C.P. No. 442
(21,155)
A.R.C. Technical Report

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by

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1959

PRICE 2s. 6d. NET

C.P. No. 442

U.D.C. No. 533.6.013.422:533.69.033:533.6.011.6

Report No. Structures 249

April, 1959

ROYAL AIRCRAFT ESTABLISHMENT

FLUTTER OF AN UNTAPERED WING ALLOWING FOR THERMAL EFFECTS

by

E. G. Broadbent

SUMMARY

The flutter problem of a thin rectangular solid steel wing of aspect ratio 3 is considered with allowance for reduction in stiffnesses due to thermal effects. The change in camber associated with wing bending gives rise to a destabilising aerodynamic coupling which leads to a critical flutter Mach number of just over $3\frac{1}{2}$; no other coupling in the same sense exists with the assumptions made, so that without this effect the flutter speed would be infinite. The root constraint is not important for a wing of the aspect ratio considered.

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ILLUSTRATION

1 INTRODUCTION

In reference 1 a simple example of aeroelastic deformation at high temperatures was presented. The calculation was for a solid steel biconvex wing of rectangular planform, and used the large deflection analysis due to Mansfield[2]. Mansfield assumes that the heating leads to a temperature distribution that is parabolic across the wing chord and then derives exact two-dimensional relationships between the torque and rate of twist, and between the bending moment and the curvature. These non-linear equations were solved by a process of successive approximation[1] to obtain the equilibrium deformation of the wing when set at incidence in a supersonic airstream. The effects of the thermal stresses are to reduce the torsional and bending stiffnesses and to increase the anticlastic curvature that accompanies any bending deformation. This last effect is stabilising in the static problem mentioned above but it seemed to the writer that it would very likely be destabilising in an aeroelastic vibration. This point is illustrated in the present paper in which the flutter speed of the same wing is calculated with an allowance for thermal effects, although in this case only the linear part of the equations is retained. The aerodynamic assumptions are such that the flutter speed would have been infinite with no chordwise curvature, but with the curvature included the critical flutter condition is represented by a Mach number of just over $3\frac{1}{2}$.

2 FLUTTER ANALYSIS

The wing is illustrated in Fig.1. It is a rectangular wing clamped at the root and with a panel aspect ratio of $1\frac{1}{2}$ (i.e. the net tip to tip aspect ratio would be 3) and with a symmetric biconvex section of 2% thickness chord ratio in solid steel. The flutter analysis is in two degrees of freedom, one of wing torsion in which the rate of twist is assumed to fall linearly to zero from root to tip and the other of wing bending in which the spanwise curvature also falls linearly to zero from root to tip; these modes satisfy the tip condition of zero bending moment and zero torque. Lagrange's equations are used, so that the elastic coefficients are obtained from the strain energy, the inertia coefficients are obtained from the kinetic energy and the aerodynamic coefficients (based in this case on piston theory) are obtained by considering the virtual work done in a small displacement.

2.1 The strain energy

The linearised equations can be obtained from the exact relationships[2] by neglecting the second order terms* which leads to:-

$$\frac{\hat{T}}{\theta} = 1 - \hat{\sigma}(1 + \nu) \tag{1}$$

$$\frac{\hat{M}}{\kappa} = (1 - \hat{\sigma}(1 + \nu))(1 + (1 - \nu)\hat{\sigma}) = \frac{\hat{T}}{\theta}\{1 + \hat{\sigma}(1 - \nu)\} \tag{2}$$

and

$$\frac{\kappa'}{\kappa} = -\{\nu + (1 - \nu^2)\hat{\sigma}\} \tag{3}$$

---

* They are also given by Mansield[3] in an earlier paper, but it is convenient here to use the notation of ref.1 which is mostly taken from ref.2. Work similar to Mansfield's has been published independently by Kochanski and Argyris[4].

where T is the torque on a section

θ is the rate of twist at the section

$\bar{M}$ is the bending moment on a section

κ is the spanwise curvature at the section

κ' is the chordwise curvature at the section

and the circumflex accent denotes a non-dimensional form of the quantity i.e.

$$\left.\begin{aligned}
\hat{\theta} &= \frac{c^2\,\theta}{4\sqrt{5}\,t_o} \\
\hat{\kappa} &= \frac{c^2\,\kappa}{4\sqrt{5}\,t_o} \\
\hat{M} &= \frac{c^2\,\bar{M}}{4\sqrt{5}\,t_o\,B} \\
\hat{T} &= \frac{c^2\,T}{4\sqrt{5}\,t_o\,C}
\end{aligned}\right\} \qquad (4)$$

where B is the flexural rigidity $(E\,I)$ and C is the torsional rigidity $(G\,J)$, without thermal effects

and t_o is the wing thickness at mid chord.

Also ν is Poisson's ratio

and
$$\hat{\sigma} = \frac{c^2\,\bar{\alpha}\,\bar{T}}{10\,t_o^{\,2}} \qquad (5)$$

where c is the wing chord

$\bar{T}$ is the temperature difference between the average temperature of the leading and trailing edges and the mid chord

$\bar{\alpha}$ is the coefficient of thermal expansion.

The non-dimensional parameter $\hat{\sigma}$ gives a measure of the thermal strain (or thermal stress) and in the present example has a value of 0.4. This value, taken from ref.1, is based on a temperature difference of 133° which gives a reasonable simulation of conditions reached by acceleration to a Mach number of about 3 at 20,000 ft altitude. It will perhaps be as well to state here that the method of solution of the flutter equations is to calculate the critical Mach number for flutter; if this critical Mach number had been found to be very different from 3 then a process of trial and error would have been followed, but in fact such a process proves to be unnecessary in this particular example.

The modes of deformation are given by:-

$$\frac{\partial\Theta}{\partial\eta} = s\,\theta = (1 - \eta)\, q_1 \tag{6}$$

and

$$\frac{\partial\phi}{\partial\eta} = s\,\kappa = (1 - \eta)\, q_2 \tag{7}$$

where s is the distance from root to tip

Θ is the incidence of a section relative to the root

$\eta = y/s$ where y is the spanwise variable

ϕ is the slope of the mid-chord line relative to the corresponding slope at the root

and q_1 and q_2 are the generalised co-ordinates.

The strain energy $\bar{V}$ is given by:-

$$2\,\bar{V} = s\int_0^1 \theta\, T\, d\eta + s\int_0^1 \kappa\, \bar{M}\, d\eta \quad . \tag{8}$$

From equations (1), (2), (6) and (7) and the relations (4), this expression for the strain energy can be evaluated in terms of the generalised co-ordinates q_1 and q_2, viz:-

$$2\,\bar{V} = \frac{1}{3s}\left\{1 - \hat{\sigma}\,(1 + \nu)\right\}\left\{C\, q_1^{\,2} + B\left[1 + \hat{\sigma}\,(1 - \nu)\right] q_2^{\,2}\right\} \quad . \tag{9}$$

2.2 The kinetic energy

The downward deflection of a point P on the wing mid chord AB (see Fig. 1) is given by:-

$$z_p = s\int_0^\eta \phi\, d\eta = \frac{\eta^2}{2}\left(1 - \frac{\eta}{3}\right) s\, q_2 \tag{10}$$

by equation (7). The deflection of a point Q in the chord D E (see Fig. 1) relative to P_1 is:-

$$z_{Q-P} = \int_0^x x\,\kappa'\, dx + \int_0^x \Theta\, dx$$

where x is the chordwise variable measured aft from the mid chord; i.e. x = PQ. Let

$$x = c\xi$$

then
$$z_{Q-P} = c^2\int_0^\xi \xi\kappa'\, d\xi + c\int_0^\xi \Theta\, d\xi = \frac{c^2\,\xi^2}{2}\,\kappa' + c\,\Theta\xi \tag{11}$$

since κ' and Θ are constant over the chord. If we substitute for Θ using equation (6) and for κ' using equations (3) and (7), then by (10) and (11) we have:-

$$z(\xi,\eta) = c\, q_1\, \xi \left(\eta - \frac{\eta^2}{2}\right) + s\, q_2 \left[\frac{\eta^2}{2}\left(1 - \frac{\eta}{3}\right) - \frac{c^2}{2s^2}\, \xi^2\, (1-\eta)\, \{\nu + (1-\nu^2)\hat{\sigma}\}\right]. \quad (12)$$

The kinetic energy $\tilde{T}$ is now given by

$$2\,\tilde{T} = \rho_s\, c\, s\, t_o \int_0^1 d\eta \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - 4\,\xi^2)\, \dot{z}^2\,(\xi,\eta)\, d\,\xi \quad (13)$$

where ρ_s is the density of steel, since the wing thickness at a point (ξ,η) is $t_o\,(1-4\,\xi^2)$.

2.3 The aerodynamic forces

In the present example piston theory[5] is used with the thickness terms neglected. This leads to the very simple expression for the pressure difference between the lower and upper surfaces:-

$$\Delta p = 2\,\rho a \left(\frac{\partial z}{\partial t} + V\,\frac{\partial z}{\partial x}\right) \quad (14)$$

where ρ is the air density

and a is the speed of sound.

We let $$z = \bar{z}\, e^{\lambda\,\tau} \quad (15)$$

where $\tau = \dfrac{ta}{c}$ is a non-dimensional time variable. Hence, if M is the

Mach number, $$\Delta p = 2\,\frac{\rho a^2}{c}\left(\lambda\, z + M\,\frac{\partial z}{\partial \xi}\right), \quad (16)$$

and the work done in a small displacement δz is given by:-

$$W = -sc \int_0^1 d\eta \int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta p\ \delta z\ (\xi,\eta)\ d\xi\ . \quad (17)$$

2.4 Application of Lagrange's equations

Lagrange's equations are given by:-

$$\frac{d}{dt}\,\frac{\partial \tilde{T}}{\partial \dot{q}_r} + \frac{\partial \bar{V}}{\partial q_r} = W_{\delta q_r} \quad (18)$$

where the suffix r is given in turn the values 1 and 2, and $W_{\delta q_r}$ is the coefficient of δq_r in the expression for W given by equation (17); i.e. $W_{\delta q_r}$

is the generalised force in the degree of freedom q_r. We have two equations of the form (18) which are made non-dimensional by dividing through by the factor $\rho a^2 \ sc^2$. The inertia terms will then have the form in the ith Lagrangian equation:-

$$\frac{A_{ij}\ \ddot{q}_j}{\rho a^2\ sc^2} \equiv \frac{A_{ij}\ \lambda^2 q_j}{\rho sc^4} \equiv a_{ij}\ \lambda^2 q_j \tag{19}$$

by the definition of λ in equation (15). The aerodynamic terms in the ith Lagrangian equation will have the form:-

$$(\lambda b_{ij} + Mc_{ij})\ q_j \tag{20}$$

where b_{ij} and c_{ij} are non-dimensional aerodynamic coefficients, and the elastic coefficients in the ith Lagrangian equation will have the form:-

$$\frac{E_{ij}}{\rho a^2\ sc^2}\ q_j \equiv e_{ij}\ q_j \tag{21}$$

since there are two co-ordinates i and j take the values 1 and 2 and the two flutter equations can be written in matrix form:-

$$[a\ \lambda^2 + b\lambda + cM + e]\ \ q = 0 \tag{22}$$

where a, b, c and e are square matrices of non-dimensional coefficients and q is a matrix column. Equation (22) is solved by equating the determinant to zero and solving for the Mach number M.

2.5 Solution of the flutter equation

We may remark from the form of the expression for potential energy, equation (9), that no elastic coupling exists, i.e. $e_{ij} = 0$, $i \neq j$. Also in the expression for the kinetic energy, equations (12) and (13), since the integral of an odd power of ξ between the limits of $-\frac{1}{2}$ and $\frac{1}{2}$ is zero we again have no coupling term and $a_{ij} = 0$, $i \neq j$. This implies that the two co-ordinates are in fact normal co-ordinates, which can also be inferred from physical considerations. Finally, in the expression for the aerodynamic work from equations (16) and (17), we again have that the part which is proportional to λ is an integral of z^2 (similar to the integral for the kinetic energy) and hence $b_{ij} = 0$, $i \neq j$. It also follows that the direct aerodynamic dampings which are proportional to b_{11} and b_{22} are positive, and only the aerodynamic stiffnesses can supply the couplings which would lead to flutter. It is instructive here to consider the test functions for stability; the flutter determinant can be expanded to give

$$p_o\ \lambda^4 + p_1\ \lambda^3 + p_2\ \lambda^2 + p_3\ \lambda + p_4 = 0 \tag{23}$$

and it is necessary and sufficient for stability that all the p's shall be positive and that

$$T_3 \equiv p_1 p_2 p_3 - p_o p_3^2 - p_1^2 p_4 > 0 \quad . \tag{24}$$

Suppose, for the moment that the camber term, κ', were zero, as would generally be assumed in a flutter calculation that did not take account of the thermal stresses. In this case c_{11}, c_{12} and c_{22} would all be zero; hence the motion would inevitably be stable since no (1,2) coupling term would exist and the direct dampings would be positive. We know, however, that the effect of the camber κ' is stabilising as regards static divergence (see ref.1, for example, in which the analogy with a swept back wing is drawn) so that the value of c_{12} introduced by κ' must increase p_4 since this coefficient vanishes at the divergence speed. Alternatively, it can be seen from the expression for W (or from physical considerations) that c_{21} is positive and that c_{12}, which depends on κ' is negative, and hence p_4 is increased. The magnitude of this increase is, moreover, proportional to M^2 whereas p_o, p_1, p_2 and p_3 are all constant, and it follows from the form of T_3 in expression (24) that this must lead to instability for sufficiently large values of M.

The critical Mach number for flutter is found most simply by using the test function T_3 and solving for the limiting case $T_3 = 0$. This is a linear equation in M^2.

3 NUMERICAL RESULT AND CONCLUSIONS

The structural data assumed are given by:-

$$E = 29.5 \times 10^6 \text{ lb/sq in.}$$

$$G = 11.5 \times 10^6 \text{ lb/sq in.}$$

$$\nu = 0.28$$

$$\rho_s = 490 \text{ lb/cub. ft}$$

$$GJ = 4\ GI$$

and

$$\vartheta = 0.4$$

No allowance for reduction in the elastic moduli due to thermal effects has been made. The aerodynamic data are

$$\rho = 0.533 \times 0.002378 \text{ slug/cub. ft (at 20,000 ft)}$$

$$a = 1038 \text{ ft/sec.}$$

These data lead to the following matrices of coefficients:-

$$a = \begin{bmatrix} 1.067 & 0 \\ 0 & 9.273 \end{bmatrix}$$

$$b = \begin{bmatrix} 0.02222 & 0 \\ 0 & 0.1146 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 & -0.009037 \\ 0.1621 & 0 \end{bmatrix}$$

$$e = \begin{bmatrix} 0.1066 & 0 \\ 0 & 0.0882 \end{bmatrix}$$

and the solution of $T_3 = 0$ gives $M = 3.6$. This is in sufficiently close agreement with the assumed conditions at a Mach number of 3 to be taken as the critical flutter Mach number. It may be noted that this value of 3.6 compares with the value of infinity if there is no camber change and $c_{12} = 0$. It may perhaps be mentioned here that the reason why finite flutter speeds are often predicted for symmetric supersonic sections using piston theory is that if the thickness terms are included a small negative value is introduced for b_{12} (and also for c_{22} although this is much less important) and this provides the required coupling. The coupling would be very small for a wing of only 2% thickness/chord ratio.

It may be thought that the flutter Mach number of 3.6 is unrealistic because of the root constraint, but this is not in fact so. Mansfield[3] has shown that the root constraint would be effective to a distance of about $\frac{1}{4}c$ (for a parabolic section as used in our example) from the root. The magnitude of this restraint has been estimated by neglecting the effect of κ' for $0 < \eta < 0.2$. This led to a reduction in the numerical value of c_{12} by about 15% and an increase in the value of e_{22} (since there is now no loss in bending stiffness near the root) by about 20%. The reduction in c_{12} raises the flutter speed, but the increase in e_{22} lowers it again so that the net effect is that the critical flutter speed is increased only slightly: in fact the solution gave $M = 3.8$.

The general implications of these calculations are that thermal stresses can seriously lower the flutter speed of a solid wing at high Mach numbers. For a thicker wing, however, the effects would be much less, because of greater intrinsic stiffness, less drop in stiffness and less adverse camber change. It is also apparent that if the wing is designed to keep the thermal stresses to a low level by using a torsion box, for example, instead of a solid section, then again the flutter speed would remain high.

LIST OF SYMBOLS

A_{ij}	a dimensional inertia coefficient
B	the flexural rigidity (EI)
C	the torsional rigidity (GJ)
E	Young's modulus

LIST OF SYMBOLS (Cont'd)

E_{ij}	a dimensional stiffness coefficient
G	shear modulus
I	section moment of inertia
J	effective polar moment of inertia
M	Mach number
$\bar{M}$	bending moment
$\hat{M}$	non-dimensional form of $\bar{M}$ (see equation 4)
T	section torque
T_3	stability test function
$\bar{T}$	temperature difference
$\hat{T}$	non-dimensional form of T (see equation 4)
$\tilde{T}$	kinetic energy
V	forward speed
$\bar{V}$	strain energy
W	work
a	speed of sound
a_{ij}	non-dimensional inertia coefficient
b_{ij}	non-dimensional aerodynamic damping coefficient
c	wing chord
c_{ij}	non-dimensional aerodynamic stiffness coefficient
e_{ij}	non-dimensional structural stiffness coefficient
p	pressure
p_i	coefficient in λ-polynomial
q_i	generalised co-ordinate
s	distance from root to tip
t	time
t_o	wing thickness at half chord
x	chordwise variable
y	spanwise variable
z	vertical downward deflection

LIST OF SYMBOLS (Cont'd)

Θ	rotation of chordwise section relative to root
$\bar{\alpha}$	coefficient of thermal expansion
η	non-dimensional spanwise variable = y/s
θ	rate of twist
$\hat{\theta}$	non-dimensional rate of twist (see equation 4)
κ	spanwise curvature
$\hat{\kappa}$	non-dimensional spanwise curvature (see equation 4)
κ'	chordwise curvature
λ	coefficient of τ in the exponent
ν	Poisson's ratio
ξ	non-dimensional chordwise variable = x/c
ρ	density of air
ρ_s	density of steel
$\hat{\sigma}$	non-dimensional thermal stress (see equation 4)
τ	non-dimensional form of time = $t\ a/c$
ϕ	bending slope

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Wt.2078.C.P.422.K3 - Printed in Great Britain

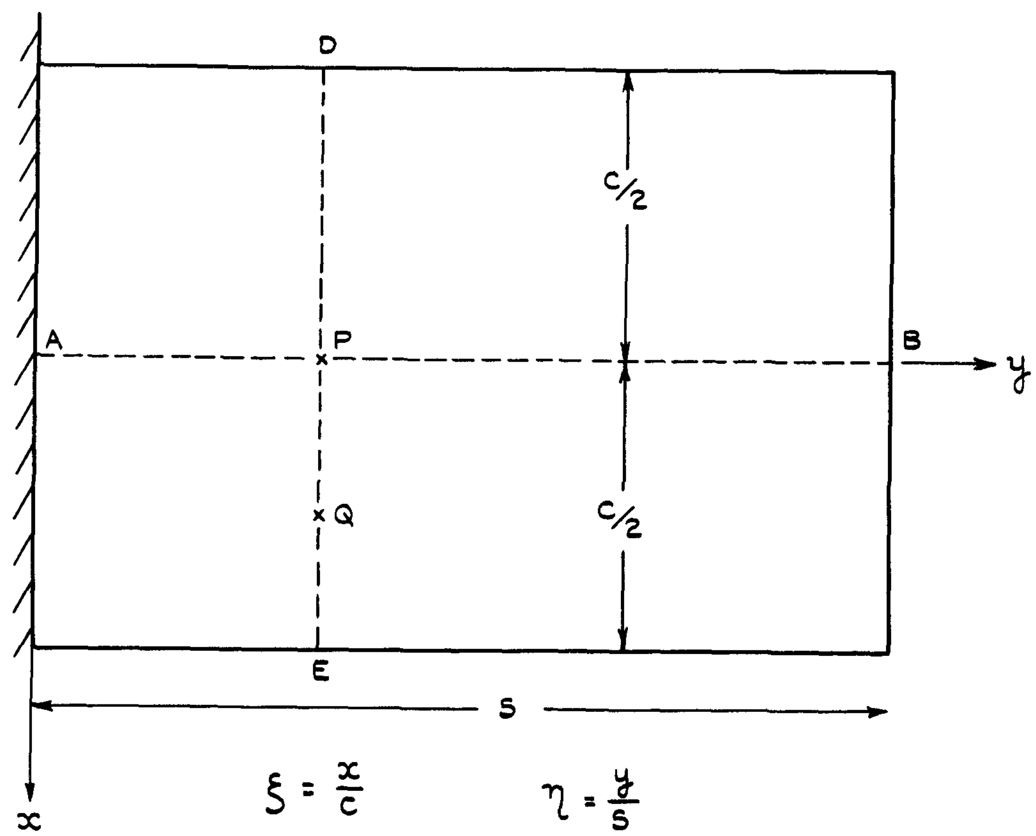

FIG. 1. PLANFORM OF WING CONSIDERED.

C.P. No. 442
(21,155)
A.R.C. Technical Report

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S.O. Code No. 23-9011-42

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