2 & a_1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ a_3 & a_2 & a_1 & 0 & \cdot & \cdot & \cdot & \cdot \\ a_4 & a_3 & a_2 & a_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Since $A\,A^{-1} = I$, matrix multiplication immediately yields the set of equations

$$a_1\,d_1 = 1,$$

a_2 /

* This corresponds to 'serial division' in Ref. 1.

$$a_2 d_1 + a_1 d_2 = 0$$

$$a_3 d_1 + a_2 d_2 + a_1 d_3 = 0, \qquad \ldots(6)$$

and so on

When the elements d_1, d_2, d_3 etc. are known the elements a_1, a_2, a_3, etc. can then be determined successively, and $\{e\}$ can be calculated from (3) when $\{r\}$ is known. Similarly when $\{r\}$ and $\{e\}$ are known, $\{d\}$, the response due to unit Δ input, is given by (4). It should be noted that (3) is the numerical equivalent of the differential equation defining the motion, say $f(p)r = e$, where $p \equiv \frac{d}{dt}$. The above procedure avoids 'serial division' as carried out by Tustin.

In general the response $\{r\}$ and the input $\{e\}$ are known, or can be measured, and one is faced with the problem of determining $\{d\}$. This is given directly by (4) or it can be derived from the expanded form of (1). To ensure accuracy δ must be small, and this means heavy numerical work, particularly when several degrees of freedom are involved. In the next section an alternative method is suggested which may in certain cases reduce the amount of computation without introducing inaccuracies.

3. Alternative Method

The exact formula for the response $r(t)$ due to any input function $e(t)$ is

$$r(t) = \int_0^t d(t-\tau)\; e(\tau)d\tau, \qquad \ldots(7)$$

where $d(\tau)$ now represents the response due to a unit impulse.

It is assumed that $r = \frac{dr}{dt} = 0$ at $t = 0$.

Let us suppose that the range of integration is divided into n equal intervals δ. Then, in serial numbers the integrand $d(t-\tau)\;e(\tau)$ is expressible in the form

$$\{ d[(n-s)\delta]\; e(s\delta) \} = \{ d_n e_0,\; d_{n-1} e_1,\; \ldots.,\; d_0 e_n \} \qquad \ldots(8)$$

By the use of Simpson's integration rules✗ the value of $r(n\delta)$ can be

calculated /

✗ The $1/3$ rule is mainly used but for r_5, r_7, r_9, etc. the integration is rounded off by using the $3/8$ th rule as shown by (9).

calculated accurately for any value of n greater than unity. The serial numbers r_2, r_3, r_4 etc. are given by the following set of equations

$$r_2 = \frac{\delta}{3}(d_2e_o + 4d_1e_1 + d_oe_2)$$

$$r_3 = \frac{3\delta}{8}(d_3e_o + 3d_2e_1 + 3d_1e_2 + d_oe_3)$$

$$r_4 = \frac{\delta}{3}(d_4e_o + 4d_3e_1 + 2d_2e_2 + 4d_1e_3 + d_oe_4) \qquad \ldots(9)$$

$$r_5 = \frac{\delta}{3}(d_5e_o + 4d_4e_1 + \frac{17}{8}d_3e_2 + \frac{27}{8}d_2e_3 + \frac{27}{8}d_1e_4 + \frac{9}{8}d_oe_5)$$

and so on.

In matrix notation (9) yields

$$\begin{aligned}\{r_2, r_3, r_4 \ldots\} &= \delta M(d) \{e_o, e_1, e_2 \ldots\ldots\} \\ &= \delta M(e) \{d_o, d_1, d_2 \ldots\ldots\}\end{aligned} \qquad \ldots(10)$$

where

$$M(q) = \begin{bmatrix} \frac{q_2}{3} & \frac{4q_1}{3} & \frac{q_o}{3} & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{3q_3}{8} & \frac{9q_2}{8} & \frac{9q_1}{8} & \frac{3q_o}{8} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{q_4}{3} & \frac{4q_3}{3} & \frac{2q_2}{3} & \frac{4q_1}{3} & \frac{q_o}{3} & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{q_5}{3} & \frac{4q_4}{3} & \frac{17q_3}{24} & \frac{9q_2}{8} & \frac{9q_1}{8} & \frac{3q_o}{8} & 0 & \cdot & \cdot & \cdot \\ \frac{q_6}{3} & \frac{4q_5}{3} & \frac{2q_4}{3} & \frac{4q_3}{3} & \frac{2q_2}{3} & \frac{4q_1}{3} & \frac{q_o}{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \ldots(11)$$

and /

and q denotes d or e as the case might be. In response problems, however, when the time dependent variables correspond to displacements, the initial value of $d(t)$ is zero ($d_o = 0$), and, except for inputs of the unit step type, $e_o = 0$ can also be assumed. Equations (10) reduce to

$$\begin{aligned} \{r_2, r_3, ..\} &= \bar{M}(d)\ \{e_1, e_2 ....\} \\ &= \bar{M}(e)\ \{d_1, d_2 ....\}, \end{aligned} \qquad ...(12)$$

where the modified matrix operator

$$\bar{M}(q) \equiv \begin{bmatrix} \frac{4q_1}{3} & 0 & 0 & 0 & 0 & . & . \\ \frac{9q_2}{8} & \frac{9q_1}{8} & 0 & 0 & 0 & . & . \\ \frac{4q_3}{3} & \frac{2q_2}{3} & \frac{4q_1}{3} & 0 & 0 & . & . \\ \frac{4q_4}{3} & \frac{17q_3}{24} & \frac{9q_2}{8} & \frac{9q_1}{8} & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \qquad ...(13)$$

and q = d or e according to which form of (12) is the most convenient to use.

Equations (12) and (1) correspond. Both A and $\bar{M}$ are triangular, but the elements in the latter matrix operator are multiplied by certain factors. It should be noted that the element r_1 of the response function $\{ r(t) \}$ is omitted in (12).

If the input is zero after a finite time the serial numbers will all be zero after a certain value. Let e_n correspond to the first zero value. The value of r_m for $m > n$ will then be given by

$$r_m = \frac{\delta}{3} \left[4d_{m-1}\, e_1 + 2d_{m-2}e_2 + 4d_{m-3}e_3 + ... 4d_{m-n+1}\, e_{n-1}\right] \qquad ...(14)$$

when n is even, and by

r_m = /

$$r_m = \frac{\delta}{3}\Big[4d_{m-1}\, e_1 + 2d_{m-2}e_2 + 4d_{m-3}\, e_3 + \cdots$$
$$+ \frac{17}{8}\, d_{m-n+3}\, e_{n-3} + \frac{27}{8}\, d_{m-n+2}\, e_{n-2} + \frac{27}{8}\, d_{m-n+1}e_{n-1}\Big] \qquad \ldots(15)$$

when n is odd. The corresponding matrix operator $\bar{M}(q)$ would for such a case have to be modified. For instance, if $e(t) = \{e_1, e_2, e_3, 0,0,0,0\ldots\}$ the response $\{r\}$ will be given by (12) with $\bar{M}(d)$ and $\bar{M}(e)$ replaced respectively by

$$\bar{M}_1(d) \equiv \begin{bmatrix} \frac{4d_1}{3} & 0 & 0 \\ \frac{9d_2}{8} & \frac{9d_1}{8} & 0 \\ 8 & & \\ \frac{4d_3}{3} & \frac{2d_2}{3} & \frac{4d_1}{3} \\ \frac{4d_4}{3} & \frac{2d_3}{3} & \frac{4d_1}{3} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \qquad \ldots(16)$$

and

$$\bar{M}_1(e) \equiv \begin{bmatrix} \frac{4e_1}{3} & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{9e_2}{8} & \frac{9e_1}{8} & 0 & 0 & 0 & \cdot & \cdot \\ \frac{4e_3}{3} & \frac{2e_2}{3} & \frac{4e_1}{3} & 0 & 0 & \cdot & \cdot \\ 0 & \frac{4e_3}{3} & \frac{2e_2}{3} & \frac{4e_1}{3} & 0 & \cdot & \cdot \\ 0 & 0 & \frac{4e_3}{3} & \frac{2e_2}{3} & \frac{4e_1}{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \text{and so on.} & & & \end{bmatrix} \qquad \ldots(17)$$

When/

When $\{r\}$ and $\{e\}$ are known, $\{d\}$ can either be obtained from the set of equations represented by

$$\{r_2, r_3 \ldots.\} = \delta \bar{M}_1(d) \{e_1, e_2, e_3\}, \qquad \ldots(18)$$

or, directly, from

$$\{d\} = \frac{1}{\delta} [\bar{M}_1(e)]^{-1} \{r\}, \qquad \ldots(19)$$

4. <u>Integration and Differentiation</u>

Let us consider the curve $y(t) = \{o, y_1, y_2, \ldots\}$ shown below.

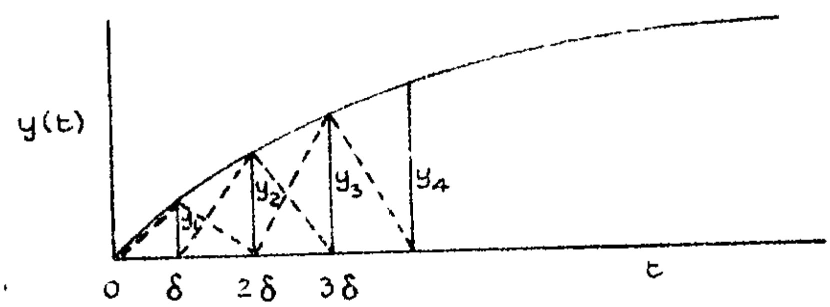


FIG 2

In terms of Δ units it immediately follows that the integral

$$r(t) = \int_0^t y\, dt = \delta \left\{ \frac{y_1}{2}, y_1 + \frac{y_2}{2}, y_1 + y_2 + \frac{y_3}{2}, \ldots.. \right\},$$

and this is expressible in the matrix form.

$$\{r\} = \frac{\delta}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 2 & 1 & 0 & \cdot & \cdot & \cdot \\ 2 & 2 & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \{y\} \qquad \ldots(20)$$

Hence, /

Hence, if $p \equiv \frac{d}{dt}$, the differential operator, it can be shown that the integrating operator,

$$\frac{1}{p} \equiv \frac{\delta}{2} \left[T\{1,2,2,.2..\}\right], \qquad ...(21)$$

where $[T \{a_1, a_2, a_3, ...\}]$ represents, in general, a triangular matrix with equal elements along the principal diagonal and along any line parallel to it. It can also be shown that

$$[T \{1,2\ 2\ 2\ .....\}] = [T \{1,1,0\ 0\ ...\}]\ [T\{1,1,1,1,1,1\ ...\}]$$
$$= [T \{1,1,0,0,0\ ...\}]\ [T\{1,-1,0\ 0\ 0..\}]^{-1}$$

$$\text{Hence} \quad \frac{1}{p} \equiv \frac{\delta}{2} [T \{1,1,\ 0\ 0\ ...\}]\ [T\{1,-1,\ 0\ 0\ ......\}]^{-1}.$$

$$\text{and} \quad p \equiv \frac{2}{\delta}[T \{1,-1,\ 0\ 0\ ...\}][T\{1,1,\ 0\ 0\ .....\}]^{-1} \qquad ...(22)$$

From (22) it follows that

$$p^2 = \frac{4}{\delta^2} [T \{1-2,\ 1\ 0\ 0\ ...\}]\ [T\{1,2,1,\ 0\ 0\ .....\}]^{-1} \qquad ...(23)$$

and, in general, when n is odd for instance,

$$p^n = \left(\frac{2}{\delta}\right)^n \frac{[T \{1,\ -n,\ \frac{n.n-1}{2!},\ ....\ n,\ -1,\ 0\ 0\ ......\}]}{[T \{1,\ n.\ \frac{n.n-1}{2!}\ ....\ n,\ 1,\ 0\ 0\ ......\}]} \qquad ...(24)$$

where the elements in the first columns of the numerator and the denomenater are the coefficients of x in $(1-x)^n$ and $(1+x)^n$ respectively. By the use of (24) any differential equation of the type

$$(a_0p^n + a_1p^{n-1} + ...\ a_n)\ r = e \qquad ...(25)$$

can be expressed in serial form by substitution for p and its powers.

It is, however, clear that the integral of y will not be given accurately by (20) unless δ is small. If use is made of Simpson's integration rules the following alternative form may be deduced, namely

equation /

$$\{r_1, r_2, r_3 \ldots\} = \delta \begin{bmatrix} \frac{19}{24}, & \frac{-5}{24}, & \frac{1}{24} & 0 & 0 & 0 & \cdot & \cdot \\ \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 & \cdot & \cdot \\ \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & \cdot & \cdot \\ \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & \cdot & \cdot \\ \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \{y_1, y_2 \ldots\} \qquad \ldots(26)$$

In the case, when $y_0 \neq 0$, there is an additional column on the left hand side of the matrix and the analysis would have to be extended as shown later.

Equation (26) can be expressed more conveniently as

$$\{r\} = \delta S \{y\} . \qquad \ldots(27)$$

Premultiplication of $\{r\}$ by B where

$$B \equiv \begin{bmatrix} -9, & -9, & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \ldots(28)$$

yields /

yields

$$B\{r\} = \delta \begin{bmatrix} -18 & 0 & 0 & 0 & 0 & . & . & . \\ \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & . & . & . \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & . & . & . \\ \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & . & . & . \\ \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{3} & . & . & . \\ & & & \text{and so on} & & & & \end{bmatrix} \{y\}$$

which may be represented more concisely by

$$B\{r\} = \delta K \{y_1, y_2, y_3 \ldots\} \cdot \qquad \ldots(29)$$

Since K is triangular, K^{-1} is readily determined, and from (29) the relation

$$\{y_1, y_2 \ldots\} = \frac{1}{\delta} K^{-1} B\{r_1, r_2 \ldots\} \qquad \ldots(30)$$

can be deduced, From (27) and (30), it follows that

$$\frac{1}{p} \equiv \delta S, \quad p \equiv \frac{1}{\delta} K^{-1} B, \qquad \ldots(31)$$

where S represents the matrix in (26).

In expanded form

p ≡ /

$$p = \frac{1}{\delta}\begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & -\frac{1}{18} & 0 & 0 & 0 & . & . \\ -2, & 1, & \frac{2}{9} & 0 & 0 & 0 & . & . \\ \frac{9}{2}, & \frac{-9}{2}, & \frac{13}{6} & 0 & 0 & 0 & . & . \\ -16 & 14 & \frac{-80}{9} & 3 & 0 & 0 & . & . \\ \frac{73}{2} & \frac{-193}{6} & \frac{359}{18} & -9 & \frac{8}{3} & 0 & . & . \\ -130, & \frac{344}{3}, & \frac{-638}{9}, & 30, & \frac{22}{3}, & 3 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \qquad \ldots(32)$$

It should be remembered that the above operator has been derived on the assumption that $y_0 = 0$. One should not therefore expect to get correct slopes with the above form of p for terms of lower order than t^2. In serial form

$$\{t^2\} = \delta^2 \{1, 4, 9, \ldots\}$$

and, by (32), it follows that

$$p\ \{t^2\} = \delta\ \{2, 4, 6, 8, \ldots\} \qquad \ldots(33)$$

$$= \{2t\}\ .$$

The above result is correct and it can be shown that, in general $p^{n-1}\ \{t^m\}$ is accurately represented provided $m \geqslant n$. For example, let $m = 4$.

Then $\{t^4\} \equiv \delta^4\ \{1, 16, 81, 256, \ \ldots\}$

$$p\ \{t^4\} \equiv \delta^3\ \{4, 32, 108, \qquad \}$$

$$\equiv \{4t^3\}$$

$$p^2\ \{t^4\} = 4p\ \{t^3\} = 4p\delta^3 \{1, 8, 27, 64 \ldots\}$$

$$= 4\delta^2 \{3, 12, 27 \ldots\} \qquad \ldots(34)$$

$$= 12\ \{t^2\}$$

$$p^3\ \{t^4\} = 12p\delta^2 \{1, 4, 9, 16 \ldots\}$$

$$= 24\delta\ \{1, 2, 3, 4 \ldots\}$$

$$= 24\ \{t\}$$

A11 /

All the above results are correct but the process breaks down on further differentiation. It is found that

$$p\{t\} = p\delta\{1, 2, 3, 4 \ldots\}$$
$$= \{\tfrac{4}{3}, \tfrac{2}{3}, 2 \ldots\ldots \} \neq \{1, 1, 1,\} \qquad \ldots(35)$$

This is because (26) is not true when $y_0 \neq 0$.

In Ref. 1, it is suggested that a differential equation of the form

$$(a_0p^n + a_1p^{n-1} + \ldots a_n)r = e \qquad \ldots(36)$$

can be represented in the serial form $U\{r\} = \{e\}$, where U is an equivalent matrix operator formed by substitution for p^n, p^{n-1} etc. and summation. It seems to the writer, in view of the preceding results, that such a representation might not be valid in general. This criticism also applies to the operators used by Tustin since he obtains the result.

$$p\{t\} = p\delta\{1, 2, 3, 4 \ldots\}$$
$$= \{2, 0, 2, 0, 2. \} \qquad \ldots(37)$$

by the use of (22). Further differentiation makes matters even worse.

This difficulty can, however, be partly overcome if $y_0 \neq 0$ is assumed, and (26) is replaced by

$$\{r_0, r_1, r_2, \ldots\} = \delta \begin{bmatrix} Y & -3Y & 3Y & -Y & 0 & 0 & 0 & . & . & . \\ \frac{9}{24} & \frac{19}{24} & \frac{-5}{24} & \frac{1}{24} & 0 & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & . & . & . \\ \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & . & . & . \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix} \qquad \ldots(38)$$

$$= \delta R \{y_0, y_1, y_2 \ldots\}$$

where Υ is an arbitrary factor which is assumed to have the value $\Upsilon = 1$ in the subsequent analysis. The above form of R gives the correct value of r when $\{y\} = \{1, 1, 1 ..\}$, $\{y\} = \{t\}$, $\{y\} = \{t^2\}$. For $\{y\} = \{t^3\}$, the value of r_o is in error* but all the other values are correct. Hence, if we write

$$\frac{1}{\bar{p}} = \delta R, \quad \text{and} \quad \bar{p} = \frac{1}{\delta} R^{-1}, \qquad \ldots(39)$$

where R is defined above and R^{-1} is the inverse matrix, it follows from (38) that

$$\bar{p} \{r\} = \{y\}, \qquad \ldots(40)$$

and $\bar{p} \{t\} = \{1, 1 \ldots, 1\}$. Unfortunately, however, when $\Upsilon = 1$ is assumed in (38)

$$\bar{p}^2 \{t\} = \{2, \frac{1}{4}, -\frac{1}{12}, \frac{1}{12}, \ldots.\} \qquad \ldots(41)$$

instead of zeros. However, it seems that if repeated differentiation of r in (36) never leads to a function which is approximately constant over a period of time, the differential equation (36) can be represented in numerical form provided $\bar{p} \equiv \frac{d}{dt}$ is of the form given by (38).

The matrix R^{-1} corresponding to R as defined by (38) is

$$R^{-1} = \begin{bmatrix} \frac{1}{4\Upsilon}, & 3, & -\frac{3}{2}, & \frac{1}{3}, & 0 & 0 & . & . & . \\ -\frac{1}{12\Upsilon}, & -\frac{1}{2}, & 1, & -\frac{1}{6}, & 0 & 0 & . & . & . \\ \frac{1}{12\Upsilon}, & -1, & \frac{1}{2}, & \frac{1}{3}, & 0 & 0 & . & . & . \\ -\frac{1}{4\Upsilon}, & \frac{3}{2}, & -3, & \frac{11}{6}, & 0 & 0 & . & . & . \\ \frac{11}{12}\Upsilon, & -5, & \frac{17}{2}, & -\frac{23}{3}, & 3, & 0 & . & . & . \\ -\frac{41}{12\Upsilon}, & \frac{23}{2}, & -\frac{59}{3}, & \frac{103}{6}, & -9, & \frac{8}{3}, & . & . & . \\ & & \text{and so on.} & & & & & & \end{bmatrix}$$

In /

* N.B. The error in r_o can be eliminated by taking more terms in the first row. The elements are the coefficients in the expansion of $\Upsilon(1-x)^m$.

In practice, however, it would perhaps be more convenient to express R and R^{-1} as ratios. This can readily be done, for premultiplication of R by T $\{1, -1, 0\ 0\ 0 \ldots\}$ leads to a triangular matrix of simpler form with zero elements in the bottom left hand corner. The corresponding form for p, however, is not quite as simple as that given in (22).

5. Many degrees of freedom

The analysis of paragraphs 2 and 3 can be extended to include cases where many degrees of freedom are involved as would normally be the case in aircraft response and flutter research. For simplicity, let two degrees of freedom be assumed. The dynamical equations of motion for such a system can be expressed in the form

$$\begin{aligned}(a_{11}p^2 + b_{11}p + c_{11})z + (a_{12}p^2 + b_{12}p + c_{12})\theta &= F(t) , \\ (a_{21}p^2 + b_{21}p + c_{21})z + (a_{22}p^2 + b_{22}p + c_{22})\theta &= G(t) ,\end{aligned} \qquad \ldots(42)$$

where z and θ represent time dependent variables and F and G represent external forces or inputs.

In matrix notation (42) reduces to

$$(ap^2 + bp + c)\ r(t) = e(t) \qquad \ldots(43)$$

where $r(t) \equiv \{z, \theta\}$, and $e(t) \equiv \{F, G\}$

Now let it be supposed that $z(t)$ and $\theta(t)$ have been measured in flight for particular forms of $F(t)$ and $G(t)$. Then, if Δ represents the matrix operator corresponding to unit Δ inputs (or unit impulses) relations of the following form are valid for linear systems.

$$\begin{aligned}\{z\} &= \Delta_{11}\{F\} + \Delta_{12}\{G\} \\ \{\theta\} &= \Delta_{21}\{F\} + \Delta_{22}\{G\}\end{aligned} \qquad \ldots(44)$$

where $$\Delta \equiv \begin{bmatrix}\Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22}\end{bmatrix} ,$$

and Δ_{11} etc. are triangular sub-matrices corresponding in form to the operators $A(d)$ or $\bar{M}(d)$ defined by (1) and (12) respectively. In a more concise form (44) is expressed as

$$\{r\} = \Delta\ \{e\} , \qquad \ldots(45)$$

and hence $$\{e\} = \Delta^{-1}\{r\}. \qquad \ldots(46)$$

Equation (46) is the numerical equivalent of (43) and it can be shown in this case that

$$\{e\} = \left|\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}\right|^{-1} \begin{bmatrix}\Delta_{22}, & -\Delta_{12} \\ -\Delta_{21}, & \Delta_{11}\end{bmatrix} \{r\} \qquad \ldots(47)$$

and /

and that the following operators are approximately equivalent,

$$ap^2 + bp + c \equiv \left|\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}\right|^{-1}\begin{bmatrix}\Delta_{22} & -\Delta_{12}\\ -\Delta_{21} & \Delta_{11}\end{bmatrix} \quad \ldots(48)$$

Extension of the above analysis is relatively straight forward. The numerical work increases rapidly with the number of degrees of freedom and if the response over a long period were required, it would be almost prohibitive unless δ could be taken reasonably large. It may be, however, that the operator Δ can be represented as a ratio of two simpler matrices. This would probably be possible for the elements in each column of the sub-matrices satisfy certain recurrence relations. If this were so, the numerical work would be reduced and the accuracy of the analysis improved (see paragraph 6).

6. Simple Applications

(i) Calculation of Response

In order to try out numerically the Tustin method and the alternative scheme suggested, the following equation for an undamped system was considered, namely,

$$(p^2 + \pi^2)r = e, \quad \ldots(49)$$

where $p \equiv \frac{d}{dt}$. It can readily be established that the response $d(\Delta)$ due to a Δ unit input is in this case given by

$$d(\Delta) = \{R_1, R_2-2R_1, R_3-2R_2 + R_1, \ldots R_n-2R_{n-1}+R_{n-2}, \ldots\} \quad \ldots(50)$$

where $R_n \equiv \frac{1}{\delta}\left(\frac{n\delta}{\pi^2} - \frac{\sin n\pi\delta}{\pi^3}\right)$. The response due to a unit impulse is simply

$$d(t) \equiv \frac{\sin \pi t}{\pi} \quad \ldots(51)$$

The response due to a general input $e(t)$ is then expressible in a form similar to (1), namely,

$$r = [T\{d(\Delta)\}]\, e, \quad \ldots(52)$$

or in the alternative form givdn by (12), namely,

$$r = \delta\, \bar{M}\, (d)\, e, \quad \ldots(53)$$

where $\bar{M}(d)$ is defined by (13) and (51). Approximate values of r given by (52) and (53) for particular inputs are compared with the true response given by

$$r = \frac{1}{\pi}\int_0^t \sin \pi\,(t-\tau)\; e\,(\tau)\; d\tau \quad \ldots(54)$$

in Figs. 4 and 5. Two cases are considered, namely,

(a) /

$$\left.\begin{aligned}
&(a) \quad e(t) = \sin \pi t, \quad \ldots\ldots \quad 0 \leqslant t \leqslant 1 \\
&\qquad\qquad = 0 \qquad\qquad\qquad t \geqslant 1 \\
&(b) \quad e(t) = \sin \pi t, \quad \ldots\ldots \quad t \geqslant 0
\end{aligned}\right\} \ldots(55)$$

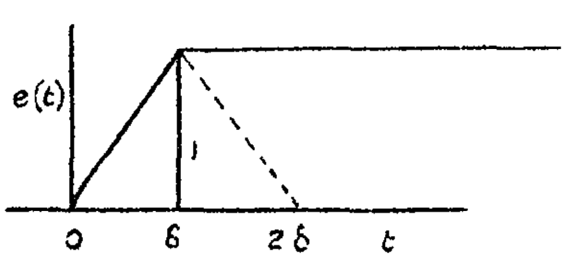
for which the exact solutions are

$$\left.\begin{aligned}
&(a) \quad r(t) = \frac{\sin \pi t}{2\pi^2} - \frac{t \cos \pi t}{2\pi} \ldots\ldots \quad 0 \leqslant t \leqslant 1 \\
&\qquad\qquad = -\frac{\cos \pi t}{2\pi}, \ldots\ldots\ldots\ldots\ldots\ldots \quad t \geqslant 1 \\
&(b) \quad r(t) = \frac{\sin \pi t}{2\pi^2} - \frac{t \cos \pi t}{2\pi} \ldots\ldots\ldots \quad t > 0
\end{aligned}\right\} \ldots(56)$$

As far as the calculation of the response due to a particular input is concerned, the alternative method suggested in this note appears to give good agreement with the exact values and to be slightly better then the Tustin method, but for all practical purposes the latter scheme seems to be sufficiently accurate. It was also found that the responses $d(\Delta)$ and $d(t)$ due to a Δ unit and a unit impulse respectively could be determined with reasonable accuracy from (52) and (53), when the exact values of $r(t)$ and $e(t)$ were assumed, as shown in Figs. 6 and 7. In flight tests, both $e(t)$ and $r(t)$ would be measured and the problem would be to determine $d(\Delta)$ (or $d(t)$) so that the response due to any general input could be estimated. Slight errors in $r(t)$ and $e(t)$ might, however, lead to trouble due to the form of the simultaneous equations which determine the serial ordinates representing $d(\Delta)$. The expanded form of (52) is

$$\left.\begin{aligned}
r_1 &= d_1 e_1 \\
r_2 &= d_2 e_1 + d_1 e_2 \\
r_3 &= d_3 e_1 + d_2 e_2 + d_1 e_3 \qquad \text{and so on.}
\end{aligned}\right\} \ldots(57)$$

and it is clear that an error in d_1, for instance, would affect the value of d_2 and all the other ordinates. When r and e are approximately proportional, as might well be the case, the above set of equations becomes ill-conditioned as $r - d_1 e$ would tend to zero and the value obtained for d_2, for instance, would probably be inaccurate. These troubles could to some extent be avoided if an input approximating closely to the form shown below could be applied in flight and the response measured



$$\begin{aligned}
e(t) &= \frac{t}{\delta}, \ldots 0 \leqslant t \leqslant \delta \\
&= 1, \ldots t \geqslant \delta
\end{aligned}$$

FIG 3

If/

If $R(t)$ represents the response due to such an input, the reponse due to a Δ input would be given by

$$d(\Delta) = R(t) - R(t-\delta), \qquad \ldots(58)$$

but even in this case $d(\Delta)$ would be given as a difference and would in the limit correspond to the slope of $R(t)$. By drawing smooth curves through the measured values of $R(t)$ before taking differences, one might, however, be able to get reasonably accurate values for $d(\Delta)$. It is thought, however, that the response at time $t = n\delta$, $n > 2$, due to any transient input $e(t)$ such that $\int_0^{2\delta} e(t)\,dt = \delta$ would correspond closely to the response due to a Δ unit. In flight it may therefore be unnecessary to apply inputs of a pure Δ form to get a good estimate of the response due to a Δ unit input. If this reponse could be measured directly, the numerical difficulties arising from inversion would be avoided. The reliability of the results obtained could be checked by making use of the estimated $d(\Delta)$'s to calculate the measured response due to some more practical form of $e(t)$ which could be applied in flight. A possible form of input might be

$$\text{(c)} \qquad e(t) = \frac{1-\cos\pi t}{2}, \quad \ldots\ldots \; 0 < t \leqslant 1$$
$$= 1 \qquad\qquad t \geqslant 1 \qquad \ldots(59)$$

and for the simple system considered here the response to such an input is given to reasonable accuracy by the Tustin method (see Fig.4). The true response for this case is

$$r(t) = \frac{1}{2\pi}\left[\frac{1-\cos\pi t}{\pi} - \frac{t}{2}\sin\pi t\right], \quad \ldots\; 0 \leqslant t \leqslant 1.0$$
$$= \frac{1}{\pi^2} - \frac{\sin\pi t}{4\pi}, \quad \ldots\ldots \qquad t \geqslant 1.0 \qquad \ldots(60)$$

(ii) <u>Characteristic roots</u>

In general the free motion of a system in any of its degrees of freedom can be represented in the form

$$d(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + \ldots\ldots \qquad \ldots(61)$$

where λ_1, λ_2, etc. are the characteristic roots and A_1, A_2, etc. constants determined by the initial conditions. The corresponding serial form of solution is

$$\{d_n\} = A_1\{x^n\} + A_2\{y^n\} + A_3\{z^n\} + \ldots \qquad \ldots(62)$$

where $x = e^{\lambda_1\delta}$, $y = e^{\lambda_2\delta}$ etc., and the curly brackets denote columns of the values for $n = 1, 2, 3, \ldots$ etc. It follows from (62) that

equation /

$$x d_n - d_{n+1} = A_2 y^n (x-y) + A_3 z^n (x-y) + \text{etc.}$$

$$y(x d_n - d_{n+1}) - (x d_{n+1} - d_{n+2}) = A_3 z^n [y(x-y)-z] + \text{etc.} \qquad \ldots(63)$$

and so on.

Hence, in general, when the number of roots is finite the ordinates d_n will be linearly related. When there are only two roots, for instance(63) gives

$$d_{n+2} - (x+y) d_{n+1} - xy d_n = 0 \qquad \ldots(64)$$

It follows from this that if the serial numbers d_n satisfy a relation of the form

$$a_o d_{n+m} + a_1 d_{n+m-1} + \ldots \quad a_m d_n = 0 \qquad \ldots(65)$$

which has solutions $d_n = \rho^n$, then x, y, z, etc. will be the roots of

$$a_o \rho^m + a_1 \rho^{m-1} \; \ldots\ldots \; a_m = 0, \qquad \ldots(66)$$

and since $x = e^{\lambda_1 \delta}$, $y = e^{\lambda_2 \delta}$, etc. the characteristics roots λ_1, λ_2, etc. can be determined provided δ is sufficiently small. If some of the modes are highly damped, however, the order of (66) may be reduced since, in practice, δ would not be infinitesimal.

For the simple example considered $\delta = 0.2$ was assumed, and it was found that the ordinates of $d(\Delta)$ given by (50) satisfied the relation

$$d_{n+2} - 1.618\, d_{n+1} + d_n = 0 \qquad \ldots(67)$$

The roots of the corresponding characteristic equation

$$\rho^2 - 1.618\rho + 1 = 0 \qquad \ldots(68)$$

were found to be

$$\rho = 0.8090 \pm 0.5878\, i = e^{\pm 0.2 \pi i} \qquad \ldots(69)$$

as was expected.

(iii) Differential Equation

When the differential equation defining the motion is known, as for instance in (49), it can be represented in time series form by substituting for p. By (22)

$$p = \frac{2}{\delta} \; \frac{T\{1, -1, 0\,0\,0 \ldots\}}{T\{1, 1\,0\,0\,0 \ldots\}}, \qquad \ldots(70)$$

and on substitution (49) yields in serial form

$$\left[\frac{4}{\delta^2} \; \frac{T\{1, -2, 1, 0\,0 \ldots\}}{T\{1, 2, 1\,0\,0 \ldots\}} + \pi^2 \right] \{r\} = \{e\}, \qquad \ldots(71)$$

where /

where $\{r\}$ and $\{e\}$ represent columns of serial ordinates. Premultiplication of both sides of (71) by $T\{1, 2, 1, 0\ 0\ 0 \ldots\}$ leads to an equation of the form

$$\left[\frac{4}{\delta^2} T\{1, -2, 1, 0\ 0 \ldots\} + \pi^2 T\{1, 2, 1, 0\ 0 \ldots\}\right]\{r\} = \{e'\}, \quad \ldots(72)$$

which may be written

$$\left[T\left\{\frac{4}{\delta^2} + \pi^2, -\frac{8}{\delta^2} + 2\pi^2, \frac{4}{\delta^2} + \pi^2, 0, 0, 0 \ldots\right\}\right]\{r\} = \{e'\}, \quad \ldots(73)$$

A typical equation of the set represented by (73) is

$$\left(\frac{4}{\delta^2} + \pi^2\right) r_{n+2} - \left(\frac{8}{\delta^2} - 2\pi^2\right) r_{n+1} + \left(\frac{4}{\delta^2} + \pi^2\right) r_n = e'_{n+2} \quad \ldots(74)$$

and, when $e'_{n+2} = 0$ is assumed and $r_n = \rho^n$ is substituted, (74) reduces to the quadratic

$$\rho^2 - \frac{8 - 2\pi^2\delta^2}{4 + \pi^2\delta^2}\rho + 1 = 0 \quad \ldots(75)$$

with the roots $\rho = 0.8203 \pm 0.5719i$ for $\delta = 0.2$.

It will be noticed that the roots obtained differ from the exact values given by (69). Since the coefficient of ρ in (68) is $2 \cos \pi\delta$, for exact agreement, the relation

$$\operatorname{Cos}\pi\delta = \frac{4 - \pi^2\delta^2}{4 + \pi^2\delta^2} \quad \ldots(76)$$

must be satisfied, and this is the case when $\delta \to 0$.

In practice, however, the differential equation defining the motion of a linear system is usually unknown and one is faced with the problem of determining its characteristics from a knowledge of the reponses due to known inputs. For the particular example considered the response and the input are related in terms of Δ units by (1), and it is shown in Fig. 6 that the $d(\Delta)$ response due to unit Δ input can be estimated with reasonable accuracy. If the exact values of $d(\Delta)$ as given by (50) are substituted in $A(d)$ the resulting triangular matrix can be expressed fully in the form

A (d) = /

$$A(d) = \begin{bmatrix} 0.00654 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ 0.0362 & 0.00654 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ 0.0585 & 0.0362 & 0.00654 & 0 & 0 & 0 & \cdot & \cdot \\ 0.0585 & 0.0585 & 0.0362 & 0.00654 & 0 & 0 & \cdot & \cdot \\ 0.0362 & 0.0585 & 0.0585 & 0.0362 & 0.00654 & 0 & \cdot & \cdot \\ 0 & 0.0362 & 0.0585 & 0.0585 & 0.0362 & 0.00654 & \cdot & \cdot \\ -0.0362 & 0 & 0.0362 & 0.0585 & 0.0585 & 0.0362 & \cdot & \cdot \\ -0.0585 & -0.0362 & 0 & 0.0362 & 0.0585 & 0.0585 & \cdot & \cdot \\ -0.0585 & -0.0585 & -0.0362 & 0 & 0.0362 & 0.0585 & \cdot & \cdot \\ -0.0362 & -0.0585 & -0.0585 & -0.0362 & 0 & 0.0362 & \cdot & \cdot \\ 0 & -0.0362 & -0.0585 & -0.0585 & -0.0362 & 0 & \cdot & \cdot \\ & & \text{and so on} & & & & & \end{bmatrix}$$

$$\equiv [T\{0.00654,\ 0.0362,\ 0.0585,\ 0.0585\ \ 0.0362,\ 0,\ \ -0.0362, \text{etc.}\}] \qquad \ldots(77)$$

The inverse of the above matrix* is

$$[A(d)]^{-1} = [T\{153,\ -847,\ 3323,\ -12180,\ 44440,\ -162200,\ 591000,\ -2155000\ \text{etc.}\}] \qquad \ldots(78)$$

and the fact that the elements increase and alternate in sign should be noted.

The numerical equivalent of (49) is

$$[A(d)]^{-1}\{r\} = \{e\} \qquad \ldots(79)$$

where $[A(d)]^{-1}$ is defined above. If e is assumed to be zero after a finite time, say $e_r = 0,\ r \geqslant 3$, then (79) yields the following set of equations

$$\begin{aligned} 3323r_1 - 847r_2 + 153r_3 &= 0, \\ -12180r_1 + 3323r_2 - 847r_3 + 153r_4 &= 0, \\ 44440r_1 - 12180r_2 + 3323r_3 - 847r_4 + 153r_5 &= 0, \end{aligned} \qquad \ldots(80)$$

and so on.

If /

* More significant figures were kept in the actual calculations.

If $r = e^{\lambda t}$ is assumed and ρ is substituted for $e^{\lambda\delta}$, the above equations yield a set of polynomial equations which should lead to the characteristic equation of the system namely

$$\rho^2 - 1.618\rho + 1 = 0 \qquad \ldots(81)$$

It should be noted that the characteristic roots are not given directly by the polynomial form of (80). However, if r_1 and r_2 are first eliminated the true recurrence relation is obtained, namely

$$r_3 - 1.618r_4 + r_5 = 0. \qquad \ldots(82)$$

Similarly, the n^{th} equation in (80) reduces after elimination to

$$r_n - 1.618r_{n+1} + r_{n+2} = 0. \qquad \ldots(83)$$

It then follows that the characteristic roots would be given by (81).

Alternatively, $A(d)$ can be expressed as a ratio of two simpler matrices and (81) can be derived directly. It can be shown that

$$A(d) \equiv 0.00654 \frac{[T\{1, 3.921, 1,0,0,0, \ldots\}]}{[T\{1, -1.618, 1,0,0,0 \ldots\}]} \qquad \ldots(84)$$

and that

$$[A(d)]^{-1} \equiv 153 \frac{T\{1, -1.618, 1, 0, 0, 0, \ldots\}}{T\{1, 3.921, 1, 0, 0, 0, \ldots\}} \qquad \ldots(85)$$

When this expression is substituted for $[A(d)]^{-1}$ in (79) and the whole equation is premultiplied by the denominator, the following equation is derived, namely,

$$T\{1, -1.618, 1,0,0,0, \ldots\}\{r\} = 0.00654T\{1, 3.921, 1, 0, 0, 0, \}\{e\} \qquad \ldots(86)$$

This equation leads directly to (81).

7. Concluding Remarks

The simple example considered reveals some of the difficulties which arise in the numerical analysis of the behaviour of a system and shows the advantages of using matrix notation. Before general conclusions can be drawn as to the advisability of using this technique in the study of aircraft stability, however, further work will have to be done. It is suggested that a detailed numerical study of the lateral stability of a particular aircraft be made where the stability derivatives are assumed to be known and where the responses due to assigned inputs could be calculated. The inputs would be chosen to correspond to such as can be applied in practice and the calculated responses could be regarded as

corresponding /

corresponding to the responses measured in flight. An attempt could then be made to determine the stability characteristics of the aircraft from a knowledge of the responses due to certain specified inputs as one would have to do in analysing flight test results. In this case, however, the true characteristics would be known and the accuracy of the method of serial representation could be checked. Suitable data for such a check calculation are given in Ref. 2.

REFERENCES

NO.	AUTHOR.	TITLE, etc.
1	A. Tustin	A method of analysing the behaviour of linear systems in terms of time series. Vol. 94. Part 11A. No. 1, Journal of the Institution of Electrical Engineers. 1947.
2	R. W. Gandy	The response of an aeroplane to the application of aileron and rudders. R. & M. 1915.

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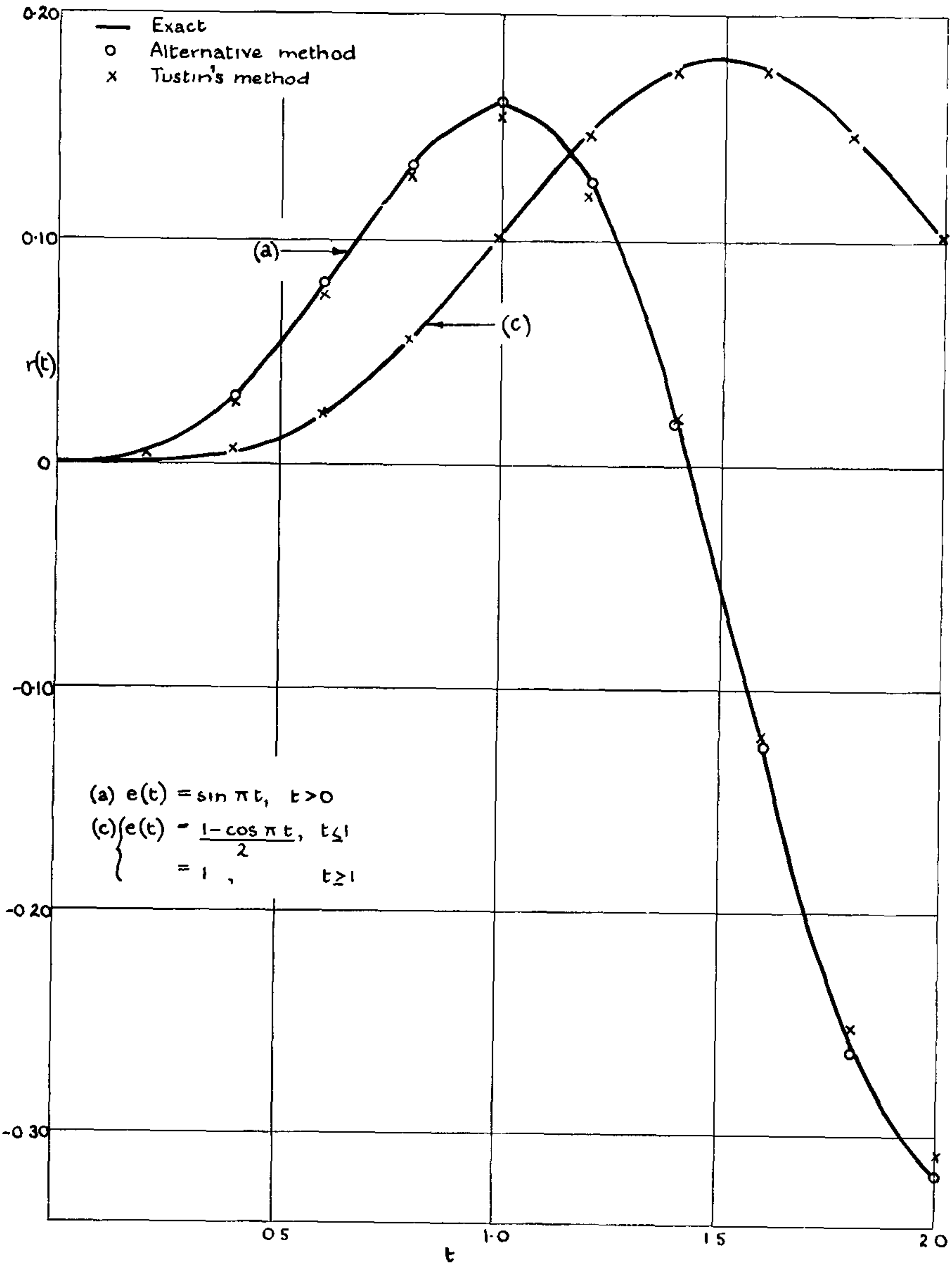
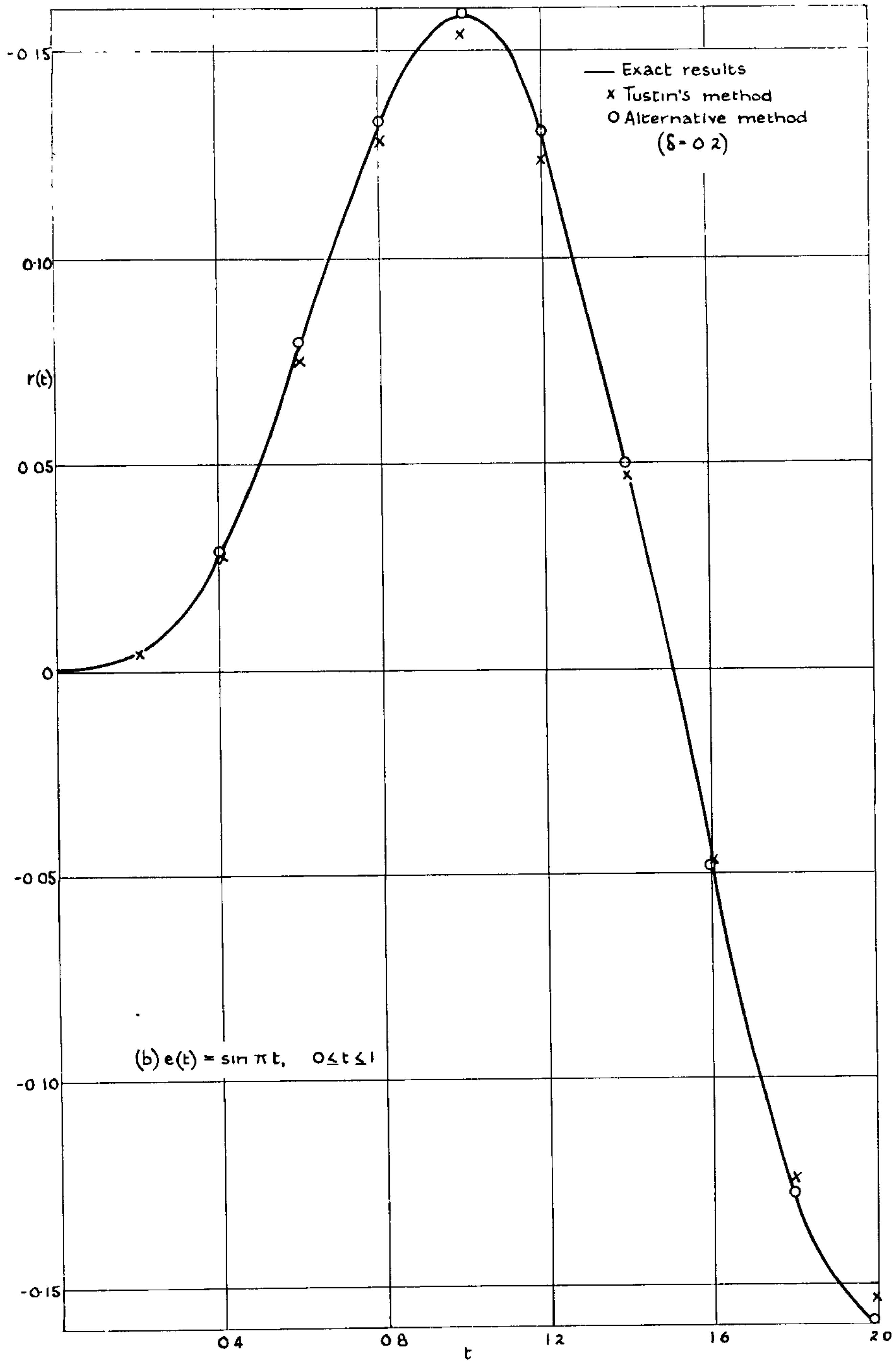


Response due to input e(t)



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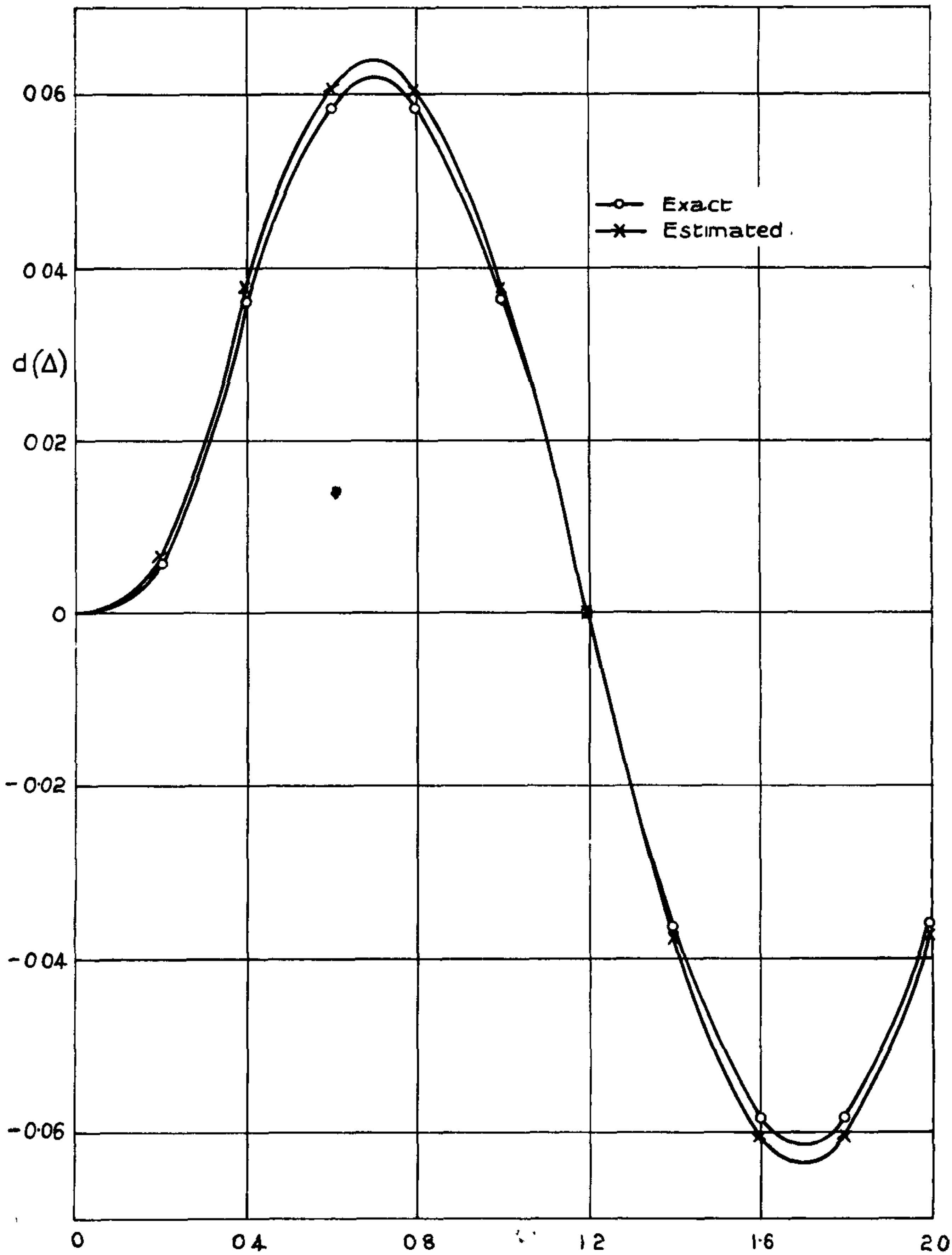
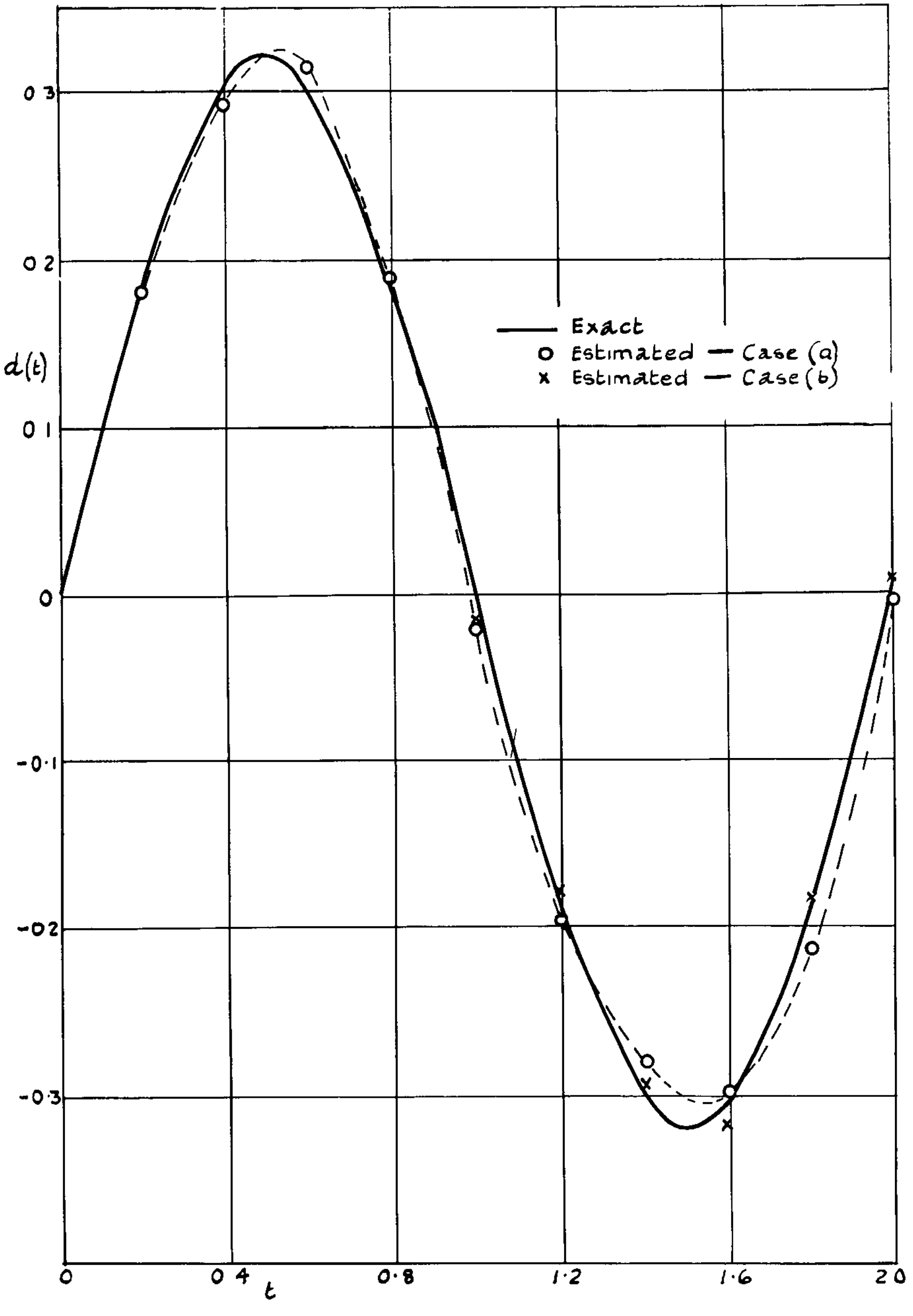


Response due to a Δ unit input



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