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Some Numerical Methods for obtaining Harmonic and Subharmonic Solutions of Duffing's Equation

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Some Numerical Methods for obtaining Harmonic and Subharmonic Solutions of Duffing's Equation

By M. Newby, B.Sc.

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Summary.
Harmonic and subharmonic solutions are determined numerically for the forced oscillations of a system governed by Duffing's equation, and the stability of these oscillations is discussed.

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1. *Introduction.*

The non-linear ordinary differential equation

$$d^2x/dt^2 + \alpha x + \beta x^3 = E \cos \omega t \quad (1)$$

is the well-known Duffing's equation[1]. It describes the forced oscillations of several mechanical and electrical systems (e.g. a circuit consisting of an iron-core inductance in series with a linear capacitance driven by a source of alternating voltage).

There are many solutions of this equation[2], but the ones of interest here are the periodic oscillations which have the fundamental frequency equal to the frequency of the forcing term (harmonic response) or equal to an integral factor of it (subharmonic response).

Methods are discussed in this Report for determining the response curves for both the harmonic and subharmonic oscillations of (1), and for examining the stability of these oscillations. The response curves were determined numerically to check some theoretical work by Neumark[3].

Equation (1) is sometimes used to describe the forced oscillations of a mass on the end of a spring; in this case, if $\alpha > 0$, $\beta < 0$ then the spring is said to be 'soft', and if $\alpha > 0$, $\beta > 0$ then the spring is said to be 'hard'.

This report deals only with the case of a 'soft' spring, although the methods described can be applied to the case of a 'hard' spring.

Using Neumark's notation, since $\alpha > 0$, $\beta < 0$, equation (1) can be written as

$$d^2x/dt^2 + \omega_0^2 x - \omega_0^2 x^3/c^2 = \omega_0^2 f \cos \omega t\,, \quad (2)$$

where $\omega_0^2 = \alpha$, $c^2 = -\alpha/\beta, f = E/\alpha$.

The results in this Report were obtained using a digital computer; they agree closely with results obtained using an analogue computer[4,5].

2. *Determination of Response Curves for Harmonic Oscillations.*

2.1. *Equilibrium Positions.*

It is assumed that the solution of (2) can be written as a Fourier series

$$x = c \sum_{n=0}^{\infty} (a_n \cos n\,\omega t + b_n \sin n\,\omega t)\;;$$

substituting this expansion in (2) gives

$$a_n = 0 \text{ if } n \text{ is even}$$

and

$$b_n = 0 \text{ for all } n\,.$$

The solution of (2) can therefore be written simply as $x = c\sum a_n \cos n\,\omega t$ (n odd), with the a_n satisfying

$$-\omega^2 c \sum_{n=1}^{\infty} n^2 a_n \cos n\,\omega t + \omega_0^2 c \sum_{n=1}^{\infty} a_n \cos n\,\omega t - \omega_0^2 c \left(\sum_{n=1}^{\infty} a_n \cos n\,\omega t \right)^3 = \omega_0^2 f \cos \omega t\,.$$

Putting $a = c \sum_{n=1}^{\infty} a_n$ which, in most cases, is the amplitude of the oscillation, we obtain at $t = 0$

$$-\omega^2 c \sum_{n=1}^{\infty} n^2 a_n + \omega_0^2 a - \omega_0^2 a^3/c^2 = \omega_0^2 f,$$

or

$$(\omega/\omega_0)^2 = -(a^3/c^2 - a + f) \Big/ c \sum_{n=1}^{\infty} n^2 a_n . \tag{3}$$

When $\omega = 0$, corresponding to equilibrium, (3) gives

$$a^3 - ac^2 + fc^2 = 0 . \tag{4}$$

Equation (4) has three real roots if the discriminant D of (4) given by $D = -4(-c^2)^3 - 27 f^2 c^4$ is non-negative, i.e. if $|f/c| \leqslant 2\sqrt{3}/9$.

Hence, if $|f/c| < 2\sqrt{3}/9$, there are three static equilibrium positions given by the roots of (4). However, if $|f/c| > 2\sqrt{3}/9$, there is only one equilibrium position given by the real root of (4). Finally if $|f/c| = 2\sqrt{3}/9$, there are two equilibrium positions given by $a = c/\sqrt{3}$, which is a repeated root of (4), and $a = -2c/\sqrt{3}$.

The roots of (4) have been calculated for various values of f/c using Newton's method, and are given in the Table.

2.2. *Solution by Simple Iteration.*

In general, when $\omega \neq 0$, a solution of (2) can be found by iterating on an approximation x_n to the solution x. The iterative scheme is defined by

$$\ddot{x}_n + \omega_0^2 x_n = \omega_0^2 f \cos \omega t + \omega_0^2 x_{n-1}^2 / c^2 . \tag{5}$$

We first observe that if x is known to be a Fourier series of the form

$$x = c \sum_{i=1}^{\infty} a_i \cos i\,\omega t \ (i \text{ odd})$$

then we have

$$x^3 = c^3 \sum_i \sum_j \sum_k a_i\, a_j\, a_k \cos i\,\omega t \cos j\,\omega t \cos k\,\omega t \ ; \tag{6a}$$

but

$$\cos i\,\omega t \cos j\,\omega t \cos k\,\omega t = \tfrac{1}{4}\left[\cos(i+j+k)\,\omega t + \cos(i+j-k)\,\omega t + \cos(i-j+k)\,\omega t + \cos(i-j-k)\,\omega t\right] \tag{6b}$$

so that

$$x^3 = c^3 \sum_{i=1}^{\infty} b_i \cos i\,\omega t \ (i \text{ odd}),$$

and each b_i can be obtained in terms of the a_i's from (6).

Inserting these expressions for x_n, and x_{n-1}^3 into (5) gives the set of linear equations

$$\left.\begin{aligned} [-(\omega_n/\omega_0)^2+1]a_1^{(n)} &= f/c+b_1^{(n-1)} \\ [-(3\omega_n/\omega_0)^2+1]a_3^{(n)} &= b_3^{(n-1)} \\ [-(5\omega_n/\omega_0)^2+1]a_5^{(n)} &= b_5^{(n-1)} \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ [-(p\omega_n/\omega_0)^2+1]a_p^{(n)} &= b_p^{(n-1)}; \end{aligned}\right\} \tag{7}$$

where $a_k^{(n)}$ is the nth approximation to the coefficient a_k; ω_n the nth approximation to the frequency ω; and the infinite series for both x_n and x_{n-1}^3 have been truncated after $\frac{1}{2}(p+1)$ terms.

Now, if a_1 is given and held fixed throughout the iteration, ω_n can be calculated from the first equation of (7), which can be written as

$$(\omega_n/\omega_0)^2 = -f/ca_1 - b_1^{(n-1)}/a_1 + 1\,. \tag{8}$$

The iteration is started by giving a_1 a fixed value and specifying initial values for $a_3^{(0)},\ldots, a_p^{(0)}$. The initial values $b_1^{(0)},\ldots, b_p^{(0)}$ are then obtained from (6), and ω_1 from (8). Using this value of ω_1, $a_3^{(1)},\ldots, a_p^{(1)}$ are calculated from the remaining equations in (7). This process is continued until ω, $a_3,\ldots, a_p$ are all sufficiently accurate.

In practice, the iteration was carried out with $p = 9$, as it was found that there was no significant difference between the results with $p = 9$ and $p = 27$. Although this method converged for most values of a_1, it was found that in certain cases it did not. The regions in which this method fails to converge are shown shaded in Fig. 1. In these regions other methods were used and these are described in the following sections. Fig. 4 shows the lower parts of the harmonic response curves, and, as can be seen, certain solutions obtained by the above method do not lie on the expected response curves; this agrees with certain solutions obtained using an analogue computer[4].

2.3. *Particular Method for $\omega = \omega_0/m$.*

It is found that when $\omega = \omega_0/m$ (where m is an odd integer), the iterative scheme defined by (5) fails, due to the mth equation in (7) becoming

$$0\,.\,a_m^{(n)} = b_m^{(n-1)}\,. \tag{9}$$

From (9) it follows that for $\omega = \omega_0/m$,

$$b_m = 0 \text{ for all } n\,. \tag{10}$$

Now, since b_m is a Fourier coefficient of x^3, it follows that b_m is a cubic in a_m, where a_m is the corresponding Fourier coefficient of x. Hence (10) can be written as

$$Aa_m^3+Ba_m^2+Ca_m+D = 0\,, \tag{11}$$

where A, B, C, D can be calculated from $a_1,\ldots, a_{m-2}, a_{m+2},\ldots, a_p$.

The linear equations corresponding to (7) are now given by

$$\left.\begin{aligned}
a_1^{(n)} &= [f/c + b_1^{(n-1)}] \,/\, (1-m^2) \\
a_2^{(n)} &= b_3^{(n-1)} \,/\, (9-m^2) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
a_{m-2}^{(n)} &= b_{m-2}^{(n-1)} \,/\, (4-4m) \\
0 &= b_m \\
a_{m+2}^{(n)} &= b_{m+2}^{(n-1)} \,/\, (4+4m) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
a_p^{(n)} &= b_p^{(n-1)} \quad (p^2-m^2)\,.
\end{aligned}\right\} \tag{12}$$

A numerical solution is sought as follows: a first approximation x_0 is used to calculate the coefficients A, B, C, D in (11). Equation (11) is solved for a root $a_m^{(0)}$ which is used, together with the old values $a_1^{(0)}, \ldots,$ $a_{m-2}^{(0)}, a_{m+2}^{(0)}, \ldots, a_p^{(0)}$, to find the coefficients $b_1^{(0)}, \ldots, b_p^{(0)}$ and these are inserted into (12) to obtain a second approximation x_1. This process is continued until the coefficients a_k are of the required accuracy.

Using this modified iterative method, solutions hre obtained which lie on the lower parts of the response curves. However, when $\omega = \frac{1}{3}$, the solutions obtained do not lie on the expected response curves and an example of this kind of solution is shown in Fig. 7.

2.4. *Variation of Rauscher's Method.*

It is also found that when a_1 is increased beyond a certain value (approximately 0·8) the simple iterative procedure fails, with (8) giving a negative value for ω^2 throughout the iteration.

In order to determine these parts of the response curves, a variation of Rauscher's method[6] is used and the iteration now proceeds as follows.

The equation of free oscillation

$$\ddot{x} + \omega_0^2 x - \omega_0^2 x^3/c^2 = 0 \tag{13}$$

is integrated with the initial conditions $x(0) = a/c$, $\dot{x}(0) = 0$ where a is, in most cases, the amplitude of the oscillation. Assuming the motion to be periodic with frequency ω_1, the time taken to reach the point $x = 0$ is $\pi/2\omega_1$, from which ω_1 can be calculated.

The iteration is continued by integrating

$$\ddot{x}+\omega_0^2 x-\omega_0^2 x^3/c^2 = \omega_0^2 f\cos\omega_1 t$$

with the initial conditions $x(0) = a/c$, $\dot{x}(0) = 0$, and with ω_2 then given by $x(\pi/2\omega_2) = 0$. The iteration proceeds in this way until ω is of the required accuracy. The convergence of this method depends on f/c; if this is sufficiently small, then four or five iterations suffice. However, in general, convergence is very slow and a simple acceleration process is used. This process uses the three previous iterates to estimate a new value for ω which is given by

$$(\omega-\omega_{n+1})/(\omega-\omega_n) = (\omega_{n+1}-\omega_n)/(\omega_n-\omega_{n-1}),$$

where the three previous iterates are ω_{n-1}, ω_n, ω_{n+1}.

If $a > c$, it is known that (13) has no periodic solution[6], so in this case ω_1 is taken to be zero.

Rauscher's original method depends on the solution being monotonic over quarter of a period, but the present method does not suffer from this limitation and in fact solutions have been obtained which are non-monotonic over this range. However, if the first zero of the solution occurs after a time less than a quarter of a period, then the sequence $\{\omega_n\}$, obtained as described above, oscillates. In this case a solution can usually be found by assuming that the second zero occurs after a quarter of a period.

On the lower part of the response curve for $f > 0$, the above method failed to give any results with ω/ω_0 less than a certain value (approximately 0·25). It did, however, produce some unexpected solutions and one of these is shown in Fig. 11.

Fig. 3 shows the response curves for the harmonic oscillations, with a/c plotted against ω/ω_0 using f/c as parameter. As mentioned previously, Fig. 4 shows the lower parts of the response curves for $f > 0$, indicating which solutions were produced by the different methods. The most interesting feature of this figure is the set of solutions obtained using Rauscher's method with a less than the smaller positive root of (4).

3. *Determination of Response Curves for Subharmonic Oscillations.*

As well as oscillations having fundamental frequency equal to the forcing frequency, oscillations having fundamental frequency equal to an integral factor of the forcing frequency have been observed in systems governed by Duffing's equation. These are known as subharmonic oscillations.

Methods of solution similar to those described above were tried for these subharmonic oscillations. The methods, however, failed to converge in some cases and the corresponding response curves are thus incomplete.

Fig. 5 shows these response curves with a/c plotted against ω/ω_0 using f/c as parameter.

3.1. *Solution by Simple iteration.*

This method is similar to the one described in Section 2.2., except that (2) is now assumed to have a solution of the form

$$x = c\sum_{i=1}^{\infty} a_i \cos(i\,\omega t/m)\,,\ i, m \text{ odd},$$

where m is the order of the subharmonic.

The set of linear equations of the iteration corresponding to this solution is

$$\left.\begin{aligned}
[-(\omega_n/m\omega_0)^2+1]a_1^{(n)} &= b_1^{(n-1)} \\
[-(3\omega_n/m\omega_0)^2+1]a_3^{(n)} &= b_3^{(n-1)} \\
&\ \ \vdots \\
[-(\omega_n/\omega_0)^2+1]a_m^{(n)} &= f/c+b_m^{(n-1)} \\
&\ \ \vdots \\
[-(p\omega_n/m\omega_0)^2+1]a_p^{(n)} &= b_p^{(n-1)} .
\end{aligned}\right\} \tag{14}$$

To ensure that a subharmonic oscillation is obtained, a_1 is specified as a non-zero constant. ω_k can now be calculated from the first equation of (14), which is

$$(\omega_k/\omega_0)^2 = m^2(1-{}_1^{(k-1)}/\ a_1)\,. \tag{15}$$

Using this value of ω_k, the iteration proceeds, as described in Section 2.2., until ω_k, $a_1^{(k)},\ldots, a_p^{(k)}$ are sufficiently accurate.

3.2. *Variation of Rauscher's Method.*

This method is described in Section 2.4., and the only difference here is in the end condition in each iteration.

The general iteration proceeds as follows: the equation

$$\ddot{x}+\omega_0^2x-\omega_0^2x^3/c^2 = \omega_0^2 f\cos\omega_n t$$

is integrated with the initial conditions $x(0) = a/c$, $\dot{x}(0) = 0$ and with the end condition $x(m\pi/2\omega_{n+1}) = 0$, from which ω_{n+1} is determined. This process is continued until ω_n is sufficiently accurate. However, in many cases, it is found in practice that the sequence $\{\omega_n\}$ oscillates, and the method, mentioned in Section 2.4., has to be used to obtain solutions in this case.

4. *Some Peculiar Solutions.*

In the response curves, the ordinate a/c is usually the maximum displacement of the motion. However for some solutions it was observed that this was not the case. The conditions under which these solutions arise are discussed below.

Suppose

$$x = c\sum_{n=1}^{\infty} a_n \cos(n\omega t/m)\,,\ m = 1,3,5,\ n \text{ odd},$$

is a solution of (2), then

$$\ddot{x} = -\left(c\omega^2 \sum_{n=1}^{\infty} n^2 a_n \cos(n\omega t/m)\right) \Big/ m^2 .$$

At $t = 0$,

$$x = c \sum_{n=1}^{\infty} a_n = a \text{ (assumed always positive)}$$

and

$$\ddot{x} = -\left(c\omega^2 \sum_{n=1}^{\infty} n^2 a_n\right) \Big/ m^2 .$$

For A to be the maximum displacement, we require $\ddot{x} < 0$, and thus $\sum_{n=1}^{\infty} n^2 a_n > 0$.
Now, inserting the expressions for $\ddot{x}$ and x into (2) for $t = 0$ gives

$$(\omega/m\omega_0)^2 = -(a^3/c^2 - a + f)/c \sum_{n=1}^{\infty} n^2 a_n , \tag{16}$$

and ω exists if and only if the numerator and denominator of (16) have different signs.

Hence, if a is to be the maximum displacement we require

$$a^3/c^2 - a + f < 0 .$$

If a_2, a_3 are the two positive roots of (4) with $a_2 \leqslant a_3$ for $0 < f 2c\sqrt{3}/9$, and if a_1 is the positive root of (4) for $f < 0$, then it follows that a is the maximum displacement if $a_2 < a < a_3$ for the case when $0 < f < 2c\sqrt{3}/9$, and if $a < a_1$ for the case $f < 0$.

It can therefore be seen that a is not the maximum displacement in the following cases:

(a) when $0 < f < 2c\sqrt{3}/9$, a is not the maximum displacement if $0 < a < a_2$ or $a > a_3$;

(b) when $f > 2c\sqrt{3}/9$ (subharmonics only), is never the maximum displacement; and

(c) if $f < 0$, then a is not the maximum displacement if $a > a_1$.

However, in some cases (e.g. subharmonics with $F < 0$) the point $t = 0$, $x = a$ is a local maximum, but a is not the maximum displacement (*see* Fig. 10).

Not all of the above oscillations lie on the expected response curves; examples of this and other oscillations satisfying (2) are given in Figs. 6 to 11.

5. *Stability of Oscillations.*

Before discussing the stability of oscillations, it is necessary to define what is meant by stability in this context.

Let $x(t)$ be a solution of (2), and $x(t) + u(t)$ be a slightly perturbed solution, then inserting $x(t) + u(t)$ into (2) and neglecting powers of u above the first gives a linear differential equation in u. If all solutions $u(t)$ of this equation are bounded, then $x(t)$ is said to be stable, otherwise it is said to be unstable.

In the case of Duffing's equation (2) the related linear equation is

$$\ddot{u} + \omega_0^2(1 - 3x^2/c^2)u = 0 . \tag{17}$$

Now put

$$x = c \sum_{n=1}^{\infty} a_n \cos n\omega t \, , \; n \text{ odd},$$

so that

$$x^2 = c^2 \sum_{n=0}^{\infty} b_n \cos 2n\, \omega t \, ,$$

where the b_n can be obtained from the a_n. Putting $z = \omega t$ and the series for x^2 into (17) gives

$$d^2u/dz^2 + \omega_0^2 \left(1 - 3 \sum_{n=0}^{\infty} b_n \cos 2nz\right) u/\omega^2 = 0 \, .$$

Further, let

$$d_0 = \omega_0^2(1-3b_0)/\omega^2$$

and

$$d_n = -3\omega_0^2 b_n/2\omega^2 \, , \; n \neq 0,$$

so that finally we have

$$d^2u/dz^2 + \left(d_0 + 2 \sum_{n=1}^{\infty} d_n \cos 2nz\right) u = 0 \, . \qquad (18)$$

This is a form of Hill's equation.

5.1. *Hill's Method of Solution.*

The method used to examine the stability is based on a method used by Hill[7] to find a solution of (18). Defining $d_{-n} = d_n$, we suppose

$$u = e^{\mu z} \sum_{n=-\infty}^{\infty} v_n \, e^{2niz}$$

is a solution of (18). Substituting this in (18) gives

$$\sum_{n=-\infty}^{\infty} (\mu + 2ni)^2 v_n e^{(\mu+2ni)z} + \left(\sum_{n=-\infty}^{\infty} d_n e^{2niz}\right)\left(\sum_{n=-\infty}^{\infty} v_n e^{(\mu+2ni)z}\right) = 0 \, ,$$

multiplying out and equating coefficients of e^{2iz} gives the following system of equations

$$(\mu + 2ni)^2 v_n + \sum_{m=-\infty}^{\infty} d_m v_{n-m} = 0 \; (n = \ldots, -2, -1, 0, 1, 2, \ldots) \, .$$

For the existence of a non-trivial solution of this system, the matrix of coefficients of the v_n must be singular. After dividing each equation of the system by $(d_0 - 4n^2)$ to secure convergence the following infinite determinant is obtained (known as Hill's determinant):

$$
\Delta(i\mu) = \begin{vmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \frac{(i\mu+4)^2-d_0}{16-d_0} & \frac{-d_1}{16-d_0} & \frac{-d_2}{16-d_0} & \frac{-d_3}{16-d_0} & \frac{-d_4}{16-d_0} & \cdots \\
\cdots & \frac{-d_1}{4-d_0} & \frac{(i\mu+2)^2-d_0}{4-d_0} & \frac{-d_1}{4-d_0} & \frac{-d_2}{4-d_0} & \frac{-d_3}{4-d_0} & \cdots \\
\cdots & \frac{-d_2}{-d_0} & \frac{-d_1}{-d_0} & \frac{(i\mu)^2-d_0}{-d_0} & \frac{-d_1}{-d_0} & \frac{-d_2}{-d_0} & \cdots \\
\cdots & \frac{-d_3}{4-d_0} & \frac{-d_2}{4-d_0} & \frac{-d_1}{4-d_0} & \frac{(i\mu-2)^2-d_0}{4-d_0} & \frac{-d_1}{4-d_0} & \\
\cdots & \frac{-d_4}{16-d_0} & \frac{-d_3}{16-d_0} & \frac{-d_2}{16-d_0} & \frac{-d_1}{16-d_0} & \frac{(i\mu-4)^2-d_0}{16-d_0} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{vmatrix}
$$

The equation for μ is therefore

$$\Delta(i\mu) = 0\,. \tag{19}$$

It can be shown[7] that

$$\Delta(i\mu) = \Delta(0) - \sin^2(\tfrac{1}{2}\pi i\mu)/\sin^2(\tfrac{1}{2}\pi d_0^{\frac{1}{2}})\,,$$

and therefore (19) is equivalent to

$$\sin^2(\tfrac{1}{2}\pi i\mu) = \Delta(0)\sin^2(\tfrac{1}{2}\pi d_0^{\frac{1}{2}})\,. \tag{20}$$

Now, for the solution u of (18) to be bounded for all t, $Re\,(\mu) \equiv 0$ so that $\frac{1}{2}\pi iu$ must be real, and hence

$$0 \leqslant \sin^2(\tfrac{1}{2}\pi i\mu) \leqslant 1\,,$$

which from (20) can be expressed as

$$0 \leqslant \Delta(0)\sin^2(\tfrac{1}{2}\pi d_0^{\frac{1}{2}}) \leqslant 1\,.$$

Assuming $\Delta(0) > 0$ (which it was always found to be in practice), this becomes

$$0 \leqslant \sin^2(\tfrac{1}{2}\pi d_0^{\frac{1}{2}}) \leqslant 1/\Delta(0)\,.$$

If $\frac{1}{2}\pi d_0^{\frac{1}{2}}$ is real, than $\sin^2(\frac{1}{2}\pi d_0^{\frac{1}{2}}) \geqslant 0$. However, if $\frac{1}{2}\pi d_0^{\frac{1}{2}}$ is pure imaginary then $\sin^2(\frac{1}{2}\pi d_0^{\frac{1}{2}}) < 0$; therefore u is unbounded if $d_0 < 0$, i.e. if $\omega_0^2(1-3b_0)/\omega^2 < 0$, or $3b_0 > 1$. Now, from above, we have

$$b_0 = \tfrac{1}{2}\sum_{n=1}^{\infty} a_n^2\,,\ n \text{ odd},$$

therefore u is unbounded if

$$\sum_{n=1}^{\infty} a_n^2 \leqslant 2/3\,. \tag{21}$$

The other condition states that u is bounded if

$$\sin^2(\tfrac{1}{2}\pi\, d_0^{\frac{1}{2}}) \leqslant 1/\Delta(0)$$

or

$$-(\Delta(0))^{-\frac{1}{2}} \leqslant \sin(\pi\omega_0(1-3b_0)^{\frac{1}{2}}/2\omega) \leqslant (\Delta(0))^{-\frac{1}{2}}\,. \tag{22}$$

When $a_1 = a_3 = \ldots. = 0$, then $b_0 = 0$, $\Delta(0) = 1$ and $a = 0$. These conditions will give the limits of the unstable regions as $a \to 0$. Equation (22) is now

$$-1 \leqslant \sin(\pi\omega_0/2\omega) \leqslant 1\,.$$

Limits of the unstable regions are therefore given by

$$\sin(\pi\omega_0/2\omega) = \pm 1\,,$$

or

$$\omega/\omega_0 = 1/(2p+1)\,, p = \dots, -2, -1, 0, 1, 2, \dots$$

When the coefficients a_n have been obtained by the simple iterative method above (Section 2.2.), the quantities b_0, b_1, b_2,.... can be determined, and hence $\Delta(0)$ evaluated. These values are then used to see if either of the conditions (21), (22) are violated. If they are not, then the oscillation is stable; if they are, then it is unstable.

5.2. *Use of Floquet Theory.*

As stated previously, for a particular oscillation of (2) to be stable, all solutions of the corresponding Hill's equation (18) must be bounded.

Since (18) is linear, every solution of it is a linear combination of its fundamental solutions[8].

Consider the two fundamental solutions $u_1(z)$, $u_2(z)$ of (18), and let us suppose they have the initial conditions

$$\left.\begin{array}{ll} u_1(0) = 1, & u_1'(0) = 0 \;\; (a) \\ u_2(0) = 0, & u_2'(0) = 1, \;\; (b) \end{array}\right\} \qquad (23)$$

where $'$ denotes differentiation *w.r.t.* z. Then

$$u_1(z+\pi) = u_1(\pi)\,u_1(z) + u_1'(\pi)\,u_2(z)$$

and

$$u_2(z+\pi) = u_2(\pi)\,u_1(z) + u_2'(\pi)\,u_2(z)\,,$$

or, in matrix notation,

$$\begin{pmatrix} u_1(z+\pi) \\ u_2(z+\pi) \end{pmatrix} = U \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}, \qquad (24)$$

where

$$U = \begin{pmatrix} u_1(\pi) & u_1'(\pi) \\ u_2(\pi) & u_2'(\pi) \end{pmatrix}.$$

The characteristic polynomial of U is

$$\lambda^2 - (u_1(\pi) + u_2'(\pi))\lambda + \det U = 0\,,$$

$$\text{where } \det U = u_1(\pi)\,u_2'(\pi) - u_2(\pi)\,u_1'(\pi)\;;$$

but this latter is independent of z and hence we have $\det U = 1$ (the value at $z = 0$). The characteristic polynomial is therefore

$$\lambda^2 - D\lambda + 1 = 0\,,$$

where $D = u_1(\pi) + u_2'(\pi)$.

Hence, the eigenvalues of U are given by

$$\lambda = D/2 \pm (D^2/4 - 1)^{\frac{1}{2}}.$$

From (24), it can be seen that

$$\begin{pmatrix} u_1(z+n\pi) \\ u_2(z+n\pi) \end{pmatrix} = U^n \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}.$$

For the solutions of (18) to be bounded, the eigenvalues of U must have moduli less than or equal to unity. (In fact, the eigenvalues must have moduli equal to unity, since their product is unity.) That is, we require $|D| < 2$. Hence if $|D| < 2$, the corresponding oscillation is stable, but if $|D| \geqslant 2$, the oscillation is unstable.

Since the coefficients a_i are known from the simple iterative method, the b_i, and hence the d_i, may be determined. Equation (18) is now integrated for both sets of initial conditions (23); $u_1(\pi)$ and $u_2'(\pi)$ evaluated, and the value of D obtained is used to examine the stability of the motion.

5.3. *Discussion of Results.*

The two methods described above were used to examine the stability of the harmonic oscillations obtained previously by the method of Section 2.2.

When $\sum_{n=1}^{\infty} a_n^2 > 2/3$ (n odd), such oscillations are found to be unstable (*see* Section 5.1.), and this first occurs when $a_1 \doteq 0{\cdot}8$ which is near the limit of convergence of the simple iterative method.

Another unstable region was found to be bounded by the locus of vertical tangents of the response curves for $f > 0$, and by the response curve for $f = 0$. The region's limit point is $a/c = 0$, $\omega/\omega_0 = 1$, as expected from the results of Section 5.1. Other regions of unstable oscillations were found for the response curves for $f > 0$, and these regions have limit points $a/c = 0$, $\omega/\omega_0 = 1/n$ (n odd). Only a few solutions were found in these regions because in most cases the simple iterative method fails to converge. Rauscher's method will give solutions here, but these cannot be conveniently used to check stability, as the method defines the solution only as a numerical function of time, and does not provide the necessary Fourier coefficients.

As mentioned previously, the free oscillations of Duffing's equation are found to be unstable; this is due to the definition of stability used here: if Poincaré's definition is used instead, they are found to be orbitally stable. A full discussion of orbital stability may be found in Ref. 6.

The approximate boundaries of the regions of stability and instability are shown in Fig. 2; the unstable regions being shaded.

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TABLE

Static Equilibrium Positions of the Forced Oscillations of Duffing's Equation with 'Soft' Restoring Force.

f/c	Values of	a/c
−0·50		1·1915
−0·45		1·1759
−0·40		1·1597
−0·35		1·1429
−0·30		1·1254
−0·25		1·1072
−0·20		1·0880
−0·15		1·0679
−0·10		1·0467
−0·05		1·0241
0·00	0·0000	1·0000
0·05	0·0501	0·9740
0·10	0·1010	0·9457
0·15	0·1536	0·9143
0·20	0·2092	0·8789
0·25	0·2696	0·8376
0·30	0·3389	0·7865
0·35	0·4289	0·7140

The positive value of f/c for which the system has only one equilibrium position is $2\sqrt{3}/9 \simeq 0{\cdot}3849$. This position is given by $\omega/\omega_0 = 0$, $a/c = 1/\sqrt{3} \simeq 0{\cdot}5774$.

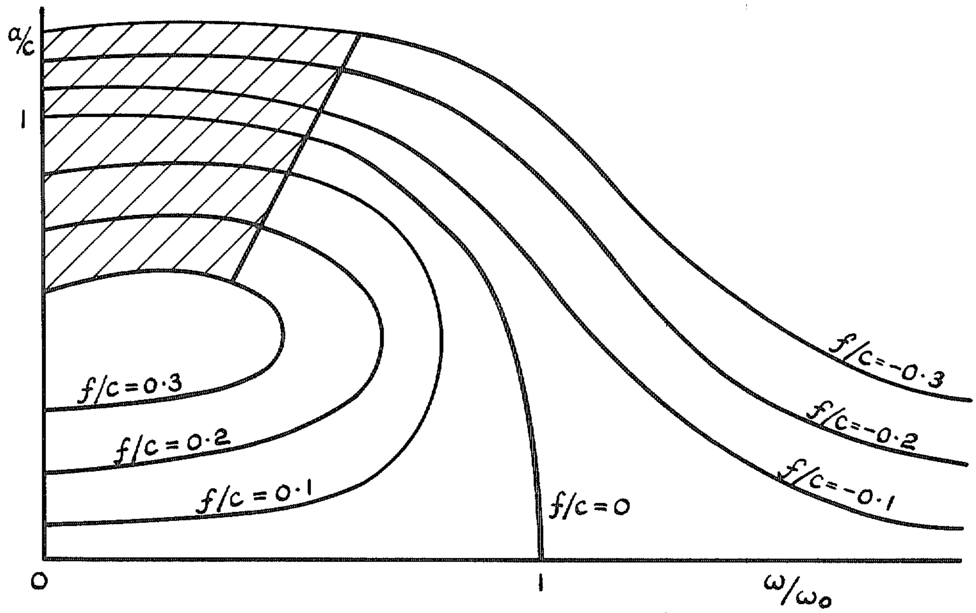

FIG. 1. Regions of convergence of simple iterative method. (Non-convergence in shaded region).

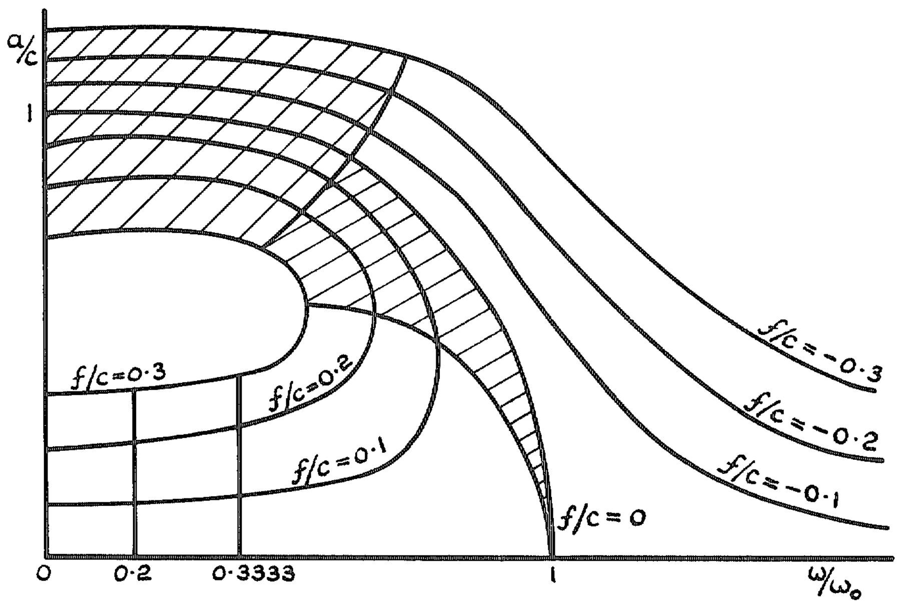

FIG. 2. Approximate boundaries of stability regions. (Unstable in shaded regions).

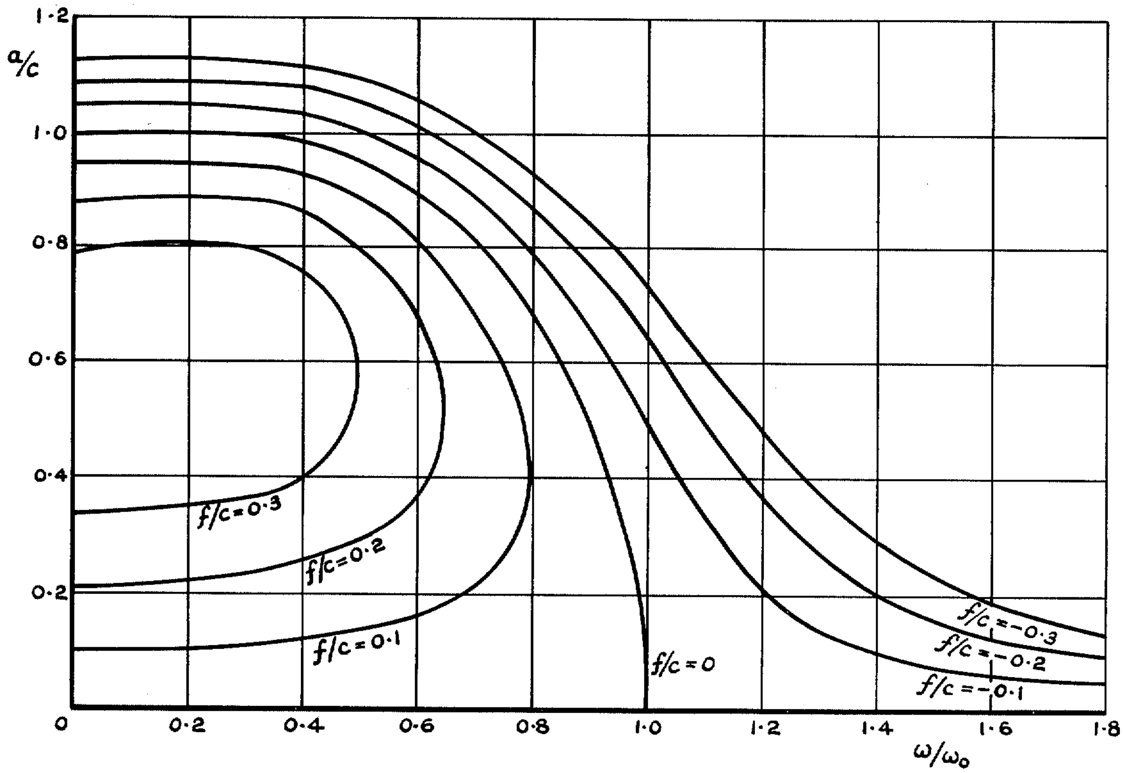

FIG. 3. Response curves for harmonic oscillations.

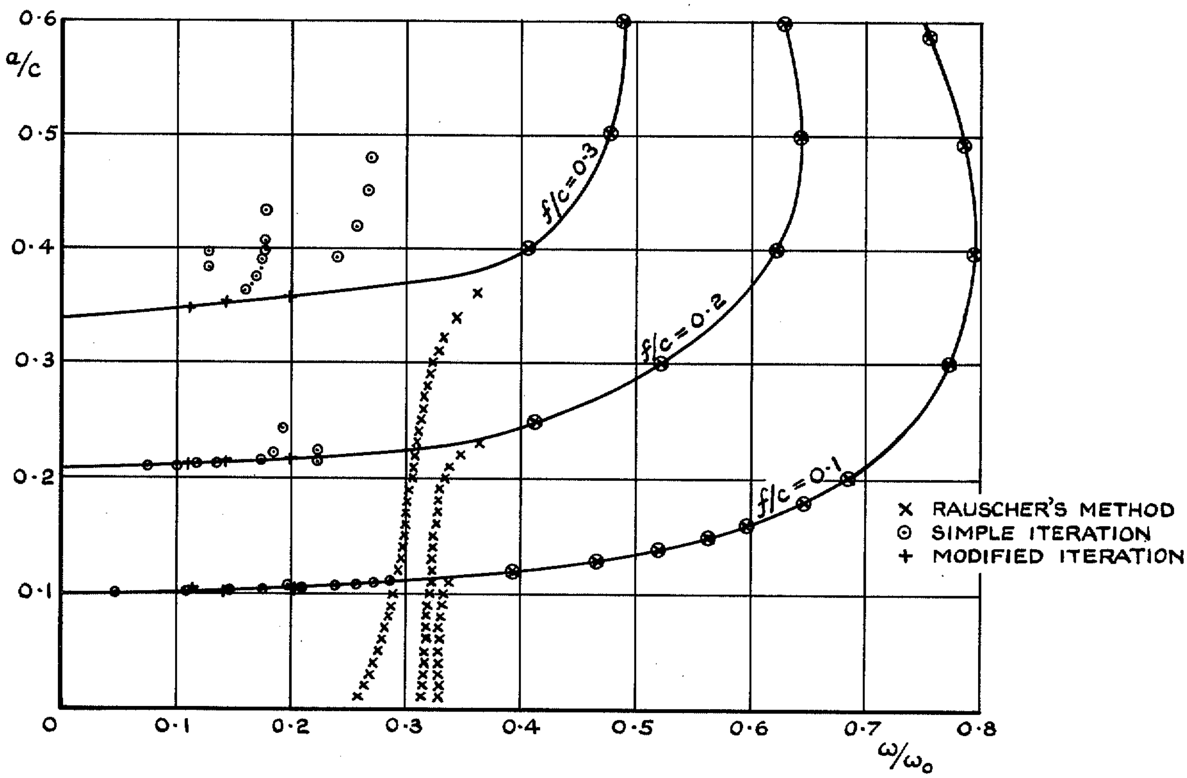

FIG. 4. Lower parts of harmonic response curves.

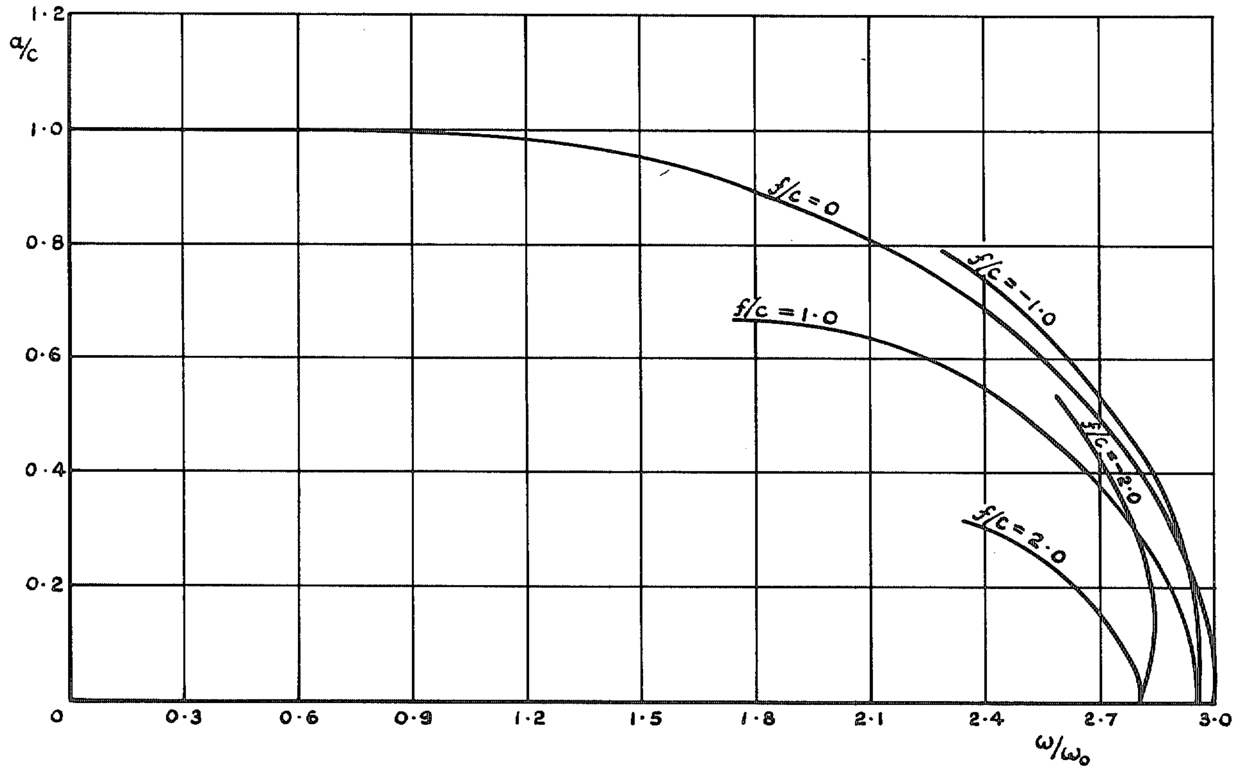

FIG. 5. Response curves for subharmonic oscillations ($m = 3$).

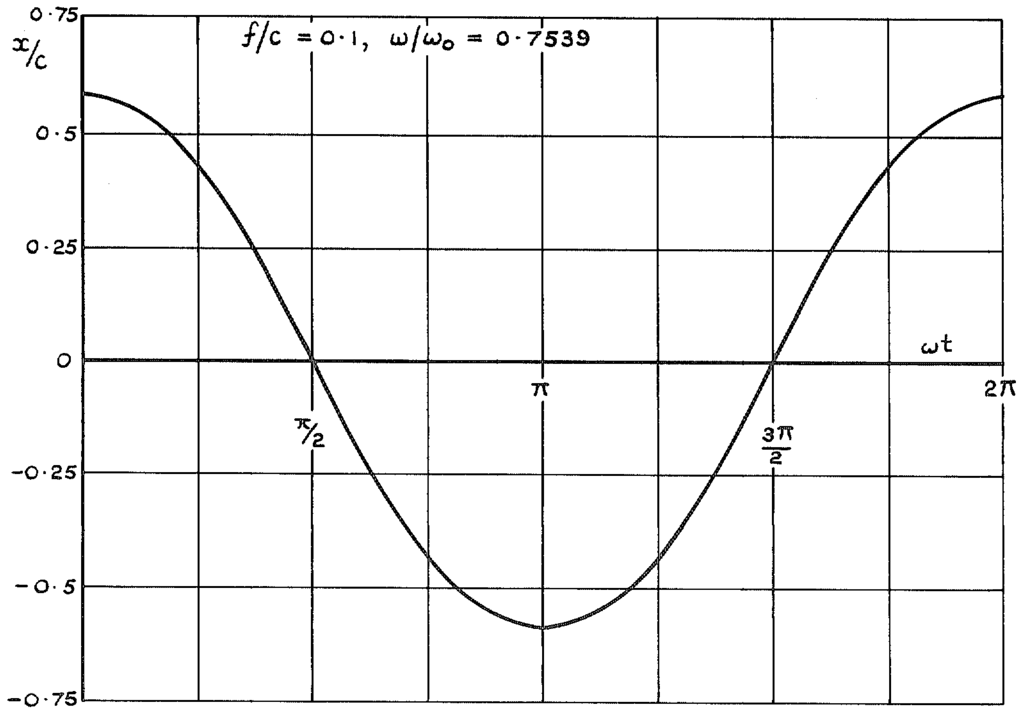

FIG. 6. Example of harmonic oscillation.

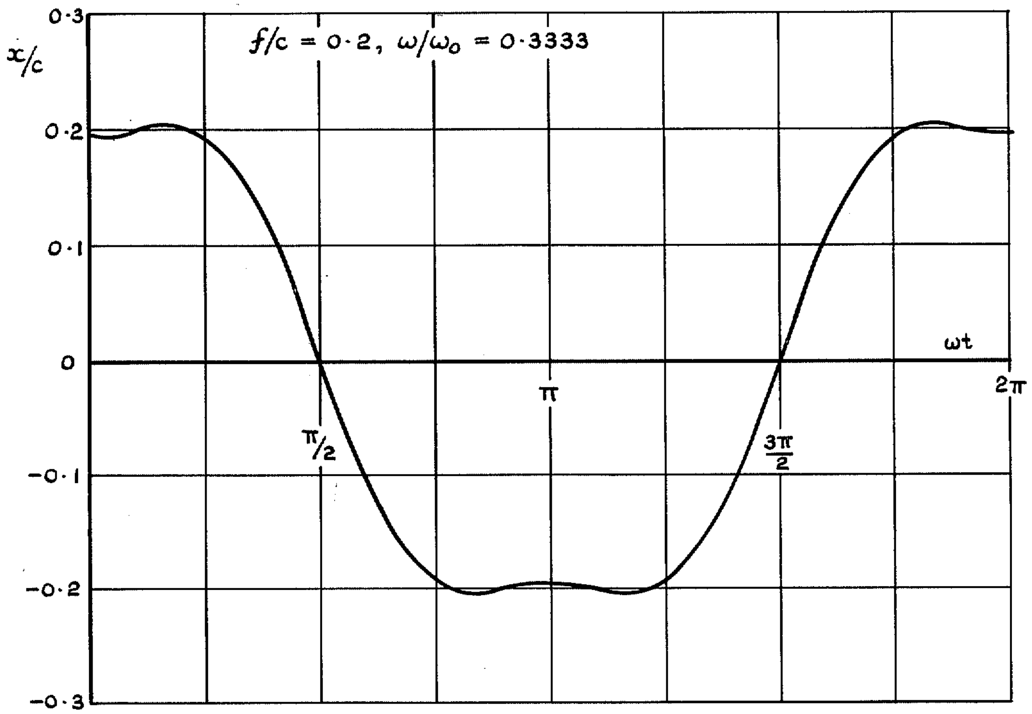

FIG. 7. Harmonic oscillation obtained by modified iteration.

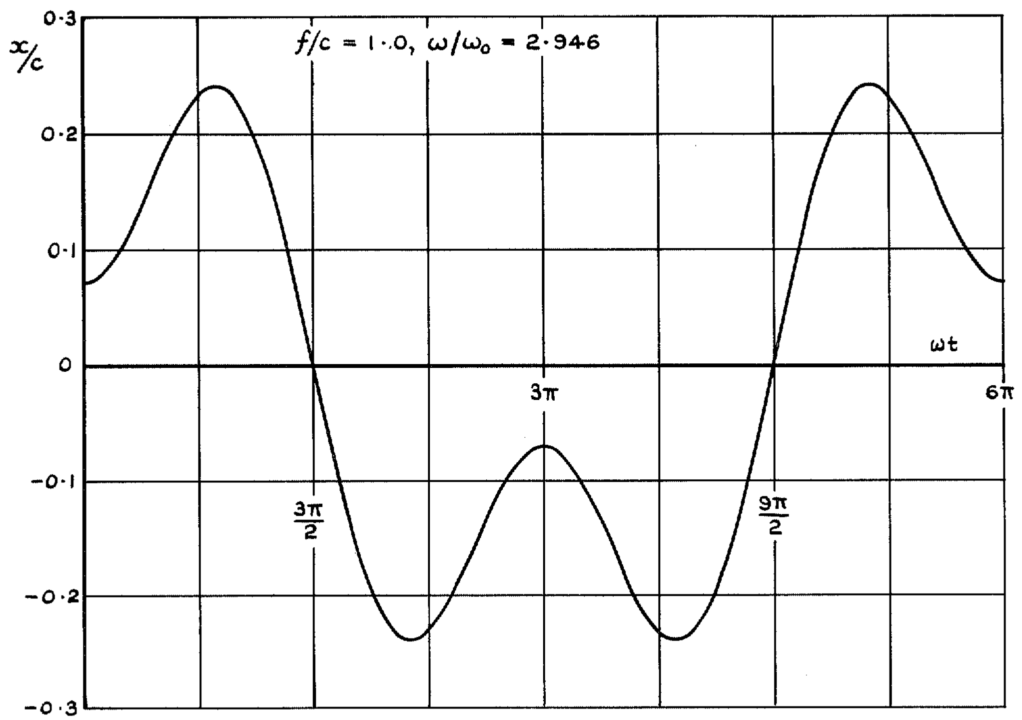

FIG. 8. Subharmonic oscillation with $f/c > 0$ and $m = 3$.

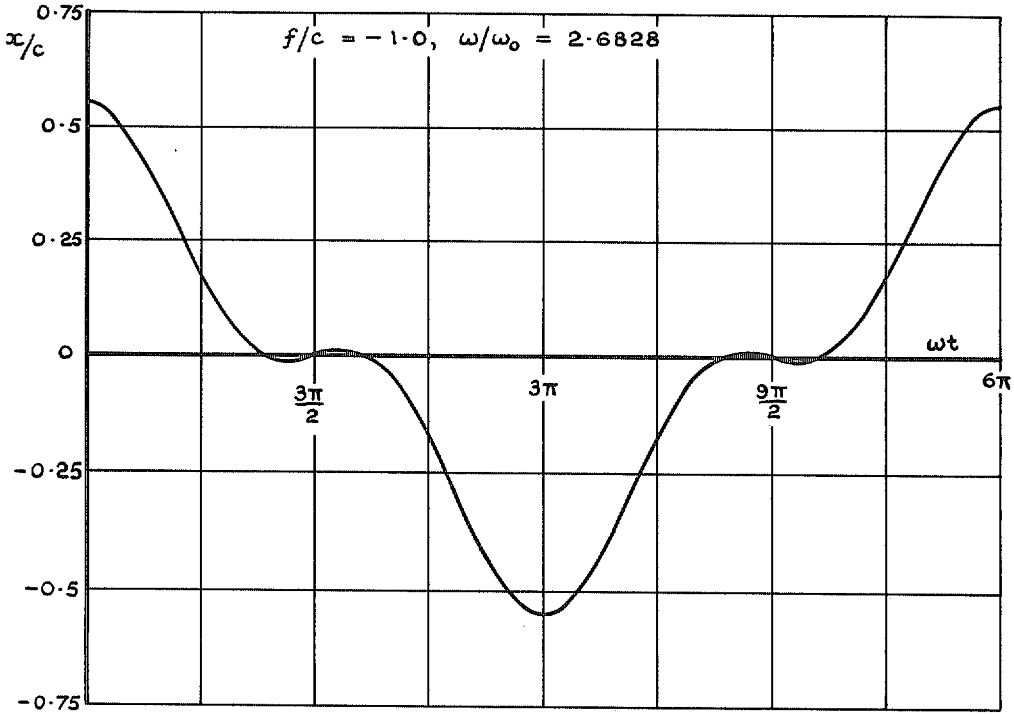

Fig. 9. Subharmonic oscillation with $f/c < 0$ and $m = 3$.

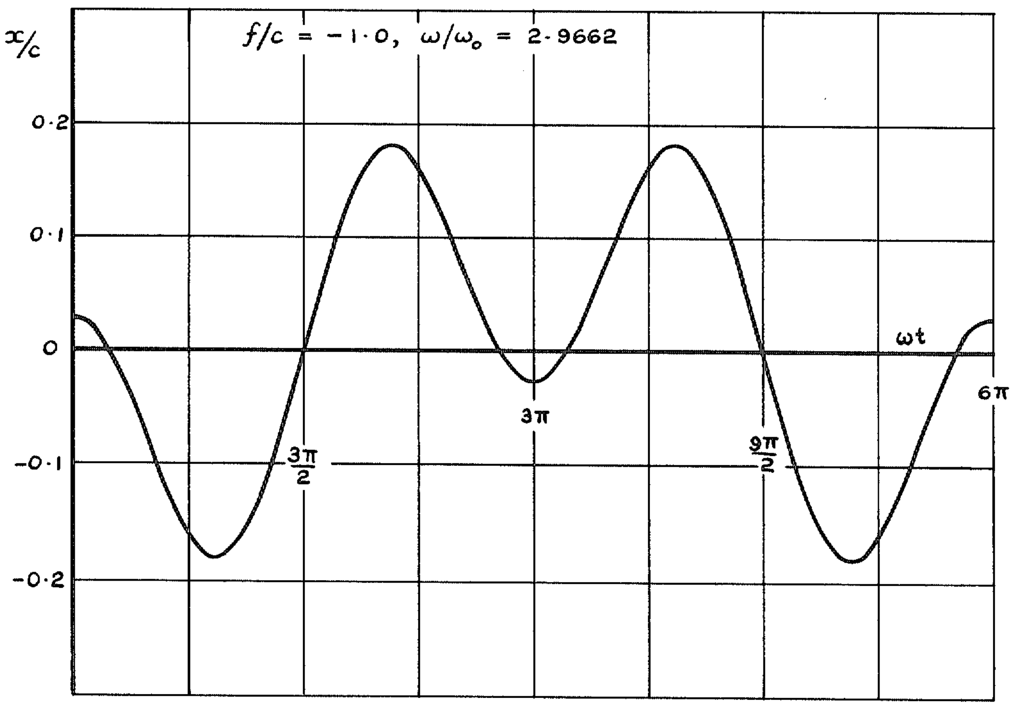

Fig. 10. Subharmonic oscillation with $f/c < 0$ and $m = 3$.

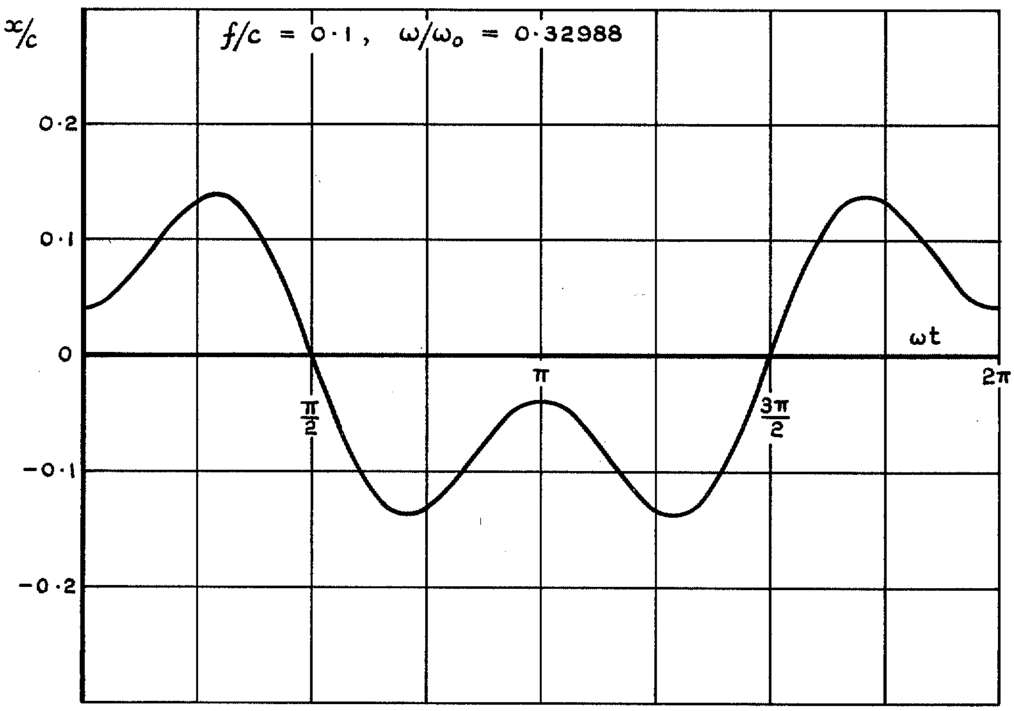

FIG. 11. Harmonic oscillation obtained by Rauscher's method.

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