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The Buckling of an Axially Loaded Circular Cylinder with Initial Imperfections

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Summary.

An extension of the Donnell and Wan analysis for the buckling of a shell with initial imperfections is presented for a mode of arbitrary profile. Results for a mode with 26 arbitrary parameters are given with charts of the stress-strain characteristics for cylinders with initial displacements of between one-eighth and five times the skin thickness of the cylinder. A direct optimisation of the energy function is used. Almroth's minimum post-buckled stress for the ideal cylinder ($0 \cdot 0656\ Et/R$) is lowered to $0 \cdot 0518\ Et/R$ and a critical assessment of the value of such figures provided.

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1. *Introduction.*

The difficulty of reconciling observed experimental results on the buckling of thin elastic shells in axial compression with the available theoretical estimates has, for a long time, puzzled students of applied mechanics. The ideal conditions analysed in theory are not met in practice and numerous attempts to refine the theory to allow for experimental difficulties have at best provided a qualitative explanation of the disparity between calculation and observation. At the present time it is known[1] that quite small amounts of initial geometric imperfection can lead to significant changes in the load-deflection characteristics for a cylinder but an adequate quantitative solution is not available. The persistence of the lack of agreement in results has been largely ascribed to inadequate experimental technique and in particular to the inability of experimenters to produce cylinders of sufficiently small initial imperfection. Recently, however, some centrifugally cast Hysol cylinders of very low initial deformation have been tested[2] in strain-increment loading and have failed at about 90% of the classical value.

Improvements in the experimental procedure are unlikely to be very marked but the analysis of the buckling phenomena may be extended considerably now that digital computers are available. Since the analysis, using large-displacement theory, of Karman and Tsien[3], successive workers[4 to 8] have included more terms in the approximate mode used with consequent lowering of the minimum post-buckled stress. Progress in the solution of the problem is indicated by the following table.

Year	Author	Ref. No.	Number of parameters used	Minimum post-buckled stress
1939	Karman and Tsien . .	3	4	No appropriate figure can be given owing to interpretation
1942	Leggett and Jones . .	4	4	$0 \cdot 190\ Et/R$
1948	Michielsen	5	4	$0 \cdot 195\ Et/R$
1954	Kempner	6	5	$0 \cdot 1824\ Et/R$
1962	Sobey	7	7	$0 \cdot 1496\ Et/R$
1962	Almroth	8	11	$0 \cdot 0656\ Et/R$

All these efforts have been directed to solving the problem of the ideal cylinder and have been concerned primarily with the minimum post-buckled stress. With the exception of Almroth's work the minimum post-buckled stress occurs at a strain around $0 \cdot 4\ t/R$ whereas in Almroth's analysis it occurs at about $1 \cdot 2\ t/R$. This considerable difference is due to the inclusion by Almroth, for the first time, of a term [the (2, 2) term[7]] which is known to be important[7]. The absence of this term in modes previously used results in a premature stiffening of the cylinder, due to an artificial constraint, once buckling has taken place.

Almroth's paper is a significant landmark in the theory, for his minimum post-buckled stress is the first one achieved which is consistent with experimental values of the buckling stress. He claims that the inclusion of further terms in the modal representation fails to lower the minimum post-buckled stress. However an analysis of the constraint system[7] used to maintain the mode showed that in order to refine the theory over the range $0 \cdot 3\ t/R$ to $1 \cdot 5\ t/R$ (i.e. from the region of minimum post-buckled strain to that of minimum post-buckled stress) a further 8 terms are needed of which Almroth includes only 3. Furthermore the position and value of the minimum post-buckled stress is no criterion of buckling. As Donnell and Wan[1] emphasise, the ideal characteristic is of little consequence in practice. Eccentricity effects must be included if practical buckling criteria are to be established.

In the earlier work of the writer[7], a direct minimisation technique is used to obtain the modal parameters. If a general mode is to be examined, with an arbitrary number of modal parameters, a direct minimisation technique is highly desirable so that by changing only a few parameters in a programme (such as the number of degrees of freedom) entirely separate analyses can be evaluated. A technique is established whereby the computer produces the logical flow for a general mode which is then used as a basis for a high-speed arithmetic programme for modes with up to 26 degrees of freedom.

Direct minimisation of an energy function is a very convenient technique when the effects of initial eccentricity of the Donnell and Wan[1] type are included. Only a few programme changes are necessary to make this important amendment to the analysis, whereby allowance is made for the introduction of an initial displacement which is of the same distribution as the final deflection under load but reduced in amplitude. The presence of an initial eccentricity of such a form is unlikely in practice although local imperfections in an imperfect cylinder may be represented adequately by this form. It is significant that long axial wavelengths are associated with the solution to this problem so that initial imperfections arising from imperfect development of a cylinder from the flat sheet can be treated by the Donnell and Wan artifice. The solution for a general mode of initial displacement when loaded would be exceedingly difficult to analyse and the inherent approximations involved in the analysis would make such refinement absurd.

The numerical analysis of the problem shows that the region requiring careful but extensive analysis is that which lies between strains of $0 \cdot 13\ t/R$ and $0 \cdot 2\ t/R$ for fairly high stress levels. In this region the stress-strain characteristics for a given initial displacement tend to reach their maxima over a wide range of initial eccentricities, whereas the minimum post-buckled stress occurs at a strain of about $1 \cdot 5\ t/R$ (or $2\frac{1}{2}$ times the classical buckling strain). Whilst investigations of the stress-strain characteristics for strains above about $0 \cdot 4\ t/R$ have some academic interest, characteristics of practical value are obtained only for small values of the strain. It is noteworthy that the cylinder continues to unload (stress falling with increase in strain) for a very considerable range of strain (from $0 \cdot 2\ t/R$ to $1 \cdot 5\ t/R$) and long before the minimum is reached other considerations, such as the inadmissibility of the controlling equations, become serious. For this reason, and because the cylinder with initial eccentricity is of greatest practical interest, only the restricted region referred to above is analysed intensively.

Further progress along the lines indicated in their original paper by Karman and Tsien[3] is unlikely to be made. Furthermore, if such an investigation were undertaken, it is unlikely to be fruitful. A new approach to this problem is required if a fuller understanding of the very complex mechanics of the buckling of a cylinder, particularly under stress (dead-weight) loading is to be obtained. Such a new procedure would, of necessity, have to include the effects of non-uniformity of material, loading and geometry as well as to allow for very large displacements under load.

2. *Development of Energy Function.*

In this section the approximate solution for a mode of general profile is developed and an energy approach is adopted to find the optimum mode. To obtain the modal parameters, an energy function (strain or potential energy) is required for variation and the analysis of Ref. 7 is extended to provide a general expression for this energy for a given mode shape.

2.1. *Basic Equations.*

The equations of Ref. 7 are modified to include the effects of initial eccentricity. The strain displacement relations become

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w_l}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w_l}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w_l}{\partial x}\frac{\partial w_l}{\partial y} - \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{aligned}\right\} \tag{1}$$

where w_0 is the radial deflection of the cylinder under no load (i.e. initial eccentricity) and w_l is the radial displacement under load. The deflection produced by the load is

$$w = w_l - w_0 . \tag{2}$$

The analysis of the buckling of the cylinder for any initial form w_0 is involved, being dependent on the specific number of waves round the cylinder. The particular case where w_0 is a submultiple of w reduces the mathematical complexity enormously[1] and this assumption is adopted. It must be emphasized that this assumption is made for mathematical simplification only and that no practical cylinder conforms to this assumption. Nevertheless the behaviour of a practical cylinder can be interpreted from among the results obtained with a wide range of w_0/w, as explained below.

Following Donnell and Wan[1] it is convenient to introduce an eccentricity factor

$$K = 1 + \frac{2w_0}{w} \tag{3}$$

so that on substitution equations (1) become

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{K}{2}\left(\frac{\partial w}{\partial x}\right)^2 , \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{K}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \text{and} \qquad \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + K\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} . \end{aligned}\right\} \tag{4}$$

The stress equilibrium equations[7] in the tangent plane are unchanged by the presence of displacements w and are given by

and
$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0.\end{aligned}\right\} \tag{5}$$

These equations are automatically satisfied if an Airy stress function F is introduced where

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \tag{6}$$

and the equation governing F, which is found by eliminating u, v from (4) takes the form

$$\frac{1}{E} \nabla^4 F = K \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} - \frac{1}{R} \frac{\partial^2 w}{\partial x^2}. \tag{7}$$

2.2. *Development of Energy Function for an Arbitrary Mode $w(x,y)$.*

The deflection $w(x,y)$ is given a general representation of the form

$$w(x,y) = t \sum_m \sum_n w_{m,n} \cos \frac{m\pi x}{\lambda_x} \cos \frac{n\pi y}{\lambda_y} + \text{constant term} \tag{8}$$

or equally

$$w(x,y) = t \sum_i w_i \cos \frac{m\pi x}{\lambda_x} \cos \frac{n\pi y}{\lambda_y} + \text{constant term}, \tag{8$'$}$$

where, with each value of i, a specific pair of integers m, n are to be associated. We seek an expression for the total potential energy of the loaded system in terms of the parameters $w_{m,n}$ and the wavelengths λ_x and λ_y.

If the axial strain is fixed, variation of these parameters will lead to variations in the strain energy which consists of two parts, U_1 and U_2, where U_1 is the strain energy due to bending, and U_2 is the strain energy due to stretching of the shell.

$$U_1 = \frac{D}{2} \int_0^L \int_0^{2\pi R} \left[(\nabla^2 w)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} \right] dx\, dy \tag{9}$$

which reduces, for the mode given in equation (8) to the form

$$U_1 = \frac{D}{2} \int_0^L \int_0^{2\pi R} (\nabla^2 w)^2 \, dx\, dy$$

where L is a complete number of axial waves, λ_x.

The extensional strain energy is

$$U_2 = \frac{t}{2E} \int_0^L \int_0^{2\pi R} \left[(\nabla^2 F)^2 + 2(1+\nu) \left\{ \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2 - \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} \right\} \right] dx\, dy. \tag{10}$$

When $w(x,y)$ has the form of equation (8) the corresponding form for F consists of a periodic component of similar form to w together with terms which represent the mean stresses in the unbuckled state. In consequence the terms in { } in (10) can be ignored.

In developing the energy expression U_2, it is necessary only to evaluate $\nabla^2 F$ and not F itself. From equation (7) we find on substituting from equation (8′) for $w(x,y)$

$$\frac{\lambda_x^2\lambda_y^2\nabla^4 F}{\pi^4 t^2 EK} = -\frac{1}{2}\sum_{i=1}^{N} m^2 n^2 w_i^2\left(\cos\frac{2m\pi x}{\lambda_x}+\cos\frac{2n\pi y}{\lambda_y}\right)+\frac{1}{K\eta}\sum_{i=1}^{N} m^2 w_i\cos\frac{m\pi x}{\lambda_x}\cos\frac{n\pi y}{\lambda_y}-$$

$$-\frac{1}{4}\sum_{i=1}^{N}\sum_{j=i+1}^{N} w_i w_j\left[[mq-np]^2\left\{\cos\frac{(m-p)\pi x}{\lambda_x}\cos\frac{(n-q)\pi y}{\lambda_y}+\right.\right.$$

$$\left.+\cos\frac{(m+p)\pi x}{\lambda_x}\cos\frac{(n+q)\pi y}{\lambda_y}\right\}+$$

$$+[mq+np]^2\left\{\cos\frac{(m-p)\pi x}{\lambda_x}\cos\frac{(n+q)\pi y}{\lambda_y}+\right.$$

$$\left.\left.+\cos\frac{(m+p)\pi x}{\lambda_x}\cos\frac{(n-q)\pi y}{\lambda_y}\right\}\right]$$

where $\eta = \pi^2 Rt/\lambda_y^2$.

This equation can be integrated to give

$$(RKEt\mu^2\eta)\left(\nabla^2 F-\frac{Et\sigma^*}{R}\right) = \frac{1}{8}\sum_{\substack{i=1\\ m\,\&\,n\neq 0}}^{N} w_i^2\left(m^2\cos\frac{2n\pi y}{\lambda_y}+\frac{n^2}{\mu^2}\cos\frac{2m\pi x}{\lambda_x}\right)-$$

$$-\sum_{i=1}^{N}\frac{m^2 w_i}{K\eta(m^2\mu^2+n^2)}\cos\frac{m\pi x}{\lambda_x}\cos\frac{n\pi y}{\lambda_y}+$$

$$+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=i+1}^{N} w_i w_j\times$$

$$\times\left\{\frac{(mq-np)^2}{(m-p)^2\mu^2+(n-q)^2}\cos\frac{(m-p)\pi x}{\lambda_x}\cos\frac{(n-q)\pi y}{\lambda_y}+\right.$$

$$+\frac{(mq-np)^2}{(m+p)^2\mu^2+(n+q)^2}\cos\frac{(m+p)\pi x}{\lambda_x}\cos\frac{(n+q)\pi y}{\lambda_y}+$$

$$+\frac{(mq+np)^2}{(m-p)^2\mu^2+(n+q)^2}\cos\frac{(m-p)\pi x}{\lambda_x}\cos\frac{(n+q)\pi y}{\lambda_y}+$$

$$\left.+\frac{(mq+np)^2}{(m+p)^2\mu^2+(n-q)^2}\cos\frac{(m+p)\pi x}{\lambda_x}\cos\frac{(n-q)\pi y}{\lambda_y}\right\} \tag{11}$$

where $\mu = \lambda_y/\lambda_x$ and the Fourier series in $w(x, y)$ has been summed under a single summation. To each index i (or j) are associated periodicities m and n (or p and q). The mean axial stress in the undeflected condition is σ or $\sigma^*\ Et/R$ where σ^* is a reduced stress coefficient. When all the w_i are zero, equation (11) reduces to $\nabla^2 F = Et\sigma^*/R$, as it should.

On the right-hand side of equation (11) all terms are periodic in either x or y or both and equation (11) can be written in the form

$$\frac{R}{KEt\mu^2\eta}\left(\nabla^2 F-\frac{Et\sigma^*}{R}\right) = \sum_{i=1}^{N'} f_i\cos\frac{m'\pi x}{\lambda_x}\cos\frac{n'\pi y}{\lambda_y} \tag{12}$$

where the range of summation is over every term in equation (11) and N' may be much larger than N. Each f_i is the sum of terms in equation (11) which have like periodicities in x and in y and the parameters m' and n' are ranged over all the combinations of $m \pm p$ and $n \pm q$ respectively.

Equation (12) is a convenient representation of $\nabla^2 F$ since we require in forming U_2 to evaluate

$$\frac{t}{2E}\int_0^L \int_0^{2\pi R} (\nabla^2 F)^2 dx\, dy$$

and every term in this expression other than those which involve squares of the f_i is zero.

Whence

$$U_2 = \frac{\pi L E t^3}{4R}\left\{4\sigma^{*2} + K^2\mu^4\eta^2 \sum_{\substack{i=1 \\ m' \,\&\, n' \neq 0}}^{N'} f_i^2 + 2K^2\mu^4\eta^2 \sum_{\substack{i=1 \\ m' \text{ or } n' = 0}}^{N'} f_i^2\right\}. \tag{13}$$

Substituting in

$$U_1 = \frac{D}{2}\int_0^L \int_0^{2\pi R} (\nabla^2 w)^2 dx\, dy \text{ for } w,$$

$$U_1 = \frac{\pi L E t^3}{R}\,\frac{\eta^2}{48(1-\nu^2)}\left\{2 \sum_{\substack{i=1 \\ m \text{ or } n=0}}^{N} (m^2\mu^2 + n^2)^2 w_i^2 + \sum_{\substack{i=1 \\ m \,\&\, n \neq 0}}^{N} (m^2\mu^2 + n^2) w_i^2\right\}. \tag{14}$$

The total strain energy is $U = U_1 + U_2$ as given by equations (13) and (14).

2.3. *Restriction on the Variables.*

The variables λ_x, λ_y, w_i and the mean axial stress and strain are not free to take arbitrary values since the displacements are continuous round the cylinder and the mean axial strain introduces a restriction on the variables.

The mean inward contraction is determined by the condition of continuity of displacements around the cylinder and is not considered here. The mean axial strain, however, is given by

$$e = e^* \frac{t}{R} = -\frac{1}{2\pi L R}\int_0^L \int_0^{2\pi R} \frac{\partial u}{\partial x}\, dx\, dy. \tag{15}$$

On substituting from equations (1), (4), (6) and (8′) this reduces to

$$e^* - \sigma^* = \frac{1}{8}\mu^2\eta K\left\{2 \sum_{\substack{i=1 \\ \text{All } n \text{ zero}}}^{N} m^2 w_i^2 + \sum_{\substack{i=1 \\ \text{All } n \neq 0}}^{N} m^2 w_i^2\right\}. \tag{16}$$

When the mean axial strain is given, corresponding to a specific value of the coefficient e^*, the stress coefficient σ^* is given in terms of e^*, μ^2, η and the w_i by equation (16). The total strain energy is found by substituting this value of σ^* into equation (13) and we can write U in the form

$$U = U(\mu^2, \eta, w_i, \ldots\ldots, w_N) \tag{17}$$

where the explicit value of U corresponding to a given set of the parameters μ^2, η, w_i can be determined by synthesising the f_i.

2.4. *Approximate Solution for Buckled Form of Cylinder.*

In Section 2.1 the equations of equilibrium in the tangent plane, compatibility of strain and stress strain were developed and the resulting stress function F ensures that conditions in the tangent plane of the cylinder are correct. Any trial mode w of equation (8′) is, however, only an approximate solution since equilibrium conditions in planes normal to the surface of the cylinder are not satisfied. To satisfy, in an approximate manner, the condition of radial equilibrium the parameters μ, η and the w_i are chosen so as to minimise the energy. The other conditions arising from the third dimension are easily satisfied.

Two equivalent forms of the approximate solution of the problem are obtained by minimising U with respect to μ, η and w_i for e^* and by minimising $U - 2\pi LEt^3\sigma^*e^*/R$ when σ^* is given. Both problems lead to identical equations for the stationary value of the appropriate energy function. The energy function in the latter problem includes the potential energy of the moving load source.

3. *Numerical Analysis.*

3.1. *Optimisation of Energy Function.*

The two wavelength parameters μ and η and the amplitudes w_i of the mode $w(x, y)$ must be chosen so that the total potential energy of the cylinder, U, is a minimum. The values of these variables can be obtained by deriving the stationary conditions

$$\frac{\partial U}{\partial \mu} = \frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial w_i} = 0 \qquad (i = 1, 2, \ldots, N) \tag{18}$$

and by solving these equations (which are non-linear in the variables) by trial and error using, for example, the Newton-Raphson method[8]. A superior technique is to find the minimum of U directly with an optimisation routine. Such a procedure was adopted in Ref. 7 where the function minimised had 7 degrees of freedom and the Rosenbrock[9] process was used. When the number of variables is large, say 20 or more, the Rosenbrock process is not suitable partly because the gradient vector of U is found by finite difference methods and partly because the search strategy down the line of steepest descent is uneconomical in computing time. Convergence towards the minimum is initially fast but progress becomes increasingly more sluggish as the procedure continues since the gradient is ill-defined in the neighbourhood of the minimum point.

3.2. *The Fletcher-Powell Method.*

The difficulties encountered with the Rosenbrock process are avoided when the gradient vector is defined analytically. Even so the difficulties experienced by the poor definition of the gradient vector near the minimum point will persist because in that neighbourhood the second-order terms in the Taylor expansion of U about its minimum point will dominate. A method due to Fletcher and Powell[10] incorporates the influence of these second-order terms by constructing, by means of a clever synthesis, the inverse Hessian matrix of U (that is the inverse of the matrix of second derivatives of U). For a quadratic form in n variables, the process converges in exactly n steps. As a consequence, a complex function like U will converge initially rapidly towards a minimum point and progress towards the minimum will continue progressively more slowly until the function U is dominated by its second-order terms when the final convergence on to the minimum itself will be very rapid.

3.3. *Serial Computation.*

The considerable merit of the Fletcher-Powell optimisation routine is enhanced by the fact that if a single parameter, say K or e^*, is changed but slightly, the constructed inverse Hessian matrix is an excellent starting approximation to that of the subsequent case, so that convergence in this latter case is accelerated by the use of information derived in the former. In this way a whole series of cases may be analysed in sequence to enable a complete stress-strain characteristic to be evaluated. Such economy in calculation is of considerable value when many variables are used and contrasts markedly with the Rosenbrock or Newton-Raphson process even when excellent trial starting values for the variables are available.

3.4. *Analysis of a K-Characteristic.*

A K-characteristic is defined as the stress-strain locus for a fixed value of K, the eccentricity parameter {equation (3)}. When $K = 1$, the cylinder is perfectly formed. For values of $K > 1$, each point of a K-characteristic corresponds to a point of a stress-strain locus, or W-characteristic for a cylinder with specific initial displacement. As the K-characteristic is followed from the origin to large values of strain the corresponding initial displacement w_0 increases steadily. The W-characteristics are formed by interpolation from among the K-characteristics.

Each K-characteristic consists of 3 distinct parts. The first, corresponding to very small total displacements, joins the origin in the stress-strain plane to a point (P) of locally maximum stress and strain and this portion of the characteristic lies only slightly below the line of uniform pre-buckling compression of a perfect cylinder. The second portion of the K-characteristic is associated with falling stress and strain to the minimum post-buckled strain (Q). In the third part the characteristic is continued from Q through the region of minimum post-buckled stress and on into a region of positive stiffness with stress rising with increasing strain. As K increases from unity the K-characteristics nest within each other and fill the region between the zero stress axis and the characteristic for the ideal cylinder.

Of the three parts of a K-characteristic, the third is the most easy to evaluate since, for a given large strain, the solution sought is one with absolute minimum energy. The characteristic can be followed all the way back to Q by serial computation even though, in the region of Q, there are equilibrium states with lower strain energy (corresponding to points on the first and second part of the K-characteristic). In the first two parts of a K-characteristic minima occur of lower energy than the solution sought and special artifices have to be employed to recover some of these parts. However a large part of the second region of a K-characteristic can be recovered using stress as independent variable and minimising the total potential energy. Even so, it is advisable to introduce restraints into the optimising process (*see* Section 3.5) to prevent the search procedure from seeking false minima.

3.5. *Use of Penalty Functions.*

Those parts of a K-characteristic for which the minimum sought is not the absolute minimum can be recovered using penalty functions. These are functions introduced to compel a variable to lie within a certain range. Suppose, for example, that the variable w_j is to be confined to the region $a \leqslant w_j \leqslant b$. Then U is changed to the function $U + C_1(w_j - a)^2 + C_2(w_j - b)^2$, where C_1 is zero if $w_j > a$ and C_2 is zero if $b > w_j$. Otherwise C_1 and C_2 are made positive and large enough to ensure that U has a true minimum in the range (a, b). It is possible that a solution for w_j very close to a or b may be found in which case it is most likely not to be a true minimum and the indicated solution is rejected. The limits (a, b) have to be changed empirically to trap the solution. This is frequently difficult, particularly with small amplitudes since the vanishing of all the w_i is a theoretically possible minimum which must be constrained not to appear.

In practice the variables μ and η must be restrained from becoming too small and at least one of the w_i must be bounded. The most significant of the w_i is the (1, 1) term (which occurs in the linearised solution as the only periodic one) and this is bounded above and below to recover points on the PQ part of the characteristic. Owing to the intrinsic difficulty of recovering those points whose minimum energy is not the lowest possible energy state for the given stress or strain, only such points of K-characteristics as are needed to define the W-characteristics for low values of w_0 have been computed.

4. *Results.*

From the aggregate of points which comprise the set of K-characteristics, the W-characteristics are derived. Each K-characteristic is a mathematical locus corresponding to no real structure but the W-characteristics which are developed from them correspond to the stress-strain loci of a practical cylinder.

For very large w_0, say for w_0 greater than about $7t$, the W-characteristic rises steadily with no apparent maximum in the strain range computed (i.e. up to $1 \cdot 5\ t/R$). For intermediate values of w_0, say $7t$ down to $0 \cdot 25t$, the stress-strain characteristic rises to a maximum stress continues to a minimum and rises again so that for any given strain there is one and only one value of the stress. For very small values of w_0 ($\leqslant 0 \cdot 25t$) the W-characteristic is reflex (i.e. there are intermediate values of strain for which there are 3 values of the stress).

4.1. *The Ideal Cylinder.*

The stress-strain characteristic for the perfect cylinder using a mode with $24 w_i$ is shown in Fig. 2 together with the characteristics of Almroth[8] and an earlier one of the writer[7]. It will be noticed that the minimum post-buckled strain and minimum post-buckled stress are both reduced, and in particular the region which bounds the available space occupied by W-characteristics is much reduced in area. The minimum post-buckled strain is a little under $0 \cdot 21\ t/R$ and the minimum post-buckled stress is $0 \cdot 0518\ Et/R$ or about 79% of Almroth's value of $0 \cdot 0656\ Et/R$. The point of minimum post-buckled stress occurs at a strain of about $1 \cdot 5\ t/R$ compared with $1 \cdot 2\ t/R$ in Almroth's case.

The characteristic falls for a large range of strain ($0 \cdot 21\ t/R$ to $1 \cdot 5\ t/R$) and throughout the whole of this region the cylinder has negative stiffness. The associated displacements increase from about $10t$ to $60t$ over this region. Because of these very large displacements the basic Donnell equations are suspect unless the value of R/t is very large. It is generally accepted that about 4 complete waves must occur peripherally in the buckled state if the Donnell approximations are valid, so that R/t must exceed 400.

However for cylinders of smaller R/t the characteristic will be valid for a restricted range of strain. For example for $R/t = 200$ the solution is valid up to $e = 0 \cdot 4\ t/R$ and for $R/t = 150$ the solution is only just applicable in the region of minimum post-buckled strain. When the imperfect cylinder is analysed, it is clear that the validity of the controlling equations and the analytical procedure is suspect only for cylinders of such low R/t that inelastic considerations may become important.

Figs. 3 to 6 show the variation with buckle amplitude of the modal parameters used. Some of the 24 amplitudes used in the mode are not significant at any point of the characteristic but it is interesting to note how several of those predicted[7] as likely to be significant are important. In particular the (2, 2) and (4, 0) terms are large, as also are the (5, 1) (4, 2) (3, 3) and (5, 3). The importance of some of these terms was also indicated by Cox[11] from Fourier analyses of a polyhedral mode. In particular the order of significance of modal parameters in the region of minimum post-buckled strain (Amplitude of $8t$ to $20t$, say) is quite different from that which prevails in the moderately developed form.

The mode of the cylinder in the region of minimum post-buckled strain (Q) is shown in Fig. 7. The portions of the skin which move radially inward are associated with nearly circular contours and well rounded profile whilst the parts of the skin which move outwards are of lozenge shape. The nodal lines are curvilinear, running almost axially and circumferentially. No comparison with observed modes is available.

4.2. *Cylinders with Initial Imperfections.*

Results are presented, in Fig. 8, for the W-characteristics of cylinders whose initial deformation is a proportion of the final mode. Such an initial deformation is most unlikely to occur for cylinders of practical value but nevertheless results presented here are believed applicable to cylinders whose initial deformations are of the same order but of different distribution[1, 12, 13].

As expected, only a small amount of initial deformation is necessary to produce significant reductions in the buckling load and to eliminate the reflex character of the stress-strain curves. For a given strain, e^*, in excess of $0 \cdot 21$ there will be three stress levels corresponding to zero initial deformation and for stress levels between the highest two there will correspond small initial deformations. It follows that for some value of the initial displacement w_0 the variation of stress with displacement will have a vertical tangent at the point of inflection. This appears (Fig. 8) to occur for a strain of $0 \cdot 20\ t/R$ and is associated with initial displacements of the order of $0 \cdot 25t$. For initial irregularities of smaller amplitude the W-characteristic is reflex and qualitatively is like that of the ideal cylinder. In particular, arguments about the possibility of 'snap-through' buckling which have been adduced for the perfect cylinder apply to cylinders with initial displacements of less than $0 \cdot 25t$. For all cylinders with greater initial displacement than $0 \cdot 25t$ snapping will not occur in strain-increment loading. Furthermore for dead-weight loading the maximum stress is also the upper limit of the buckling load in dead-weight loading[13].

If the (σ^*, w_0) loci for given e^* are plotted (Fig. 9) the curves have an envelope which will be the locus of the peak stress which a cylinder with given w_0 can carry versus w_0. This locus gives the stress at buckling for all cylinders with initial deformation greater than $0 \cdot 25\ t/R$ and for smaller deformations will give an upper limit to the stress on buckling[7]. It is interesting to note that the stress levels at which buckling occurs are all very low as the following table shows:

w_0/t	σ_{buckling}	R/Et
0·125	0·200	.... (With possible snap through at a lower strain level in strain-increment loading.)
0·25	0·176	
0·5	0·154	
0·75	0·139	
1	0·1275	
1·5	0·109	
2	0·0950	
3	0·0749	
4	0·0621	
5	0·0534	

Of all the numerous tests conducted on cylinders in axial compression, which have been collocated by several writers[1, 12], the range of buckling stresses shown embraces nearly every cylinder tested under stress loading conditions. Few specimens have failed at stresses as high as those at the top of the table and few have been of such poor quality that they have failed at stresses like those at the bottom of the table. Initial deformations as great as $5t$ are, of course, unlikely to be found in nominally ideal cylinders. The variation of buckling stress with initial displacement is shown in Fig. 10.

In Fig. 11, the strain energy of the cylinder with given initial eccentricity is shown as a function of the square of the mean axial strain. For cylinders immune from snap-through in strain-increment loading ($w_0 > 0 \cdot 25t$) the cylinder behaves as a non-linear spring whose stiffness falls with increasing load in the range of strain shown. Once snapping is possible the double cusped energy characteristics of the perfect cylinder begin to appear and it is notable that the energy level for the ideal cylinder lies well above that for the $0 \cdot 25t$ initial displacement. That it appears disproportionately so is due to the very large spike on the reflex stress-strain characteristic and the removal of the cusp on the energy characteristic to a very large strain value. No accurate data could be interpolated for this region.

5. *Conclusions.*

Improved stress-strain characteristics for the initially imperfect cylinder loaded in axial compression are presented and represent a comprehensive analysis of the buckled cylinder. The inclusion of additional terms in the mode may produce very small changes in the characteristics in the low strain ($0 \cdot 13t \sim 0 \cdot 2t$) region with consequent adjustments to the W-characteristics presented but such requirements are not likely to be worth consideration. Modifications to the analysis, particularly for cylinders of low R/t, are more pressing and the results presented here represent a limiting synthesis of the characteristics of a buckled cylinder as initiated by Karman and Tsien and by Donnell and Wan.

The presentation of an optimal solution to an essentially non-linear mechanical problem has been given in detail and may well be useful to students of mechanics in other fields. The use of direct minimisation ensures a physically meaningful solution and avoids spurious solutions of the non-linear equations of stationary energy that might occur with other techniques. Application to other problems in the structural field is anticipated and the computational restriction which has hindered analysis of this and other problems is overcome.

6. *Acknowledgements.*

The author wishes to acknowledge with gratitude his indebtedness to Dr. M. J. D. Powell of A.E.R.E., Harwell for providing a copy of his optimisation programme and to Mr. R. T. Robinson of Mathematics Department, Royal Aircraft Establishment for translating this programme from Fortran into Mercury Autocode.

SYMBOLS

X, Y, Z	Cartesian coordinates (Fig. 1)
ν	Poisson's ratio (assumed equal to $0 \cdot 3$)
E	Young's modulus
t	Thickness of cylinder
R	Radius of cylinder
ϵ_x, ϵ_y, γ_{xy}	Direct and shear strains in tangent plane of cylinder
u, v	Tangential displacements in x, y directions
w_0	Initial radial displacement under no load
w_l	Total radial displacement under load
w	$= w_l - w_0$, deflection produced during loading
K	$= 1 + \dfrac{2w_0}{w}$, initial deformation parameter
σ_x, σ_y, τ_{xy}	Direct and shear stresses in tangent plane
F	Airy stress function
$w_{m,n}$, w_i, w_j	Coefficients in the arbitary mode $w(x, y)$
λ_x, λ_y	Wavelengths in x, y directions
U, U_1, U_2	Total, bending and extensional strain energies of loaded cylinder
D	Flexural rigidity $= Et^3/12(1-\nu^2)$
L	An axial length of several wavelengths λ_x
∇^2	The operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$
η	$= \pi^2 Rt/\lambda_y{}^2$
μ	$= \lambda_y/\lambda_x$
N, N'	Limits of summation in the series expansions for w and for F
σ	Mean axial stress
σ^*	Reduced stress coefficient $R\sigma/Et$
e	Mean axial strain
e^*	Reduced strain coefficient Re/t
m, n, p, q	Periodicities of terms in $w(x, y)$
m', n'	Periodicities of terms in F
f_i	The regrouped Fourier coefficients in the development of F {equation (12)}

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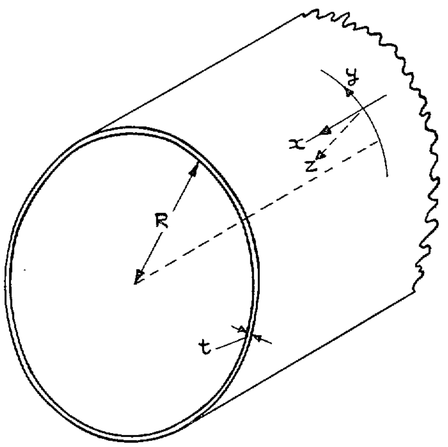

FIG. 1. Cylinder and coordinate system.

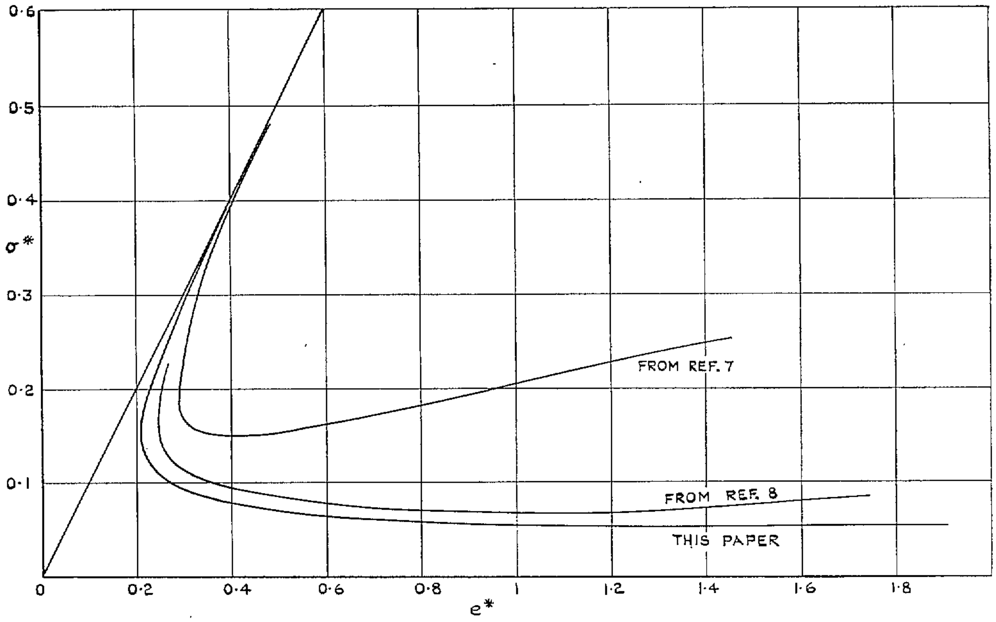

FIG. 2. Comparison of stress-strain characteristics for an ideal cylinder.

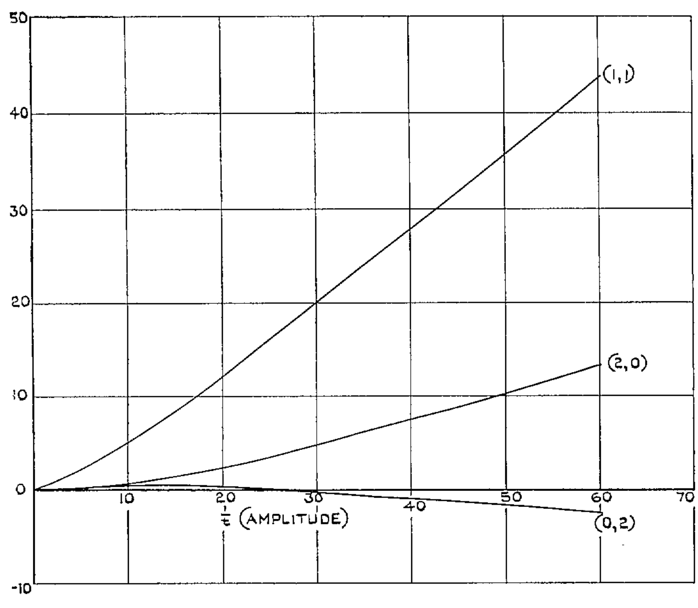

FIG. 3. Modal parameters for ideal cylinder for $m + n = 2$.

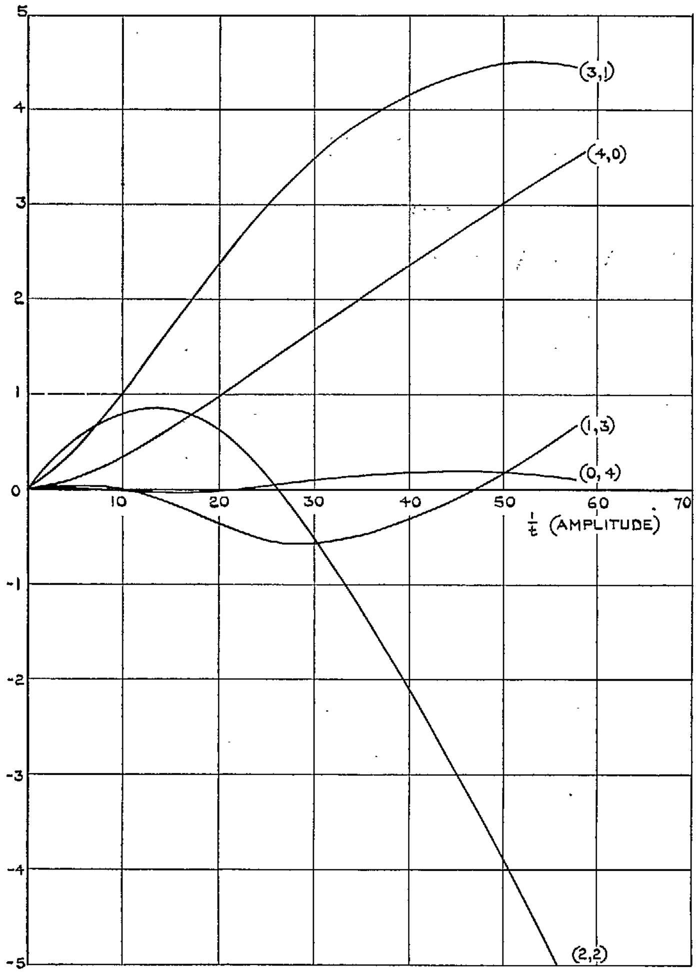

FIG. 4. Modal parameters for ideal cylinder for $m + n = 4$.

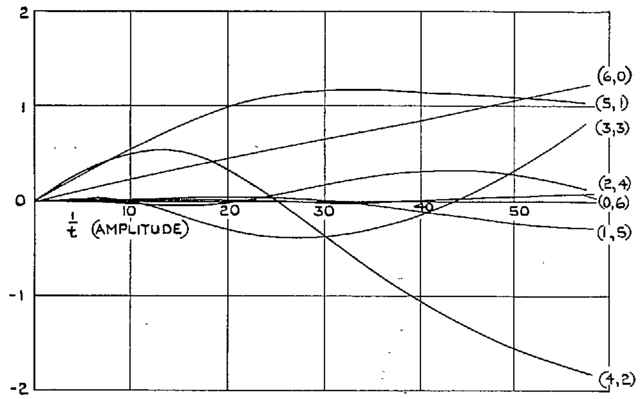

FIG. 5. Modal parameters for ideal cylinder for $m + n = 6$.

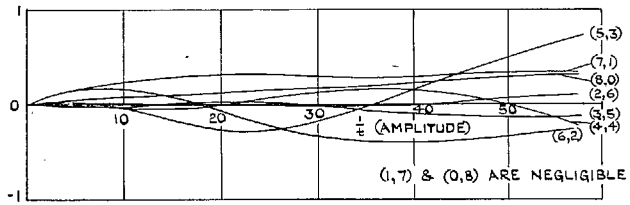

FIG. 6. Modal parameters for ideal cylinder for $m + n = 8$.

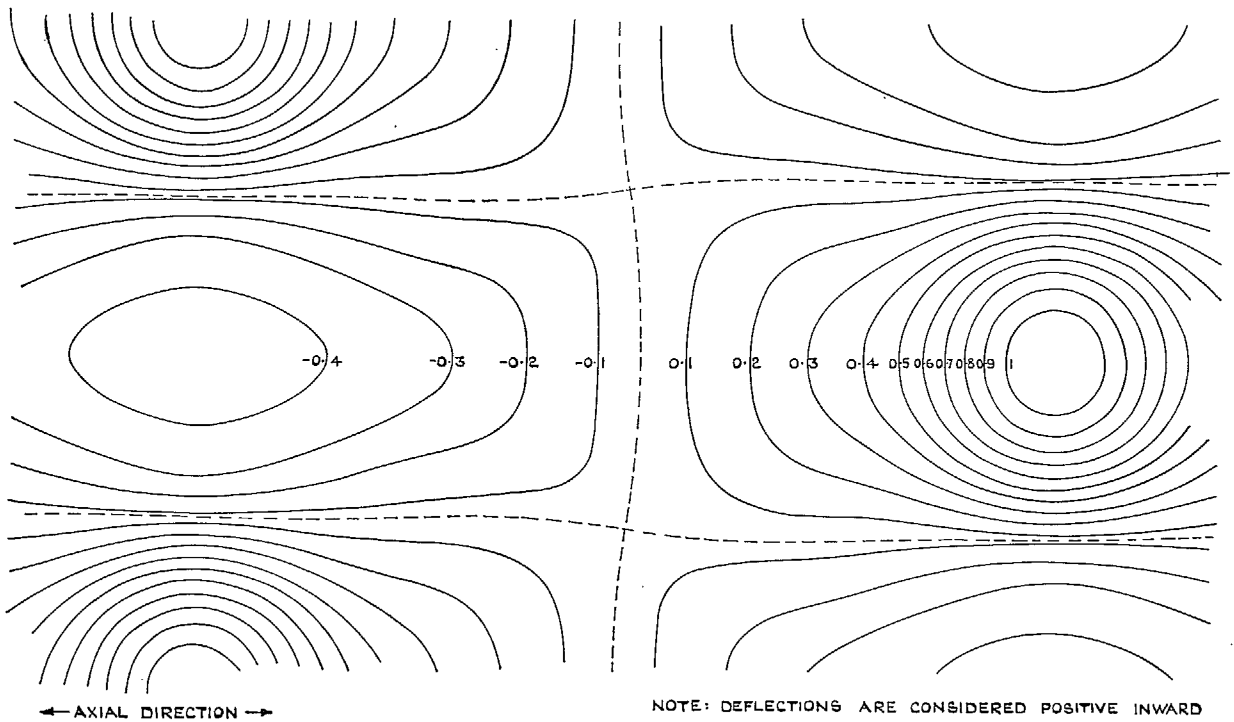

FIG. 7. Contours of deflected cylinder in region of minimum post-buckled strain.

(92365)

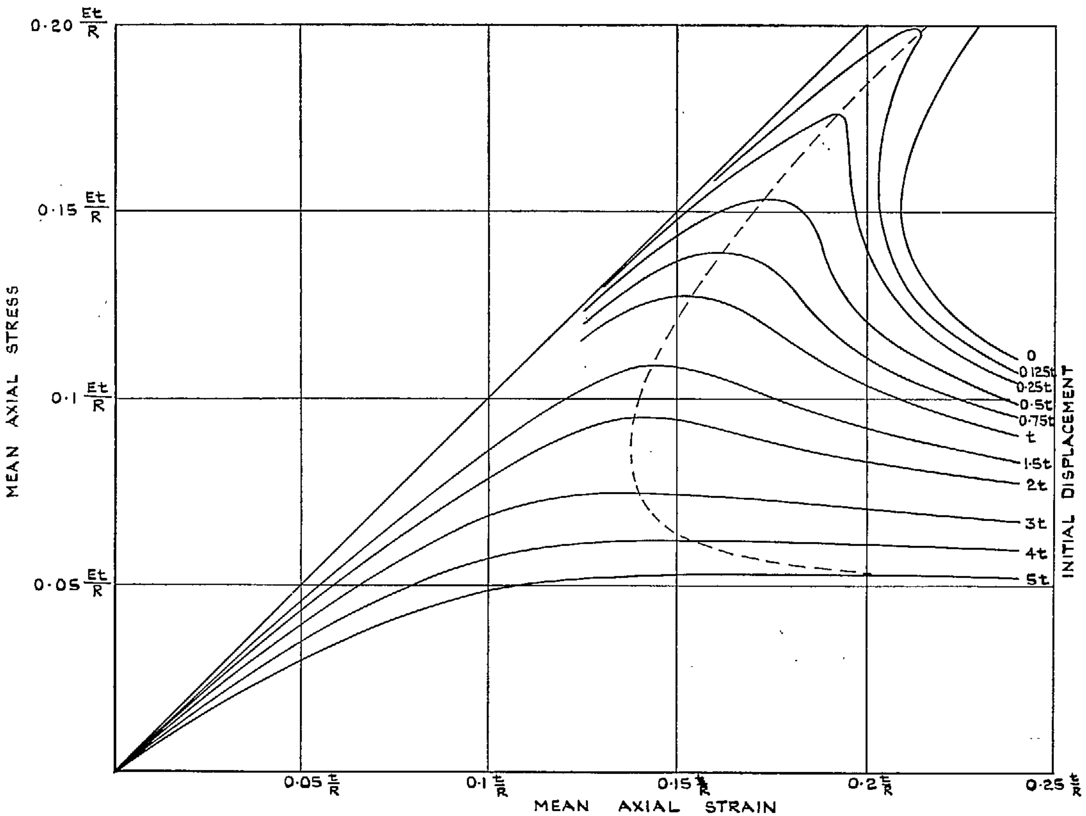

FIG. 8. Stress-strain characteristics for cylinders with initial displacement.

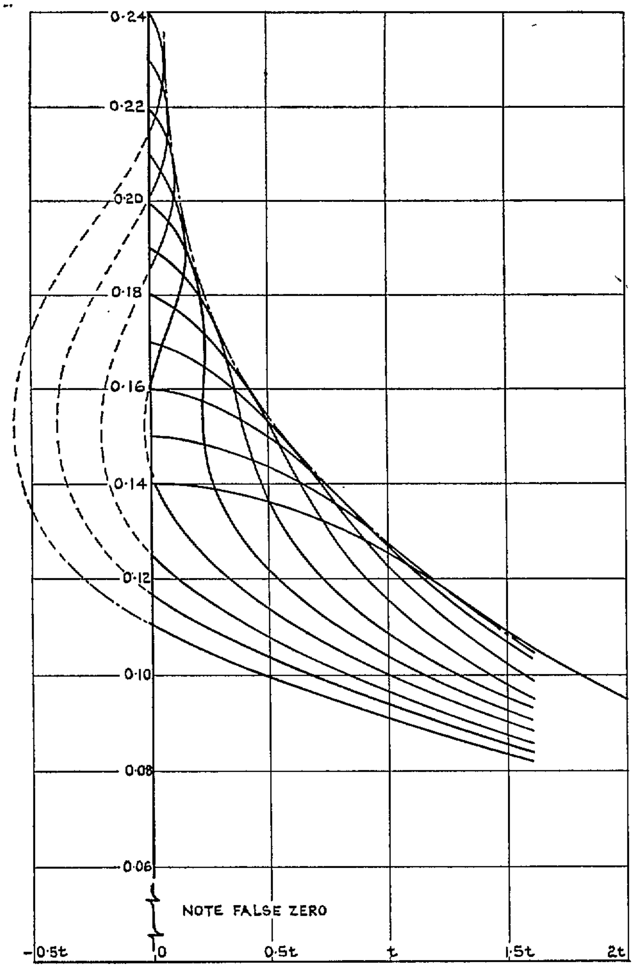

FIG. 9. Generation of buckling stress: initial displacements envelope for small initial displacements.

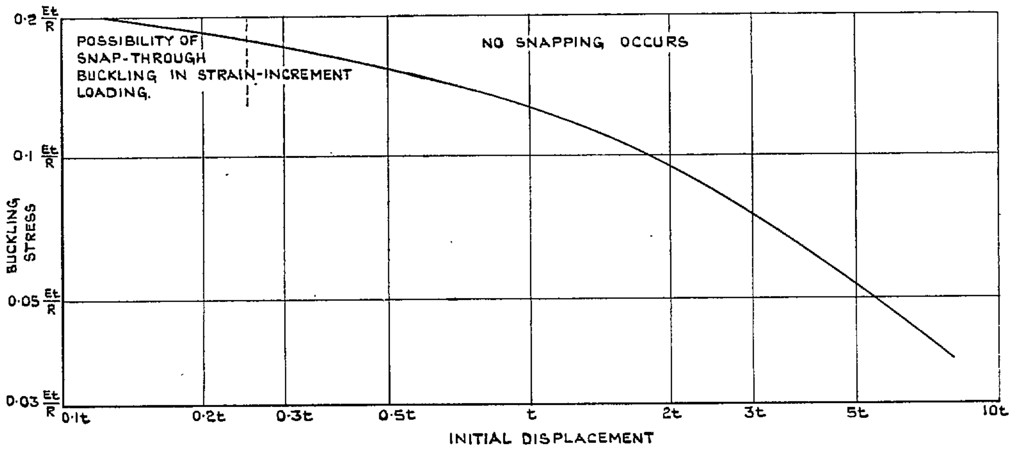

FIG. 10. Variation of buckling stress with initial displacement of imperfect cylinder.

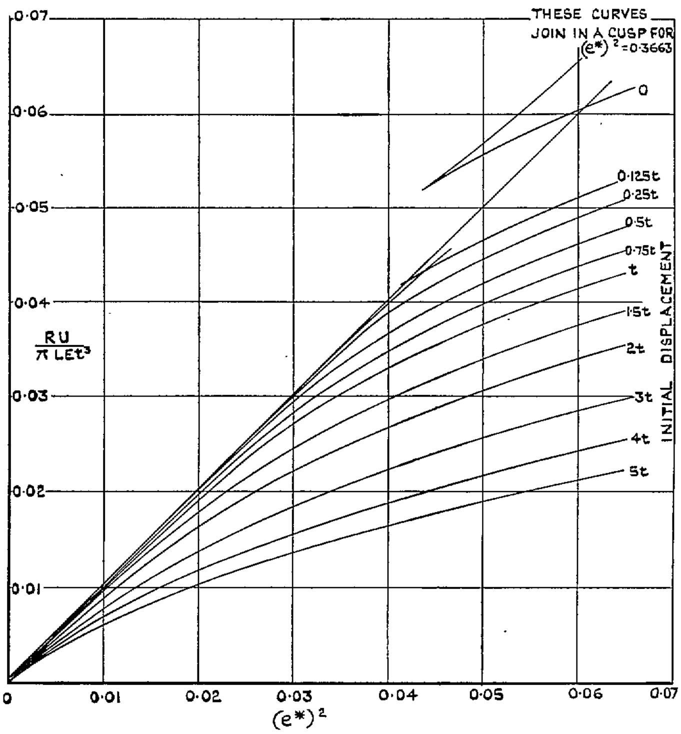

FIG. 11. Variation of strain energy with strain for varying initial displacement.

(92365) Wt. 67/7689 K.5 6/66 Hw.

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