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COMPLEX VARIABLE APPLICATIONS TO CERTAIN COUPLED SYSTEMS

By

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Complex Variable Applications to Certain Coupled Systems

- By -

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February, 1957

SUMMARY

It is shown that the solutions of similar differential equations which are coupled together can be expressed in terms of the solutions of a single differential equation, possibly containing complex parameters, but of the same order as each separate equation. Some implications of this result are discussed, and Nyquist's criterion is generalized to study the stability of constant parameter systems of this type.

Contents

1. Introduction

It is frequently possible to represent a pair of linear coupled equations by a single equation containing complex quantities, the conditions on the real and imaginary parts separately giving the original relations. This gives a compact way of handling the equations and can be a convenient method of obtaining analytical solutions.

This Memorandum shows that it is possible to use complex numbers to simplify the analysis of any number of linear systems which are coupled together provided that the separate systems are alike, and the couplings of similar form. The behaviour of the coupled systems can be written as a superposition of the response of a number of uncoupled systems each of the same order as one of the original separate systems, though possibly containing complex parameters. The method is a generalization of the transformation to normal coordinates used in the dynamical theory of small oscillations.

The theoretical study of systems containing such complex parameters is no more difficult than if the parameters were purely real. Any analytical solution has only to be extended to the complex plane, and the knowledge that it will be an analytic function of the complex parameter satisfying the Cauchy-Riemann relations may assist in an

understanding/

understanding of its properties. Constant parameter systems whose exponential solutions are well known for complex arguments are particularly easy to treat, and Nyquist's criterion readily extended to discuss their stability.

While this paper was being written, the author's attention was directed to work by Merson (1954) and Jeffrey (1955) (the latter unpublished) which is related; but the present treatment is different and may be said in some ways to unify the two earlier approaches.

2. The Complex Variable Concept

It will be useful to study first some examples of the way in which complex variable notation can simplify the mathematical formulation of a problem.

Consider the pair of differential equations

$$x \cos \phi + y \sin \phi = T \frac{d}{dt}(r - x)$$
$$-x \sin \phi + y \cos \phi = T \frac{d}{dt}(s - y) \qquad \ldots(1)$$

The orthodox method of solving these equations is to eliminate y between them solving the resulting second order equation for x, and then repeating the process for y. But by defining

$$z = x + iy \; ; \; q = r + is$$

the equations may be identified with the equations for the real and imaginary parts of

$$ze^{-i\phi} = T \frac{d}{dt}(q - z)$$

or, writing $Te^{i\phi} = S$

$$\left(1 + S \frac{d}{dt}\right) z = S \frac{dq}{dt} \qquad \ldots(2)$$

If a sinusoidal input is applied to one plane only so that s vanishes and $q = r = \sin \omega t$, it is easily verified that the solution of equation (2) with $z = 0$ at $t = 0$ is

$$(1 + \omega^2 S^2)\, z = \omega S \, [\omega S \sin \omega t + \cos \omega t - \exp(-t/S)] \qquad \ldots(3)$$

and/

and this is true whether S is real or complex. In this example, the output signals x and y will be the real and imaginary parts, respectively, of z. These are seen to be given by

$$(1 + 2\omega^2 T^2 \cos 2\phi + \omega^4 T^4)\, x$$

$$= \omega T \{\omega T (\cos 2\phi + \omega^2 T^2) \sin \omega t + (1 + \omega^2 T^2) \cos \phi \cos \omega t$$

$$- \exp(- t \cos \phi/T) [\cos (\phi + t \sin \phi/T) + \omega^2 T^2 \cos (\phi - t \sin \phi/T)]\} \quad \ldots(4)$$

$$(1 + 2\omega^2 T^2 \cos 2\phi + \omega^4 T^4)\, y = \omega T \{\omega T \sin 2\phi \sin \omega t + (1 - \omega^2 T^2) \sin \phi \cos \omega t$$

$$- \exp(- t \cos \phi/T) [\sin (\phi + t \sin \phi/T) - \omega^2 T^2 \sin (\phi - t \sin \phi/T)]\}$$

The coupling operation in this example, a transformation between error and torque axes, is particularly simply represented by using complex variables. In general, if signals r and s corresponding to motion in one set of rectangular axes are resolved into signals r' and s' in another set of axes making an angle ϕ (possibly time varying) with the first

$$r' + is' = q' = e^{-i\phi} q = e^{-i\phi} (r + is) \quad \ldots(5)$$

The resolved signals might each be passed through a linear filter represented by $A(D)$, a polynominal function of the differential operator $D = d/dt$ and then resolved back to the original axes as outputs x and y. Working back through the system these operations may be represented by

$$x + iy = z = e^{i\phi} z' = e^{i\phi} A(D) q' = e^{i\phi} A(D) e^{-i\phi} q = A'(D) q ,$$

where $A'(D)$ is readily found when the time variation of ϕ is known. For example, if $A(D) = D^3$ and $\phi = \Omega t$ where Ω is constant

$$z = e^{+i\Omega t} D^3 (e^{-i\Omega t} q) = [D^3 - 3i\Omega D^2 - 3\Omega^2 D + i\Omega^3]\, q = [D - i\Omega]^3\, q$$

Equating real and imaginary parts

$$x = D^3 r + 3\Omega D^2 s - 3\Omega^2 Dr - \Omega^3 s$$

$$y = D^3 s - 3\Omega D^2 r - 3\Omega^2 Ds + \Omega^3 r$$

3. <u>Transformation of Coupled Systems</u>

A set of linear systems may be coupled in many ways. Some of these, such as combination of the inputs before any element introducing a time dependence and addition of the outputs in groups, are trivial in that they can be dealt with by the superposition principle. Feed-back

coupling/

coupling, on the other hand, where quantities related to each output may be added to all the inputs presents more difficulty. Typical equations governing such coupling between linear systems which are otherwise identical may be written.

$$x_j = A(D,t)\left[s_j + \sum_k^N B(D,t)\, b_{jk}\, x_k\right] \qquad \ldots(6)$$

The suffices take on as many values as there are systems coupled, x_j being the response of the jth system to its stimulus s_j, and the real numerical coefficients b_{jk} giving the proportion of the output from the kth system which is fed-back to the input of the jth system. The operators $A(D,t)$ and $B(D,t)$ may be any function of time, t, and the differential operator, D. It will be shown that the solution of these equations can be written in terms of the solutions of single uncoupled equations.

Define a new set of variables by adding the x_j together in various proportions whose magnitudes will be determined later.

$$z_i = \sum_j^N c_{ij}\, x_j \qquad \ldots(7)$$

The c_{ij} determine the weights in the inverse transformation

$$x_j = \sum_k^N d_{jk}\, z_k$$

through the N sets of N simultaneous equations

$$\sum_j^N c_{ij}\, d_{jk} = \delta_{ik} = 1 \quad i = k$$

$$= 0 \quad i \neq k \qquad \ldots(8)$$

Substituting in equation (6) gives the N equations

$$z_i = A(D,t)\left[\sum_j^N c_{ij}\, s_j + B(D,t) \sum_j^N \sum_k^N \sum_\ell^N c_{ij}\, b_{jk}\, d_{k\ell}\, z_\ell\right] \qquad \ldots(9)$$

Now fix the values of the c_{ij} so that

$$\sum_j^N \sum_k^N c_{ij}\, b_{jk}\, d_{k\ell} = \mu_\ell\, \delta_{i\ell} = \mu_\ell, \quad i = \ell$$

$$= 0, \quad i \neq \ell \qquad \ldots(10)$$

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This implies that the $d_{k\ell}$ are the solutions of N sets of N homogeneous simultaneous equations (each value of ℓ gives one set of equations)

$$\sum_{k}^{N} (b_{jk} - \mu_\ell \, \delta_{jk}) \, d_{k\ell} = 0 \qquad j = 1 \text{ to } N \qquad \ldots(11)$$

and can only be non-zero if the determinant of the coefficients vanishes,

$$\left| b_{jk} - \mu_\ell \, \delta_{jk} \right| = 0 \qquad \ldots(12)$$

Equation (12) determines the N values of μ_ℓ as the roots of the Nth degree polynomial obtained by expanding the determinant. They are called the latent roots of the matrix of the b_{jk}, and by solving successively the sets of simultaneous equations (11) and (8) lead to the values of the $d_{k\ell}$ and the c_{ij} which satisfy equation (10). Such values can always be found if all the μ_ℓ are different and using them equation (9) may be written.

$$z_i = A(D,t) \left[\sum_{j}^{N} c_{ij} \, s_j + B(D,t) \, \mu_i \, z_i \right] \qquad \ldots(13)$$

There are N equations like (13) corresponding to the N values of i, but they are all independent, each one representing a system like one of the original coupled systems with feed back from its own output only. If $F(r,w)$ is used to represent the output from such a system with feed back coefficient w when the input is r, so that

$$F(r,w) = A(D,t) \left[r + B(D,t) \, wF(r,w) \right] \qquad \ldots(14)$$

it will be seen from equation (13) that

$$z_i = \sum_{j}^{N} c_{ij} \, F(s_j, \mu_i)$$

by the superposition principle. Hence

$$x_i = \sum_{j}^{N} \sum_{k}^{N} d_{ij} \, c_{jk} \, F(s_k, \mu_j) \qquad \ldots(15)$$

As initial conditions for equations (14) it is convenient to choose

$$\left[F(s_k, \mu_j) \right]_{t=t_o} = \left[x_k \right]_{t=t_o} \quad \text{for all } j \qquad \ldots(16)$$

These/

These last three equations, (14), (15) and (16), are completely equivalent to the original set of equations (6) with their initial conditions. Physically it may be said that the feed-back cross couplings of the original system which make it difficult to analyse have been replaced by cross couplings between the inputs only, and between the outputs only, so that the superposition principle may be used.

Transformation of the type used here are familiar in the study of the equations of motion of dynamical systems and, following the nomenclature used there, equation (13) may be called the "normal equation" representing one of the "normal systems" derived from the original coupled systems. Since it is of the same order as each of the coupled equations, solution of the problem through the normal equations is considerably easier than solving the high order system obtained by eliminating all but one variable. In the same way properties such as stability of the coupled systems can be discussed through the properties of the normal system.

Although the stimulus r and the initial values of $F(r,w)$ are real, equation (14) will in general be complex. This is because the latent roots of an arbitrary matrix are complex, though since here the matrix is real such latent roots must occur in complex conjugate pairs. Thus for generally coupled systems, some of the normal equations may have real values of w and hence have real solutions, while in others the parameter may be complex so that their solutions and the coefficients c and d will also be complex. Such complex numbers do not hinder an analytical solution unless it is required to evaluate it numerically for functions which are not well tabulated for complex arguments. Physically, however, although equation (14) can be represented by a single system with a feed-back path when w is real, this is impossible for complex w. But from Section 2 it will be realized that it can be represented by coupling two systems in the appropriate way and identifying one system with the real part of the solution and the other with the imaginary part. This is shown in Fig. 1 with $B(D,t) = 1$ and

$$F = G + iH \; ; \; w = u + iv$$

so that

$$\begin{aligned} G(r;w) &= A(D,t)\,[r + uG(r;w) - vH(r;w)] \\ H(r;w) &= A(D,t)\,[vG(r;w) + uH(r;w)] \end{aligned} \qquad \ldots(17)$$

To illustrate the application of the method, consider the pair of coupled systems shown in Fig. 2. The feed-back operator $B(D,t)$ is taken as unity, and for convenience the parameters are defined:

$$b_{11} = \alpha + \delta \; ; \; b_{12} = (\beta^2 - \delta^2)/\gamma \; ; \; b_{21} = \gamma \; ; \; b_{22} = \alpha - \delta$$

Hence the two sets of equations corresponding to 11 are obtained by putting

e/

$$\ell = 1 \text{ or } 2 \text{ in}$$

$$(\alpha + \delta - \mu_\ell)\, d_{1\ell} + (\beta^2 - \delta^2)\, \gamma\, d_{2\ell} = 0$$

$$\gamma\, d_{1\ell} + (\alpha - \delta - \mu_\ell)\, d_{2\ell} = 0 \qquad \ldots(18)$$

The condition that the determinant of the coefficients of the $d_{k\ell}$ shall vanish gives

$$\mu_1 = \alpha + \beta \, , \; \mu_2 = \alpha - \beta$$

By substituting these values in (18) and solving, the $d_{k\ell}$ may be taken as:

$$d_{11} = \beta + \delta \; ; \; d_{21} = \gamma \; ; \; d_{12} = -(\beta - \delta) \; ; \; d_{22} = \gamma$$

The c coefficients can now be obtained by solving equation (8) and substituting in equation (15)

$$2\beta\gamma\, x = \gamma\, [(\beta + \delta)\, F(r; \alpha + \beta) + (\beta - \delta)\, F(r; \alpha - \beta)] + (\beta^2 - \delta^2)\, [F(s; \alpha + \beta) - F(s; \alpha - \beta)]$$

$$\ldots(19)$$

$$2\beta\gamma\, y = \gamma\, [F(r; \alpha + \beta) - F(r; \alpha - \beta)] + (\beta - \delta)\, F(s; \alpha + \beta) + (\beta + \delta)\, F(s; \alpha - \beta)$$

The transformations can be used to draw Fig. 3 which is equivalent to Fig. 2. The c and d coefficients respectively determine the summing sections which precede and follow the two "normal systems".

When

$$(b_{11} - b_{22})^2 + 4b_{12}b_{21} < 0$$

in this example, β^2 must be negative and the μ_ℓ become complex. A system equivalent to Fig. 2 may then be built round Fig. 1 (with $u = \alpha$, $v = |\beta|$) the two summing sections being as in Fig. 3 except that β must be replaced by its modulus. In this case equation (19) may be written in terms of real quantities

$$|\beta|\gamma\, x = \gamma\, [\, |\beta| G\, (r; \alpha + i\, |\beta|) + \delta H\, (r; \alpha + i\, |\beta|)] - (|\beta|^2 + \delta^2)\, H\, (s; \alpha + i\, |\beta|) \qquad \ldots(20)$$

$$|\beta|\; y = \gamma H\, (r; \alpha + i\, |\beta|) + |\beta| G\, (s; \alpha + i\, |\beta|) - \delta H\, (s; \alpha + i|\beta|)$$

by/

by using the relations

$$F(r;w) = G(r;w) + iH(r;w) = F^*(r;w^*) = G(r;w^*) - iH(r;w^*)$$

which are implied by equation (17); here $*$ is used to denote the complex conjugate.

It may be verified by differentiating equation (17) that $\frac{\partial G}{\partial u}$ and $\frac{\partial H}{\partial v}$ satisfy a common differential equation, and so do $\frac{\partial G}{\partial v}$ and $-\frac{\partial H}{\partial u}$, each of these equations being of order $2N$, where N is the order of equation (13). Hence provided that the initial values of G and H and their time derivatives to order $2N - 1$ satisfy the Cauchy-Riemann relations as a function of w they will continue to do so for all time. This means that both G and H satisfy Laplace's equation with respect to the variable u and v so that knowledge of the variation of G with u enables an estimate of the effect of v on both G and H to be made. Expressed analytically:

$$\left.\begin{aligned} G(\alpha + i\beta) &= G(\alpha) + \sum_{r=1}^{\infty} (-)^r \frac{\beta^{2r}}{2r!} \left[\frac{\partial^{2r} G}{\partial u^{2r}}\right]_{u=\alpha} \\ H(\alpha + i\beta) &= \sum_{r=0}^{\infty} (-)^r \frac{\beta^{2r+1}}{(2r+1)!} \left[\frac{\partial^{2r+1} G}{\partial u^{2r+1}}\right]_{u=\alpha} \end{aligned}\right\} \quad \ldots(21)$$

which may be compared with the Taylor expansion for real values:

$$\left.\begin{aligned} G(\alpha + \beta) &= G(\alpha) + \sum_{r=1}^{\infty} \frac{\beta^{r}}{r!} \left[\frac{\partial^{r} G}{\partial u^{r}}\right]_{u=\alpha} \\ H(\alpha + \beta) &= 0 \end{aligned}\right\} \quad \ldots(22)$$

4. Stability of Constant Parameter Systems

If coupled systems of the type considered here are to be stable, all the normal systems derived from them must also be stable since the relationship of equation (15) can only affect the coefficient and not the exponent of any exponentially increasing term.

The methods of investigating the stability of systems with constant coefficients are based on the application of complex variable theory (Nyquist's criterion, for example), so it is not surprising that they are readily extended to treat constant complex parameter equations which may arise from the normal systems.

A constant parameter differential equation has stable solutions if each term of its complementary function contains an exponential with a negative real part in its exponent. When the

constant/

constant coefficient operators corresponding to A and B in equation (14) are the ratios of polynomials, the equation may be re-written as a true differential equation,

$$[K(D) - wL(D)]\, F = J(D)r \qquad \ldots(23)$$

J, K and L being polynomials in D. The complementary function of this equation will contain terms like $F = E_1 \exp \lambda_1 t$ where E_1 and λ_1 are complex constants, the λ_1 being the roots of the auxiliary polynomial

$$P(\lambda) = [K(\lambda) - wL(\lambda)] = 0 \qquad \ldots(24)$$

This is quite independent of whether w is a real number or another complex constant. To find if any roots of this equation have positive real parts, a standard process is to examine the path traced in the complex plane by some function in (24) as λ is taken round the contour of Fig. 4(a), i.e., from $-i\infty$ up the imaginary axis to $+i\infty$ and then clockwise round a large semicircle in the right half plane. The path traced by the whole left-hand side of equation (24) must not encircle the origin if all the roots of (24) are to have negative real parts. Since that part of the locus corresponding to the large semicircle in the λ plane will turn clockwise through $2n$ quadrants (n is the highest power of λ in (24)), the path corresponding to the imaginary axis of the λ plane must turn anti-clockwise through $2n$ quadrants. Loci for stable equations of second, third and fourth order are shown in Figs. 4(b), (c) and (d) and from their shape the Appendix derives inequalities which must be satisfied by the complex coefficients in the differential equations. Because the coefficients are complex the locus corresponding to negative frequencies is not the mirror image in the real axis of that for positive frequencies.

A more useful technique when it is required to find the range of values of w for which the system is stable is to rewrite equation (24) as

$$wK(\lambda) \left[\frac{L(\lambda)}{K(\lambda)} - \frac{1}{w}\right] = 0 \qquad \ldots(25)$$

and consider the locus of

$$\frac{L(\lambda)}{K(\lambda)} = A(\lambda)\, B(\lambda)$$

as λ traces the same closed contour. Provided that $K(\lambda)$ has no zeros in the right half plane this case is the same as the previous one but with the origin shifted to the point $1/w$ which must not be enclosed by the locus. Remembering that usually $K(\lambda)$ is of higher order than $L(\lambda)$ so that the locus corresponding to λ on the large semi-circle collapses to the origin, the locus considered is that of the "open loop transfer function" and the stability criterion the same as Nyquist's, substituting the point $1/w$ for the point $(1,0)$. It should be noted that when the feed-back loop has a variable real gain it is customary to consider the open loop locus as expanding when the gain is increased, the critical point remaining at $(1,0)$. When the gain is complex it is more convenient

to/

$$r(s) = s^n + ia_{n-1}\, s^{n-1} + a_{n-2}\, s^{n-2} + \ldots .$$

and/

to keep the locus fixed and move the critical point, otherwise the locus is rotated about the origin by the phase angle of w as well as being multiplied by its modulus.

and expand $r(s)/q(s)$ as a continued fraction by dividing to a remainder, then inverting the division and repeating:

$$r(s)/q(s) = b_1 s + \beta_1 + \cfrac{1}{b_2 s + \beta_2 + \cfrac{1}{b_3 s + \beta_3 + \cfrac{1}{b_n s + \beta_n}}}$$

The solutions of the equation will be stable if all the b are real and positive, and all β purely imaginary.

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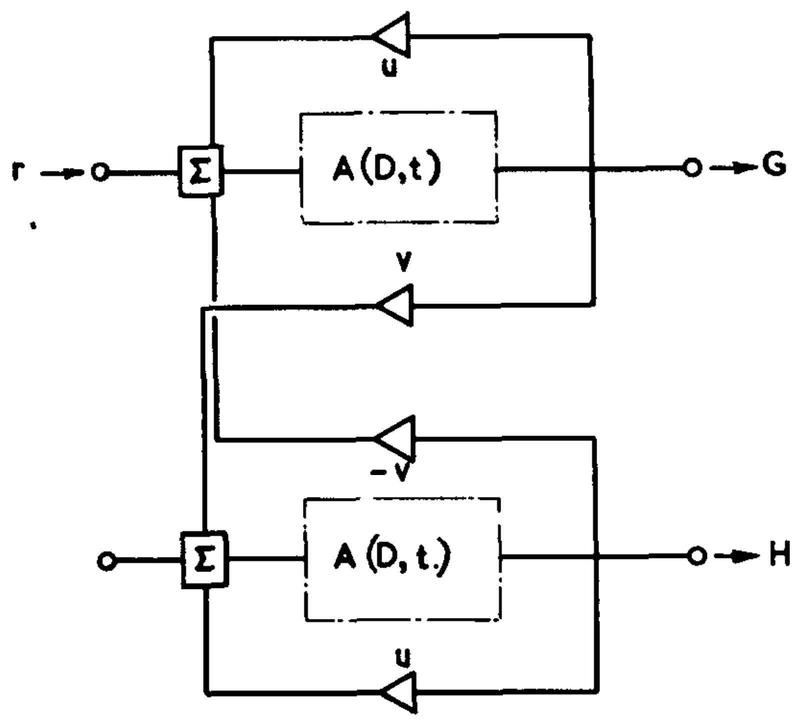

FIGURE 1.

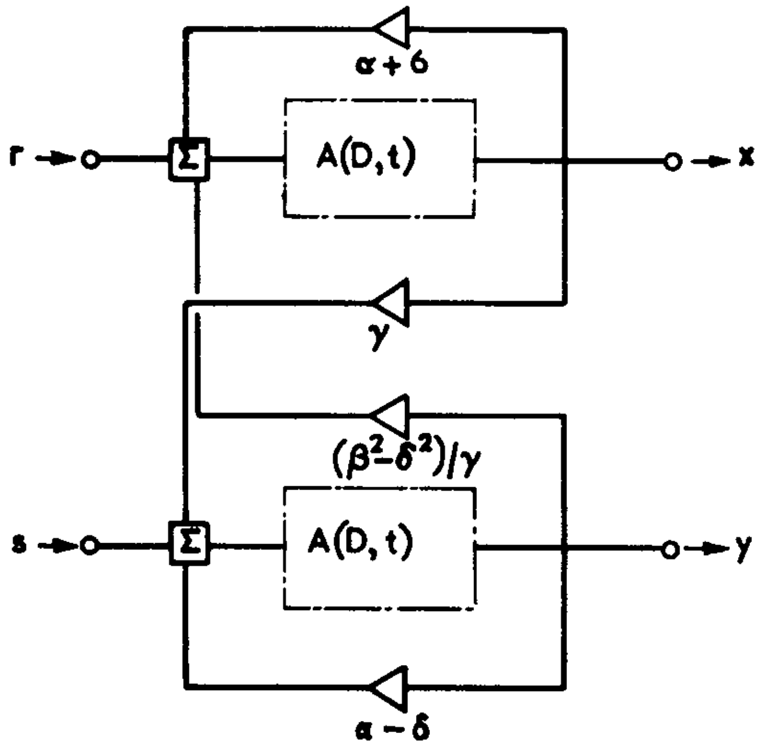

FIGURE 2

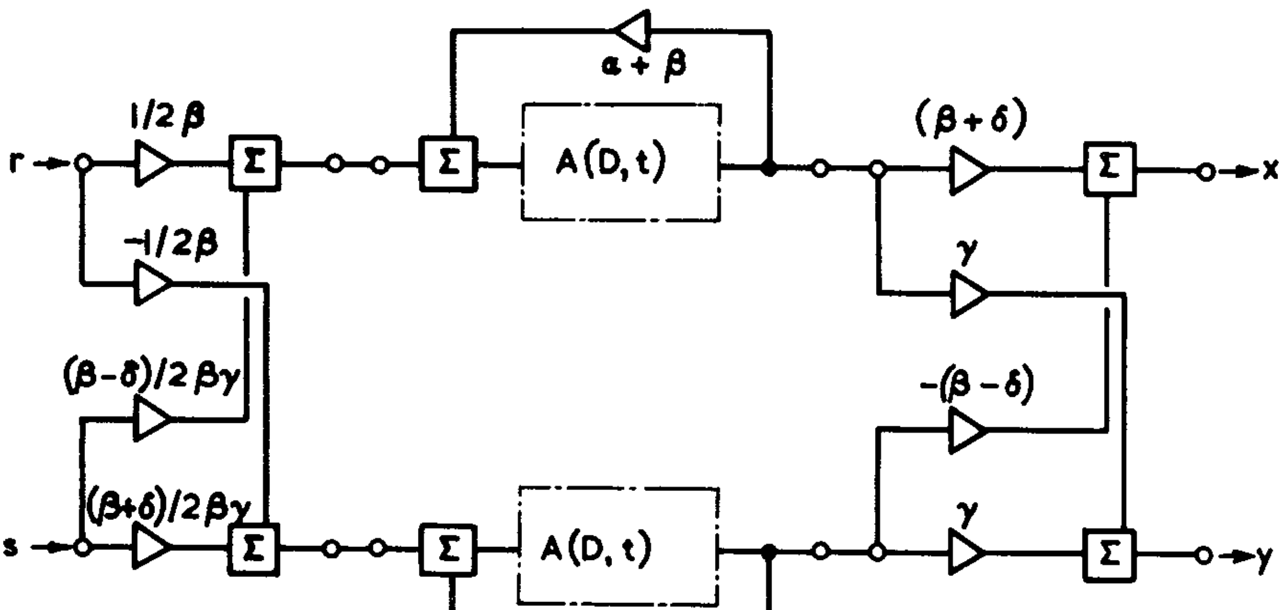

FIGURE 3

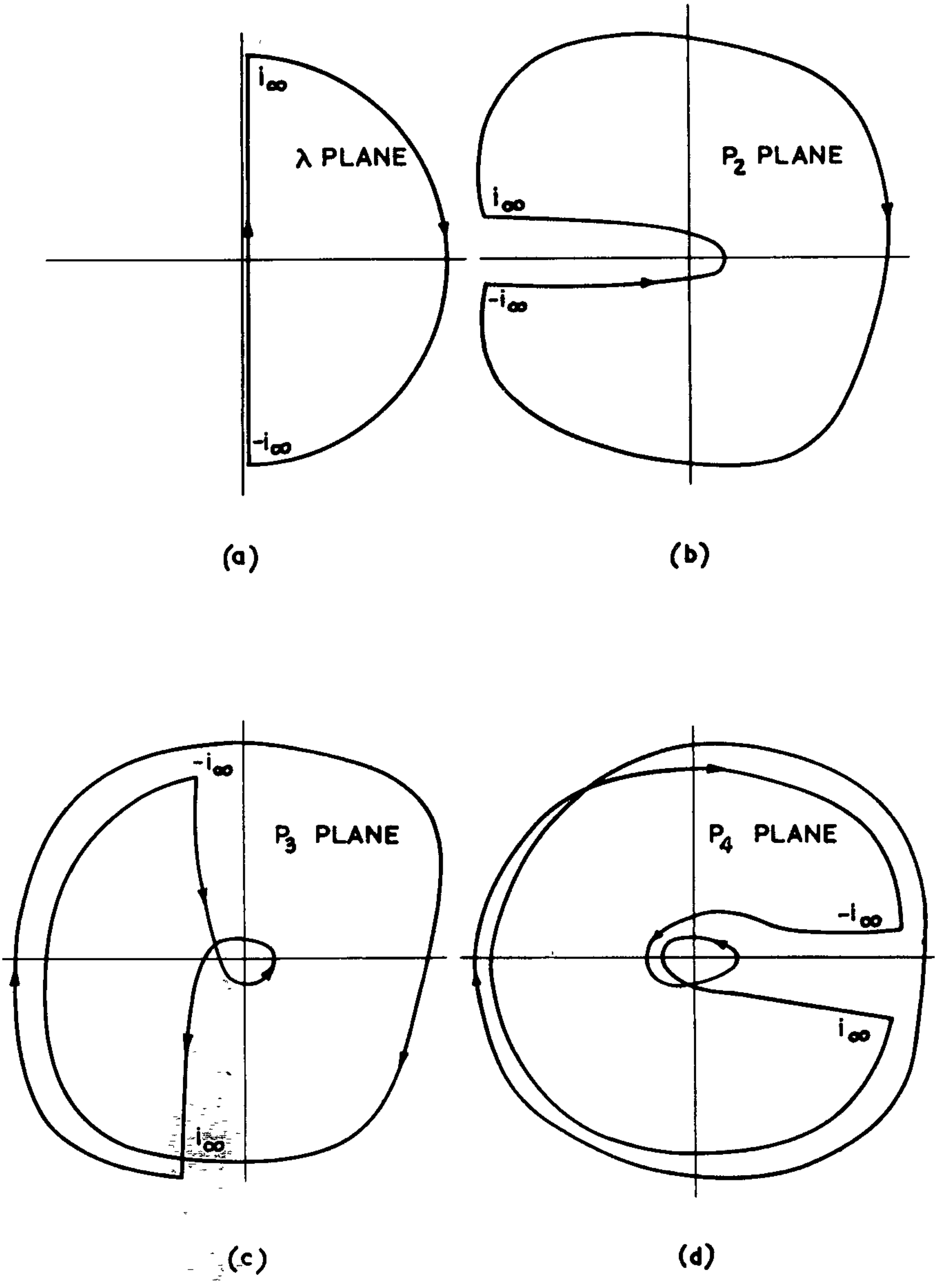

FIGURE 4.

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