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The Centre Section Shape of Swept Tapered Wings with a Linear Chordwise Load Distribution

by

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THE CENTRE SECTION SHAPE OF SWEPT TAPERED WINGS WITH A LINEAR CHORDWISE LOAD DISTRIBUTION

by

J.C. Cooke, M.A.

SUMMARY

The method of Weber[1] is used to find the shape of the centre section of a swept tapered wing to produce a load distribution which changes linearly from unity at the leading edge to zero at the trailing edge along any chord. An approximation for this distribution is used in order to make the integrals tractable. These integrals are evaluated and the downwash calculated. A reasonably accurate approximate formula for the downwash is derived, and the results are illustrated by a few examples, giving downwash and angles of twist.

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1 INTRODUCTION

Weber[1] gave formulae for the shape of the centre section of a swept untapered wing to produce a given load distribution at supersonic speeds. In particular she dealt with a constant load and also a constant spanwise but linearly varying chordwise load distribution. Similar information is required for tapered wings.

In this Note we obtain formulae for centre section downwash distribution due to a swept-back tapered wing with subsonic leading and trailing edges and a linear chordwise load distribution. Standard linear theory methods lead to a singularity at the centre section. This difficulty is avoided here by the method given in Ref.1. Some of the integrals involved are very complicated, so an approximation is used for the load distribution and tested for the case $M_o = 1$ for which the integrals for the exact form of the load distribution can be evaluated. The error is found to be small, even for a fairly extreme sweep-back and taper, and it decreases with increasing Mach number.

The algebra is tedious and much of it is omitted here. The result is complicated but we give an approximate form which had slight error in the examples tested. Results were worked out for a series of wings with leading edge sweep varying from 55° to 75° and various tapers. Angles of twist were also worked out for the same series.

We do not deal here with a constant load distribution, because in this case the results do not depend on taper, but only on the leading edge angle of sweep, and are worked out in Ref.1.

2 THE VELOCITY POTENTIAL AT THE CENTRE SECTION

We take x,y,z as a right handed co-ordinate system, the x axis in the direction of the undisturbed stream, y being spanwise. If the load coefficient is $\ell(x,y)$ the velocity potential at the centre section is given by

$$\phi(x,0,z) = \frac{3V_o}{2\pi} \iint \frac{\ell(x',y')(x-x')\,dx'\,dy'}{(y'^2+z^2)\{(x-x')^2 - \beta^2 y'^2 - \beta^2 z^2\}^{\frac{1}{2}}},$$

where $\beta^2 = M_o^2 - 1$, V_o is the velocity at infinity and M_o the free stream Mach number.

The integral is to be taken over that part of the wing for which $y' > 0$ and $(x-x') > \beta(y'^2+z^2)^{\frac{1}{2}}$, that is over the part of the wing lying between the leading edge and the Mach fore cone from the point $(x,0,z)$.

The range of integration is divided into two parts, the limits being

y' lying between 0 and $x'/\tan\varphi_{LE}$,

x' lying between 0 and x_1

for the first part, and

y' lying between 0 and $\{(x-x')^2 - \beta^2 z^2\}^{\frac{1}{2}}/\beta$,

x' lying between x_1 and $x - \beta z$

for the second part, where x_1 is given by

$$\beta^2 x_1^2/\tan^2 \varphi_{LE} = (x-x_1)^2 - \beta^2 z^2 ,$$

φ_{LE} being the angle of sweep at the leading edge.

We suppose here that, within the area covered by the range of integration, the leading edge is straight.

3 THE LOAD DISTRIBUTION

We shall suppose that the value of ℓ changes linearly from unity at the leading edge to zero at the trailing edge as we travel along any chord. We take the length of the chord at the centre section to be unity. Hence we have

$$\ell(x,y) = 1 - \xi/c(y)$$

where $c(y)$ is the local chord and $\xi = x - |y|\tan \varphi_{LE}$. Since $c(y) = 1 + |y|\tan \varphi_{TE} - |y|\tan \varphi_{LE}$, where φ_{TE} is the trailing edge angle of sweep, we find that

$$\ell(x,y) = 1 - \frac{x - |y|\tan \varphi_{LE}}{1 - |y|(\tan \varphi_{LE} - \tan \varphi_{TE})} . \tag{1}$$

It was found that the form (1) led to integrals of such a complicated nature that they could not be evaluated simply except in the case $M_o = 1$. Consequently the approximation

$$\ell(x,y) = 1 - (x - |y|\tan \varphi_{LE})(1 + \varepsilon|y| + \varepsilon^2 y^2) \tag{2}$$

was adopted, where $\varepsilon = \tan \varphi_{LE} - \tan \varphi_{TE}$.

The forms (1) and (2) agree along the centre line and along the leading edge, but for extreme tapers the expression ε is not small. The greatest value of εy within the range of integration for $M_o = 1$ will be $(\tan \varphi_{LE} - \tan \varphi_{TE})/\tan \varphi_{LE}$, but it will be less for higher Mach numbers. Fig.1 shows curves of constant ℓ for leading edge and trailing edge sweeps of 70° and 45° respectively. Only the part of the wing between the leading edge and the Mach fore cone at the centre section trailing edge need be considered, and it is seen that for $M_o = 1.2$ the error is small.

In the case $M_o = 1$ it is possible to work out the integrals for both load distributions. Fig.2 shows the error introduced in the downwash by taking the approximate value (2) for ℓ instead of the true value (1) for a wing of section R.A.E. 101, 6% thickness chord ratio, with $\varphi_{LE} = 70°$, $\varphi_{TE} = 45°$. Fig.1 suggests that for $M_o = 1.2$ the error will be considerably less than that shown in Fig.2, since most of this error would seem to come from the triangular area between the Mach lines for $M_o = 1.0$ and $M_o = 1.2$.

Even with the approximate load distribution the labour of evaluating the integrals is heavy though straightforward.

4 EVALUATION OF THE INTEGRALS

We write

$$\beta^2 g^2 = (x-x')^2 - \beta^2 z^2, \qquad y' = g \sin\theta, \qquad T = \tan\varphi_{LE}, \quad (3)$$

and we have

$$\frac{2\pi\beta\phi}{z V_o} = \int_0^{x_1} (x-x')dx' \int_0^{\sin^{-1}(x'/gT)} \frac{\ell(x',y')d\theta}{g^2 \sin^2\theta + z^2}$$

$$+ \int_{x_1}^{x-\beta z} (x-x')dx' \int_0^{\frac{1}{2}\pi} \frac{\ell(x',y')d\theta}{g^2 \sin^2\theta + z^2} .$$

We let

$$G_n = \int_0^{\sin^{-1}(x'/gT)} \frac{y'^n d\theta}{g^2 \sin^2\theta + z^2}, \qquad G_n' = \int_0^{\frac{1}{2}\pi} \frac{y'^n d\theta}{g^2 \sin^2\theta + z^2} . \quad \ldots\ldots (4)$$

By Appendix 1 we have

$$G_o = \frac{\beta}{z(x-x')} \tan^{-1} \frac{x'(x-x')}{z S}, \quad \text{where } S^2 = T^2\{(x-x')^2 - \beta^2 z^2\} - \beta^2 x'^2,$$

$$G_1 = \frac{\beta}{2(x-x')} \left\{ \log \frac{(x-x')T - S}{(x-x')T + S} - \log \frac{(x-x') - \beta g}{(x-x') + \beta g} \right\},$$

$$G_2 = \sin^{-1}(x'/gT) - z^2 G_o ,$$

$$G_3 = g - (S/\beta T) - z^2 G_1 ,$$

$$G_o' = \frac{\beta\pi}{2z(x-x')} ,$$

$$G_1' = -\frac{\beta}{2(x-x')} \log \frac{x - x' - \beta g}{x - x' + \beta g} ,$$

$$G_2' = \tfrac{1}{2}\pi - z^2 G_0' ,$$

$$G_3' = g - z^2 G_1' .$$

Putting in the value of $\ell(x',y')$ we have

$$\frac{2\pi\beta\phi}{zV_0} = I_0 + I_0' + \varepsilon(I_1 + I_1') + \varepsilon^2(I_2 + I_2') ,$$

where

$$I_0 = \int_0^{x_1} (x-x')\{T G_1 + (1-x')G_0\}dx' ,$$

$$I_1 = \int_0^{x_1} (x-x')(T G_2 - x' G_1)\, dx' ,$$

$$I_3 = \int_0^{x_1} (x-x')(T G_3 - x' G_2) dx' , \qquad (5)$$

and I_0', I_1', I_2' are the same as I_0, I_1 and I_2, with the limits changed to x_1 and $x - \beta z$, and with G replaced by G'. We have

$$I_0 = \beta z^{-1} \{\tfrac{1}{2} T z(M_0 - N_0) + L_0 - L_1\} ,$$

$$I_0' = \tfrac{1}{2}\beta z^{-1}(-T z N_0' + F_0' - F_1') ,$$

$$I_1 = T P_1 - T\beta z L_0 - \tfrac{1}{2}\beta (M_1 - N_1) ,$$

$$I_1' = \tfrac{1}{2} T\{(x-\beta z)F_0' - F_1'\} + \tfrac{1}{2}\beta N_1' ,$$

$$I_2 = TR - \beta^{-1} V - \tfrac{1}{2}\beta z^2 T (M_0 - N_0) + P_2 - x P_1 + \beta z L_1 ,$$

$$I_2' = TR' + \tfrac{1}{2}\beta z^2 T N_0 + \tfrac{1}{2}(F_2' - x F_1') + \tfrac{1}{2}\beta z F_1' ,$$

where

$$L_n = \int_{o}^{x_1} x'^n \tan^{-1} \frac{x'(x-x')}{zS} \, dx' \; ,$$

$$M_n = \int_{o}^{x_1} x'^n \log \frac{(x-x')T - S}{(x-x')T + S} \, dx' \; ,$$

$$N_n = \int_{o}^{x_1} x'^n \log \frac{x - x' - \beta g}{x - x' + \beta g} \, dx' \; ,$$

$$N_n' = \int_{x_1}^{x-\beta z} x'^n \log \frac{x - x' - \beta g}{x - x' + \beta g} \, dx' \; ,$$

$$P_n = \int_{o}^{x_1} (x-x')^n \sin^{-1} (x'/T g) \, dx' \; ,$$

$$F_n' = \pi \int_{x_1}^{x-\beta z} x'^n \, dx' \; ,$$

$$R = \int_{o}^{x_1} (x-x') g \, dx' \; ,$$

$$R' = \int_{x_1}^{x-\beta z} (x-x') g \, dx' \; ,$$

$$V = \int_{o}^{x_1} (x-x') S \, dx' \; . \qquad (6)$$

We note that N_o and N_o' only occur in the combination $N_o + N_o'$. The same applies to N_1 and N_1', R and R'. These integrals are evaluated in Appendix 3.

We shall also make use of integrals

$$J_n = \int_0^{x_1} \frac{x'^{n-1}\,dx'}{(x'^2 + z^2 T^2) S}, \qquad K_n = \int \frac{x'^n\,dx'}{S}. \tag{7}$$

Weber[1] evaluated J_1 and J_2. K_n is evaluated in Appendix 3. We give in Appendix 4 derivatives of some of these integrals which will be required in what follows.

5 DOWNWASH

To find the downwash we differentiate ϕ with respect to z.

We shall divide the result into three parts, expressed as the coefficients of ε^o, ε^1 and ε^2. The coefficient of ε^o will be the same as for an untapered wing with the load distribution $\ell = 1 - \xi$ as worked out in Ref.1, but will be repeated here for completeness.

5.1 The ε^o term

We have

$$\frac{2\pi\phi}{V_o} = z\beta^{-1}(I_o + I_o')$$

$$= \tfrac{1}{2} T z(M_o - N_o) + L_o - L_1 + \tfrac{1}{2}(-T z N_o' + F_o' - F_1') .$$

Hence we have, differentiating with respect to z and inserting the values of the various quantities from the appendices,

$$\frac{2\pi}{V_o}\frac{\partial\phi}{\partial z} = (1-x)(T^2-\beta^2)K_o - T^2(x J_2 + z^2 T^2 J_1) - T E$$

$$- \tfrac{1}{2} T x D + \beta^2 z Q + z^2 T^4 (x J_1 - J_2) ,$$

which agrees with Weber's result. E, Q and D are given in equations (8) below.

5.2 The ε^1 term

This term's contribution is given by

$$\frac{2\pi\phi}{\varepsilon V_o} = z\beta^{-1}(I_1 + I_1')$$

$$= z\beta^{-1} T P_1 - T z^2 L_o - \tfrac{1}{2} z(M_1 - N_1) + \tfrac{1}{2} z N_1' + \tfrac{1}{2} z\beta^{-1} T \{(x-\beta z)F_o' - F_1'\} .$$

Hence we have

$$\frac{2\pi}{\varepsilon V_o}\frac{\partial \phi}{\partial z} = \tfrac{1}{2} TK_o\{x^2 - 3z^2(T^2-\beta^2)\} + \tfrac{3}{2} z^2 T^2 (x J_2 + z^2 T^2 J_1)$$

$$- \pi z x T + \frac{5x}{4} E + \tfrac{1}{4}(x^2 + \tfrac{3}{2}\beta^2 z^2) D + 2 z x Q .$$

5.3 The ε^2 term

This term's contribution is given by

$$\frac{2\pi\phi}{\varepsilon^2 V_o} = z\beta^{-1}(I_2 + I_2')$$

$$= z\beta^{-1} T (R+R') - \frac{zV}{\beta^2} - \tfrac{1}{2} z^3 T(M_o - N_o - N_o') + \frac{z}{\beta}(P_2 - xP_1)$$

$$+ z^2 L_1 + \tfrac{1}{2} z\beta^{-1}(F_2' - xF_1') + \tfrac{1}{2} z^2 F_1' .$$

Hence we have

$$\frac{2\pi}{\varepsilon^2 V_o}\frac{\partial \phi}{\partial z} = K_o\left\{\frac{\beta^2 x^3}{6(T^2-\beta^2)} + \frac{3xz^2}{2}(2T^2-\beta^2)\right\} - 2T^4 z^4 (xJ_1 - J_2)$$

$$+ \frac{1}{6}\pi z(3x^2 + 2\beta^2 z^2) + E\left\{\frac{-Tx^2}{6(T^2-\beta^2)} + 2Tz^2\right\}$$

$$+ \frac{3}{2} T x z^2 D - \tfrac{1}{2} z (3x^2 + 2\beta^2 z^2) Q .$$

In these results

$$E = (x^2 - \beta^2 z^2)^{\frac{1}{2}}, \qquad Q = \tan^{-1}\frac{zT}{E}, \qquad D = \log\frac{x-E}{x+E}, \qquad (8)$$

K_o is given in equation (10) below, and J_1 and J_2 are evaluated in the Appendix to Ref.1.

6 APPROXIMATIONS

J_1 and J_2 are complicated functions. However, in the cases considered, namely section R.A.E. 101 with thickness/chord ratio of 6%, and $M_o = 1.2$, it was found sufficient to use the approximations given by Weber, valid when z/x is small.

It was also found sufficient to ignore terms of order $z^2 \log z$ and higher orders in E, D and Q. The only term where it was not safe to make the approximation z/x small was in K_o. It led to large errors near the leading edge, and so K_o was evaluated in full.

If these approximations are made the final result for the downwash is given by the equation

$$\frac{2\pi}{V_o}\frac{\partial\phi}{\partial z} = (1-x)T\log\frac{\beta|z|}{2x} - xT + (1-x)(T^2-\beta^2)K_o + \tfrac{1}{2}\pi(T^2-\beta^2)z$$

$$+ \varepsilon\left\{\tfrac{1}{2}x^2\log\frac{\beta|z|}{2x} + \frac{5}{4}x^2 + \tfrac{1}{2}x^2 T K_o - \pi x T z\right\}$$

$$+ \varepsilon^2\left\{-\frac{T x^3}{6(T^2-\beta^2)} + \frac{\beta^2 x^3}{6(T^2-\beta^2)}K_o + \tfrac{1}{2}\pi z x^2\right\}, \qquad (9)$$

where

$$K_o = \frac{1}{(T^2-\beta^2)^{\frac{1}{2}}}\log\frac{\beta\{x^2+(T^2-\beta^2)z^2\}}{Tx-(T^2-\beta^2)^{\frac{1}{2}}(x^2-\beta^2 z^2)^{\frac{1}{2}}}. \qquad (10)$$

For constant unit load distribution the result is

$$\frac{2\pi}{V_o}\frac{\partial\phi}{\partial z} = (T^2-\beta^2)K_o - T^2(xJ_2 + z^2T^2J_1)$$

$$\simeq (T^2-\beta^2)K_o + T\log\frac{\beta|z|}{2x} \qquad (11)$$

and so the result for a load distribution of the form $A + B\xi/c$ may be obtained from equations (9) and (11) by writing

$$\ell = A + B - B(1-\xi/c).$$

7 THE ANGLE OF TWIST

The angle of twist α_T is obtained from the relations

$$\alpha_T = \tan^{-1}\{-z(1,0)\}, \qquad z(x,0) = \int_0^x \frac{1}{V_o}\frac{\partial\phi}{\partial z}\,dx'.$$

We show these angles in Fig.4.

8 DISCUSSION

The main purpose of this paper is to put on record the values of integrals required in the computation of the downwash at the centre section of tapered swept wings with constant spanwise load distribution, where the standard methods of linear theory lead to singularities and so break down. Unfortunately the exact linear load distribution cannot be used without producing very complicated integrals. Consequently an approximation has been made by means of which the integrals are calculable, though the work is tedious. The approximation could have been further extended to terms of higher than the second order in ε ($= \tan \varphi_{LE} - \tan \varphi_{TE}$) and the integrals could still have been evaluated, but the extensions seemed scarcely necessary at the Mach numbers and sweeps and tapers of interest in this connection. Indeed, Fig.3 shows that over the first 40% of the chord the untapered case ($\varepsilon = 0$) gives a sufficiently accurate approximation. The second correction term proportional to ε^2 only gives a correction greater than 0.5% of the untapered value at points very close to the trailing edge, where there is a logarithmic singularity in any case. The first correction term (proportional to ε) gives very much greater corrections amounting to about 50% at 0.9 chord, and rising onwards from there. It accounts almost entirely for the divergencies of the curves from the untapered case.

Fig.3 shows the downwash for an R.A.E. 101 section with thickness/chord ratio of 6%, at a Mach number of 1.2 for various sweeps and tapers. Fig.4 shows the corresponding angles of twist required. It appears that taper increases the angle of twist required to maintain the linear load distribution for a given mean sweep.

LIST OF SYMBOLS

a	$T^2 - \beta^2$
b	$-T^2 x$
c	$T^2(x^2 - \beta^2 z^2)$
$c(y)$	local chord
D	$\log \frac{x - E}{x + E}$
E	$(x^2 - \beta^2 z^2)^{\frac{1}{2}}$
F'	defined by equations (6)
g	$\beta^{-1}\{(x-x')^2 - \beta^2 z^2\}^{\frac{1}{2}}$
G_n, G_n'	defined by equations (4)
I_o, I_1, I_2	defined by equations (5)
J_n	defined by equations (7)
K_n	defined by equations (7)
$\ell(x,y)$	load coefficient
L	defined by equations (6)
M_o	Mach number of the free stream
M, N, P	defined by equations (6)

LIST OF SYMBOLS (Contd.)

Q	$\tan^{-1}(zT/E)$
R, R'	defined by equations (6)
S	$[T^2\{(x-x')^2-\beta^2 z^2\} - \beta^2 x'^2]^{\frac{1}{2}}$
t	$\tan\theta$
T	$\tan\varphi_{LE}$
V	defined by equations (6)
V_o	velocity of the free stream
x, y, z	Cartesian co-ordinates, x axis along the centre section chord, y axis spanwise, z axis upwards
x_1	defined by $\beta^2 x_1^2 = T^2\{(x-x_1)^2 - \beta^2 z^2\}$
$z_t(x)$	thickness distribution along the centre section
α	$x \mp \beta z$
α_T	angle of twist at wing root
β	$(M_o^2-1)^{\frac{1}{2}}$
ε	$\tan\varphi_{LE} - \tan\varphi_{TE}$
θ	defined by equation (3)
ξ	$x - \lvert y\rvert \tan\varphi_{LE}$
ϕ	velocity potential
φ_{LE}	angle of sweep at leading edge
φ_{TE}	angle of sweep at trailing edge.

REFERENCE

No.	Author	Title, etc.
1	Weber, J.	The shape of the centre part of a sweptback wing with a required load distribution. R. & M. 3098. May 1957.

APPENDIX 1

THE G INTEGRALS

$$G_n = \int \frac{g^n \sin^n \theta d\theta}{g^2 \sin^2 \theta + z^2}, \qquad \beta^2 g^2 = (x-x')^2 - \beta^2 z^2$$

where the limits are 0 to $\sin^{-1}(x'/gT)$ for G_n and 0 to $\frac{1}{2}\pi$ for G_n'.

Hence, on writing $t = \tan\theta$, we have

$$G_o = \frac{1}{z(z^2+g^2)^{\frac{1}{2}}} \tan^{-1} \frac{(z^2+g^2)^{\frac{1}{2}} t}{z} .$$

On putting in the limits we have

$$G_o = \frac{\beta}{z(x-x')} \tan^{-1} \frac{x'(x-x')}{zS}, \qquad G_o' = \frac{\beta\pi}{2z(x-x')},$$

where

$$S^2 = T^2\{(x-x')^2 - \beta^2 z^2\} - \beta^2 x'^2 .$$

$$G_1 = \int \frac{g \sin\theta d\theta}{g^2 \sin^2 \theta + z^2} = \int \frac{g \sin \theta d\theta}{g^2 + z^2 - g^2 \cos^2 \theta}$$

$$= \frac{\beta}{2(x-x')} \log \frac{(g^2+z^2)^{\frac{1}{2}} - g\cos\theta}{(g^2+z^2)^{\frac{1}{2}} + g\cos\theta} .$$

Hence we have

$$G_1 = \frac{\beta}{2(x-x')} \left\{ \log \frac{(x-x')T - S}{(x-x')T + S} - \log \frac{x - x' - \beta g}{x - x' + \beta g} \right\},$$

$$G_1' = - \frac{\beta}{2(x-x')} \log \frac{x - x' - \beta g}{x - x' + \beta g} .$$

$$G_2 = \int \frac{g^2 \sin^2 \theta d\theta}{g^2 \sin^2 \theta + z^2} = \int \left(1 - \frac{z^2}{g^2 \sin^2 \theta + z^2}\right) d\theta = \theta - z^2 G_o .$$

Hence

$$G_2 = \sin^{-1}(x'/Tg) - z^2 G_o, \qquad G_2' = \tfrac{1}{2}\pi - z^2 G_o'.$$

Similarly, for the indefinite integral we have

$$G_3 = -g\cos\theta - z^2 G_1$$

and hence

$$G_3 = g - (S/\beta T) - z^2 G_1, \qquad G_3' = g - z^2 G_1'.$$

APPENDIX 2

THE INTEGRALS W

These integrals will be required in what follows.

We define

$$W_{1,2} = \int_0^{x_1} \frac{d x'}{(x \mp \beta z - x')S} ,$$

where the suffixes 1 and 2 are associated with the upper and lower signs respectively, and

$$S^2 = T^2\{(x-x')^2 - \beta^2 z^2\} - \beta^2 x'^2 = a x'^2 + 2 b x' + c$$

where

$$a = T^2 - \beta^2, \quad b = -T^2 x, \quad c = T^2(x^2 - \beta^2 z^2) .$$

It can be shown by the substitution $y = 1/(\alpha - x)$ that

$$\int \frac{d x}{(\alpha-x)(a x^2 + 2 b x + c)^{\frac{1}{2}}} = \frac{1}{\beta_1} \sin^{-1} \frac{(a \alpha + b)(\alpha - x') + \beta_1^2}{(\alpha - x')(b^2 - a c)^{\frac{1}{2}}} ,$$

where $\beta_1 = (-a \alpha^2 - 2 b \alpha - c)^{\frac{1}{2}}$, if terms under the square root signs are positive, as they are here.

If we suppose $\alpha = x \mp \beta z$, and note that in our integral the limits are from 0 to x_1, where x_1 satisfies $a x_1^2 + 2 b x_1 + c = 0$, and in fact

$$x_1 = \frac{-b - (b^2 - ac)^{\frac{1}{2}}}{a} ,$$

we find that $\beta_1 = \alpha \beta$, and

$$W_{1,2} = \frac{1}{\alpha \beta} \left\{ \sin^{-1} \frac{(a \alpha + b)(\alpha - x_1) - a \alpha^2 - 2 b \alpha - c + a x_1^2 + 2 b x_1 + c}{(\alpha - x_1)(b^2 - a c)^{\frac{1}{2}}} \right.$$

$$\left. - \sin^{-1} \frac{- b \alpha - c}{\alpha (b^2 - a c)^{\frac{1}{2}}} \right\} .$$

In the first term we have inserted the term $a x_1^2 + 2 b x_1 + c$ for convenience. It has of course the value zero.

Hence

$$W_{1,2} = \frac{1}{\alpha\beta}\left\{\sin^{-1}\frac{-a x_1 - b}{(b^2 - ac)^{\frac{1}{2}}} - \sin^{-1}\frac{-b\alpha - c}{(b^2 - ac)^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\alpha\beta}\left\{\tfrac{1}{2}\pi \pm \sin^{-1}\frac{zT}{\{x^2 + (T^2-\beta^2)z^2\}^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\alpha\beta}\left\{\tfrac{1}{2}\pi \pm \tan^{-1}\frac{zT}{(x^2-\beta^2 z^2)^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\alpha\beta}\left(\tfrac{1}{2}\pi \pm Q\right),$$

where

$$Q = \tan^{-1}\frac{zT}{(x^2-\beta^2 z^2)^{\frac{1}{2}}} .$$

APPENDIX 3

THE INTEGRALS K, L, M, N, P, F, R AND V

$$K_o = \int_o^{x_1} \frac{dx'}{S} = \int_o^{x_1} \frac{dx'}{(ax'^2 + 2bx' + c)^{\frac{1}{2}}}$$

$$= a^{\frac{1}{2}} \log \{ax' + b + a^{\frac{1}{2}}(ax'^2 + 2bx' + c)^{\frac{1}{2}}\}_o^{x_1}$$

$$= \frac{1}{(T^2-\beta^2)^{\frac{1}{2}}} \log \frac{\beta\{x^2 + (T^2-\beta^2)z^2\}}{xT - (T^2-\beta^2)^{\frac{1}{2}}(x^2-\beta^2 z^2)^{\frac{1}{2}}} .$$

If we use the formula

$$I_m = \int \frac{x^m dx}{(ax' + 2bx + c)^{\frac{1}{2}}}$$

$$= \frac{1}{ma} x^{m-1} (ax^2 + 2bx + c)^{\frac{1}{2}} - \frac{(2m-1)b}{ma} I_{m-1} - \frac{(m-1)c}{ma} I_{m-2} ,$$

we find that

$$K_1 = \frac{-T(x^2-\beta^2 z^2)^{\frac{1}{2}}}{T^2 - \beta^2} + \frac{xT^2}{T^2 - \beta^2} K_o,$$

$$K_2 = \frac{T^2}{2(T^2-\beta^2)} \{3xK_1 - (x^2-\beta^2 z^2)K_o\} ,$$

$$K_3 = \frac{T^2}{3(T^2-\beta^2)} \{5xK_2 - 2(x^2-\beta^2 z^2)K_1\} .$$

$$L_o = \int_o^{x_1} \tan^{-1} \frac{x'(x-x')}{zS} \, dx'$$

$$= \tfrac{1}{2}\pi x_1 - zT^2 \int \frac{x'(x-x')dx}{(x'^2 + z^2 T^2)S} - \beta^2 z \int \frac{x'^2 \, dx'}{\{(x-x')^2 - \beta^2 z^2\}S}$$

on integrating by parts, and rearranging the right hand side. The first integral may be written $xJ_2 - J_3$, where

$$J_n = \int_o^{x_1} \frac{x'^{n-1} \, dx'}{(x'^2 + z^2 T^2)S} \; .$$

J_1 and J_2 are evaluated in the Appendix to Ref.1. J_3 may be expressed in terms of J_1 and K_o; in fact it is easy to see that

$$J_3 = K_o - z^2 T^2 J_1 \; .$$

In the second integral the coefficient of $1/S$ is split into partial fractions and Appendix 2 used. We obtain

$$L_o = \tfrac{1}{2}\pi x_1 - zT^2(x J_2 + z^2 T^2 J_1) + \tfrac{1}{2}\pi \beta z - xQ + z(T^2 - \beta^2)K_o \; ,$$

where Q is given by equations (8).

L_1 may be evaluated in the same way as L_o. The result is

$$L_1 = \tfrac{1}{2}\pi x_1^2 - \tfrac{1}{2} zTE + \tfrac{1}{2} z^3 T^4 (x J_1 - J_2) + \tfrac{1}{2}\pi\beta z x - \beta^2 z x K_o - \tfrac{1}{2}(x^2 + \beta^2 z^2) Q \; ,$$

where E and Q are given by equations (8).

In order to find M_o we consider

$$\int_o^{x_1} \log \{(x-x')T \pm S\} \, dx' \; .$$

On integrating by parts and rearranging the right hand side this integral may be written

$$x_1 \log \{(x-x_1)T\} \pm \int_0^{x_1} \frac{x' T\, dx'}{S} \pm \beta^2 \int_0^{x_1} \frac{x'^2\, dx'}{\{(x-x')T \pm S\}S} .$$

Hence we have

$$M_o = \int_0^{x_1} \log \frac{(x-x')T - S}{(x-x')T + S}\, dx'$$

$$= -2T \int \frac{x'\, dx'}{S} - 2T \int \frac{x'^2 (x-x')\, dx'}{S(x'^2 + z^2 T^2)}$$

$$= -2xT \int \frac{dx'}{S} + 2z^2 T^3 \int \frac{(x-x')\, dx'}{S(x'^2 + z^2 T^2)}$$

$$= -2xTK_o + 2z^2 T^3 (xJ_1 - J_2) .$$

If we similarly integrate by parts we find that

$$M_1 = -xTK_1 + z^2 T^3 (xJ_2 - K_o + z^2 T^2 J_1).$$

$$N_o + N_o' = \int_0^{x-\beta z} \log \frac{x - x' - \beta g}{x - x' + \beta g}\, dx' .$$

On making the substitution $x - x' = \beta z \cosh\theta$ we obtain in a straightforward way

$$N_o + N_o' = Dx + 2E ,$$

where D and E are given in equations (8).

In a similar way we obtain

$$N_1 + N_1' = \tfrac{1}{4} D\,(2x^2 + \beta^2 z^2) + \tfrac{3}{2}\,E\,x\,.$$

On integrating by parts and rearranging the right hand side we have

$$P_1 = \int_0^{x_1} (x-x')\,\sin^{-1}(x'/T\,g)\,d\,x'$$

$$= -\tfrac{1}{4}\pi(x-x_1)^2 + \tfrac{1}{2}\beta\int \frac{x^2-\beta^2 z^2 - x\,x'}{S}\,d\,x' + \tfrac{1}{2}\beta\int \frac{\beta^2 z^2\,x(x-x') - \beta^4 z^4}{\{(x-x')^2 - \beta^2 z^2\}S}\,d\,x'.$$

We split the last integral into partial fractions and use Appendix 2, and obtain

$$P_1 = -\tfrac{1}{4}\pi(x-x_1)^2 + \tfrac{1}{2}\beta\{(x^2-\beta^2 z^2)K_o - xK_1\} + \tfrac{1}{4}\pi\beta z^2\,.$$

In a similar way we find that

$$P_2 = -\frac{1}{6}\pi(x-x_1)^3 + \frac{1}{3}\beta\{x^3 K_o + (\beta^2 z^2 - 2x^2)K_1 + x\,K_2\} + \frac{1}{3}\,\beta^3 z^3 Q\,.$$

We obtain at once

$$R + R' = \frac{1}{\beta}\int_0^{x-\beta z} (x-x')\{(x-x')^2 - \beta^2 z^2\}^{\frac{1}{2}}\,d\,x'$$

$$= E^3/3\,\beta,$$

where E is given in equation (8).

$$V = \int_0^{x_1} (x-x')\,S\,d\,x'\,,$$

where $S^2 = ax'^2 + 2bx' + c$, $\quad a = T^2 - \beta^2$, $\quad b = -xT^2$, $\quad c = (x^2 - \beta^2 z^2)T^2$.

Hence

$$V = \int_0^{x_1} \frac{(ax'^2 + 2bx' + c)(x - x')\,dx'}{(ax'^2 + 2bx' + c)^{\frac{1}{2}}}$$

$$= -aK_3 + (ax - 2b)K_2 + (2bx - c)K_1 + cxK_o,$$

$$V = ET^3 \left\{ \frac{x^2 - \beta^2 z^2}{3(T^2 - \beta^2)} - \frac{\beta^2 x^2}{2(T^2 - \beta^2)^2} \right\} + \frac{T^4 \beta^4 x}{2(T^2 - \beta^2)^2} \{x^2 + (T^2 - \beta^2) z^2\} K_o,$$

using the values of the K's found at the beginning of this Appendix, E being given in equation (8).

APPENDIX 4

DERIVATIVES OF K, L, M, N, P, R AND V

By differentiating the integral and rearranging the right hand side we have

$$\frac{\partial L_o}{\partial z} = \tfrac{1}{2}\pi\frac{\partial x_1}{\partial z} + \beta^2 \int \frac{x'(x-x')\,d\,x'}{\{(x-x')^2 - \beta^2 z^2\}\,S} - T^2 \int \frac{x'(x-x')\,d\,x'}{(x'^2 + z^2 T^2)S} .$$

On splitting the first integrand into partial fractions and using Appendix 2, we obtain

$$\frac{\partial L_o}{\partial z} = \tfrac{1}{2}\pi\frac{\partial x_1}{\partial z} + (T^2-\beta^2)K_o + \tfrac{1}{2}\pi\beta - T^2(x\,J_2 + z^2 T^2 J_1) .$$

In a similar way we find

$$\frac{\partial L_1}{\partial z} = \tfrac{1}{2}\pi x_1 \frac{\partial x_1}{\partial z} - TE - \beta^2 x K_o + \tfrac{1}{2}\pi\beta x - \beta^2 z\,Q + T^4 z^2 (x J_1 - J_2) .$$

By differentiation we find immediately, from the integrals, that

$$\frac{\partial M_o}{\partial z} = 2T^3 z\,(x J_1 - J_2) ,$$

$$\frac{\partial M_1}{\partial z} = 2T^3 z\,(x J_2 - K_o + z^2 T^2 J_1)$$

We may differentiate the results of Appendix 3 to obtain at once

$$\frac{\partial}{\partial z}(N_o + N_o') = \frac{2E}{z}, \qquad \frac{\partial}{\partial z}(N_1 + N_1') = \tfrac{1}{2}\beta^2 z D + \frac{xE}{z} .$$

By differentiating the integral we have

$$\frac{\partial P_1}{\partial z} = \tfrac{1}{2}\pi(x-x_1)\frac{\partial x_1}{\partial z} + \beta^3 z \int \frac{(x-x')\,x'\,dx'}{\{(x-x')^2 - \beta^2 z^2\}S} .$$

On splitting the integrant into partial fractions we find, using Appendix 2,

$$\frac{\partial P_1}{\partial z} = \tfrac{1}{2}\pi(x-x_1)\frac{\partial x_1}{\partial z} - \beta^3 z K_o + \tfrac{1}{2}\pi\beta^2 z .$$

Similarly we obtain

$$\frac{\partial P_2}{\partial z} = \tfrac{1}{2}\pi(x-x_1)^2\frac{\partial x_1}{\partial z} + \beta^3 z K_1 + \beta^3 z^3 Q .$$

We have directly

$$\frac{\partial}{\partial z}(R+R') = -\beta z E .$$

On differentiating the integral we find that

$$\frac{\partial V}{\partial z} = -\int_o^{x_1} \frac{(x-x')\beta^2 T^2 z}{(a x'^2 + 2 b x' + c)^{\frac{1}{2}}} \, d x'$$

$$= -\beta^2 T^2 z\,(x K_o - K_1) .$$

Finally by direct differentiation we find that

$$\frac{\partial K_o}{\partial z} = -\frac{z x T}{E\{x^2 + (T^2-\beta^2)z^2\}} ,$$

$$\frac{\partial K_1}{\partial z} = -\frac{z T E}{x^2 + (T^2-\beta^2)z^2} .$$

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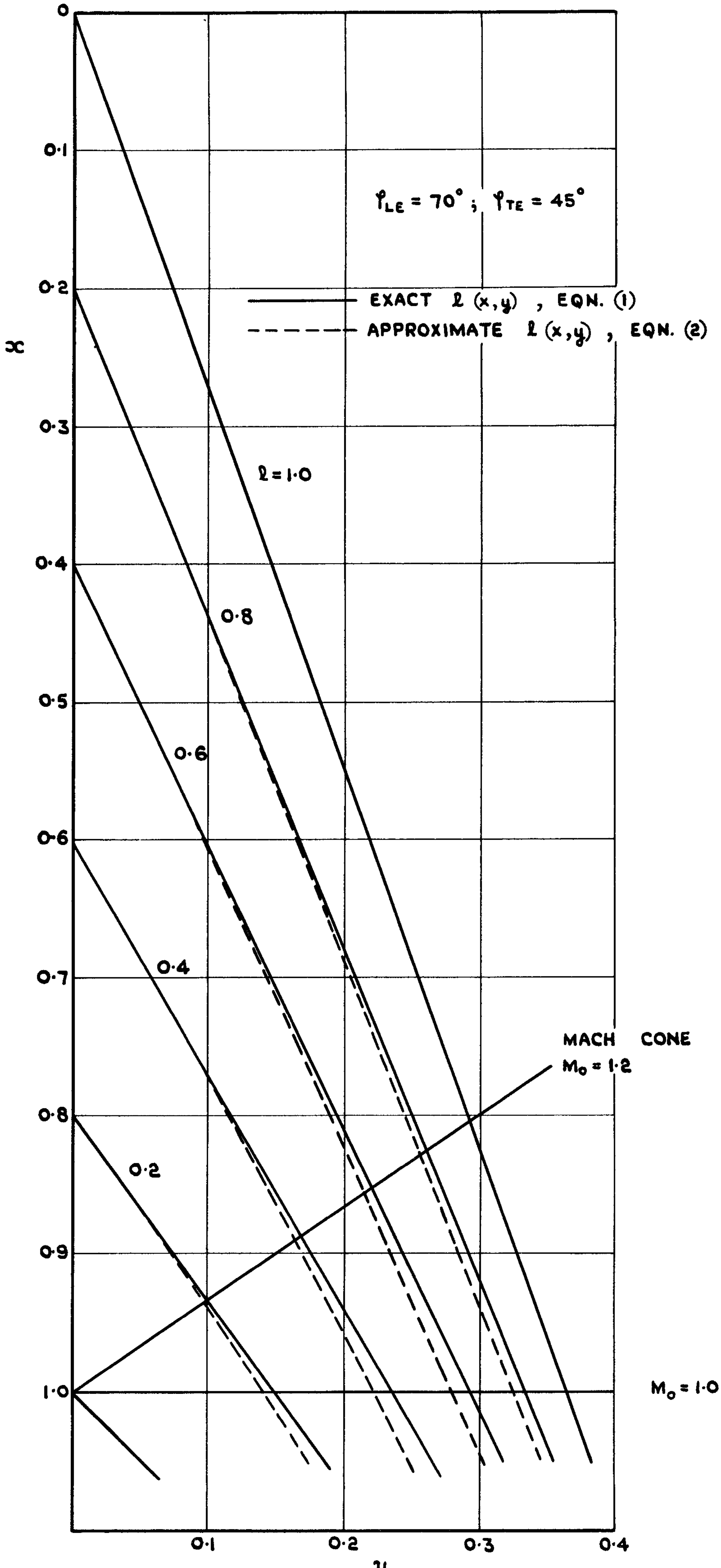


FIG. 1. CURVES OF CONSTANT ℓ.

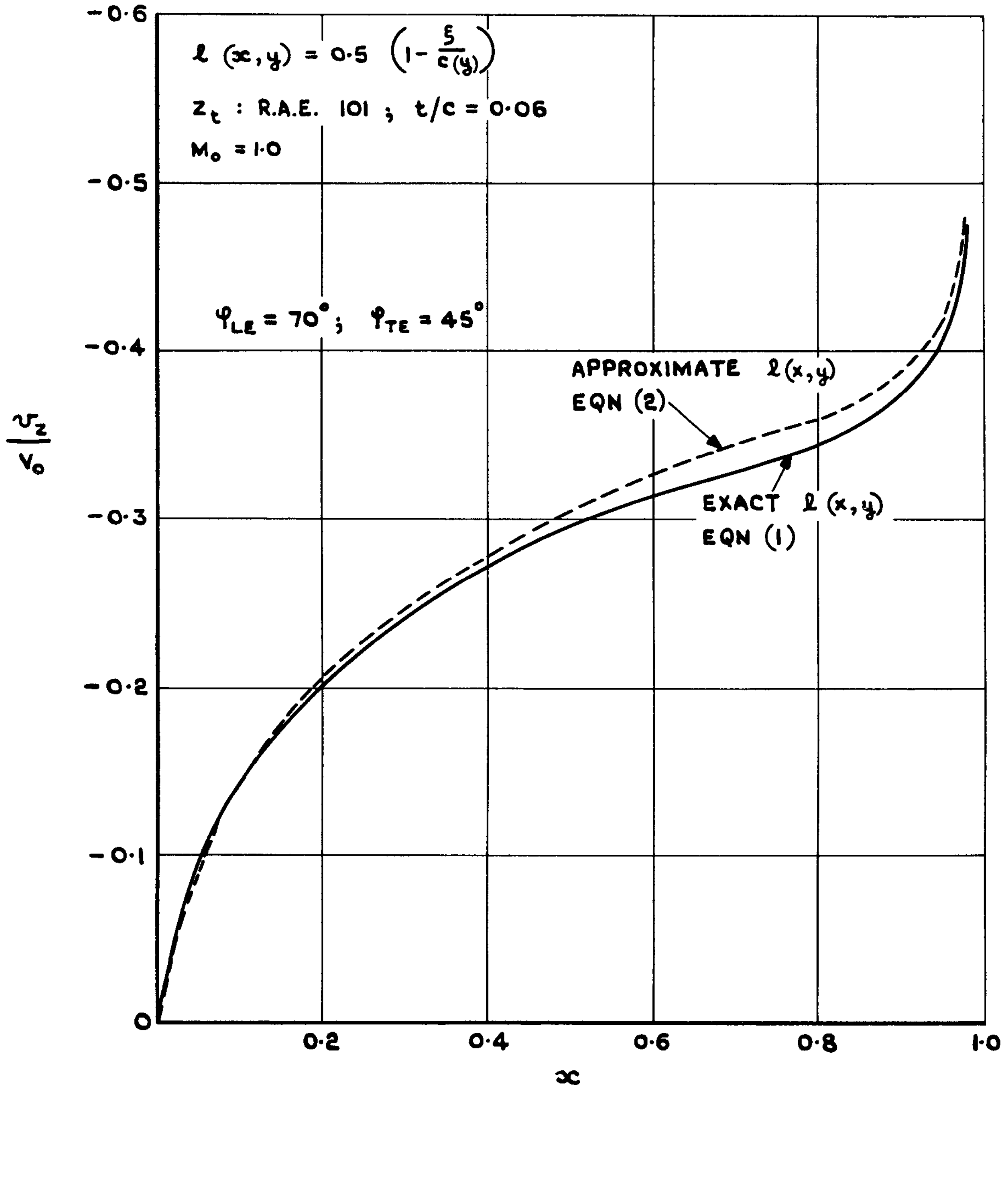
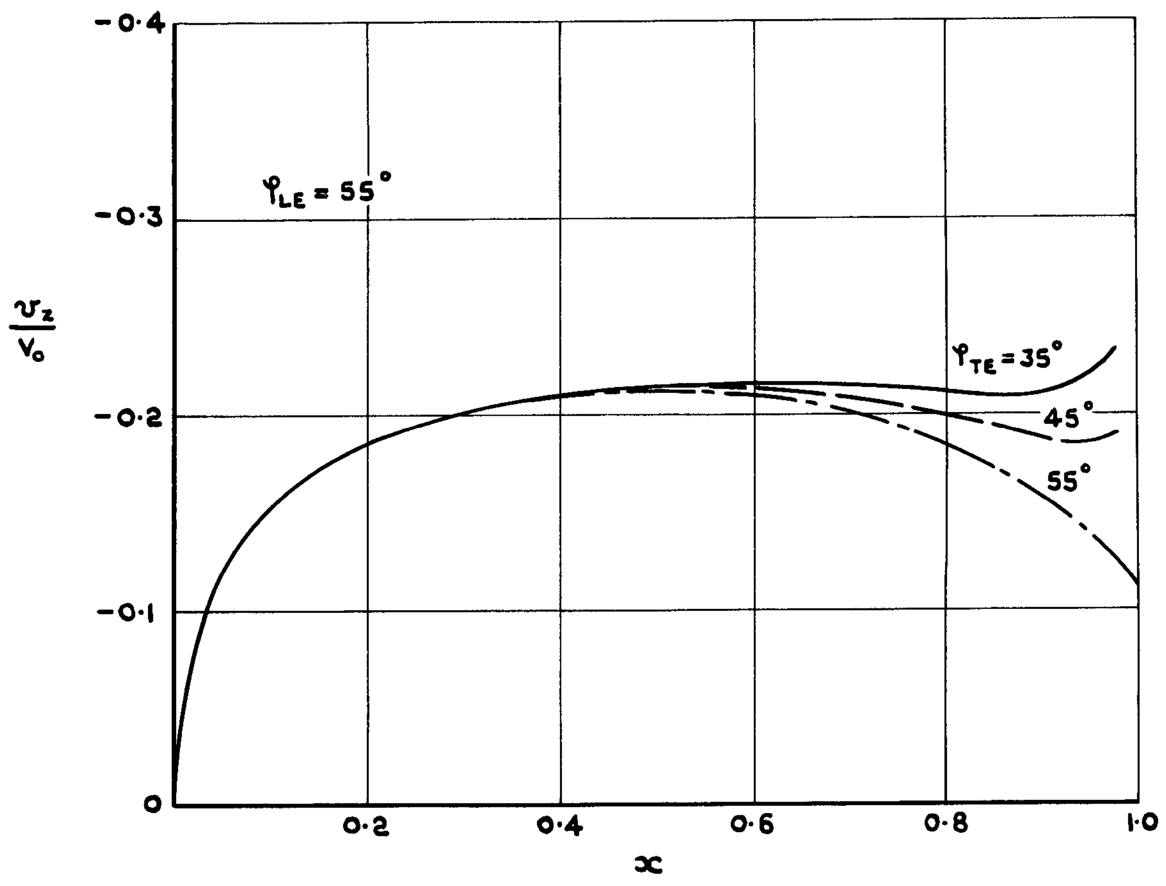
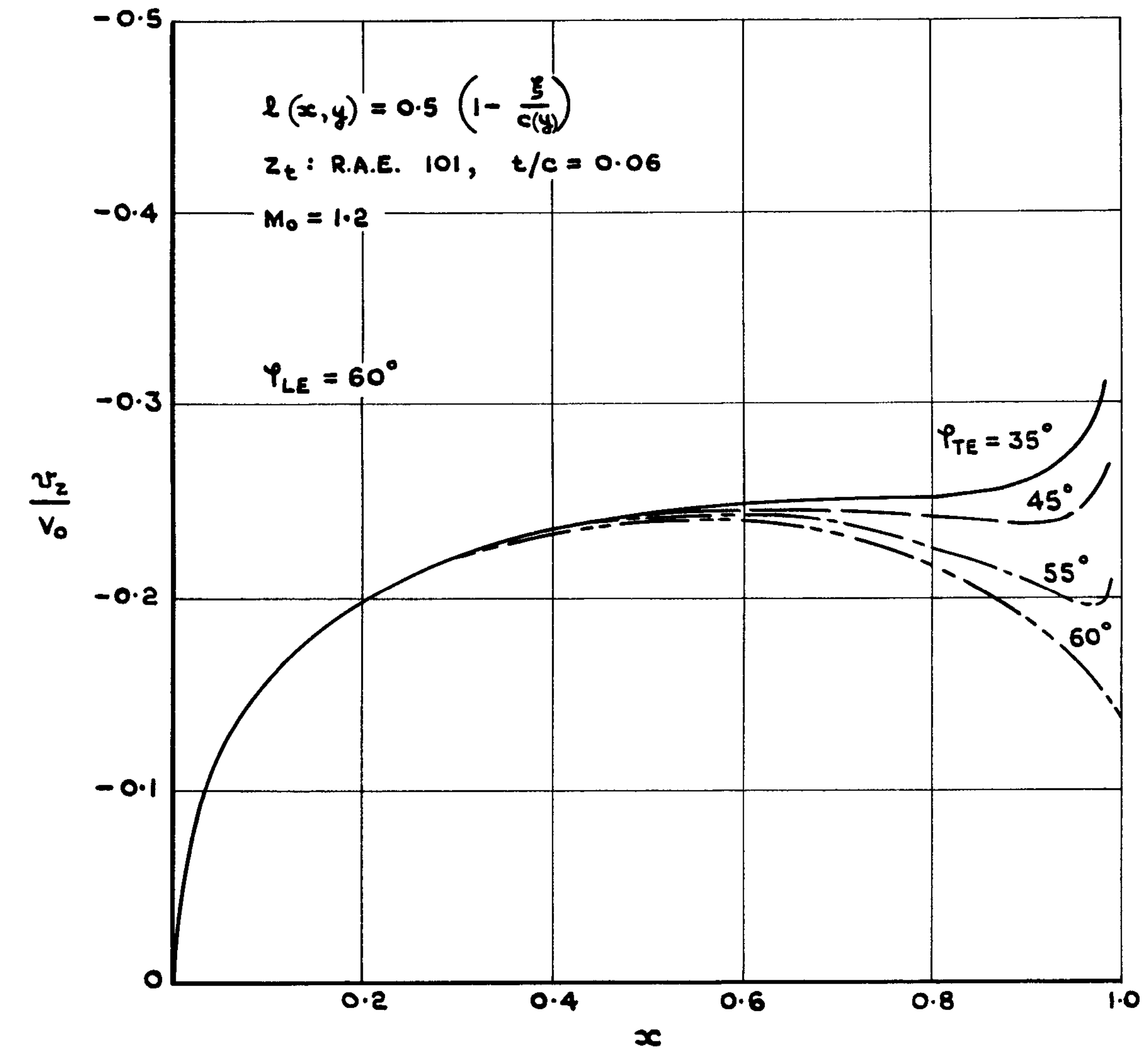


FIG. 2. DOWNWASH AT $M_0 = 1$ FOR EXACT AND APPROXIMATE LOAD DISTRIBUTIONS.



(a)



(b)

FIG. 3(a & b). CENTRE SECTION DOWNWASH DISTRIBUTIONS.

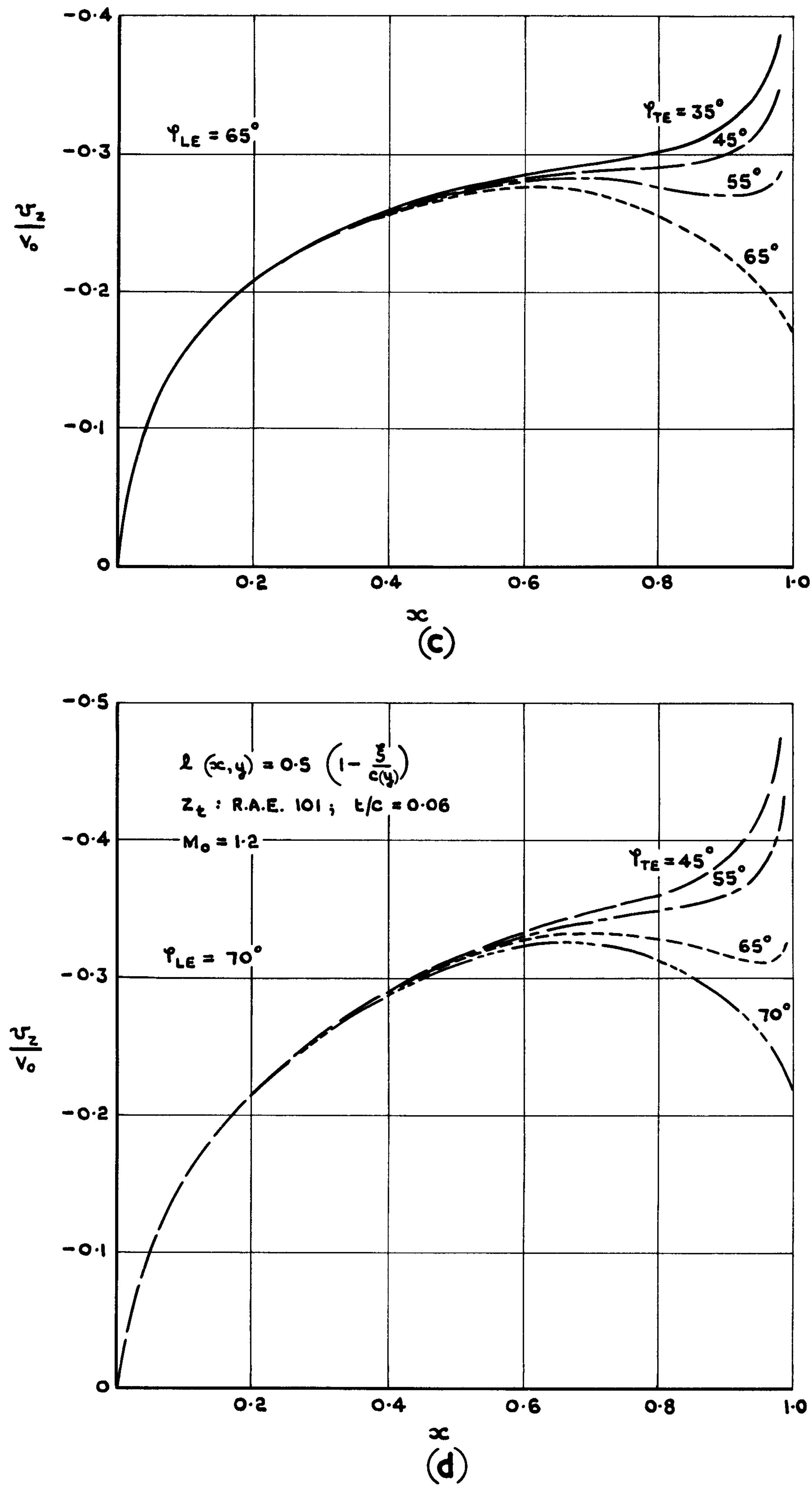


FIG. 3(c & d). CENTRE SECTION DOWNWASH DISTRIBUTIONS.

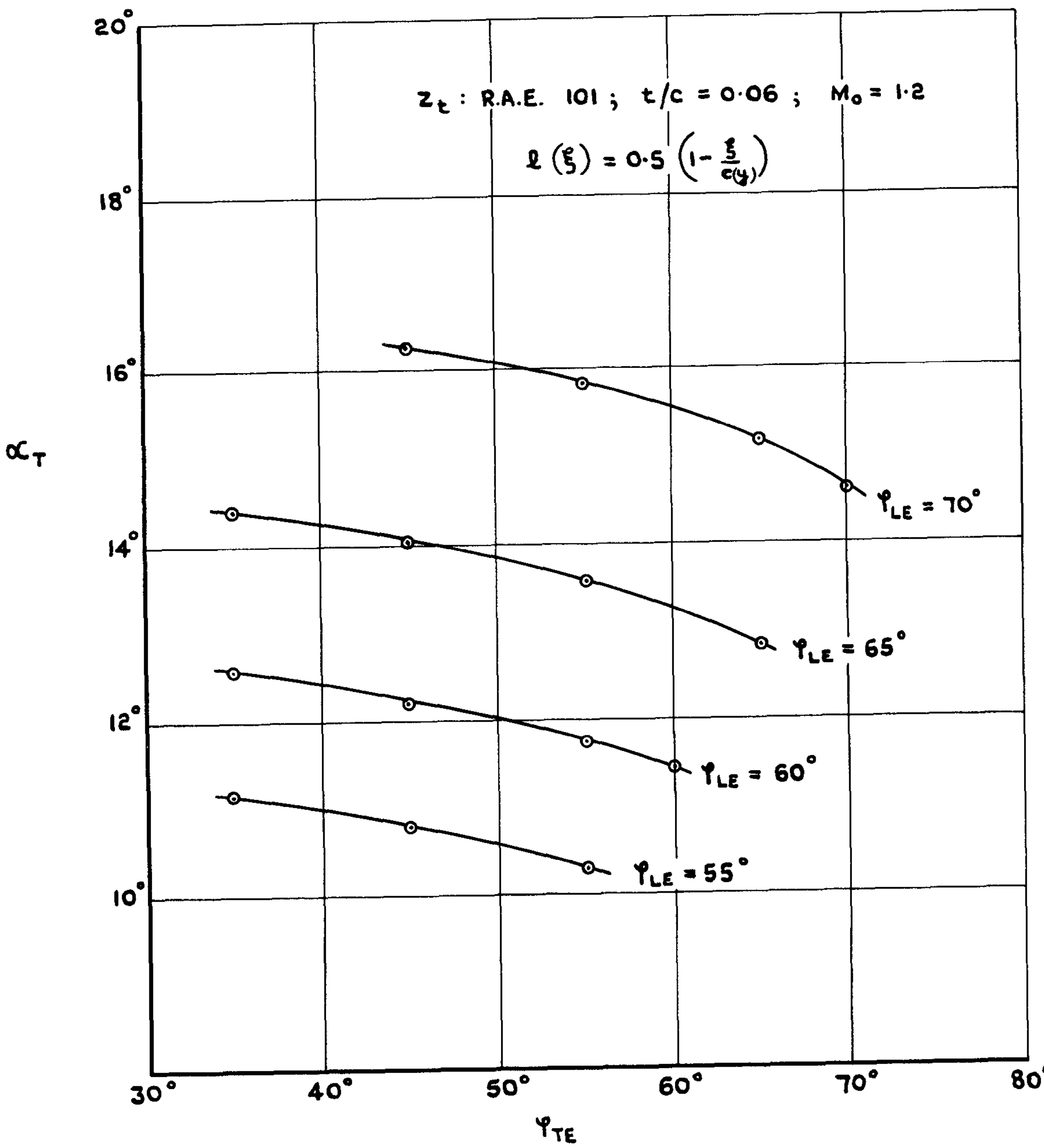


FIG. 4. ANGLE OF TWIST.

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