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The Radiation Sound Field of a Rectangular Panel set in an Infinite Wall and Excited by a Turbulent Boundary Layer

By D. E. Davies

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Summary.

A flexible panel, set in the an infinite wall, is subject to an excitation force distribution which can be described by a correlation function. The panel vibrates and radiates sound. Expressions are derived for the intensity and pressure power spectrum in the sound field at points which are far away from the panel in comparison with the superficial dimensions of the panel.

If the excitation of the panel is due to the flow, over one of its faces, of a turbulent boundary layer, the thickness of which is small in comparison with the superficial dimensions of the panel, then an approximate expression for the correlation function of the excitation force distribution can be used in the evaluation of the expressions obtained. This evaluation has been made to determine the intensity and pressure power spectrum at points on or near to the normal through the centre of the undisturbed panel and on the other side of the panel from the one containing the boundary-layer flow. In this way we can determine the way in which the intensity and pressure power spectrum at these points depends on the plate dimensions and material properties, on the speed of the flow containing the boundary layer, on the thickness of the boundary layer and on the density of and speed of sound in the fluid medium around the panel.

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1. *Introduction.*

Turbulent boundary-layer pressure fluctuations excite vibration in an aeroplane fuselage skin and the skin in turn radiates sound into the inside of the fuselage. The higher the speed of the aircraft, the greater are the boundary-layer pressure fluctuations and consequently the higher is the intensity of the sound radiated from this source into the inside of the fuselage.

Several writers have been concerned with the determination of the intensity of sound inside the fuselage. Owing to the complexity of the problem a great deal of idealisation has been resorted to. The fuselage skin is replaced by a flat surface and the boundary-layer pressure fluctuation is assumed to be independent of the vibration of the surface. The vibration of the surface will be influenced by the back pressure resulting from the radiation of sound on both sides of the surface so that, strictly, vibration of the surface and radiation of sound are coupled phenomena. However, it is assumed that for a surface vibrating in air the back pressure can be taken into account by incorporating a damping term into the equation of vibration of the surface.

Ribner[1] considers the surface to be a flat plate without any supports and predicts the sound intensity by considering travelling ripples in the plate and assuming that the spatial pattern of correlation in the turbulent boundary layer is rigidly convected. Corcos and Liepmann[2] consider the same problem as Ribner[1] but their method is more general and allows for a more general description of the boundary-layer fluctuations.

Kraichnan[3] considers the flat surface to be made up of an array of equal rectangular panels, each simply supported at its edges. The intensity of the radiation is obtained by multiplying the velocity of a point on the vibrating panel by the pressure at that point and integrating over the panel. Several approximations are made in the ensuing analysis in order to get results.

The problem considered by Dyer[4] is that of the radiation of sound into a rectangular box of which a flexible rectangular panel forms one side and the other sides are pressure release surfaces. The flexible rectangular panel is excited by a turbulent boundary layer. The rectangular box is filled with water and in this case coupling of the plate vibration and sound radiation is considered.

Experimental work has been carried out by Ludwig[5] when the flat surface consists of one rectangular flexible panel in a rigid surface. The sound pressure level in a reverberant chamber enclosing the panel was measured and this was related to the total sound power radiated by the panel.

In this report the flat surface again consists of one rectangular flexible panel in a rigid surface. The nature of the pressure and the intensity of the sound radiated are investigated at large distances from the panel and on the other side from the one in which the turbulent boundary layer is present. The turbulent boundary layer is assumed to be flowing in the direction of one of the panel edges. Explicit expressions for the pressure power spectrum and the intensity are obtained for points on or near to the normal through the centre of the undisturbed panel.

There is not a great deal of agreement between the results given in the papers mentioned above. In Refs. 1, 2 and 3 the intensity of sound radiation is equal to the power per unit area radiated. In Ref. 4 the total power radiated is measured. In the present report the intensity near to the normal to the panel is obtained, and this is not simply related to the total power radiated. The results are therefore not directly comparable. Nevertheless we do find qualitative agreement between the results of Corcos and Liepmann[2] and the present paper.

2. *Radiation Field of a Vibrating Panel.*

A set of right-handed rectangular cartesian coordinates x, y, z are chosen with x and y-axis along two adjacent sides of the rectangular panel and z-axis normal to the plane of the panel when it is undisturbed. The origin is taken as any convenient point, for example the bottom left-hand corner of the panel.

Let the excess air pressure over the undisturbed pressure be $p(x,y,z,t)$ at a point (x,y,z) at time t. Then the auto-correlation function $\phi(x,y,z,\tau)$ of the pressure at the point (x,y,z) is defined by

$$\phi(x,y,z,\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(x,y,z,t)\, p\,(x,y,z,t+\tau)\, dt \tag{1}$$

and this is an even function of τ.

The power spectrum $P(x,y,z,\omega)$ of the pressure at the point (x,y,z) is then defined by

$$P(x,y,z,\omega) = \int_{-\infty}^{\infty} \phi(x,y,z,\tau) \exp\,(i\omega\tau)\, d\tau \tag{2}$$

and since $\phi(x,y,z,\tau)$ is an even function of τ this is a real even function of ω which alternatively may be defined by

$$P(x,y,z,\omega) = 2\int_{0}^{\infty} \phi(x,y,z,\tau) \cos\,(\omega\tau)\, d\tau. \tag{3}$$

By Fourier inversion of (2) we get

$$\phi(x,y,z,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(x,y,z,\omega) \exp\,(-i\omega\tau)\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} P(x,y,z,\omega) \cos \omega\tau\, d\omega. \tag{4}$$

The mean square excess pressure at the point (x,y,z) is then

$$\overline{p^2} = \phi(x,y,z,{}^0) = \frac{1}{\pi}\int_{0}^{\infty} P(x,y,z,\omega)\, d\omega. \tag{5}$$

Instrumentation is available for measuring the power spectrum of the pressure, so for that reason we shall investigate its nature in the present problem.

We shall also be interested in the intensity of the sound radiated at large distances from the panel.

If the fluid particle velocities at the point (x,y,z) at time t in the directions of x,y, and z are respectively $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$, then the average flux of energy in the directions of x,y and z are given by γ_x, γ_y and γ_z, respectively, where

$$\left.\begin{aligned}
\gamma_x &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p(x,y,z,t)\, u(x,y,z,t)\, dt \\
\gamma_y &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p(x,y,z,t)\, v(x,y,z,t)\, dt \\
\gamma_z &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p(x,y,z,t)\, w(x,y,z,t)\, dt
\end{aligned}\right\} \tag{6}$$

These fluxes of energy are the components of the intensity vector at the point x,y,z. The magnitude $\Upsilon(x,y,z)$ of the untensity veotor is therefore given by

$$\Upsilon(x,y,z) = \sqrt{\gamma_x^2+\gamma_y^2+\gamma_z^2} \tag{7}$$

If the displacement at time t of a point (x_0,y_0) on the panel is given by the function $Z(x_0,y_0,t)$, then according to Rayleigh (Ref. 6 page 107) the velocity potential $\varphi(x,y,z,t)$ at the point (x,y,z) at time t is given by the formula

$$\varphi(x,y,z,t) = -\frac{1}{2\pi}\iint_{\text{panel}} \frac{\partial}{\partial t} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right)\frac{dx_0\,dy_0}{r_0} \tag{8}$$

where

$$r_0 = \sqrt{(x-x_0)^2+(y-y_0)^2+z^2} \tag{9}$$

and a_0 is the speed of propagation of sound.

The pressure $p(x,y,z,t)$ is obtained from the linearised Beroulli equation and is therefore given by

$$\begin{aligned}
p(x,y,z,t) &= -\rho_0 \frac{\partial}{\partial t}\ (x,y,z,t) \\
&= \frac{\rho_0}{2\pi}\iint_{\text{panel}} \frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right)\frac{dx_0\,dy_0}{r_0}
\end{aligned} \tag{10}$$

where ρ_0 is the density of air.

The pressure auto-correlation function, defined in equation (1), is then

$$\phi(x,y,z,\tau) = \frac{\rho_0^2}{4\pi^2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \iint_{\text{panel}} \iint_{\text{panel}} \frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right) \times$$

$$\times \frac{\partial^2}{\partial t^2} Z\left(x_0',y_0',t+\tau-\frac{r_0'}{a_0}\right) \frac{1}{r_0 r_0'} dx_0\, dy_0\, dx_0\, dy_0 \tag{11}$$

where

$$r_0' = \sqrt{(x-x_0')^2+(y-y_0')^2+z^2}. \tag{12}$$

Now

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right) \frac{\partial^2}{\partial t^2} Z\left(x_0',y_0',t+\tau-\frac{r_0'}{a_0}\right) dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T-\frac{r_0}{a_0}}^{T-\frac{r_0}{r_a}} \frac{\partial^2}{\partial t^2} Z(x_0,y_0,t) \frac{\partial^2}{\partial t^2} Z\left(x_0',y_0',t+\tau+\frac{r_0-r_0'}{a_0}\right) dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{\partial^2}{\partial t^2} Z(x_0,y_0,t) \frac{\partial^2}{\partial t^2} Z\left(x_0',y_0',t+\tau+\frac{r_0-r_0'}{a_0}\right) dt$$

$$= \chi\left(x_0,y_0,x_0',y_0',\ \tau+\frac{r_0-r_0'}{a_0}\right) \tag{13}$$

where we define

$$\chi(x_0,y_0,x_0',y_0',\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{\partial^2}{\partial t^2} Z(x_0,y_0,t) \frac{\partial^2}{\partial t^2} Z(x_0',y_0',t+\tau)\, dt. \tag{14}$$

If therefore we perform first the integration with respect to t in (11) we obtain

$$\phi(x,y,z,\tau) = \frac{\rho_0^2}{4\pi^2} \iint_{\text{panel}} \iint_{\text{panel}} \chi\left(x_0,y_0,x_0',y_0',\tau+\frac{r_0-r_0'}{a_0}\right) \frac{dx_0\, dy_0}{r_0}\ \frac{dx_0'\, dy_0'}{r_0'}. \tag{15}$$

The particle velocity is given by the gradient of the velocity potential. Its components are therefore obtained from (8) as

$$u(x,y,z,t) = \frac{1}{2\pi} \iint_{\text{panel}} \left[\frac{1}{a_0 r_0} \frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right) + \frac{1}{r_0^2} \frac{\partial}{\partial t} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right)\right] \frac{(x-x_0)}{r_0} dx_0\, dy_0 \tag{16}$$

$$v(x,y,z,t) = \frac{1}{2\pi}\iint\limits_{\text{panel}} \left[\frac{1}{a_0 r_0}\frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right) + \frac{1}{r_0^2}\frac{\partial}{\partial t} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right)\right] \frac{(y-y_0)}{r_0} dx_0\, dy_0 \qquad (17)$$

$$w(x,y,z,t) = \frac{1}{2\pi}\iint\limits_{\text{panel}} \left[\frac{1}{a_0 r_0}\frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right) + \frac{1}{r_0^2}\frac{\partial}{\partial t} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right)\right] \frac{z}{r_0} dx_0\, dy_0 \qquad (18)$$

The intensity if obtained by substituting expressions (16), (17) and (18) for $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ and the expression (10) for $p(x,y,z,t)$ into equation (6) and then using (7).

If $\sqrt{x^2+y^2+z^2}$ is much greater than the diagonal of the rectangular panel much simplification occurs since then r_0 changes only little over the area of the panel and certain terms in the integrands can be taken as effectively constant. The expression for the intensity may then be approximated by

$$\Upsilon(x,y,z) = \frac{\rho_0}{4\pi^2}\left[\frac{1}{a_0 r^2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt \iint\limits_{\text{panel}}\iint\limits_{\text{panel}} \frac{\partial^2}{\partial t^2} Z\left(x_0,y_0,t-\frac{r_0}{a_0}\right)\times\right.$$

$$\times\frac{\partial^2}{\partial t^2} Z\left(x'_0,y'_0,t-\frac{r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0 + \frac{1}{r^3}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt \iint\limits_{\text{panel}}\iint\limits_{\text{panel}} \frac{\partial}{\partial t}\left(Zx_0,y_0,t-\frac{r_0}{a_0}\right)\times$$

$$\left.\times\frac{\partial^2}{\partial t^2} Z\left(x'_0,y'_0,t-\frac{r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0\right]$$

$$= \frac{\rho_0}{4\pi^2}\left[\frac{1}{a_0 r^2}\iint\limits_{\text{panel}}\iint\limits_{\text{panel}} \chi\left(x_0,y_0,x'_0,y'_0,\frac{r_0-r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0 + \right.$$

$$+\frac{1}{r^3}\iint\limits_{\text{panel}}\iint\limits_{\text{panel}} \Delta\left(x_0,y_0,x'_0,y'_0,\frac{r_0-r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0 \qquad (19)$$

where

$$\Delta(x_0,y_0,x'_0,y'_0,\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{\partial}{\partial t} Z(x_0,y_0,t)\frac{\partial^2}{\partial t^2} Z(x'_0,y'_0,t+\tau)\, dt \qquad (20)$$

and

$$r = \sqrt{x^2+y^2+z^2} \qquad (21)$$

Very far from the panel

$$\frac{1}{a_0 r^2}\iint\limits_{\text{panel}}\iint\limits_{\text{panel}} \chi\left(x_0,y_0,x'_0,y'_0,\frac{r_0-r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0 \qquad (22)$$

will dominate over

$$\frac{1}{r^3}\iint\limits_{\text{panel}} \iint\limits_{\text{panel}} \Delta\left(x_0,y_0,x'_0,y'_0,\frac{r_0-r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0 \tag{23}$$

and the second term in equation (19) may be neglected. In this case we shall say that we are in the far field. When $\sqrt{x^2+y^2+z^2}$ is much greater than the diagonal of the rectangular panel, and the second term in equation (19) may not be neglected, we shall say that we are in the intermediate field. In the far field the intensity is then given by

$$\Upsilon(x,y,z) = \frac{\rho_0}{4\pi^2 a_0 r^2}\iint\limits_{\text{panel}} \iint\limits_{\text{panel}} \chi\left(x_0,y_0,x'_0,y'_0,\frac{r_0-r'_0}{a_0}\right) dx_0\, dy_0\, dx'_0\, dy'_0. \tag{24}$$

On comparing equations (15) and (24) we get, in the far field, the relation

$$\begin{aligned}\Upsilon &= \frac{1}{\rho_0 a_0}\,\phi(x,y,z,0)\\ &= \frac{1}{\rho_0 a_0}\,\overline{p^2}\end{aligned} \tag{25}$$

so that the intensity is closely related to the mean-square pressure. In the intermediate field no such simple expression holds.

3. *Vibrations of the Panel.*

The classical partial differential equation governing the vibration of the panel is

$$M\frac{\partial^2 Z}{\partial t^2} + D\,\nabla^4 Z = f(x,y,t). \tag{26}$$

In this equation M is the mass per unit area of the panel, $f(x,y,t)$ is the exciting force per unit area and D is the rigidity coefficient defined by

$$D = \frac{Eh^3}{12(1-\sigma^2)} \tag{27}$$

where h is the panel thickness, E is Young's modulus of the plate material and σ is the Poisson ratio.

The exciting force per unit area $f(x,y,t)$ arises from the pressure fluctuations in the boundary layer and also from the unsteady pressure distribution arising from the vibration of the panel in air. We shall assume that the pressure fluctuations in the boundary layer are not affected by the vibration of the panel. The contribution to $f(x,y,t)$ of the unsteady pressure distribution arising from the vibration of the panel is a complicated integral expression in Z, and substitution of this into (26) would lead to a complicated integro-differential equation. To make the problem tractable we shall make the assumption that the contribution of this unsteady pressure can be taken into account by adding a virtual mass to M and bringing in a damping term $b\frac{\partial Z}{\partial t}$ on the left-hand-side of equation (27). The $f(x,y,t)$ on the right-hand-side will then arise entirely from the pressure fluctuations in the boundary layer.

The values to be ascribed to the virtual mass and to the damping coefficient b are difficult to estimate. However the virtual mass can be expected to be small in comparison with M so that its effect is small and can be neglected. For a rigidly oscillating infinite plate the acoustic damping coefficient b would have the value $2\rho_0 a_0$. This value may need modification for application to a finite vibrating plate. There will also be a contribution to b from the structural damping and this contribution will depend on the plate thickness.

We take therefore, as the governing equation of the panel vibrating in air:

$$M\frac{\partial^2 Z}{dt^2}+b\frac{\partial Z}{\partial t}+D\,\nabla^4 Z = f(x,y,t) \tag{28}$$

where $f(x,y,t)$ is the exciting force arising entirely from the turbulent boundary-layer pressure fluctuations.

The natural modes of oscillation of the panel satisfy the differential equation

$$M\frac{\partial^2 Z}{\partial t^2}+D\,\nabla^4 Z = 0 \tag{29}$$

and also certain conditions at the edges of the panel.

Let

$$Z = \varepsilon(x,y)\exp(i\omega t) \tag{30}$$

satisfy equation (29), and the conditions at the edges of the panel.

Then

$$\nabla^4\,\varepsilon(x,y)-\lambda^2\,\varepsilon(x,y) = 0 \tag{31}$$

where

$$\lambda = \sqrt{\frac{M}{D}}\,.\,\omega\,. \tag{32}$$

Equation (31) is satisfied for only a discrete set of values of λ for the given edge conditions, and to each of these values of λ there corresponds a function $\varepsilon(x,y)$ which we call a modal function.

The discrete set of values λ may be numbered and then the mth member is denoted by λ_m and the corresponding modal function is denoted by $\varepsilon_m(x,y)$. The mth natural circular frequency ω_m is obtained from (32) and is

$$\omega_m = \sqrt{\frac{D}{M}}\cdot\lambda_m. \tag{33}$$

The modal functions $\varepsilon_m(x,y)$ are orthogonal for clamped or simply supported edge conditions, and we normalise them so that

$$\iint\limits_{\text{panel}} \varepsilon_m(x,y)\,\varepsilon_n(x,y)\,dx\,dy = \delta_{m,n} \tag{34}$$

where $\delta_{m,n}$ is Kronecker's delta.

If, for example, the panel is simply supported at its edges the natural circular frequencies are given by

$$\omega_m = \pi^2\sqrt{\frac{D}{M}}\left(\frac{m_1^2}{c^2}+\frac{m_2^2}{d^2}\right) \tag{35}$$

and the corresponding modal functions are

$$\varepsilon_m(x,y) = \frac{2}{\sqrt{cd}} \sin\left(\frac{m_1 \pi x}{c}\right) \sin\left(\frac{m_2 \pi y}{d}\right) \tag{36}$$

where c and d are the lengths of the sides of the panel parallel to the x and y axes respectively.

The integers m_1 and m_2 are associated in a one-to-one correspondence with the integer m.

If the panels are clamped at their edges then there is no analytic expression for the natural frequencies and the modal functions. The natural frequencies and modal functions may nevertheless be obtained to good accuracy by an approximate procedure such as the method of Raleigh-Ritz, provided the mode shape does not have too many peaks and troughs over the area of the panel.

Following Powell[7] we shall write the solution of equation (28) as an infinite series in the modal functions:

$$Z = \sum_m \varepsilon_m(x,y)\, \xi_m(t) \tag{37}$$

where $\xi_m(t)$ are functions of time only and may be regarded as generalised coordinates.

Substituting (37) into the differential equation (28) and making use of (31) and (33) leads to

$$\sum_m \varepsilon_m(x,y) \left\{ M\, \ddot{\xi}_m(t) + b\, \dot{\xi}_m(t) + \omega_m^2\, M\, \xi_m(t) \right\} = f(x,y,t) \tag{38}$$

and then using the orthonormal property (34) we get

$$\ddot{\xi}_m(t) + \beta\, \dot{\xi}_m(t) + \omega_m^2\, \xi_m(t) = f_m(t) \tag{39}$$

where

$$\beta = \frac{b}{M} \tag{40}$$

and

$$f_m(t) = \frac{1}{M} \iint\limits_{\text{panel}} f(x,y,t)\, \varepsilon_m(x,y)\, dx\, dy. \tag{41}$$

The function $f(x,y,t)$ is a complicated function and there is no hope of determining it either experimentally or theoretically. However the correlation function of the excitation

$$\psi(x,y,x',y',\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x,y,t)\, f(x',y',t+\tau)\, dt \tag{42}$$

is believed to be a well defined function which is well behaved, and which can be measured. We shall assume that the function $\psi(x,y,x',y',\tau)$ is given as the description of the turbulent boundary layer excitation.

The power spectrum $Q(x,y,x',y',\omega)$, sometimes called the cross power spectrum, of the excitation is defined by

$$Q(x,y,x',y',\omega) = \int_{-\infty}^{\infty} \psi(x,y,x',y',\tau)\exp(i\omega\tau)\,d\tau \tag{43}$$

and, alternatively, this function might be given as a description of the turbulent boundary-layer excitation.

The displacement function $Z(x,y,t)$ corresponding to the exciting function $f(x,y,t)$ is also a complicated function, but the correlation functions (14) and (20) are well behaved. We can give expressions for χ and Δ in terms of ψ or Q.

By use of equation (37) we get for the correlation functions χ and Δ the series

$$\chi(x,y,x',y',\tau) = \sum_m \sum_n \varepsilon_m(x,y)\,\varepsilon_n(x',y')\,\zeta_{m,n}(\tau) \tag{44}$$

$$\Delta(x,y,x',y',\tau) = \sum_m \sum_n \varepsilon_m(x,y)\,\varepsilon_n(x',y')\,\mu_{m,n}(\tau) \tag{45}$$

where

$$\zeta_{m,n}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2}\xi_m(t)\frac{\partial^2}{\partial t^2}\xi_n(t+\tau)\,d\tau \tag{46}$$

and

$$\mu_{m,n}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\xi_m(t)\frac{\partial^2}{\partial t^2}\xi_n(t+\tau)\,d\tau. \tag{47}$$

Define

$$\theta_{m,n}(\omega) = \int_{-\infty}^{\infty} \zeta_{m,n}(\tau)\exp(i\omega\tau)\,d\tau \tag{48}$$

$$\nu_{m,n}(\omega) = \int_{-\infty}^{\infty} \mu_{m,n}(\tau)\exp(i\omega\tau)\,d\tau \tag{49}$$

and

$$R_{m,n}(\omega) = \iint_{\text{panel}} \iint_{\text{panel}} Q(x,y,x',y',\omega)\,\varepsilon_m(x,y)\,\varepsilon_n(x',y',)\,dx\,dy\,dx'\,dy'. \tag{50}$$

It follows from equations (39), (41), (42) and (43), as is shown in Appendix I, that

$$\theta_{m,n}(\omega) = \frac{1}{M^2} \frac{\omega^4 R_{m,n}(\omega)}{[-\omega^2+i\beta\omega+\omega_m^2][-\omega^2-i\beta\omega+\omega_n^2]} \tag{51}$$

$$v_{m,n}(\omega) = -\frac{1}{M^2} \frac{i\omega^3 R_{m,n}(\omega)}{[-\omega^2+i\beta\omega+\omega_m^2][-\omega^2-i\beta\omega+\omega_n^2]} \tag{52}$$

The pressure auto-correlation function is obtained from (15) and (44) and is

$$\phi(x,y,z,\tau) = \sum_m \sum_n \frac{\rho_0^2}{4\pi^2} \iint_{\text{panel}} \iint_{\text{panel}} \frac{\varepsilon_m(x_0,y_0)}{r_0} \frac{\varepsilon_n(x_0',y_0')}{r_0'} \zeta_{m,n}\left(\tau+\frac{r_0-r_0'}{a_0}\right) dx_0\,dy_0 \times dx_0'\,dy_0' \tag{53}$$

The power spectrum of the pressure in the far and intermediate fields is obtained by taking the Fournier integral of $\phi(x,y,z,\tau)$ according to equation (2) and is

$$P(x,y,z,\omega) = \sum_m \sum_n \frac{\rho_0^2}{4\pi^2} \iint_{\text{panel}} \iint_{\text{panel}} \frac{\varepsilon_m(x_0,y_0)}{r_0} \frac{\varepsilon_m(x_0',y_0')}{r_0} dx_0\,dy_0\,dx_0'\,dy_0' \times \int_{-\infty}^{\infty} \zeta_{m,n}\left(\tau+\frac{r_0-r_0'}{a_0}\right) e^{i\omega\tau}\,d\tau$$

$$= \sum_m \sum_n \frac{\rho_0^2}{4\pi^2} \theta_{m,n}(\omega) \iint_{\text{panel}} \iint_{\text{panel}} \frac{\varepsilon_m(x_0,y_0)}{r_0} \frac{\varepsilon_n(x_0',y_0')}{r_0'} \exp\left[-\frac{i\omega}{a_0}(r_0-r_0')\right] \times dx_0\,dy_0\,dx_0'\,dy_0'$$

$$= \frac{\rho_0^2}{4\pi^2 M^2} \sum_m \sum_n \frac{\omega^4 R_{m,n}(\omega)}{[-\omega^2+i\beta\omega+\omega_m^2][-\omega^2-i\beta\omega+\omega_n^2]} \times$$

$$\times \iint_{\text{panel}} \frac{\varepsilon_m(x_0,y_0)}{r_0} \exp\left(-\frac{i\omega}{a_0} r_0\right) dx_0\,dy_0 \iint_{\text{panel}} \frac{\varepsilon_n(x_0',y_0')}{r_0'} \exp\left(\frac{i\omega r_0'}{a_0}\right) dx_0'\,dy_0' \tag{54}$$

The intensity of radiated sound at the point (x,y,z) in the intermediate field is obtained from (19), (44) and (45) and is

$$\Upsilon(x,y,z) = \frac{\rho_0}{4\pi^2} \left[\frac{1}{a_0 r^2} \sum_m \sum_n \iint_{\text{panel}} \iint_{\text{panel}} \varepsilon_m(x_0,y_0)\,\varepsilon_n(x_0',y_0') \times \right.$$

$$\times \zeta_{m,n}\left(\frac{r_0-r_0'}{a_0}\right) dx_0\,dy_0\,dx_0'\,dy_0' + \frac{1}{r^3} \sum_m \sum_n \iint_{\text{panel}} \iint_{\text{panel}} \varepsilon_m(x_0,y_0) \times$$

$$\left. \times\,\varepsilon_n(x_0',y_0')\,\mu_{m,n}\left(\frac{r_0-r_0'}{a_0}\right) dx_0\,dy_0\,dx_0'\,dy_0' \right]. \tag{55}$$

In order to evaluate this expression the functions $\zeta_{m,n}(\tau)$ and $\mu_{m,n}(\tau)$ must be obtained. The function $R_{m,n}(\omega)$ can be determined from equation (50) and then $\theta_{m,n}(\omega)$ and $v_{m,n}(\omega)$ are determined from equations (51) and (52). The functions $\zeta_{m,n}(\tau)$ and $\mu_{m,n}(\tau)$ are then obtained by taking the inverse Fourier integrals of $\theta_{m,n}(\omega)$ and $v_{m,n}(\omega)$. The process involves taking the Fourier integral of ψ to obtain Q at the beginning and then taking an inverse Fourier integral at the end. These processes can be effected analytically.

On taking the Fourier inverse integral of $\theta_{m,n}(\omega)$ and using (51) we get

$$\zeta_{m,n}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta_{m,n}(\omega) \exp(-i\omega\tau)\, d\omega$$

$$= \frac{1}{2\pi M^2}\int_{-\infty}^{\infty} \frac{\omega^4 R_{m,n}(\omega)\exp(-i\omega\tau)d\omega}{[-\omega^2+i\beta\omega+\omega_m^2]\,[-\omega^2-i\beta\omega+\omega_n^2]}. \tag{56}$$

We can show that (*see* Appendix II)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^4 \exp(-\omega\tau)\, d\omega}{[-\omega^2+i\beta\omega+\omega_m^2]\,[-\omega^2-i\beta\omega+\omega_n^2]} = \delta(\tau)+S_{m,n}(\tau) \tag{57}$$

where $\delta(\tau)$ is Dirac's delta function and

$$S_{m,n}(\tau) = \frac{\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4-i\beta\alpha_n(2\omega_n^2-\beta^2)}{2i\alpha_n[\omega_n^2-\omega_m^2-\beta^2-2i\beta\alpha_n]}\exp\left(-\frac{\beta}{2}-i\alpha_n\right)\tau -$$

$$-\frac{\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4+i\beta\alpha_n(2\omega_n^2-\beta^2)}{22i\alpha_n[\omega_n^2-\omega_m^2-\beta^2+2i\beta\alpha_n]}\exp\left(-\frac{\beta}{2}+i\alpha_n\right)\tau$$

$\tau>0$

$$S_{m,n}(\tau) = \frac{\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4-i\beta\alpha_m(2\omega_m^2-\beta2)}{2i\alpha_m[\omega_m^2-\omega_n^2-\beta^2-2i\beta\alpha_m]}\exp\left(\frac{\beta}{2}+i\alpha_m\right)\tau -$$

$$-\frac{\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4+i\beta\alpha_m(2\omega_m^2-\beta^2)}{2i\alpha_m[\omega_m^2-\omega_n^2-\beta^2+2i\beta\alpha_m]}\exp\left(\frac{\beta}{2}-i\alpha_m\right)\tau$$

$\tau<0$ (58)

with

$$\alpha_m = \sqrt{\omega_m^2-\frac{\beta^2}{4}}. \tag{59}$$

The right hand side of equation (56) may then be replaced by a convolution integral and we obtain

$$\zeta_{m,n}(\tau) = \frac{1}{M^2}\int_{-\infty}^{\infty} [\delta(u)+S_{m,n}(u)]\,\Gamma_{m,n}(\tau-u)\,du$$

$$= \frac{1}{M^2}\quad \Gamma_{m,n}(\tau)+\int_{-\infty}^{\infty} S_{m,n}(u)\,\Gamma_{m,n}(\tau-u)\,du \tag{60}$$

where

$$\Gamma_{m,n}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{m,n}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$= \iint_{\text{panel}} \iint_{\text{panel}} \psi(x_0,y_0,x_0',y_0',\tau)\,\varepsilon_m(x_0,y_0)\,\varepsilon_n(x_0',y_0')\,dx_0\,dy_0\,dx_0'\,dy_0'. \tag{61}$$

On taking the Fourier inverse integral of $v_{m,n}(\omega)$ and using (52) we get

$$\mu_{m,n}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v_{m,n}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$= -\frac{1}{2\pi M^2}\int_{-\infty}^{\infty} \frac{i\omega^3 R_{m,n}(\omega)\exp(-i\omega\tau)d\omega}{[-\omega^2+i\beta\omega+\omega_n^2]\,[-\omega^2-i\beta\omega+\omega_n^2]}. \tag{62}$$

We can show that (*see* Appendix II)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\omega^3\exp(-i\omega\tau)\,d\omega}{[-\omega^2+i\beta\omega+\omega_n^2]\,[-\omega^2-i\beta\omega+\omega_n^2]} = T_{m,n}(\tau) \tag{63}$$

where

$$T_{m,n}(\tau) = -\frac{(\omega_n^2-2\beta^2\omega_n^2+\frac{1}{2}\beta^4)-i\beta\alpha_n(2\omega_n^2-\beta^2)}{2\alpha_n\left(\alpha_n-\frac{i\beta}{2}\right)(\omega_n^2-\omega_m^2-\beta^2+2i\beta\alpha_n)}\exp\left(-\frac{\beta}{2}-i\alpha_n\right)\tau -$$

$$-\frac{(\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4)+i\beta\alpha_n(2\omega_n^2-\beta^2)}{2\alpha_n\left(\alpha_n+\frac{i\beta}{2}\right)(\omega_n^2-\omega_m^2-\beta^2+2i\beta\alpha_n)}\exp\left(-\frac{\beta}{2}+i\alpha_n\right)\tau$$

$\tau > 0.$

$$T_{m,n}(\tau) = \frac{(\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4)-i\beta\alpha_n(2\omega_m^2-\beta^2)}{2\alpha_m\left(\alpha_m-\frac{i\beta}{2}\right)(\omega_m^2-\omega_n^2-\beta^2+2i\beta\alpha_n)}\exp\left(\frac{\beta}{2}+i\alpha_m\right)\tau +$$

$$+\frac{(\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4)+i\beta\alpha_m(2\omega_m^2-\beta^2)}{2\alpha_m\left(\alpha_m+\frac{i\beta}{2}\right)(\omega_m^2-\omega_n^2-\beta^2+2i\beta\alpha_m)}\exp\left(\frac{\beta}{2}-i\alpha_m\right)\tau$$

$\tau < 0.$ (64)

We notice from the definitions (57) and (63) that

$$\frac{d}{d\tau} T_{m,n}(\tau) = \delta(\tau)+S_{m,n}(\tau) \tag{65}$$

and this can indeed be verified by using the expressions (58) and (64).

The right hand side of equation (62) may now be replaced by a convolution integral so that we obtain

$$\mu_{m,n}(\tau) = \frac{1}{2M^2}\int_{-\infty}^{\infty} T_{m,n}(u)\,\Gamma_{m,n}(\tau-u)\,du. \tag{66}$$

The intensity of the sound at the point (x,y,z) can then be evaluated using (55), (60) and (66). If $z>>x$ and $z>>y$ then r_0 and r_0' are practically constant for all points on the panel, and

$$r_0-r_0' \simeq 0.$$

This is true for points near the perpendicular through the centre of the undisturbed panel and then sound signals leaving any points on the panel simultaneously will arrive at the observation point simultaneously. In this case the expression (55) for the intensity simplifies to

$$(x,y,z) = \frac{\rho_0}{4\pi^2}\left[\frac{1}{a_0 r^2}\sum_m\sum_n H_m H_n \zeta_{m,n}(0)+\frac{1}{r^3}\sum_m\sum_n H_m H_n \mu_{m,n}(0)\right] \tag{67}$$

where

$$H_m = \iint_{\text{panel}} \varepsilon_m(x,y)\,dx\,dy. \tag{68}$$

For a panel with simply supported edges the expression for H_m corresponding to the modal functions given in equation (36) is

$$\begin{aligned} H_m &= \frac{2\sqrt{cd}}{m_1 m_2}[1-(-1)^{m_1}]\,[1-(-1)^{m_2}] \\ &= 0\left(\frac{1}{m_1 m_2}\right). \end{aligned} \tag{69}$$

In this case also the expression for the power spectrum given in equation (54) simplifies to

$$P(x,y,z,\omega) = \frac{\rho_0^2}{4\pi^2 M^2}\,\frac{1}{r^2}\sum_m\sum_n \frac{H_m H_n \omega^4 R_{m,n}(\omega)}{[-\omega^2+i\beta\omega+\omega_m^2]\,[-\omega^2-i\beta\omega+\omega_n^2]} \tag{70}$$

and the mean square pressure, obtained from (53) by putting $\tau = 0$, becomes

$$\overline{p^2} = \phi(x,y,z,0) = \frac{\rho_0^2}{4\pi^2}\sum_m\sum_n H_m H_n \zeta_{m,n}(0). \tag{71}$$

4. *Evaluation of the Expressions.*

In order to evaluate the integrals we must know either the correlation function $\psi(x,y,x',y',\tau)$ defined in equation (42) or the power spectrum $Q(x,y,x',y',\omega)$ defined in equation (43).

The function $\psi(x,y,x',y',\tau)$ is to be substituted into equation (61) and the integration carried out. It is not likely that this integral can be carried out analytically even if an analytical expression for $\psi(x,y,x',y',\tau)$ is known. Numerical integration is, however, possible for any given value of τ. Care must be exercised in using the numerical procedures when the modal numbers m and n are high for then the functions $\varepsilon_m(x,y)$ and $\varepsilon_n(x,y)$ become highly oscillatory over the extent of the panel. This oscillatory behaviour of $\varepsilon_n(x,y)$ and $\varepsilon_n(x,y)$ is responsible for a rapid decrease in the values of $\Gamma_{m,n}(\tau)$, at a given value of τ, when m and n increase and leads to rapid convergence of the series involved.

In a turbulent boundary layer whose thickness grows only little in a distance of order of a panel representative length the correlation function $\psi(x,y,x',y',\tau)$ may be taken to be a function of $x-x'$, $y-y'$ and τ only, i.e. the pressure field may be taken to be homogeneous. The maximum value of $\psi(x,y,x',y',\tau)$ occurs when $x = x'$, $y = y'$ and $\tau = 0$. Also $\psi(x,y,x',y',\tau)$ becomes small when $|\tau|$ becomes large. This in turn indicates that $\Gamma_{m,n}(\tau)$ becomes small when $|\tau|$ becomes large, and in fact $\Gamma_{m,n}(\tau)$ will become effectively zero outside a finite range of τ. The evaluation of the infinite integrals in equations (60) and (66) may then be accomplished numerically for the integrands become effectively zero outside a finite range of u. The sound intensity is then obtained from (55) by evaluating numerically the integrals occuring there. Also the mean-square pressure can be obtained from equation (71).

To obtain the power spectrum $P(x,y,z,\omega)$ from equation (54) or (70) we must evaluate $R_{m,n}(\omega)$. An expression for $R_{m,n}(\omega)$ is found by taking the Fourier inverse of equation (61). This is

$$R_{m,n}(\omega) = \int_{-\infty}^{\infty} \Gamma_{m,n}(\tau) \exp(i\omega\tau)\, d\tau \tag{72}$$

and this function can be evaluated numerically once $\Gamma_{mn}(\tau)$ has been obtained at sufficient appropriate values of τ.

If the power spectrum $Q(x,y,x',y',\omega)$ is given rather than the correlation function $\psi(x,y,x',y',\tau)$ then $R_{m,n}(\omega)$ may be obtained from (50) by numerical integration, so that the power spectrum $P(x,y,z,\omega)$ is obtained immediately from equation (54) or (70).

Equations (51) and (52) may be used to determine $\theta_{m,n}(\omega)$ and $\nu_{m,n}(\omega)$ and then $\zeta_{m,n}(\tau)$ and $\mu_{m,n}(\tau)$ are obtained on inversion of (48) and (49). The intensity of the sound is again obtained from (55).

It may be noted that the evaluation of the power spectrum $P(x,y,z,\omega)$ is easier starting with $Q(x_0,y_0,x'_0,y'_0,\omega)$ given rather than with $\psi(x,y,x',y',\tau)$ given. However, if $\psi(x,y,x',y',\tau)$ decreases rapidly as τ moves away from zero then $Q(x,y,x',y',\omega)$ will decrease only slowly as ω increases. In this case it is better to avoid using the power spectrum $Q(x,y,x',y',\omega)$ for evaluating the sound intensity of the mean-square pressure as the numerical processes involved become very much more lengthy than when $\psi(x,y,x',y',\tau)$ is used directly. If analytic approximations can be made this method of evaluation may become the easier.

Experiments by Willmarth[8] show that the pressure correlation in a turbulent boundary layer corresponds to a downstream convection of a spatial pattern of correlation. For two points, one downstream of the other, the correlation curve with respect to τ shows a decrease in its maximum height and a spreading out as the distance between the two points increases. For convection in the direction of the x-axis Dyer[4] has given the correlation function,

$$\psi(x_0,y_0,x'_0,y'_0,\tau) = \overline{f^2} \exp\left[-\kappa\sqrt{(\xi_0-u_0\tau)^2+\eta_0^2}-\frac{|\tau|}{\theta}\right]. \tag{73}$$

As a fit of the experimental data which incorporates the most important features of the convected pressure field. In equation (73) the symbols have the following meanings

$$\left.\begin{aligned}\xi_0 &= x'_0 - x_0\\ \eta_0 &= y'_0 - y_0\end{aligned}\right\} \qquad (74)$$

$\overline{f^2}$ is the mean-square excess pressure in the boundary layer
u_0 is the mean convection speed along the direction of the $+ve$ x-axis
κ, θ are constants.

The expression (73) can be expected to be only an approximation. The correlation curve for two points, one downstream of the other and distance ξ_0 apart, is obtained by taking $\eta_0 = 0$ in expression (64). The curves obtained for different values of ξ_0 do not represent the spreading out of the correlation curve with increasing ξ_0, and there is a cusp at $\tau = \dfrac{\xi_0}{v_0}$. Furthermore, curves of constant ψ on the ξ, η_0 plane at a given τ are circles with centre $\xi_0 = u_0\,\tau, \eta_0 = 0$, showing that (73) represents convection of an isotropic pattern of turbulence. Experiment shows that curves of constant ψ at given τ are closed curves elongated along the direction of the flow so that in fact there is not an isotropic pattern of turbulence. Multiplying η_0^2 in expression (73) by a constant would change the constant ψ circles into ellipses, and this might be an improvement. However for the further work in this paper expression (73) will be used as an idealisation which incorporates the most important features of the convected field. Kraichnan[3] uses a field of convected turbulence which is not isotropic.

From experimental measurement[8] it is known that over a wide range of Mach number

$$\sqrt{\overline{f^2}} = 0{\cdot}006 \times \frac{1}{2}\rho_0\, U_0^2 \qquad (75)$$

where U_0 is the free stream velocity. Dyer[4] has made the following estimates for κ, θ and u_0 based on experimental measurement:

$$\left.\begin{aligned}\kappa &\simeq \frac{2}{\delta}\\ \theta &\simeq \frac{30\delta}{U_0}\\ u_0 &\simeq 0{\cdot}82\, U_0\end{aligned}\right\} \qquad (76)$$

where δ is the boundary-layer displacement thickness.

In practice δ is small compared with the dimensions of the panel so that $\dfrac{1}{\kappa}$ is small in comparison with the fundamental period of the panel.

Under certain circumstances approximations can be made in integrals containing $\psi(x_0, y_0, x'_0, y'_0; \tau)$ given by expression (73). If the nearest distance from the point $(x_0 + u_0\tau, y_0)$ to an edge of the panel is very large compared with $\dfrac{1}{\kappa}$ and if $\varepsilon_n(x'_0, y'_0)$ does not vary much over distances of $O\left(\dfrac{1}{\kappa}\right)$ from the point

$(x_0+u_0\tau, y_0)$ then we can write approximately if the point $(x_0+u_0\tau, y_0)$ is on the panel

$$\iint_{\text{panel}} \exp\left[-\kappa\sqrt{(\xi_0-u_0\tau)^2+\eta_0^2}\right] \varepsilon_n(x_0', y_0')\, dx_0'\, dy_0'$$

$$\simeq \varepsilon_n(x_0+u_0\tau, y_0) \iint_{\substack{\text{whole}\\ \text{plane}}} \exp\left[-\kappa\sqrt{(\xi_0-u_0\tau)^2+\eta_0^2}\right] dx_0'\, dy_0'$$

$$= \frac{2\pi}{\kappa^2}\varepsilon_n(x_0+u_0\tau, y_0) \tag{77}$$

whereas if the point $(x_0+u_0\tau, y_0)$ is not on the panel

$$\iint_{\text{panel}} \exp\left[-\kappa\sqrt{(\xi_0-u_0\tau)^2+\eta_0^2}\right] \varepsilon_n(x_0', y_0')\, dx_0'\, dy_0' \simeq 0. \tag{78}$$

This is equivalent to replacing $\exp\left[-\kappa\sqrt{(\xi_0-u_0\tau)^2+\eta_0^2}\right]$ by $\frac{2\pi}{\kappa^2}\,\delta(\xi_0-u_0\tau)\,\delta(\eta_0)$ as far as these integrals are concerned.

Then from (61) and (73) we get

$$\Gamma_{m,n}(\tau) = \frac{2\overline{f^2}}{\kappa^2}\exp\left(-\frac{|\tau|}{\theta}\right)\iint_{\text{panel}} \varepsilon_m(x_0, y_0)\,\varepsilon_n(x_0+u_0\tau, y_0)\, dx_0\, dy_0 \tag{79}$$

where $\varepsilon_m(x,y)$ is defined to be zero for points (x,y) outside the panel. The errors introduced near the edges by using the approximations (77) and (78) are small compared with the total value provided the panel dimensions are very large compared with $\frac{1}{\kappa}$ and the functions $\varepsilon_m(x,y)$, $\varepsilon_n(x,y)$ do not change much over a distance of $0\left(\frac{1}{\kappa}\right)$.

If τ moves away from zero, then while $u_0\tau$ is still small in comparison with the panel dimensions, $-\frac{|\tau|}{\theta}$ will have become a very large number if θ is very small compared with the panel fundamental period. This means that the integral

$$\iint_{\text{panel}} \varepsilon_m(x_0, y_0)\,\varepsilon_n(x_0+u_0\tau, y_0)\, dx_0\, dy_0 \tag{80}$$

has changed very little in the interval of τ for which $\exp\left(-\frac{|\tau|}{\theta}\right)$ has a sensible value. Hence in evaluating $R_{m,n}(\omega)$ from equation (72) we can take the value of the integral (80) to be the value it has when $\tau = 0$. This gives approximately

$$R_{m,n}(\omega) = \frac{2\overline{f^2}}{\kappa^2}\iint_{\text{panel}} \varepsilon_m(x_0, y_0)\,\varepsilon_n(x_0, y_0)\, dx_0\, dy_0 \int_{-\infty}^{\infty} \exp\left(-\frac{|\tau|}{\theta}\right)\exp(i\omega\tau)\, d\tau$$

$$= 4\overline{f^2}\,\frac{\theta}{\kappa^2}\,\frac{1}{1+\omega^2\theta^2}\,\delta_{m,n} \tag{81}$$

and this does not depend on the convection speed u_0.

The factor $\exp(i\omega\tau)$ in equation (72) will not change much near $\tau = 0$ when ω is small, but when ω is large it will have an oscillatory behaviour near $\tau = 0$ and for this reason it must be retained.

From equation (54) it then follows that the power spectrum $P(x,y,z,\omega)$ of the pressure in the intermediate and far fields is given by

$$P(x,y,z,\omega) = \frac{\rho_0^2}{\pi^2}\frac{\overline{f^2}}{M^2}\frac{\theta}{\kappa^2}\frac{1}{(1+\omega^2\theta^2)}\sum_m \frac{\omega^4}{(-\omega^2+\omega_m^2)^2+\beta^2\omega^2}\iint_{\text{panel}} \frac{\varepsilon_m(x_0,y_0)}{r_0}\exp\left(-\frac{i\omega r_0}{a_0}\right) dx_0\, dy_0 \times$$
$$\times \iint_{\text{panel}} \frac{\varepsilon_n(x_0',y_0')}{r_0'}\exp\frac{i\omega r_0'}{a_0}\, dx_0'\, dy_0' \qquad (82)$$

or, near the perpendicular through the centre of the panel we have, using equation (70)

$$P(x,y,z,\omega) = \frac{\rho_0^2}{\pi^2}\frac{\overline{f^2}}{M^2}\frac{1}{r^2}\frac{\theta}{\kappa^2}\frac{1}{(1+\omega^2\theta^2)}\sum_m \frac{\omega^4 H_m^2}{(-\omega^2+\omega_m^2)^2+\beta^2\omega^2} \qquad (83)$$

The infinite series in equation (83) is rapidly convergent in the case of simply supported edges for when m is large the terms in the series behave like $\frac{1}{\omega_m^4}H_m^2$. By (69) $H_m \doteq 0\left(\frac{1}{m_1 m_2}\right)$ and by (35) $\omega_m = 0(m_1^2)+$ $+0(m_2^2)$ so the terms tend to zero rapidly as $m\rightarrow\infty$.

The term $\frac{\omega^4 H_m^2}{(-\omega^2+\omega_m^2)^2+\beta^2\omega^2}$ of the series has a maximum value of $\frac{\omega_m^2}{\beta^2}\frac{1}{\left(1-\frac{\beta^2}{4\omega_m^2}\right)}$ at $\omega = \frac{\omega_m}{\sqrt{1-\frac{\beta^2}{2\omega_m^2}}}$ and if β is very small this term will dominate all the other terms of the series. The power spectrum $P(x,y,z,\omega)$ will therefore have maxima at, or very near to, the values $\omega = \frac{\omega_m}{\sqrt{1-\frac{\beta^2}{2\omega_m^2}}}$ when β is very small.

Near these maxima just one term of the series in (83) will be a good representation for the whole series.

We can evaluate $\zeta_{m,n}(\tau)$ and $\mu_{m,n}(\tau)$ using equations (56) and (62). These functions must then be substituted into equation (55) to get the sound intensity. The result still involves quadruple integrals which then have to be evaluated numerically. A great deal of simplification occurs if the observation point is near to the perpendicular through the centre of the panel and then the expression (67) can be used for the intensity.

With the approximation procedure which we are using for dealing with the turbulent boundary-layer pressure correlation we get, from substituting the expression (81) for $R_{m,n}(\omega)$ into (56) and (62), the relations

$$\zeta_{m,n}(\tau) = 0 \quad m \neq n$$

$$\mu_{m,n}(\tau) = 0 \quad m \neq n.$$

Also, since the integrand in equation (62) is an odd function of ω when $\tau = 0$, we have

$$\mu_{m,m}(0) = 0.$$

Finally, from (56) and (81) we have

$$\zeta_{m,m}(0) = \frac{2}{\pi}\frac{\overline{f^2}}{M^2}\frac{\theta}{\kappa^2}\int_{-\infty}^{\infty}\frac{\omega^4\,d\omega}{[1+\omega^2\theta^2]\,[(\omega^2-\omega_m^2)^2+\beta^2\omega^2]}. \tag{87}$$

The integral appearing in equation (87) is evaluated in AppendixII. The result for $\zeta_{m,m}(0)$ is

$$\zeta_{m,m}(0) = 2\frac{\overline{f^2}}{M^2}\frac{1}{\kappa^2}\left\{\frac{1}{[(1+\theta^2\omega_m^2)^2-\beta^2\theta^2]}+\frac{\theta}{\beta}\,\frac{(\omega_m^2-\beta^2)+\theta^2\omega_m^4}{[1+\theta^2(2\omega_m^2-\beta^2)+\theta^4\omega_m^4]}\right\} \tag{88}$$

The final expression for the intensity, obtained from equation (67), is then

$$\Upsilon(x,y,z) = \frac{\rho_0}{2\pi^2}\frac{\overline{f^2}}{M^2}\frac{1}{\kappa^2}\frac{1}{a_0 r^2}\sum_m H_m^2\left\{\frac{1}{(1+\theta^2\omega_m^2)^2-\beta^2\theta^2}+\frac{\theta}{\beta}\,\frac{(\omega_m^2-\beta)+\theta^2\omega_m^4}{1+\theta^2(2\omega_m^2-\beta^2)+\theta^4\omega_m^4}\right\}. \tag{89}$$

In the series

$$\sum_m H_m^2\frac{1}{[(1+\theta^2\omega_m^2)^2-\beta^2\theta^2]}$$

the terms behave like $\frac{H_m^2}{\omega_m^4}$ for large m. For simply supported edges

$$\frac{H_m^2}{\omega_m^4} = \frac{1}{0(m_1^2m_2^2).\,0(m_1^2+m_2^2)^4}$$

so that the series is eventually rapidly convergent. For small values of θ summation over more terms will be required for a given accuracy than for large values of θ.

In the series

$$\sum_m H_m^2\frac{\theta}{\beta}\,\frac{(\omega_m^2-\beta^2)+\theta^2\omega_m^4}{[1+\theta^2(2\omega_m^2-\beta^2)+\theta^4\omega_m^4]}$$

the terms behave like H_m^2 for large m. For simply supported edges $H_m^2 = \frac{1}{O(m_1^2m_2^2)}$ so that the series is convergent but not very rapidly.

Under the same conditions the correlation coefficient given in equation (53) may be evaluated. A case of interest is when $\tau = 0$ and then it is found that the mean-square pressure $\overline{p^2}$ at the point (x,y,z) on or near the perpendicular is given by

$$\overline{p^2} = \phi(x,y,z,0) = \rho_0 a_0\,\Upsilon(x,y,z) \tag{90}$$

which is the same as the relation (25) obtained for the far field. Near the perpendicular through the centre of the panel the expressions in the intermediate field are in fact the same as in the far field when we use the approximate treatment of the boundary-layer pressure correlation for then $\mu_{m,n}(0)$ vanishes for all m and n.

5. *Discussion.*

Equations have been given in Section 3 which enable the power spectrum of the pressure and the value of the sound intensity to be determined in the radiation field of a single panel embedded in an infinite plane wall and subject to a turbulent boundary-layer excitation.

The procedures for obtaining these quantities from the equations are described briefly at the beginning of Section 4 for the case when either the correlation function $\psi(x,y,x',y',\tau)$ or the power spectrum $Q(x,y,x',y',\omega)$ of the excitation is given as a general function, or maybe only as a set of numerical values. These procedures could be applied directly with the correlation function given in equation (73).

Simplifications are possible in using the correlation function (73) in certain circumstances and these lead to equations (82) for the power spectrum and to the equation (89) for the sound intensity in the far field. The value of $\frac{1}{\kappa}$ must be very small in comparison with the dimensions of the panel and the value of θ must be very small in comparison with the fundamental period of the panel for the simplifications to be valid. Also convergence of the series in these equations must be sufficiently rapid so that terms of the series which contribute significantly to its sum correspond only to values of m for which the modal functions $\varepsilon_m(x,y)$ do not change very much over a distance of $0(\frac{1}{\kappa})$. It is not likely that this will occur in practical cases so the full analysis using expression (73), or, if possible, a more representative function for the correlation would need to be carried out.

In this simplified analysis the convection velocity u_0 of the spatial pattern of the excitation correlation does not appear in the results. In a more accurate analysis there would be dependence on this velocity, but in view of the result of the simplified analysis the dependence may be expected to be small. In the more accurate analysis the actual form of the modal functions $\varepsilon_m(x,y)$ will have to be known, and in the first instance the modal functions, given in equation (36), corresponding to simple support edge conditions could be used. If the number of terms required in the double series (67) and (70) are not very large, then an attempt could be made using the modal functions corresponding to other edge conditions.

The decay with time separation τ in the correlation function has been brought in by means of a factor $\exp\left(-\frac{|\tau|}{\theta}\right)$ in equation (73). This has led to a factor $\frac{2\theta}{1+\omega^2\theta^2}$, introduced first in equation (81) and appearing in the results given in equation (82) and also influencing the result given in equation (89). If the decay with time had been brought in by means of a factor $\left(1+\frac{|\tau|}{\theta}\right)\exp\left(-\frac{|\tau|}{\theta}\right)$ then the factor $\frac{2\theta}{1+\omega^2\theta^2}$ would be replaced by $\frac{4\theta}{(1+\omega^2\theta^2)^2}$. This would not make much difference to the power spectrum $P(x,y,z,\omega)$ at low values of ω, if the estimate for θ were halved, but it does make an appreciable difference at high values of ω, and consequently makes a substantial difference to the intensity calculated, for the contributions to Υ from $P(x,y,z,\omega)$ at the higher values of ω are important. This is also evident from the fact that the infinite integrals for $\zeta_{m,m}(0)$ are more powerfully convergent than formerly. It would appear therefore that the actual shape of the correlation function near $\tau = 0$ is important.

A somewhat more elaborate analysis than the simple one considered could have been carried out on the assumption that $\frac{1}{\kappa}$ was large enough for equation (79) to be valid, but θ not small enough for equation (81) to be valid. If the $\varepsilon_m(x,y)$ are given by equation (36) then it is possible to proceed from $\Gamma_{m,n}(\tau)$ to the final results without any further approximations, but the expressions involved are rather long and complicated.

No results have been worked out from the final equations (82), (83) or (89). It may be observed from equation (83) that the magnitude of the power spectrum is proportional to $\frac{1}{r^2}\frac{\rho_0}{M^2}\frac{\overline{f^2}\theta}{\kappa^2}$ at small values of ω at fixed values of $\frac{M}{D}$ and of β. According to equation (75), $\overline{f^2}$ is proportional to $\rho_0^2 U_0^4$ and according to equations (76), $\frac{\theta}{\kappa^2}$ is proportional to $\frac{\delta^3}{U_0}$. Hence the magnitude of the power spectrum at small values of ω is proportional to $\frac{1}{r^2}\frac{\rho_0^4}{M^2}\delta^2 U_0^3$.

There is no obvious factor of proportionality in the expression (89) for the intensity, but if β is very small we may write approximately

$$\Upsilon(x,y,z) \simeq \frac{\rho_0}{2\pi^2}\frac{\overline{f^2}}{M^2}\frac{1}{\kappa^2}\frac{1}{a_0 r^2}\frac{1}{\theta\beta}\sum_m H_m^2 \frac{\theta^2 \frac{D}{M}\lambda_m^2}{\left(1+\theta^2\frac{D}{M}\lambda_m^2\right)}$$

$$\simeq \frac{(0{\cdot}006)^2}{240\pi^2}\frac{1}{\beta}\left(\frac{\rho_0}{a_0}\right)\left(\frac{\rho_0}{\rho}\right)^2\frac{U_0^5\delta}{h^2}g\left(\theta\sqrt{\frac{D}{M}}\ \frac{1}{r^2}\right) \tag{91}$$

where

$$g\left(\theta\sqrt{\frac{D}{M}}\right) = \sum_m H_m^2 \frac{\theta^2\frac{D}{M}\lambda_m^2}{\left(1+\theta^2\frac{D}{M}\lambda_m^2\right)} \tag{92}$$

The result (91) is of the form of the main result given by Corcos and Liepmann[2] for the sound intensity in the radiation field from an infinite flexible sheet subject to turbulent boundary layer excitation. The form of the function $g\left(\theta\sqrt{\frac{D}{M}}\right)$ in Ref. 2 is, however, different from that given in equation (92). In the form (91) it would appear that $\Upsilon(x,y,z)$ is proportional to $\frac{U_0^5\delta}{\beta h^2}$. However, for $\frac{\theta^2 D}{M}$ small we have

$$g\left(\theta\sqrt{\frac{D}{M}}\right) \simeq \frac{\theta^2 D}{M}k\left(\theta\sqrt{\frac{D}{M}}\right) \tag{93}$$

where $k\left(\theta\sqrt{\frac{D}{M}}\right) = 0(1)$ for $\theta\sqrt{\frac{D}{M}}$ small. A better way of writing equation (91), in our case, is therefore

$$\Upsilon(x,y,z) \simeq \frac{5}{16\pi^2}(0{\cdot}006)^2\frac{1}{\beta}\left(\frac{\rho_0}{\rho}\right)^3\frac{E}{(1-\sigma^2)}U_0^3\delta^3 k\left(\theta\sqrt{\frac{D}{M}}\ \frac{1}{a_0 r^2}\right) \tag{94}$$

which shows that $\Upsilon(x,y,z)$ is proportional to $\frac{1}{\beta} U_0^3 \delta^3$ for small values of $\theta\sqrt{\frac{D}{M}}$ for a given plate material.

In general $\beta = \frac{b}{M}$ will depend on the plate thickness. If the acoustic damping dominates over the structural damping then b is virtually independent of the plate thickness so $\beta \propto \frac{1}{h}$. The intensity would then increase linearly with h. However as h increases the structural damping increases and β may be expected to tend asymptotically to a constant value. The intensity is then independent of plate thickness for small values of $\theta\sqrt{\frac{D}{M}}$.

The results obtained by Ribner[1] and Kraichnan[3] are different from our present results. The procedures used and the form taken for the correlation function of the excitation are however different from ours.

The experimental results obtained by Ludwig[5], for a single flexible rectangular panel set in a wall and excited by turbulent boundary-layer flow show a proportionality factor of $\frac{U^5 \delta^{\frac{1}{2}}}{h}$ in the total power radiated. This cannot be compared directly with the present results for the total power is obtained by integrating the intensity over a hemisphere of large radius with the wall as diametral plane, and the intensity will vary with position on the surface of this hemisphere.

To determine more accurately how the power spectrum and intensity of sound radiation depend on the boundary-layer displacement thickness δ, free stream velocity U_0, plate thickness h and convection velocity u_0, calculations should be carried out using the procedure described at the beginning of Section 4 and using an accurate correlation function, perhaps that given in equation (73). In this case the term of $0\left(\frac{1}{r^3}\right)$ will not disappear in the expression for the intensity in the intermediate field near to the perpendicular through the centre of the panel for in this case $R_{m,n}(\omega)$ is not an even function of ω giving $\mu_{m,n}(0) \neq 0$, so that the second summation in equation (67) does not vanish. The expression for the mean-square pressure will, as before, not have this term present. Graphs could then be presented to show the variation with δ, U_0, h and v_0.

The mean-square response and the mean-square stresses at points on the panel can also be obtained by the methods used in this paper in a fairly straight-forward manner. The mean-square response is of course just the mean square of $Z(x,y,t)$ but the mean-square stresses will involve mean products of $\frac{\partial^2 Z}{\partial x^2}, \frac{\partial^2 Z}{\partial y^2}, \frac{\partial^3 Z}{\partial x^3}, \frac{\partial^3 Z}{\partial x^2 \partial y}, \frac{\partial^3 Z}{\partial x \partial y^2}, \frac{\partial^3 Z}{\partial y^3}$ taken two at a time. The mean-square response can be considered by the simplified procedure described in Section 4 but this procedure will lead to divergent series in the cases of the mean-square stresses and so a more accurate procedure must be used.

The power spectrum and intensity of sound radiation from several panels vibrating in a plane may also be considered. If there is no correlation between the pressures arising from the vibration of different panels then the power spectrum and intensity are obtained as the arithmetic sum of the separate power spectra and intensities. Otherwise the correlations of the pressure arising from pairs of vibrating panels must be taken into account. Ideally the number of vibrating panels could be infinite but the numerical procedure then becomes forbiddingly lengthy.

6. *Conclusions.*

A theory of determining the power spectrum of the pressure and the intensity of radiated sound from a vibrating panel set in an infinite wall and excited by a turbulent boundary-layer flow has been given. In a simplified analysis it has been shown that the intensity is proportional to $\frac{\delta^3 U_0^3}{\beta}$, where δ is the boundary-layer displacement thickness, U_0 is the free stream velocity and β is a coefficient dependent on the damping and mass per unit area of the plate.

LIST OF SYMBOLS

a_0	Speed of sound
b	Damping coefficient in equation (28)
c	Length of panel in x-direction
d	Length of panel in y-direction
D	Rigidity coefficient
E	Young's modulus
$f(x,y,t)$	Panel exciting force per unit area
$\overline{f^2}$	Mean square of panel exciting force per unit area
$f_m(t)$	Defined in equation (41)
h	Panel thickness
H_m	Defined in equation (68)
m,n	Numerical values of modes
m_1, m_2	Integers associated with m
M	Mass per unit area of the panel
$p(x,y,z,t)$	Excess air pressure over the undisturbed pressure at the point x,y,z at time t
$\overline{p^2}$	Mean square value of the excess pressure
$P(x,y,z,\omega)$	Power spectrum of the excess pressure
$Q(x,y,x',y',\omega)$	Power spectrum of the excitation function $f(x,y,t)$
r	Defined in equation (21)
r_0	Defined in equation (9)
r'_0	Defined in equation (12)
$R_{m,n}(\omega)$	Defined in equation (50)
$S_{m,n}(\tau)$	Defined in equation (57)
t	Time
T	Time
$T_{m,n}(\tau)$	Defined in equation (63)
u,v,w	Fluid particle velocities
U_0	Free stream velocity
u_0	Convection velocity of the pressure spatial correlation pattern
x,y,z	Rectangular cartesian coordinates
$Z(x,y,t)$	Displacement at time t of a point x,y on the panel
α_m	Defined in equation (59)
$\beta = b/M$	

$\gamma_x, \gamma_y, \gamma_z$	Defined in equations (6)
$\Gamma_{m,n}(\tau)$	Defined in equation (61)
δ	Boundary-layer displacement thickness
$\Delta(x,y,x',y',\tau)$	Defined in equation (20)
$\zeta_{m,n}(\tau)$	Defined in equation (46)
η_0	Defined in equation (74)
$\theta_{m,n}(\omega)$	Defined in equation (48)
θ	Constant appearing in equation (73)
κ	Constant appearing in equation (73)
λ	Defined in equation (32)
λ_m	Defined in equation (33)
$\mu_{m,n}(\tau)$	Defined in equation (66)
$\nu_{m,n}(\omega)$	Defined in equation (49)
ξ_0	Defined in equation (74)
$\xi_m(t)$	Generalised coordinate for the mode ^{m}
ρ	Density of the plate material
ρ_0	Density of air
τ	Time difference
$\dot{\Upsilon}$	Sound intensity defined in equation (7)
$\phi(x,y,z,\tau)$	Autocorrelation function for the pressure, defined in equation (1)
(x,y,z,t)	Velocity potential given by equation (8)
$\chi(x,y,x',y',\tau)$	Defined in equation (14)
$\psi(x,y,x',y',\tau)$	Correlation function for the excitation, defined in equation (42)
ω	Circular frequency
ω_m	Natural frequency of the panel in mode m

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APPENDIX I

Formal Derivation of Dependent Power Spectra

We consider the excitation function $f(x,y,t)$ and expand it in a Fourier series in t in the interval $-T<t<T$ *i.e.* let

$$f^T(x,y,t) = \sum_{k=-\infty}^{\infty} A^T(x,y,k) \exp\left(-\frac{i\pi kt}{T}\right) \tag{95}$$

where

$$A^T(x,y,k) = \frac{1}{2T}\int_{-T}^{T} f(x,y,t) \exp\left(\frac{i\pi kt}{T}\right) dt. \tag{96}$$

Then

$$f^T(x,y,t) = f(x,y,t) \quad -T<t<T \tag{97}$$

and $f^T(x,y,t)$ has period $2T$ in t.

We shall replace $f(x,y,t)$ by $f^T(x,y,t)$ when t is outside the interval $-T<t<T$. Later we shall consider the limiting process $T\to\infty$. We have

$$\psi(x,y,x',y',\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x,y,t) f(x',y',t+\tau)\, dt$$

$$= \lim_{T\to\infty} \left[\frac{1}{2T}\int_{-T}^{T} f^T(x,y,t) f^T(x',y',t+\tau)\, dt + \frac{1}{T}0(\tau)\right]$$

$$= \lim_{T\to\infty} \left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} A^T(x,y,k)\, A^T(x',y',l) \frac{1}{2T}\int_{-T}^{T} \exp\left(-\frac{i\pi kt}{T}\right) \exp\left(-\frac{i\pi l(t+\tau)}{T}\right) dt\right]$$

$$= \lim_{T\to\infty} \sum_{l=-\infty}^{\infty} A^T(x,y,-l)\, A^T(x',y',l) \exp\left(-\frac{i\pi l\tau}{T}\right) \tag{98}$$

and passing formally to the limit $T = \infty$ we get

$$\psi(x,y,x',y',\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \left\{2T\, A^T\left(x,y,-\frac{\omega T}{\pi}\right) A^T\left(x',y',\frac{\omega T}{\pi}\right)\right\} \exp(-i\omega\tau)\, d\omega\Bigg\} . \tag{99}$$

On inverting equation (43) of the main text we get

$$\psi(x,y,x',y',\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(x,y,x',y',\omega) \exp(-i\omega\tau)\, d\omega \tag{100}$$

so on comparing (99) and (100) we get the result

$$\lim_{T\to\infty} \left\{2T\, A^T\left(x,y,-\frac{\omega T}{\pi}\right) A^T \quad x',y',\frac{\omega T}{\pi}\right)\right\} = Q(x,y,x',y',\omega). \tag{101}$$

Instead of equation (39) of the main text we now consider

$$\ddot{\xi}_m^T(t) + \beta\, \dot{\xi}_m^T(t) + \omega_m^2\, M\, \xi_m^T(t) = f_m^T(t) \tag{102}$$

where

$$f_m^T(t) = \frac{1}{M}\iint_{\text{panel}} f^T(x,y,t)\, \varepsilon_m(x,y)\, dx\, dy. \tag{103}$$

It may be noted that $f_m^T(t)$ coincides with $f_m(t)$ for $-T<t<T$. If expression (95) is substituted for $f^T(x,y,t)$ in (103) we get

$$f_m^T(t) = \sum_{k=-\infty}^{\infty} B_m^T(k) \exp\left(-\frac{i\pi kt}{T}\right) \tag{104}$$

where

$$B_m^T(k) = \frac{1}{M}\iint_{\text{panel}} A^T(x,y,k)\, \varepsilon_m(x,y)\, dx\, dy. \tag{105}$$

The function $f_m^T(t)$ is of period $2T$ and so the function $\xi_m^T(t)$ satisfying equation (102) is of period $2T$. Let the Fourier series of $\xi_m^T(t)$ be

$$\xi_m^T(t) = \sum_{k=-\infty}^{\infty} C_m^T(k) \exp\left(-\frac{T}{i\pi kt}\right. \tag{106}$$

where

$$C_m^T(k) = \frac{1}{2T}\int_{-T}^{T} \xi_m^T(t) \exp\left(\frac{i\pi kt}{T}\right) dt. \tag{107}$$

Substituting the expansions (104) and (106) for $f^T(t)$ and $\xi_m^T(t)$ into the differential equation (102) and comparing coefficients of $\exp\left(-\frac{i\pi kt}{T}\right)$ we get

$$\left[-\left(\frac{\pi k}{T}\right)^2 - i\beta\left(\frac{\pi k}{T}\right) + \omega_m^2\right] C_m^T(k) = B_m^T(k) \tag{108}$$

from which we can express $C_m^T(k)$ in terms of $B_m^T(k)$ by

$$C_m^T(k) = \frac{B_m^T(k)}{\left[-\left(\frac{\pi k}{T}\right)^2 - i\beta\left(\frac{\pi k}{T}\right) + \omega_m^2\right]}. \tag{109}$$

In the range $-T<t<T$ we may write

$$\xi_m(t) = \xi_m^T(t) + \eta_m^T(t) \tag{110}$$

where $\eta_m^T(t)$ is a complementary function of the differential equation (39) obtained by replacing the right hand side of (39) by zero. This complementary function will take account of that fact that $\xi_m(t)$ and $\xi_m^T(t)$ are not equal when $t = -T$ and neither are their first derivatives.

When the damping coefficient β is positive non-zero and small the complementary function will be a decaying oscillatory function with amplitude of oscillation decreasing exponentially as t increases from $t = -T$. This then leads to the equations

$$\zeta_{m,n}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2}\xi_m^T(t)\frac{\partial^2}{\partial t^2}\xi_n^T(t+\tau)\,dt \tag{111}$$

and

$$\mu_{m,n}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\xi_m^T(t)\frac{\partial^2}{\partial t^2}\xi_n^T(t+\tau)\,dt \tag{112}$$

instead of (46) and (47).

Using the expansion (106) in (111) we get

$$\zeta_{m,n}(\tau) = \lim_{T\to\infty}\left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} C_m^T(k)\,C_n^T(l)\left(\frac{i\pi k}{T}\right)^2\left(\frac{i\pi l}{T}\right)^2\frac{1}{2T}\int_{-T}^{T}\exp\left(-\frac{i\pi kt}{T}\right)\exp\left(-\frac{i\pi l(t+\tau)}{T}\right)dt\right.$$

$$= \lim_{T\to\infty}\sum_{l=-\infty}^{\infty} C_m^T(-l)\,C_n^T(l)\left(\frac{\pi l}{T}\right)^4\exp\left(-\frac{i\pi l\tau}{T}\right)$$

$$= \lim_{T\to\infty}\sum_{l=-\infty}^{\infty}\frac{B_m^T(-l)\,B_n^T(l)\left(\frac{\pi l}{T}\right)^4\exp\left(-\frac{i\pi l\tau}{T}\right)}{\left[-\left(\frac{\pi l}{T}\right)^2 + i\beta\left(\frac{\pi l}{T}\right)^2 + \omega_m^2\right]\left[-\left(\frac{\pi l}{T}\right)^2 - i\beta\left(\frac{\pi l}{T}\right) + \omega_n^2\right]} \tag{113}$$

and passing formally to the limit $T = \infty$ we get

$$\zeta_{m,n}(\tau) = \frac{1}{2\pi M^2} \int_{-\infty}^{\infty} \frac{\omega^4 R_{m,n}(\omega) \exp(-i\omega\tau)\, d\omega}{[-\omega^2 + i\beta\omega + \omega_m^2][-\omega^2 - i\beta\omega + \omega_n^2]} \tag{114}$$

where

$$R_{m,n}(\omega) = \lim_{T\to\infty} \left\{ 2TM^2 B_m^T \left(-\frac{T\omega}{\pi}\right) B_n^T \left(\frac{T\omega}{\pi}\right) \right\} . \tag{115}$$

Then, using the formula (105) for $B_m^T(k)$ we get from (115)

$$R_{m,n}(\omega) = \lim_{T\to\infty} 2T \iint_{\text{panel}} A\left(x,y,-\frac{T\omega}{\pi}\right) \varepsilon_m(x,y)\, dx\, dy \iint_{\text{panel}} A\left(x',y',\frac{T\omega}{\pi}\right) \varepsilon_n(x',y')\, dx'\, dy'$$

$$\lim_{T\to\infty} \iint_{\text{panel}} \iint_{\text{panel}} 2T\, A\left(x,y,-\frac{T\omega}{\pi}\right) A\left(x',y',\frac{T\omega}{\pi}\right) \varepsilon_m(x,y)\, \varepsilon_n(x',y')\, dx\, dy\, dx'\, dy'$$

$$= \iint_{\text{panel}} \iint_{\text{panel}} Q(x,y,x',y',\omega)\, \varepsilon_m(x,y)\, \varepsilon_n(x',y')\, dx\, dy\, dx'\, dy' \tag{116}$$

corresponding with the definition (50).

On inverting equation (48) we get

$$\zeta_{m,n}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta_{m,n}(\omega) \exp(-i\omega\tau)\, d\omega \tag{117}$$

and on comparing (114) and (117) we get

$$\theta_{m,n}(\omega) = \frac{1}{M^2} \frac{\omega^4 R_{m,n}(\omega)}{[-\omega^2 + i\beta\omega + \omega_m^2][-\omega^2 - i\beta\omega + \omega_n^2]} . \tag{118}$$

Similarly using the expansion (106) in (112) we get

$$\mu_{m,n}(\tau) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} C_m^T(k)\, C_n^T(k) \left(-\frac{i\pi k}{T}\right) \left(\frac{i\pi l}{T}\right)^2 \frac{1}{2T} \int_{-T}^{T} \exp\left(-\frac{i\pi kt}{T}\right) \exp\left(-\frac{i\pi l(t+\tau)}{T}\right) dt$$

$$= \lim_{T\to\infty} \sum_{l=-\infty}^{\infty} \frac{-i\, B_m^T(l)\, B_n^T(-l) \left(\frac{\pi l}{T}\right)^3 \exp\left(-\frac{i\pi l\tau}{T}\right)}{\left[-\left(\frac{\pi l}{T}\right)^2 + i\beta\left(\frac{\pi l}{T}\right) + \omega_m^2\right] \left[-\left(\frac{\pi l}{T}\right)^2 - i\beta\left(\frac{\pi l}{T}\right) + \omega_n^2\right]}$$

$$= \frac{1}{2\pi M^2} \int_{-\infty}^{\infty} \frac{-i\omega^3 R_{m,n}(\omega) \exp(-i\omega\tau)\, d\omega}{[-\omega^2 + i\beta\omega + \omega_m^2][-\omega^2 - i\beta\omega + \omega_n^2]} . \tag{119}$$

But on inversion of (49) we get

$$\mu_{m,n}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \nu_{m,n}(\omega) \exp(-i\omega\tau)\, d\omega. \tag{120}$$

Hence

$$\nu_{m,n}(\omega) = -\frac{1}{M^2} \frac{i\omega^3 R_{m,n}(\omega)}{[-\omega^2+i\beta\omega+\omega_m^2][-\omega^2-i\beta\omega+\omega_n^2]}. \tag{121}$$

APPENDIX II

Evaluation of Integrals

(i) It is required to evaluate

$$I_{m,n}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^4 \exp(-i\omega\tau)\, d\omega}{[-\omega^2+i\beta\omega+\omega_m^2][-\omega^2-i\beta\omega+\omega_n^2]} \tag{122}$$

We can write the integral as

$$I_{m,n}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[1+\frac{\omega^2(\omega_m^2+\omega_n^2-\beta^2)+i\beta\omega\,(\omega_m^2-\omega_n^2)-\omega_m^2\,\omega_n^2}{(\omega_m^2-\omega^2+i\beta\omega)\,(\omega_n^2-\omega^2-i\beta\omega)}\right] \exp(-i\omega\tau)\, dw$$

$$= \delta(\tau)+S_{m,n}(\tau) \tag{123}$$

where $\delta(\tau)$ is Dirac's delta function and $S_{m,n}(\tau)$ is given by

$$S_{m,n}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{\omega^2(\omega_m^2+\omega_n^2-\beta^2)+i\beta\omega\,(\omega_m^2-\omega_n^2)-\omega_m^2\,\omega_n^2}{(\omega_m^2-\omega^2+i\beta\omega)\,(\omega_m^2-\omega^2-i\beta\omega)}\right] \exp(-i\omega\tau)\, d\omega. \tag{124}$$

The integral for $S_{m,n}(\tau)$ will be evaluated by means of complex contour integration and an application of Cauchy's theorem of residues.

Consider the function

$$F_{m,n}(\omega) = \frac{\omega^2(\omega_m^2+\omega_n^2-\beta^2)+i\beta\omega\,(\omega_m^2-\omega_n^2)-\omega_m^2\omega_n^2}{(\omega_m^2-\omega^2+i\beta\omega)\,(\omega_n^2-\omega^2-i\beta\omega)} \exp(-i\omega\tau)$$

$$= \frac{\omega^2(\omega_m^2+\omega_n^2-\beta^2)+i\beta\omega\,(\omega_m^2-\omega_n^2)-\omega_m^2\omega_n^2}{(\omega-\omega_1)\,(\omega-\omega_2)\,(\omega-\omega_3)\,(\omega-\omega_4)} \exp(-i\omega\tau) \tag{125}$$

where $\omega_1, \omega_2, \omega_3, \omega_4$ are the zeros of the denominator given by

$$\left.\begin{aligned} \omega_1 &= \frac{i\beta}{2}+\alpha_m \qquad \omega_2 = \frac{i\beta}{2}-\alpha_m \\ \omega_3 &= -\frac{i\beta}{1}+\alpha_n \quad \omega_4 = -\frac{i\beta}{2}-\alpha_n \end{aligned}\right\} \tag{126}$$

and

$$\alpha_m = \sqrt{\omega_m^2-\frac{\beta^2}{4}} \tag{127}$$

$F_{m,n}(\omega)$ has simple poles at the zeroes $\omega_1, \omega_2, \omega_3$ and ω_4 of the denominator. Let the residues at these poles be R_1, R_2, R_3, and R_4 respectively. Then

$$R_1 = \frac{\omega_1^2(\omega_m^2+\omega_n^2-\beta^2)+i\beta\omega_1(\omega_m^2-\omega_n^2)-\omega_m^2\omega_n^2}{(\omega_1-\omega_2)(\omega_1-\omega_3)(\omega_1-\omega_4)}\exp(-i\omega_1\tau)$$

$$= \frac{\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4+i\beta\,\alpha_m(2\omega_m^2-\beta^2)}{2\alpha_m[\omega_m^2-\omega_n^2-\beta^2+2i\beta\,\alpha_m]} \quad \exp\left(\frac{\beta}{4}-i\alpha_m\right)\tau \tag{128}$$

$$R_2 = \frac{\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4-i\beta\,\alpha_m(2\omega_m^2-\beta^2)}{-2\alpha_m[\omega_m^2-\omega_n^2-\beta^2-2i\beta\,\alpha_m]} \quad \exp\left(\frac{\beta}{4}+i\alpha_m\right)\tau \tag{129}$$

$$R_3 = \frac{\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4-i\beta\,\alpha_n(2\omega_n^2-\beta^2)}{2\alpha_n[\omega_n^2-\omega_m^2-\beta^2-2i\beta\,\alpha_n]} \quad \exp\left(-\frac{\beta}{2}-i\alpha_n\right)\tau \tag{130}$$

$$R_4 = \frac{\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4+i\beta\,\alpha_n(2\omega_n^2-\beta^2)}{-2\alpha_n[\omega_n^2-\omega_m^2-\beta^2+2i\beta\,\alpha_n]} \quad \exp\left(-\frac{\beta}{2}+i\alpha_n\right)\tau \tag{131}$$

The contour chosen for the evaluation of $S_{m,n}(\tau)$ is the real axis and an infinite semicircle with centre origin and real axis as diameter. When $\tau>0$ the integral over the semicircle in the lower half-plane vanishes so

$$S_{m,n}(\tau) = -i(R_3+R_4) \qquad \tau>0. \tag{132}$$

When $\tau<0$ the integral over the semicircle in the upper half-plane vanishes so

$$= i(R_1+R_2) \qquad \tau<0. \tag{133}$$

The equations (132) and (133) lead to the equations (58) when the expressions (128) to (131) are used for the residues R_1 to R_4.

(ii) It is required to evaluate

$$T_{m,n}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-i\omega^3 \exp(-i\omega\tau)\, d\omega}{[-\omega^2 + i\beta\omega + \omega_m^2][-\omega^2 - i\beta\omega + \omega_n^2]}. \tag{134}$$

We write the integral as

$$T_{m,n}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[-\frac{i}{\omega} - i\frac{\omega^2(\omega_m^2 + \omega_n^2 - \beta^2) + i\beta\omega(\omega_m^2 - \omega_n^2) - \omega_m^2\omega_n^2}{\omega(\omega_m^2 - \omega^2 + i\beta\omega)(\omega_n^2 - \omega^2 - i\beta\omega)} \right] \exp(-i\omega\tau)\, d\omega$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \left[-\frac{i}{\omega} - i\frac{\omega^2(\omega_m^2 + \omega_n^2 - \beta^2) + i\beta\omega(\omega_m^2 - \omega_n^2) - \omega_m^2\omega_n^2}{\omega(\omega_m^2 - \omega^2 + i\beta\omega)(\omega_n^2 - \omega^2 - i\beta\omega)} \right] \exp(-i\omega\tau)\, d\omega$$

$$= H(\tau) + V_{m,n}(\tau) \tag{135}$$

where

$$H(\tau) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} -\frac{i}{\omega} \exp(-i\omega\tau)\, d\omega \tag{136}$$

and

$$V_{m,n}(\tau) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{-\varepsilon} \int_{\varepsilon}^{\infty} -i\frac{\omega^2(\omega_m^2 + \omega_n^2 - \beta^2) + i\beta\omega(\omega_m^2 - \omega_n^2) - \omega_m^2\omega_n^2}{\omega(\omega_m^2 - \omega^2 + i\beta\omega)(\omega_n^2 - \omega^2 - i\beta\omega)} \exp(-i\omega\tau)\, d\omega. \tag{137}$$

Evaluation of (136) is straightforward and gives

$$H(\tau) = \begin{cases} -\frac{1}{2} & \tau > 0 \\ \frac{1}{2} & \tau < 0 \end{cases} \tag{138}$$

The integral for $V_{m,n}(\tau)$ will be evaluated by means of complex integration and application of Cauchy's theorem of residues.

Consider the function

$$G_{m,n}(\omega) = -i\frac{\omega^2(\omega_m^2 + \omega_n^2 - \beta^2) + i\beta\omega(\omega_m^2 - \omega_n^2) - \omega_m^2\omega_n^2}{\omega(\omega_m^2 + \omega_n^2 + i\beta\omega)(\omega_n^2 - \omega^2 - i\beta\omega)} \exp(-i\omega\tau)$$

$$= -i\frac{\omega^2(\omega_m^2 + \omega_n^2 - \beta^2) + i\beta\omega(\omega_m^2 - \omega_n^2) - \omega_m^2\omega_n^2}{\omega(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)} \exp(-i\omega\tau) \tag{139}$$

where ω_1, ω_2, ω_3, ω_4 are given by equations (119).

$G_{m,n}(\omega)$ has simple poles at the origin and at ω_1, ω_2, ω_3 and ω_4. Let the residues at these poles be R_0, R_1, R_2, R_3 and R_4 respectively. Then

$$R_0 = -i \tag{140}$$

$$R_1 = -\frac{i}{2}\,\frac{(\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4)+i\beta\alpha_m(2\omega_m^2-\beta^2)}{\alpha_m\left(\frac{i\beta}{2}+\alpha_m\right)(\omega_m^2-\omega_n^2-\beta^2+2i\beta\alpha_m)}\exp\left(\frac{\beta}{2}-i\alpha_m\right)\tau \tag{141}$$

$$R_2 = \frac{i}{2}\,\frac{(\omega_m^4-2\beta^2\omega_m^2+\frac{1}{2}\beta^4)-i\beta\alpha_m(2\omega_m^2-\beta^2)}{\alpha_m\left(\frac{i\beta}{2}-\alpha_m\right)(\omega_m^2-\omega_n^2-\beta^2-2i\beta\alpha_m)}\exp\left(\frac{\beta}{2}+i\alpha_m\right)\tau \tag{142}$$

$$R_3 = -\frac{i}{2}\,\frac{(\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4)-i\beta\alpha_n(2\omega_n^2-\beta^2)}{\alpha_n\left(-\frac{i\beta}{2}+\alpha_n\right)(\omega_n^2-\omega_m^2-\beta^2-2i\beta\alpha_n)}\exp\ -\frac{\beta}{2}-i\alpha_n\ \ \tau \tag{143}$$

$$R_4 = \frac{i}{2}\,\frac{(\omega_n^4-2\beta^2\omega_n^2+\frac{1}{2}\beta^4)+i\beta\alpha_n(2\omega_n^2-\beta^2)}{\alpha_n\left(-\frac{i\beta}{2}+\alpha_n\right)(\omega_n^2-\omega_m^2-\beta^2+2i\beta\alpha_n)}\exp\left(-\frac{\beta}{2}+i\alpha_n\right)\tau \tag{144}$$

The contour chosen for the evaluation of $V_{m,n}(\tau)$ is the real axis, intended by a semicircle of radius ε and centre origin, and an infinite semicircle with centre origin and real axis as diameter. When $\tau>0$ the integral over the infinite semicircle in the lower half-plane vanishes, so we integrate over a contour in the lower half-plane and then on making $\varepsilon\to 0$ we get

$$V_{m,n}(\tau) = -i\ \ R_3+R_4-\frac{1}{2}R_0 \quad \tau>0. \tag{145}$$

When $\tau<0$ the integral over the infinite semicircle in the upper half-plane vanishes, so we integrate over a contour in the upper half-plane and then on making $\varepsilon\to 0$ we get

$$V_{m,n}(\tau) = i\left(R_i+R_2+\frac{1}{2}R_0\right) \quad \tau>0. \tag{146}$$

Hence

$$T_{m,n}(\tau) = \begin{cases} -i\,(R_3+R_4)\ \tau>0 \\ i\,(R_1+R_2)\ \tau<0. \end{cases} \tag{147}$$

The equations (147) lead to the equations (64) when the expressions (141) to (144) are used for the residues R_1 to R_4.

(iii) It is required to evaluate

$$I_m = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2\theta^2}\,\frac{\omega^4}{(-\omega^2+\omega_m^2)^2+\beta^2\omega^2}\,d\omega. \tag{148}$$

Consider the function

$$F_m(\omega) = \frac{1}{1+\omega^2\theta^2}\,\frac{\omega^4}{(-\omega^2+\omega_m^2)+\beta^2\omega^2}\,. \tag{149}$$

The denominator of this function has six simple zeroes, ω_1, ω_2, ω_3, ω_4, ω_5 and ω_6 where

$$\left.\begin{aligned}
\omega_1 &= -\frac{i}{\theta}\\
\omega_2 &= \frac{i}{\theta}\\
\omega_3 &= \frac{i\beta}{2}+\alpha_m\\
\omega_4 &= \frac{i\beta}{2}-\alpha_m\\
\omega_5 &= -\frac{i\beta}{2}+\alpha_m\\
\omega_6 &= -\frac{i\beta}{2}-\alpha_m
\end{aligned}\right\} \tag{150}$$

and α_m is given by equation (127).

In this case the integral over the infinite semicircle in either the upper or lower half-plane vanishes. If we take a contour in the upper half plane we find that the value of the integral is

$$I_m = i\,[R_2+R_3+R_4] \tag{151}$$

where R_2, R_3 and R_4 are the residues of $F_m(\omega)$ at the poles ω_2, ω_3 and ω_4 respectively.

Now

$$R_2 = -\frac{i}{2\theta}\,\frac{1}{[(1+\theta^2\omega_m^2)^2-\beta^2\theta^2]} \tag{152}$$

$$R_3 = \frac{\left(\frac{i\beta}{2}+\alpha_m\right)^3}{4i\beta\,\alpha_m\left[1+\theta^2\left(\frac{i\beta}{2}+\alpha_m\right)^2\right]} \tag{153}$$

$$R_4 = -\frac{\left(\frac{i\beta}{2}-\alpha_m\right)^3}{4i\beta\,\alpha_m\left[1+\theta^2\left(\frac{i\beta}{2}-\alpha_m\right)^2\right]}. \tag{154}$$

Hence

$$I_m = \frac{1}{2\theta}\,\frac{1}{[(1+\theta^2\omega_m^2)^2-\beta^2\theta^2]}+$$

$$+\frac{1}{2\beta}\,\frac{(\omega_m^2-\beta^2)+\theta^2\omega_m^4}{[1+\theta^2(2\omega_m^2-\beta^2)+\theta^4\omega_m^4]}. \tag{155}$$

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