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Statistical Analysis of a Particular Target Manoeuvre

by

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STATISTICAL ANALYSIS OF A PARTICULAR TARGET MANOEUVRE

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INTRODUCTION

A non-periodic manoeuvre has been devised, which will be called the "Poisson" target manoeuvre. Suppose we have three "tramlines", 10 nautical miles apart, of which the middle one is the desired course; the target moves from one to another in sequence along a path inclined at 30° to the mean course and travels distances along the "tramlines" which are randomly distributed according to the Poisson distribution with a mean of 10 nautical miles. The target is assumed to change its heading instantaneously. A refinement would be to assume that the target takes up a new heading by means of a $\frac{1}{2}$ g turn instead, but this has not been done in the analysis; it would add considerably to the difficulty of the problem and it seems unlikely that it would alter materially the results and conclusions.

Statistical analysis of the Poisson target manoeuvre

The autocorrelation and cross-correlation function of the Poisson target manoeuvre will be derived. This is most easily done by calculating the spectrum of the target manoeuvre displacement and taking its Fourier transform. A short summary of the standard theory is given.

Let $y(t)$ be a stationary time series and define $y_T(t)$ as

$$y_T(t) = y(t) \qquad -T \leqslant t \leqslant T$$

$$y_T(t) = 0 \qquad t < -T, \quad t > T .$$

The Fourier transform $A_T(f)$ of $y_T(t)$ is given by

$$A_T(f) = \int_{-\infty}^{\infty} y_T(t)\, e^{-2\pi i f t} \,.\, dt$$

$$= \int_{-T}^{T} y(t)\, e^{-2\pi i f t} \,.\, dt \tag{1}$$

From the Fourier integral theorem, it follows that

$$y_T(t) = \int_{-\infty}^{\infty} A_T(f)\, e^{2\pi i f t} \,.\, df \,. \tag{2}$$

The spectrum of $y(t)$ is defined as

$$S_y(f) = \lim_{T \to \infty} \frac{1}{2T} \left| A_T(f) \right|^2 . \tag{3}$$

The autocorrelation function $\rho(\tau)$ of $y(t)$ is defined by

$$\overline{y^2}\, \rho(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} y_T(t)\, y_T(t+\tau)\, dt \,. \tag{4}$$

Then

$$\overline{y^2}\, \rho(\tau) = \int_{-\infty}^{\infty} S_y(f)\, e^{-2\pi i f \tau} \,.\, df \,. \tag{5}$$

The above definitions are frequently given in terms of the variable

$$\omega = 2\pi f$$

and in these terms the transform of $y_T(t)$ is

$$A_T(\omega) = \int_{\infty}^{\infty} y_T(t)\, e^{-i\omega t} \,.\, dt \tag{1a}$$

with the inverse equation

$$y_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_T(\omega)\, e^{-i\omega t} \,.\, d\omega \,. \tag{2a}$$

The three results following will also be employed in the analysis:

(1) If $B(\omega) = \sum_{r=1}^{m} C_r\, e^{-i\omega x_r}$, where C_r is either +1 or -1, then

$$|B(\omega)|^2 = 2\left[\frac{m}{2} + \sum_{r=1}^{m} \sum_{s=1}^{r} C_r\, C_s \cos \omega\,(x_r - x_s)\right]. \tag{6}$$

(2) If x_s $(s = 1, \ldots, r)$ is a set of r independent random variables, with probability distributions

$$P(x_s) = \beta_s\, e^{-\beta_s x_s}$$

then the mean value of $e^{-i\omega(x_1 + \cdots + x_r)}$ is

$$\left[\overline{e^{-i\omega(x_1 + \cdots + x_r)}}\right] = \prod_s \left[\frac{\beta_s}{\beta_s + i\omega}\right]. \tag{7}$$

(3) If $Y(t)$ and $y(t)$ are stationary time series with spectral functions $S_Y(\omega)$ and $S_y(\omega)$ and if

$$Y(t) = \int_0^t y(u)\, du$$

then

$$S_Y(\omega) = -\frac{S_y(\omega)}{\omega^2} \tag{8}$$

if $\int_0^T y(u)\, du$ is bounded. (There is a factor of proportionality $\frac{1}{4\pi^2}$ when the spectrum is defined in terms of the variable f given above.)

A plot of target transverse velocity $\dot{z}_T$ against time (shown in Fig. 1a, together with a plot of z_T against time) consists of a series of positive and negative pulses of height $V_T \sin\gamma$, where γ is the inclination to the mean track of the part of the course leading from one "tramline" to another, and duration $T = \frac{s}{V_T \sin\gamma}$, where s is the separation between the "tramlines". By convention the series starts from the mean course with a single positive pulse and then the pulses are alternately two negative and two positive. The intervals between the pulses are denoted by $x_1, y_1, x_2, y_2, \ldots,$ where x_i is the interval between adjacent pulses in opposite directions (the interval spent in one of the outer tramlines) and y_i is the interval between adjacent pulses in the same direction (the interval spent on the mean course). The x_i and y_i are all independent of each other, and each is distributed according to the Poisson probability law

$$P(x) = \beta e^{-\beta x}$$

where $\frac{1}{\beta} = \bar{x} = \frac{m}{V_T}$, m is the mean distance travelled along one of the tramlines. We are interested in an infinite sequence of this type, or at least in one of considerable length, but the analysis will be clearer if we consider first how such a sequence may be built up.

Single pulse pair

Consider a single pair of pulses in the sequence: suppose $\dot{z}_T$ has a unit positive pulse, duration T followed by a unit negative pulse, duration T, and separated from it by an interval x (see Fig. 1b). Taking the Fourier transform of $\dot{z}_T$

$$A(\omega) = \int_0^T e^{-i\omega t}\,dt - \int_{x+T}^{x+2T} e^{-i\omega t}\,dt$$

$$= \frac{1}{i\omega}\left[1 - e^{-i\omega t} - e^{-i\omega(x+T)} + e^{-i\omega(x+2T)}\right].$$

This is a series of the form considered in (1) above, and hence from (6)

$$|A(\omega)|^2 = \frac{2}{\omega^2}\left[2 - 2\cos\omega T + \cos\omega x - 2\cos\omega(x+T) + \cos\omega(x+2T)\right].$$

An exponential series is easier to handle than a series of cosine terms, so this may be written as,

$$|A(\omega)|^2 = \frac{2}{\omega^2} R\left[2 - 2e^{-i\omega T} + e^{-i\omega x} - 2e^{-i\omega(x+T)} + e^{-i\omega(x+2T)}\right].$$

To obtain the spectrum, $|A(\omega)|^2$ must be divided by the duration of the sequence and the average overall values of x taken. Now the mean duration of a sequence of n pulses is $n(T+\frac{1}{\beta})$, and by (7)

$$\overline{[e^{-i\omega x}]} = \frac{\beta}{\beta + i\omega}.$$

Writing $e^{-i\omega T} = B$, $\frac{\beta}{\beta + i\omega} = C$, gives for the spectrum of $\dot{z}_T$,

$$S_{\dot{z}_T}(\omega) \sim \frac{1}{\omega^2} R \, [2(1-B) + C(1-B)^2] .$$

By (8) the spectrum of z_T is given by

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R \, [2(1-B) + C(1-B)^2] . \tag{9}$$

Then, by (5), the autocorrelation function for z_T corresponding to a single pair of velocity pulses, i.e. one "hump" of z_T, is proportional to the Fourier transform of

$$\frac{1}{\omega^4} R \, [2(1-B) + C(1-B)^2] .$$

It should be stressed that the term "n pairs of velocity pulses" describes only the length of manoeuvre which is being studied: it does not necessarily imply that the spectrum of $\dot{z}_T$ is being calculated.

Two pulse pairs

We next consider two pairs of velocity pulses (see Fig. 1c) of the kind treated in the previous paragraph. Proceeding as before, and writing $e^{i\omega T} = B$, $\frac{\beta}{\beta + i\omega} = C$, and the mean value $\overline{[e^{-i\omega y}]} = Y$, the spectrum of $\dot{z}_T$ is

$$\begin{aligned} S_{\dot{z}_T}(\omega) &\sim \frac{1}{\omega^2} R \Big[2 - 2B + \{C + \tfrac{1}{2}Y\} - \{2BC + BY\} + \{B^2C + \tfrac{1}{2}B^2Y\} \\ &\qquad - \{BCY\} + \{2B^2CY\} - \{B^3CY\} + \tfrac{1}{2}B^2C^2Y - B^3C^2Y + \tfrac{1}{2}B^4C^2Y \Big] \\ &= \frac{1}{\omega^2} R \Big[\{2(1-B) + C(1-B)^2\} + \{\tfrac{1}{2}Y(1-B)^2 \, (1-BC)^2\} \Big]. \end{aligned}$$

Then, from (8), the spectrum of z_T is

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R \Big[\{2(1-B) + C(1-B)^2\} + \{\tfrac{1}{2}Y(1-B)^2 \, (1-BC)^2\} \Big]. \tag{10}$$

Comparing this with (9) we see that the first term in (10) gives the contribution of the individual pulse pairs and the second term that of the interaction between two pulse pairs with separation y. We note that in the manoeuvre described in the Introduction adjacent pairs are in opposite directions and the separation y has the same distribution as x, so that Y = C, and the spectrum for the first four velocity pulses, one complete cycle of the manoeuvre, is

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R \left[\{2(1-B) + C(1-B)^2\} - \{\tfrac{1}{2}C(1-B)^2 \, (1-BC)^2\} \right]. \tag{10a}$$

Many pulse pairs

The method of the previous two paragraphs can be applied to a sequence of any length. For the first three pulse pairs in the sequence of (1a), it leads to the spectrum

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R \Big[\{2(1-B) + C(1-B)^2\} - \{\tfrac{2}{3}\, C(1-B)^2 \, (1-BC)^2\}$$
$$+ \{\tfrac{1}{3}\, B^2C^3(1-B)^2 \, (1-BC)^2\} \Big], \tag{11}$$

and for the first four pulse pairs

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R \Big[\{2(1-B) + C(1-B)^2\} - \{\tfrac{3}{4}C(1-B)^2(1-BC)^2\}$$
$$+ \{\tfrac{1}{2}B^2C^3(1-B)^2(1-BC)^2\} - \{\tfrac{1}{4}B^4C^5(1-B)^2(1-BC)^2\} \Big]. \tag{12}$$

In a short sequence the spectrum and the autocorrelation function are very much affected by the ends of the sequence. Since the target manoeuvre may be considered to be part of an endless series of cycles and its precise length is unknown, varying from one encounter to another, it is clear that such end effects should be ignored. We therefore calculate the spectrum for a large number, n, of pulse pairs and then let n tend to infinity.

Let n be an even integer, so that a number of complete cycles of the manoeuvre is considered. The Fourier transform, $A(\omega)$, of $\dot{z}_T$ may be set up as described earlier for a single pulse pair. Then dividing $|A(\omega)|^2$ by $2n$, the number of pulses, averaging over the x's and y's, and writing $e^{-i\omega T} = B$, $[\overline{e^{-i\omega x_i}}] = C$ and $[\overline{e^{-i\omega y_i}}] = Y$ (since though $C = Y = \dfrac{\beta}{\beta + i\omega}$ it is helpful to distinguish between the two means) we obtain the spectrum of $\dot{z}_T$ as

$$S_{z_T}(\omega) \sim \frac{1}{\omega^2} R \Big[2 - 2B + C - \frac{n-1}{n} Y - 2BC + 2\,\frac{n-1}{n} BY + B^2C - \frac{n-1}{n} B^2Y$$
$$+ \frac{2(n-1)}{n} BCY - 4\,\frac{(n-1)}{n} B^2CY + \frac{2(n-1)}{n} B^3CY$$
$$- \frac{n-1}{n} B^2C^2Y + \frac{n-2}{n} B^2CY^2 + 2\,\frac{n-1}{n} B^3C^2Y - 2\,\frac{n-2}{n} B^3CY^2$$
$$- \frac{n-1}{n} B^4C^2Y + \frac{n-2}{n} B^4CY^2 - 2\,\frac{n-2}{n} B^3C^2Y^2 + 4\,\frac{n-2}{n} B^4C^2Y^2$$
$$- 2\,\frac{n-2}{n} B^5C^2Y^2 + \frac{n-2}{n} B^4C^3Y^2 - \frac{n-3}{n} B^4C^2Y^3$$
$$- 2\,\frac{n-2}{n} B^5C^3Y^2 + 2\,\frac{n-3}{n} B^5C^2Y^3 + \frac{n-2}{n} B^6C^3Y^2$$
$$- \frac{n-3}{n} B^6C^2Y^3 + 2\,\frac{n-3}{n} B^5C^3Y^3 + \dots \Big].$$

Grouping the terms in this expression according to the power of Y

$$S_{z_T}(\omega) \sim \frac{1}{\omega^2} R\Big[\{2(1-B) + C(1-B)^2\} - \frac{n-1}{n}\{Y(1-B)^2\ (1-BC)^2\}$$

$$+ \frac{n-2}{n}\{B^2CY^2(1-B)^2\ (1-BC)^2\}$$

$$- \frac{n-3}{n}\{B^4C^2Y^3(1-B)^2\ (1-BC)^2\} + \dots\Big].$$

Here, the first term gives the contribution of each pulse pair interacting with itself; the second term gives the contribution of each pulse pair interacting with its immediate neighbour; the third term gives the contribution of each pulse pair interacting with the pulse pair next but one and so on. Now, setting $Y = C$ and using (8), the spectrum of z_T is

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R\Big[\{2(1-B) + C(1-B)^2\} - \frac{n-1}{n}\{C(1-B)^2(1-BC)^2\}$$

$$+ \frac{n-2}{n}\{B^2C^3(1-B)^2\ (1-BC)^2\}$$

$$- \frac{n-3}{n}\{B^4C^5(1-B)^2\ (1-BC)^2\} + \dots\Big]. \tag{13}$$

In the limit as $n \to \infty$, the spectrum for an infinite sequence is

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R\Big[\{2(1-B) + C(1-B^2)\} - \{C(1-B)^2(1-BC)^2\}$$

$$+ \{B^2C^3(1-B)^2(1-BC)^2\} - \{B^4C^5(1-B)^2(1-BC)^2\} + \dots\Big], \tag{14}$$

where the terms still have the significance noted above. The infinite series in (14) may be summed to give the expression

$$S_{z_T}(\omega) \sim \frac{1}{\omega^4} R\left[\frac{2(1-B)(1+BC^2)}{(1+B^2C^2)}\right], \tag{15}$$

but the inverse of (15) is not known.

We take a suitable number of terms from (14) to cover the range of τ in which we are interested: the minimum distance between adjacent pulse pairs is zero, that between pulse pairs next but one is $2T$ and hence the third and successive terms in (14) cannot contribute to the autocorrelation function $\rho_z(\tau)$ for values of $\tau < 2T$. For values of τ such that $2T < \tau < 4T$, the third term has to be included, for $4T < \tau < 6T$ the fourth term as well, and so on.

The next task is to find the Fourier inverses of successive terms in (14). Considering the first term, it may be noted that

$$\{2(1-B) + C(1-B)^2\}$$

is $O(\omega)$ near the origin, but its real part behaves like ω^4, and it is therefore clear that the real part must be taken before the expression can be inverted. Also since only real parts are involved, the required Fourier transform $f(\tau)$ of a function $F(\omega)$ is given by

$$\int_0^\infty f(\tau) \cos \omega\tau \, d\tau = F(\omega) . \tag{16}$$

The inverse of products of functions may be derived using the convolution theorem, as follows: if

$$F(\omega) = \int_0^\infty f(\tau) \cos \omega\tau \, d\tau ,$$

and

$$G(\omega) = \int_0^\infty g(\tau) \cos \omega\tau \, d\tau ,$$

then

$$F(\omega)\, G(\omega) = \int_0^\infty \int_0^\tau f(u)\, g(\tau-u)\, du \cos \omega\tau \, d\tau . \tag{17}$$

Cunningham and Hynd (J.Roy.Stat.Soc.(1946)) state that if $\bar{O}_1$ and $\bar{O}_2$ denote any two linear operators and if $X(t)$ is a random variable with variance σ_X^2 and normalized autocorrelation function ρ_X, then

$$\operatorname{cov}\,[\bar{O}_1 X(u);\ \bar{O}_2 X(v)] = \sigma_X^2\, \bar{O}_1\, \bar{O}_2\, \rho_X(u-v) . \tag{18}$$

As a particular case of this result, if $Y(t) = \int_0^t y(u)\, du$, then

$$\rho_Y(\tau) = \int_0^\tau du \int_0^u \rho_y(v)\, dv = \int_0^\tau (\tau-v)\, \rho(v)\, dv . \tag{19}$$

Now (18) states that if $Y(t) = \int_0^t y(u)\, du$ then

$$S_Y(\omega) = -\frac{S_y(\omega)}{\omega^2} ;$$

and it therefore follows that the autocorrelation function which is the inverse of $\frac{S_y(\omega)}{\omega^2}$ may be obtained from the inverse of $S_y(\omega)$ by double integration. The functions whose inverses are the successive terms of

(15) are found by a series of convolutions of simple forms and double integrations. The steps in the calculation are most easily seen by setting out the various functions with their inverses, related by equation (16) in a table. In this table δ is the Dirac delta function defined

by $\delta(x) = 0, \quad x \neq 0; \quad \int_{\infty}^{\infty} \delta(x)\, dx = 1.$

$F(\omega) = \int_0^\infty f(\tau) \cos \omega\tau \, d\tau$	$f(\tau)$
$R(1-B) = 1 - \cos \omega T$	$\delta(\tau) - \delta(T-\tau)$
$\frac{1}{\omega^2} R(1-B)$	$\begin{cases} T-\tau & 0<\tau<T \\ 0 & \tau > T \end{cases}$
$R(C) = \frac{\beta^2}{\beta^2 + \omega^2}$	$\beta e^{-\beta t}$
$R(1-B)^2$	$\delta(\tau) - 2\delta(T-\tau) + \delta(2T-\tau)$
$\frac{1}{\omega^2} R(1-B)^2$	$\begin{cases} -\tau & 0<\tau<T \\ \tau-2T & T<\tau<2T \\ 0 & \tau > 2T \end{cases}$
$\frac{1}{\omega^2} R\,[C(1-B)^2]$	$\begin{cases} -\tau + \frac{1}{\beta}(1 - e^{-\beta\tau}) & 0<\tau<T \\ -e^{-\beta\tau}\frac{1}{\beta}(1-2e^{\beta T}) - (2T+\frac{1}{\beta}) + \tau & T<\tau<2T \\ -e^{-\beta\tau}\frac{1}{\beta}(1-2e^{\beta T} + e^{2\beta T}) & \tau > 2T \end{cases}$
$\frac{1}{\omega^2} R\,[2(1-B) + C(1-B)^2]$	$\begin{cases} 2T - 3\tau + \frac{1}{\beta}(1 - e^{-\beta\tau}) & 0<\tau<T \\ -e^{-\beta\tau}\frac{1}{\beta}(1-2e^{\beta T}) - (2T+\frac{1}{\beta}) + \tau & T<\tau<2T \\ -e^{-\beta\tau}\frac{1}{\beta}(1-2e^{\beta T} + e^{2\beta T}) & \tau > 2T \end{cases}$
$\frac{1}{\omega^4} R\,[2(1-B) + C(1-B)^2]$	$\begin{cases} -\frac{1}{\beta^3} + \frac{T^2}{\beta} + \frac{2}{3}T^3 + \frac{1}{\beta^2}\tau - \frac{1}{2}(2T+\frac{1}{\beta})\tau^2 + \frac{1}{2}\tau^3 + \frac{1}{\beta^3}e^{-\beta\tau} & 0<\tau<T \\ \frac{1}{\beta^3} + \frac{2T}{\beta^2} + \frac{2T^2}{\beta} + \frac{4}{3}T^3 - \tau\left(\frac{1}{\beta^2} + 2T^2 + \frac{2T}{\beta}\right) + \frac{\tau^2}{2}(2T+\frac{1}{\beta}) - \frac{1}{6}\tau^3 + \frac{1}{\beta^3}(1-2e^{\beta T})\, e^{-\beta\tau} & T<\tau<2T \end{cases}$

$F(\omega) = \int_0^\infty f(\tau)\cos\omega\tau\, d\tau$	$f(\tau)$
$\frac{1}{\omega^4}\, R\, [2(1-B)+C(1-B)^2]$	$\frac{1}{\beta^3}\,(1-2e^{\beta T}+e^{2\beta T})\, e^{-\beta\tau} \qquad \tau > 2T$
$R\, [B(1-B)^2]$	$\delta(T-\tau) - 2\delta(2T-\tau) + \delta(3T-\tau)$
$\frac{1}{\omega^2}\, R\, [B(1-B)^2]$	$\begin{cases} 0 & 0<\tau<T \\ T-\tau & T<\tau<2T \\ \tau-3T & 2T<\tau<3T \\ 0 & \tau > 3T \end{cases}$
$\frac{1}{\omega^2}\, R\, [BC^2(1-B)^2]$	$\begin{cases} 0 & 0<\tau<T \\ e^{-\beta\tau}\{e^{\beta T}(-\frac{2}{\beta}+T) - e^{\beta T}\tau\} + (T+\frac{2}{\beta}) - \tau & T<\tau<2T \\ e^{-\beta\tau}\{e^{\beta T}(-\frac{2}{\beta}+T) + e^{2\beta T}4(\frac{1}{\beta}-T) & \\ \quad +\tau(-e^{\beta T}+2e^{2\beta T})\} - (3T+\frac{2}{\beta}) + \tau & 2T<\tau<3T \\ e^{-\beta\tau}\{e^{\beta T}(-\frac{2}{\beta}+T) + e^{2\beta T}4(\frac{1}{\beta}-T) & \\ \quad + e^{3\beta T}(-\frac{2}{\beta}+3T) & \\ \quad + \tau(-e^{\beta T}+2e^{2\beta T}-e^{3\beta T})\} & \tau > 3T \end{cases}$
$\frac{1}{\omega^2}\, R\, [B^2(1-B)^2]$	$\begin{cases} 0 & 0<\tau<2T \\ 2T-\tau & 2T<\tau<3T \\ \tau-4T & 3T<\tau<4T \\ 0 & \tau > 4T \end{cases}$
$\frac{1}{\omega^2}\, R\, [B^2C^3(1-B)^2]$	$\begin{cases} 0 & 0<\tau<2T \\ e^{-\beta\tau}\{e^{2\beta T}(-2\beta T^2+4T-\frac{3}{\beta}) + e^{2\beta T}(2\beta T-2)\tau & \\ \quad - e^{2\beta T}\,\frac{\beta}{2}\,\tau^2\} + (2T+\frac{3}{\beta}) - \tau & 2T<\tau<3T \\ e^{-\beta\tau}\{e^{2\beta T}(-2\beta T^2+4T-\frac{3}{\beta}) + e^{3\beta T}(9\beta T^2-12T+\frac{6}{\beta}) & \\ \quad + [e^{2\beta T}(2\beta T-2) + e^{3\beta T}(-6\beta T+4)]\tau & \\ \quad + \frac{\beta}{2}\,(-e^{2\beta T}+2e^{3\beta T})\tau^2\} & \\ \quad - (\frac{3}{\beta}+4T) + \tau & 3T<\tau<4T \end{cases}$

$F(\omega) = \int_0^\infty f(\tau)\cos\omega\tau \, d\tau$	$f(\tau)$
$\frac{1}{\omega^2} R \; [B^2C^3(1-B)^2]$	$\begin{cases} e^{-\beta\tau}\{e^{2\beta T}(-2\beta T^2 + 4T - \frac{3}{\beta}) + e^{3\beta T}(9\beta T^2 - 12T + \frac{6}{\beta}) \\ \quad + e^{4\beta T}(-8\beta T^2 + 8T - \frac{3}{\beta}) + \tau[e^{2\beta T}(2\beta T - 2) \\ \quad - e^{3\beta T}(-6\beta T + 4) + e^{4\beta T}(4\beta T - 2)] \\ \quad + \frac{\beta\tau^2}{2}(-e^{2\beta T} + 2e^{3\beta T} - e^{4\beta T})\} & \tau > 4T \end{cases}$
$\frac{1}{\omega^2} R \; [C(1-B)^2(1-BC)^2]$	$\begin{cases} -\frac{1}{\beta} e^{-\beta\tau} + \frac{1}{\beta} - \tau & 0 < \tau < T \\ e^{-\beta\tau}\{-\frac{1}{\beta} + e^{\beta T}(-2T + \frac{6}{\beta}) + 2e^{\beta T}\tau\} \\ \quad - (4T + \frac{5}{\beta}) + 3\tau & T < \tau < 2T \\ e^{-\beta\tau}\{-\frac{1}{\beta} + e^{\beta T}(-2T + \frac{6}{\beta}) + e^{2\beta T}(-2\beta T^2 + 12T - \frac{12}{\beta}) \\ \quad + \tau[2e^{\beta T} + e^{2\beta T}(2\beta T - 6)] - e^{2\beta T}\frac{\beta}{2}\tau^2\} \\ \quad + (8T + \frac{7}{\beta}) - 3\tau & 2T < \tau < 3T \\ e^{-\beta\tau}\{-\frac{1}{\beta} + e^{\beta T}(-2T + \frac{6}{\beta}) + e^{2\beta T}(-2\beta T^2 + 12T - \frac{12}{\beta}) \\ \quad + e^{3\beta T}(9\beta T^2 - 18T + \frac{10}{\beta}) \\ \quad + \tau[2e^{\beta T} + e^{2\beta T}(-6 + 2\beta T) + e^{3\beta T}(-6\beta T + 6)] \\ \quad + \tau^2[-\frac{\beta}{2}e^{2\beta T} + \beta e^{3\beta T}]\} \\ \quad - (\frac{3}{\beta} + 4T) + \tau & 3T < \tau < 4T \\ e^{-\beta\tau}\{-\frac{1}{\beta} + e^{\beta T}(-2T + \frac{6}{\beta}) + e^{2\beta T}(-2\beta T^2 + 12T - \frac{12}{\beta}) \\ \quad + e^{3\beta T}(9\beta T^2 - 18T + \frac{10}{\beta}) + e^{4\beta T}(-8\beta T^2 + 8T - \frac{3}{\beta}) \\ \quad + \tau[2e^{\beta T} + e^{2\beta T}(-6 + 2\beta T) + e^{3\beta T}(-6\beta T + 6) \\ \quad + e^{4\beta T}(4\beta T - 2)] + \frac{\beta\tau^2}{2}(-e^{2\beta T} + 2e^{3\beta T} \\ \quad - e^{4\beta T})\} & \tau > 4T \end{cases}$

$F(\omega) = \int_0^\infty f(\tau) \cos \omega\tau \, d\tau$	$f(\tau)$
$\frac{1}{\omega^4} R \left[C(1-B)^2(1-BC)^2\right]$	$\begin{cases} -\frac{1}{\beta^3}+\frac{1}{\beta^2}\tau-\frac{1}{2\beta}\tau^2+\frac{1}{6}\tau^3+\frac{1}{\beta^3}e^{-\beta\tau} & 0<\tau<T \\ 9\frac{1}{\beta^3}+8\frac{T}{\beta^2}+3\frac{T^2}{\beta}+\frac{2}{3}T^3 - \tau\left(7\frac{1}{\beta^2}+6\frac{T}{\beta}+2T^2\right)+\frac{1}{2}\tau^2\left(4T+\frac{5}{\beta}\right) - \frac{1}{2}\tau^3+\frac{1}{\beta^2}e^{-\beta\tau}\left\{\frac{1}{\beta}+e^{\beta T}\left(2T-\frac{10}{\beta}\right) - 2\tau e^{\beta T}\right\} & T<\tau<2T \\ -18\frac{1}{\beta^3}-30\frac{T}{\beta^2}-21\frac{T^2}{\beta}-\frac{22}{3}T^3 + \tau\left(12\frac{1}{\beta^2}+18\frac{T}{\beta}+10T^2\right) - \frac{1}{2}\tau^2\left(8T+\frac{7}{\beta}\right) + \frac{1}{2}\tau^3+\frac{1}{\beta^2}e^{-\beta\tau}\left\{\frac{1}{\beta}+e^{\beta T}\left(2T-\frac{10}{\beta}\right) + e^{2\beta T}\left(2\beta T^2-16T+\frac{27}{\beta}\right) + \tau\left[-2e^{\beta T}+e^{2\beta T}(-2\beta T+8)\right] + \frac{1}{2}\beta\tau^2 e^{2\beta T}\right\} & 2T<\tau<3T \\ 10\frac{1}{\beta^3}+24\frac{T}{\beta^2}+24\frac{T^2}{\beta}+\frac{32}{3}T^3-\tau\left(6\frac{1}{\beta^2}+12\frac{T}{\beta}+8T^2\right) + \frac{\tau^2}{2}\left(4T+\frac{3}{\beta}\right)-\frac{1}{6}\tau^3+\frac{1}{\beta^2}e^{-\beta\tau}\left\{\frac{1}{\beta}+e^{\beta T}\left(2T-\frac{10}{\beta}\right) + e^{2\beta T}\left(2\beta T^2-16T+\frac{27}{\beta}\right)+e^{3\beta T}\left(-9\beta T^2+30T-\frac{28}{\beta}\right) + \tau\left[-2e^{\beta T}+e^{2\beta T}(-2\beta T+8)+e^{3\beta T}(6\beta T-10)\right] + \tau^2\left(\frac{1}{2}\beta e^{2\beta T}-\beta e^{3\beta T}\right)\right\} & 3T<\tau<4T \\ -\frac{1}{\beta^2}e^{-\beta\tau}\left\{\frac{1}{\beta}+e^{\beta T}\left(2T-\frac{10}{\beta}\right)+e^{2\beta T}\left(2\beta T^2-16T+\frac{27}{\beta}\right) + e^{3\beta T}\left(-9\beta T^2+30T+\frac{28}{\beta}\right)+e^{4\beta T}\left(8\beta T^2-16T+\frac{10}{\beta}\right) + \tau\left[-2e^{\beta T}+e^{2\beta T}(-2\beta T+8)+e^{3\beta T}(6\beta T-10) + e^{4\beta T}(-4\beta T+4)\right]+\tau^2\left[\frac{1}{2}\beta e^{2\beta T}-\beta e^{3\beta T} + \frac{1}{2}\beta e^{4\beta T}\right]\right\} & \tau > 4T \end{cases}$

Adding the inverse functions of $\frac{1}{\omega^4} R\ [2(1-B) + C(1-B)^2]$ and $\frac{1}{\omega^4} R\ [C(1-B)^2\ (1-BC)^2]$, gives a function proportional to the autocorrelation function for the Poisson target manoeuvre, valid for $|\tau| < 2T$, and since, by definition $\rho(0) = 1$,

$$\left.\begin{aligned}
\rho_{z_T}(\tau) &= 1 - \frac{\beta T}{(1+\frac{2}{3}\beta T)}\left(\frac{\tau}{T}\right)^2 + \frac{1}{3}\ \frac{\beta T}{(1+\frac{2}{3}\beta T)}\left(\frac{\tau}{T}\right)^3 \qquad 0<\tau<T \\
\rho_{z_T}(\tau) &= -\frac{1}{3}\cdot\frac{(24+18\beta T+3\beta^2T^2-2\beta^3T^3)}{\beta^2T^2\ (1+\frac{2}{3}\beta T)} + \frac{6+4\beta T}{\beta T\ (1+\frac{2}{3}\beta T)}\left(\frac{\tau}{T}\right) - \frac{2+\beta T}{1+\frac{2}{3}\beta T}\left(\frac{\tau}{T}\right)^2 \\
&\quad + \frac{1}{3}\cdot\frac{\beta T}{1+\frac{2}{3}\beta T}\left(\frac{\tau}{T}\right)^3 + \frac{8-2\beta T}{\beta^2T^2\ (1+\frac{2}{3}\beta T)}\ e^{-\beta(\tau-T)} \\
&\quad + \frac{2}{\beta T\ (1+\frac{2}{3}\beta T)}\left(\frac{\tau}{T}\right)\ e^{-\beta(\tau-T)}. \qquad T<\tau<2T
\end{aligned}\right\} \tag{20}$$

The autocorrelation function for target velocity $\dot{z}_T$ is available from the calculations leading to $\rho_z(\tau)$. Alternatively, in view of (18), it may be obtained by differentiating (20) twice with respect to τ.

Remembering that $\rho_{\dot{z}}(0) = 1$

$$\left.\begin{aligned}
\rho_{\dot{z}}(\tau) &= 1 - \left(\frac{\tau}{T}\right) \qquad 0<\tau<T \\
\rho_{\dot{z}}(\tau) &= 1 + \frac{2}{\beta T} - \left(\frac{\tau}{T}\right) + e^{-\beta(\tau-T)}\left\{\left(1-\frac{2}{\beta T}\right) - \left(\frac{\tau}{T}\right)\right\} \qquad T<\tau<2T\ .
\end{aligned}\right\} \tag{21}$$

The functions $\rho_z(\tau)$ and $\rho_{\dot{z}}(\tau)$ are graphed in Figs. 2 and 4. The values of the parameters are,

$$\gamma = 30^\circ,$$

$$s = 10 \text{ n. miles},$$

$$m = 10 \text{ n. miles},$$

$$V_T = 1{,}300 \text{ ft/sec},$$

so that

$$T = \frac{s}{V_T \sin\gamma} = 93.5 \text{ secs}$$

and

$$\beta T = \frac{s}{V_T \sin\gamma}\cdot\frac{V_T}{m} = 2\ .$$

We need to know the values of $k = \overline{z_T^{\,2}}$ and $k' = \overline{\dot{z}_T^{\,2}}$. From Fig. 1a we have, for n pulse pairs

$$z_T^{\,2} = s^2 \left\{ \int_0^T \left(\frac{t}{T}\right)^2 dt + \int_T^{T+x_1} 1\, dt + \int_{T+x_1}^{2T+x_1} \left(\frac{2T+x_1-t}{T}\right)^2 dt + \dots \right.$$

$$+ \int_{2(n-1)T+x_1+\dots+x_{n-1}+y_1+\dots+y_{n-1}}^{(2n-1)T+x_1+\dots+x_{n-1}+y_1+\dots+y_{n-1}}$$

$$\times \left[\frac{t-(2(n-1)T+x_1+\dots+x_{n-1}+y_1+\dots+y_{n-1})}{T}\right]^2 dt$$

$$+ \int_{(2n-1)T+x_1+\dots+x_{n-1}+y_1+\dots+y_{n-1}}^{(2n-1)T+x_1+\dots+x_n+y_1+\dots+y_{n-1}} 1\, dt$$

$$+ \int_{(2n-1)T+x_1+\dots+x_n+y_1+\dots+y_{n-1}}^{2nT+x_1+\dots+x_n+y_1+\dots+y_{n-1}}$$

$$\left. \times \left[\frac{2nT+x_1+\dots+x_n+y_1+\dots+y_{n-1}-t}{T}\right]^2 dt \right\}$$

$$= s^2 \left\{2n\,\frac{1}{3}\,T + x_1 + x_2 + \dots + x_n\right\}.$$

Dividing by the length of the sequence $= 2nT + x_1 + \dots + x_n + y_1 + \dots + y_n$, and averaging over all the x and y

$$\overline{z_T^{\,2}} = \frac{s^2}{2(T+\frac{1}{\beta})}\left(\frac{2}{3}\,T + \frac{1}{\beta}\right) = \frac{s^2}{2}\cdot\frac{(1+\frac{2}{3}\,\beta T)}{(1+\beta T)} = k^2. \qquad (22)$$

Again

$$\dot{z}_T^{\,2} = (V_T \sin\gamma)^2 \left\{ \int_0^T 1\, dt + \int_{T+x_1}^{2T+x_1} 1\, dt + \dots \right.$$

$$\left. + \int_{(2n-1)T+x_1+\dots+x_n+y_1+\dots+y_{n-1}}^{2nT+x_1+\dots+x_n+y_1+\dots+y_{n-1}} 1\, dt \right\}$$

$$= (V_T \sin\gamma)^2\, 2nT.$$

Dividing by the length of the sequence and averaging over all the x and y

$$\overline{\dot{z}_T^{\,2}} = \frac{(V_T \sin \gamma)^2}{T + \frac{1}{\beta}} T$$

$$= (V_T \sin \gamma)^2 \frac{\beta T}{1 + \beta T} = k'^2 . \qquad (23)$$

Finally, it is necessary to calculate the cross-correlation of $\dot{z}_T$ and z_T. Defining the cross-correlation function $\sigma_{\dot{y}y}$ of the stationary time series $y(t)$ by

$$\overline{y^2}\, \sigma_{\dot{y}y}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} \dot{y}_T(t)\, y_T(t+\tau)\, dt ,$$

then from (2)

$$y_T(t) = \int_{-\infty}^{\infty} A_T(f)\, e^{2\pi i f t} . df ,$$

so that

$$\dot{y}_T(t) = \int_{-\infty}^{\infty} 2\pi i f\, A_T(f)\, e^{2\pi i f t} . df .$$

Let

$$F(f) = \lim_{T \to \infty} \frac{1}{2T} \left(2\pi i f\, A_T(f)\, \overline{A_T(f)}\right)$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\left(\int_{-\infty}^{\infty} \dot{y}(t)\, e^{-2\pi i f t} . dt\right) \left(\int_{-\infty}^{\infty} y(s)\, e^{2\pi i f s} . ds\right)\right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\int_{-\infty}^{\infty} e^{2\pi i f \tau} \left\{\int_{-\infty}^{\infty} \dot{y}(t)\, y(t+\tau) dt\right\} d\tau\right]$$

$$= \int_{\infty}^{\infty} \overline{y^2}\, \sigma_{\dot{y}y}(\tau)\, e^{2\pi i f \tau} . d\tau ,$$

$$\therefore \qquad \overline{y^2}\,\sigma_{\dot{y}y}(\tau) = \int_{-\infty}^{\infty} F(f)\, e^{-2\pi i f\tau} \,.\, df$$

$$= \int_{-\infty}^{\infty} 2\pi i f \;\; S_y(f)\, e^{-2\pi i f\tau} \,.\, df \qquad \text{(from (3))},$$

$$= -\frac{d}{d\tau}\int_{-\infty}^{\infty} S_y(f)\, e^{-2\pi i f\tau} \,.\, df$$

$$= -\frac{d}{d\tau}\,\overline{y^2}\,\rho(\tau)\,. \qquad \text{(from (5))}.$$

Now any autocorrelation function necessarily decreases near the origin.

$$\rho(\tau) \sim 1 + \tfrac{1}{2}\,\rho''(0)\,\tau^2 \qquad \text{for small } \tau$$

$$\therefore \qquad \frac{d}{d\tau}\,\rho(\tau) \sim +\,\rho''(0)\,\tau \qquad \text{for small } \tau$$

and since $\rho(\tau) \leqslant 1$ always, $\rho''(0)$ is negative and we have as a perfectly general result that the cross-correlation function $\sigma_{\dot{y}y}(\tau)$ is positive for small positive τ irrespective of the form of $y(t)$.

Returning to the Poisson target manoeuvre, the cross-correlation of transverse velocity x displacement is obtained by differentiating (20) with respect to τ and changing the sign of the expression:

i.e.

$$\sigma_{\dot{z}z}(\tau) = \frac{2\beta}{(1+\frac{2}{3}\,\beta T)}\left(\frac{\tau}{T}\right) - \frac{\beta}{(1+\frac{2}{3}\,\beta T)}\left(\frac{\tau}{T}\right)^2 \qquad 0 < \tau < T$$

$$\sigma_{\dot{z}z}(\tau) = -\frac{6+4\beta T}{\beta T^2(1+\frac{2}{3}\,\beta T)} + \frac{2(2+\beta T)}{T(1+\frac{2}{3}\,\beta T)}\left(\frac{\tau}{T}\right) - \frac{\beta}{(1+\frac{2}{3}\,\beta T)}\left(\frac{\tau}{T}\right)^2$$

$$+ \frac{6-2\beta T}{\beta T^2(1+\frac{2}{3}\,\beta T)}\, e^{-\beta(\tau-T)} + \frac{2}{T(1+\frac{2}{3}\,\beta T)}\, e^{-\beta(\tau-T)}\left(\frac{\tau}{T}\right) \qquad T < \tau < 2T.$$

This function is graphed in Fig. 3 again with $T = 93.5$ secs, $\beta T = 2$.

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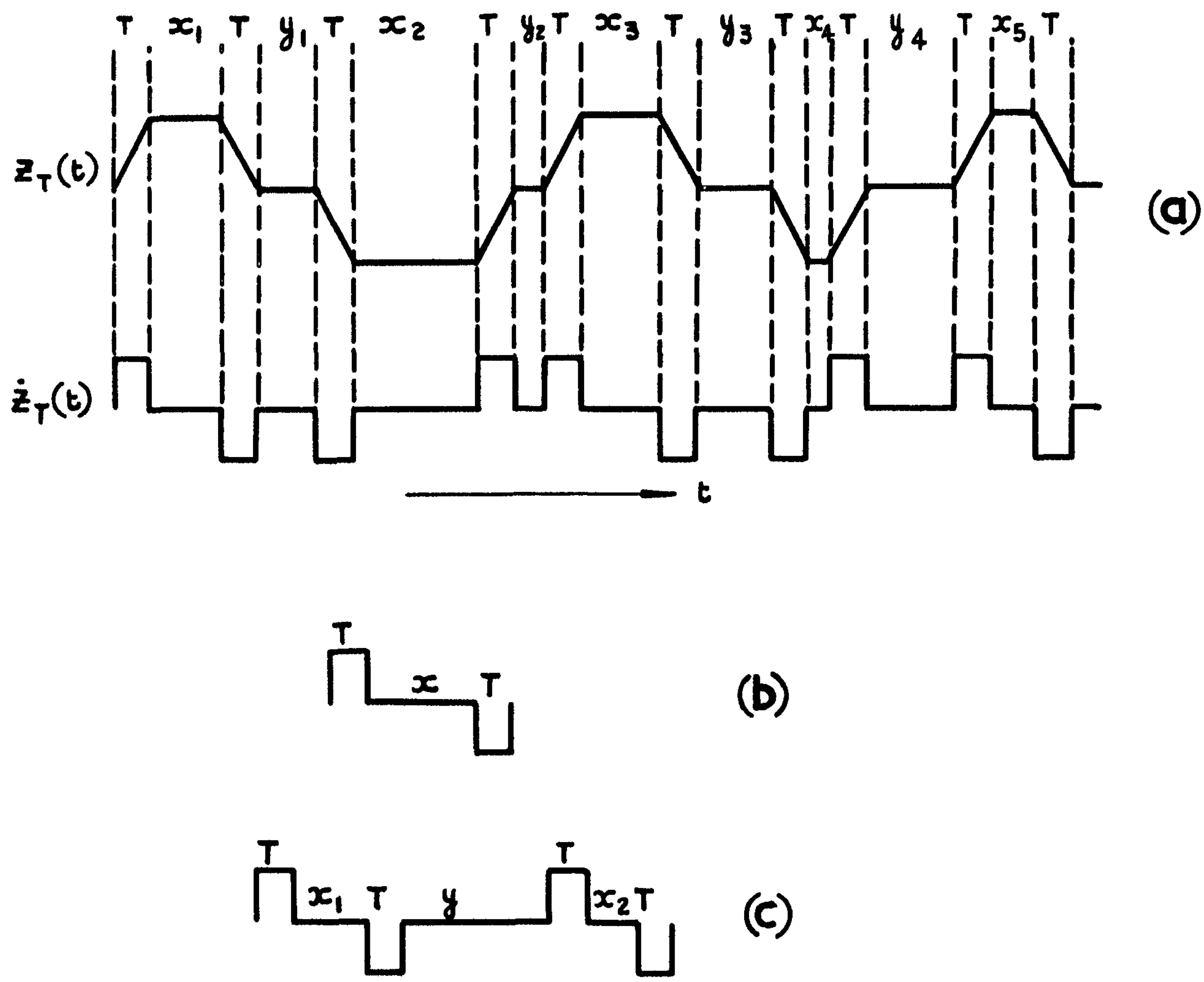

FIG. 1 (a-c) DIAGRAM OF z_T AND $\dot{z}_T$ AGAINST TIME FOR POISSON TARGET MANOEUVRE.

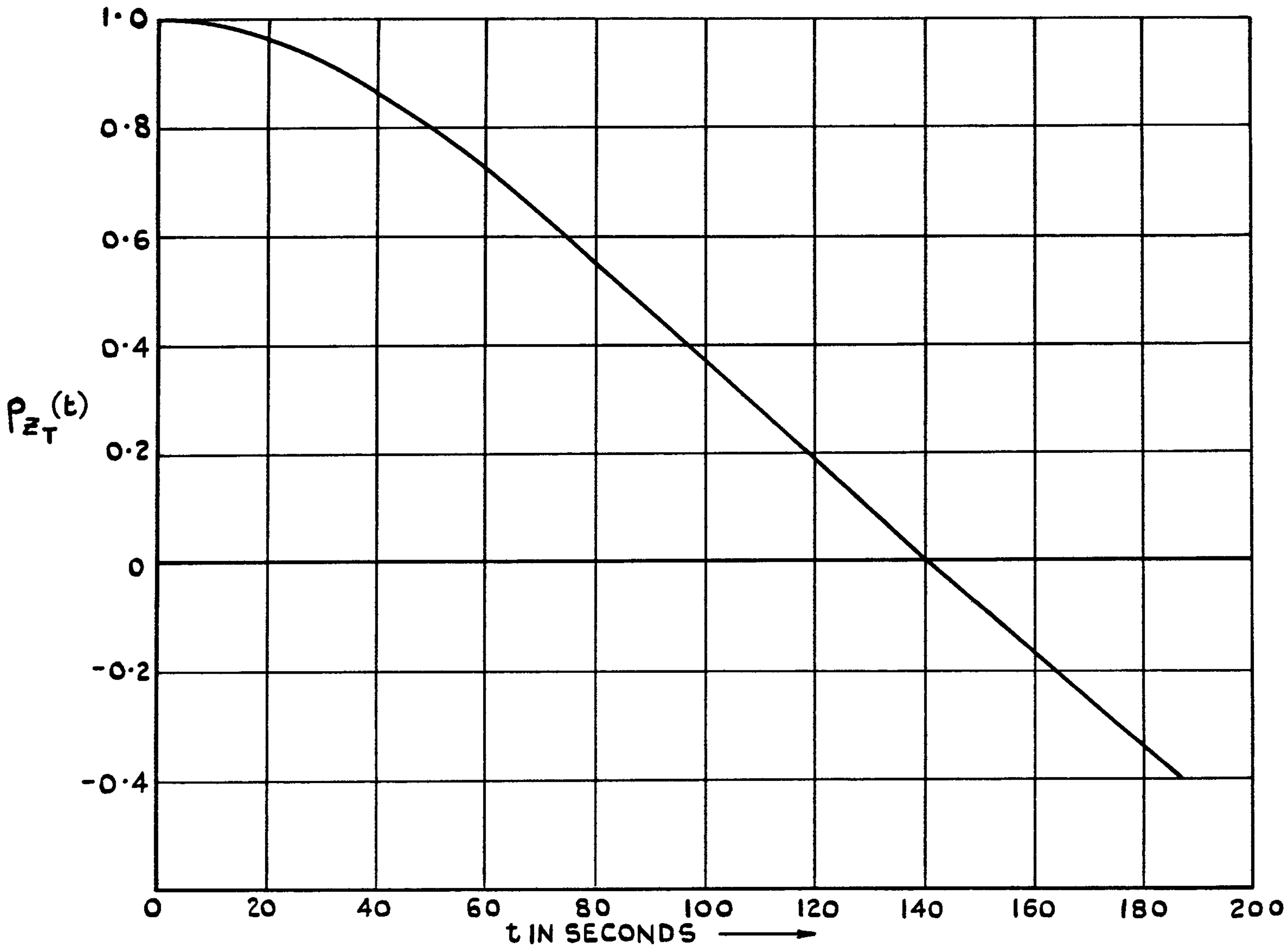

FIG. 2. AUTOCORRELATION FUNCTION OF POISSON TARGET MANOEUVRE DISPLACEMENT.

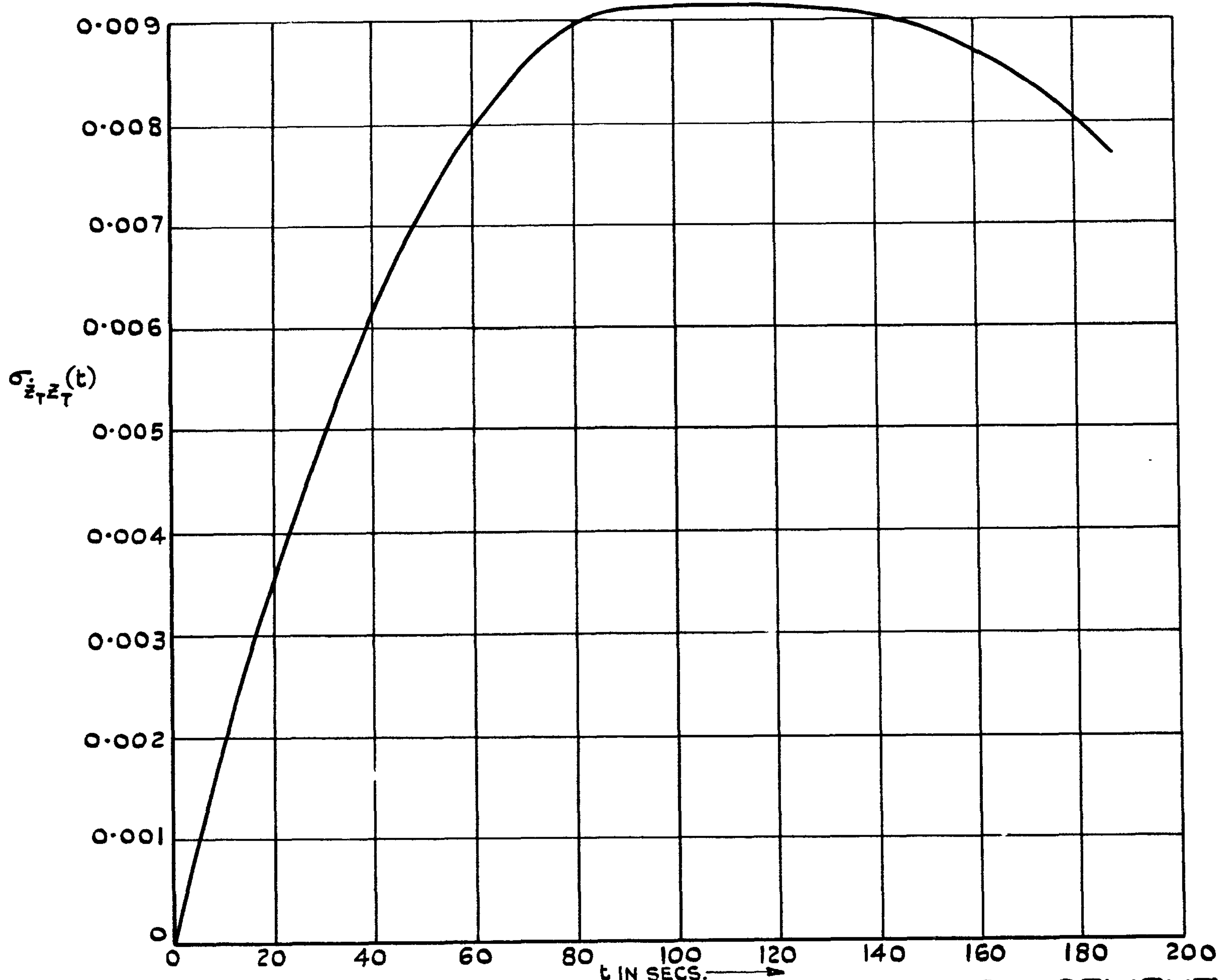

FIG. 3. CROSS-CORRELATION VELOCITY x DISPLACEMENT FUNCTION OF POISSON TARGET MANOEUVRE.

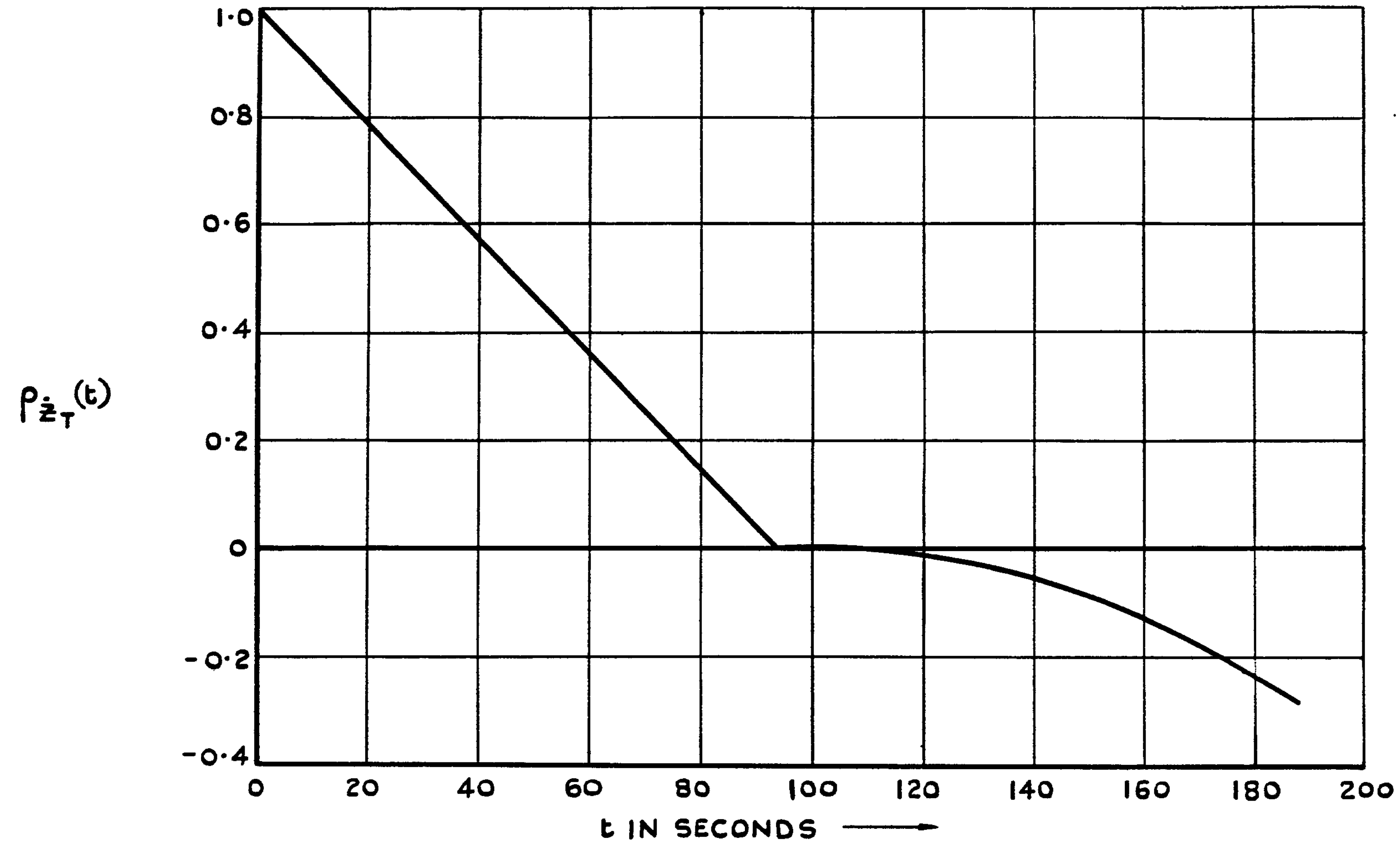

FIG. 4 AUTOCORRELATION FUNCTION OF POISSON TARGET MANOEUVRE VELOCITY.

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