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Forces on Aerofoils with both Incidence and Forward Speed Varying

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Forces on Aerofoils with both Incidence and Forward Speed Varying

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Summary.

Determination of the inviscid, incompressible flow due to a thin, two-dimensional aerofoil moving with varying incidence and forward speed requires the solution of an integral equation. This report examines the case of harmonic variation of both incidence and forward speed (same frequency, different amplitude and phase). A solution is obtained in which the first terms neglected are of order $(\nu \log \nu)^2$, where ν is the reduced frequency.

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1. *Introduction.*

The determination of the aerodynamic forces acting on a helicopter rotor is a most complicated problem, and at the present time approximate solutions only can be obtained. A common procedure is to use a 'quasi-steady' theory: at each instant of time the problem is solved by determining the forces produced by a steady flow past the configuration with the current incidence and forward speed. This clearly results in a substantial simplification, since one of the independent variables, the time, becomes merely a parameter.

The purpose of this report is to obtain an estimate of the error incurred by the use of a quasi-steady theory. A problem bearing some similarity to the determination of the flow field about a helicopter rotor is solved by two methods; one is a quasi-steady theory, the other is a more accurate theory. It is hoped that the comparison may help to answer the question whether a quasi-steady theory can adequately predict the aerodynamic forces acting on a helicopter rotor.

The problem considered is the determination of the inviscid, incompressible flow due to an infinitesimally thin, two-dimensional aerofoil that is moving forward with variable speed and oscillating in pitch at the same time. This has some connection with the motion of a helicopter blade: the forward speed of the helicopter combined with the rotatory motion of the blades means that each blade moves normal to its leading edge with variable speed; the incidence of each blade also varies. The assumption of incompressible flow and the restriction to a two-dimensional aerofoil are easily justified: the forward speed of a helicopter and the speed of rotation of the blades are normally not high enough for compressibility effects to be important; and the aspect ratio of a helicopter blade is normally large enough for two-dimensional aerofoil theory to be applicable. As usual, it is assumed that somebody else will investigate the effects of viscosity.

The procedure adopted is as follows. Section 2 treats the problem of an arbitrary aerofoil moving in a straight line with arbitrarily varying forward speed and changing its shape in an arbitrary manner; the problem is reduced to the solution of an integral equation. The analysis is based on that given in Chapter 5 of the book by Robinson and Laurmann[1]; the reader who has read the chapter may feel that the number of symbols has been reduced; he will be right. At the end of Section 2 the formulas obtained are applied to a flat-plate aerofoil. U, the speed, is assumed to be given by

$$U = U_0(1+\Upsilon \cos \omega t)\,; \tag{1}$$

U_0 corresponds to the linear speed of rotation of a cross-section of a helicopter blade and ΥU_0 to the forward speed of the helicopter. α, the instantaneous incidence, is assumed to be given by

$$\alpha = \alpha_0[1 + a \cos (\omega t+\epsilon)]\,. \tag{2}$$

U_0, Υ, ω, α_0, a and ϵ are constants, ω being the circular frequency; t is the time.

Sections 3 and 4 are concerned with the solution of the integral equation derived in Section 2. In Section 3 an approximate form of the integral equation is obtained by expanding it in powers of ν, the reduced frequency; ν is given by

$$\nu = \frac{\omega c}{U_0} = \frac{c}{r},$$

because ω is equal to U_0/r; here, c is the chord of the aerofoil and r its distance from the axis of rotation. Terms of order unity, $\nu \log \nu$, and ν are retained; terms of order $(\nu \log \nu)^2$ and higher-order terms are neglected. The approximate integral equation is solved to the same order of accuracy in Section 4. All this obviously introduces another assumption, that ν is small compared with unity. For a typical blade the chord is about 15 inches, and the length at least 12 feet; hence, the flow about a blade near its tip corresponds to the flow about the aerofoil satisfying equations (1) and (2) with c equal to 15 inches, and r equal to 12 feet. It follows that

$$\nu \doteqdot 0 \cdot 1 ;$$

the assumption that $\nu \ll 1$ is, therefore, justified. A representative value for the forward speed (ΥU_0) of the helicopter is 250 feet per second and for the tip speed (U_0) 500 feet per second, so that $\Upsilon = \frac{1}{2}$ at the tip. Throughout the report it is assumed that $\Upsilon < 1$; from equation (1) it is seen that, if $\Upsilon > 1$, the value of U is periodically negative, so that the aerofoil periodically moves back into its wake.

In Section 5 it is shown that, not surprisingly, the 'quasi-steady' solution can be obtained by retaining the terms of order unity and neglecting the terms of order $\nu \log \nu$ and ν as well as all higher-order terms. Section 5 also contains the most important formulas derived in Sections 2, 3 and 4, and a discussion of the results obtained by using these formulas. The reader who wishes to forgo the analysis of Sections 2, 3 and 4 can turn to Section 5 now.

2. *Two-Dimensional, Unsteady, Inviscid, Incompressible Flow past an Aerofoil.*

2.1. *Flow past a General Aerofoil.*

Let X and Y be rectangular Cartesian coordinates fixed in space, and let T be time; the origin of these quantities is of no importance. Suppose that the aerofoil is moving along the X axis in the negative direction with speed $U(T)$. The aerofoil is infinitesimally thin, so that, if the local incidence at any point is sufficiently small, boundary conditions on the aerofoil may be satisfied on the section of the X axis momentarily lying between the normal projections on to this axis of the leading and trailing edges. Let the X coordinate of the mid-point of this section be $X_m(T)$, so that

$$\dot{X}_m(T) = - U(T) ; \tag{3}$$

a dot denotes differentiation with respect to time. Let the density of the fluid be ρ, the local pressure be p, and the pressure at infinity be p_∞. Let the length of the aerofoil chord be c.

Suppose that the conditions (Ref. 1, page 10) for the existence of a velocity potential, ϕ, are satisfied; then the components of velocity in the X and Y directions are respectively ϕ_X and ϕ_Y, and ϕ satisfies Laplace's equation,

$$\phi_{XX} + \phi_{YY} = 0 . \tag{4}$$

Bernoulli's equation (Ref. 1, page 15) states that at any instant of time

$$\frac{p}{\rho} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial X}\right)^2 + \left(\frac{\partial \phi}{\partial Y}\right)^2\right] + \frac{\partial \phi}{\partial T} = \frac{p_\infty}{\rho} . \tag{5}$$

The assumptions made about the geometry of the aerofoil allow squares of the velocity components to be neglected in comparison with the velocity components; hence, equation (5) may be approximated by

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial T} = \frac{p_\infty}{\rho} . \tag{6}$$

The acceleration potential, Ω, is now introduced, where

$$\Omega = \frac{\partial \phi}{\partial T}; \tag{7}$$

from equation (4) it satisfies

$$\Omega_{XX} + \Omega_{YY} = 0. \tag{8}$$

From equations (6) and (7), Ω is zero at infinity. The pressure is continuous everywhere except across the aerofoil, and so the same must hold for Ω, from equations (6) and (7); ϕ, of course, is discontinuous across the wake as well, which is why the use of Ω is preferred.

Let x and y be rectangular Cartesian coordinates moving with the aerofoil and having their origin at the point $X = X_m(T)$, $Y = 0$; let the x axis have the same direction as the X axis. When the space coordinates are x and y, let t denote the time; there is then no doubt about what is being kept constant during partial differentiations. Hence,

$$x = X - X_m(T), \tag{9a}$$

$$y = Y, \tag{9b}$$

$$t = T. \tag{9c}$$

From equation (3), it follows that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}, \tag{10a}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial Y}, \tag{10b}$$

$$\frac{\partial}{\partial t} = -U(T)\frac{\partial}{\partial X} + \frac{\partial}{\partial T}. \tag{10c}$$

From equations (8), (10a) and (10b), Ω satisfies

$$\Omega_{xx} + \Omega_{yy} = 0;$$

hence

$$\Omega = \mathscr{R}\, \psi(z, t),$$

where ψ is a complex function of z, and

$$z = x + iy.$$

In the x, y coordinate system the aerofoil lies between the points $(-c/2, 0)$ and $(c/2, 0)$; this part of the x axis may be mapped on to the unit circle by the transformation

$$z = \frac{c}{4}\left(\zeta + \frac{1}{\zeta}\right). \tag{11}$$

Past experience (Ref. 1, page 125) suggests the following form for the function ψ:

$$\psi = i\left[-a_0(t)\frac{\zeta}{1+\zeta} + \sum_{n=1}^{\infty}\frac{a_n(t)}{\zeta^n}\right]. \tag{12}$$

From the antisymmetry of the problem,

$$\Omega(y) = -\Omega(-y).$$

The aerofoil is the only part of the x axis where Ω is discontinuous; therefore, Ω must be zero along the rest of the x axis and, hence, along the part of the real axis of ζ lying outside the unit circle. It follows that ψ is a function whose real part is zero along the real axis of ζ; hence, the $a_n (n \geqslant 0)$ in equation (12) are all real.

Let the equation of the aerofoil be

$$y = F(x, t). \tag{13}$$

The boundary condition is that the velocity component normal to the aerofoil should be zero. The component of velocity in the x direction is $U + \phi_x$, which is approximately U, and the component of velocity in the y direction is ϕ_y; so the boundary condition is approximately

$$\left(\frac{\partial \phi}{\partial y}\right)_{y=0} = U\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t}. \tag{14}$$

On the unit circle, which corresponds to the aerofoil, write

$$\zeta = e^{i\mu}, \tag{15a}$$

so that, from equation (11),

$$x = \frac{c}{2}\cos \mu. \tag{15b}$$

The right-hand side of equation (14) can now be expanded as $b_0/2 + \sum_{n=1}^{\infty} b_n \cos n\mu$, where the b_n are known functions of t; the boundary condition on the aerofoil becomes

$$\left(\frac{\partial \phi}{\partial y}\right)_{y=0} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos n\mu. \tag{16}$$

From equation (7),

$$\phi(X, Y, T) = \int_{-\infty}^{T} \Omega \, dT_1 = \mathscr{R}\int_{-\infty}^{T} \psi(z_1, T_1) dT_1; \tag{17}$$

here, T_1 is the variable of integration and, from equation (9)

$$z_1 = X - X_m(T_1) + iY; \tag{18}$$

the lower limit of the integral can be replaced by any value of the time early enough for Ω to be zero at the point X, Y. From equations (17) and (18),

$$\frac{\partial \phi}{\partial Y} = -\mathscr{I}\int_{-\infty}^{T} \psi'(z_1, T_1) dT_1, \tag{19}$$

where $\psi'(z, T)$ has been written for $\partial\psi/\partial z$. From equations (18) and (3),

$$dz_1 = -\dot{X}_m(T_1) dT_1 = U(T_1) dT_1;$$

hence, equation (19) becomes

$$\frac{\partial \phi}{\partial Y} = -\mathscr{I}\int_{-\infty}^{z} \psi'(z_1, T_1) \frac{dz_1}{U(T_1)}; \tag{20}$$

here, T_1 is to be regarded as a function of z_1, given by equation (18); and z is given by

$$z = X - X_m(T) + iY.$$

On the aerofoil equations (16), (10b) and (20) give

$$\mathscr{I}\int_{-\infty}^{x}\psi'(z_1,\,T_1)\,\frac{dz_1}{U(T_1)} = -\frac{b_0}{2}-\sum_{n=1}^{\infty} b_n\cos n\mu, \tag{21}$$

where $-c/2 \leqslant x \leqslant c/2$. In the integral X is kept constant; x is given by equation (9a), and T_1 (or, in an obvious notation, t_1) is given in terms of z_1 by

$$z_1 = X - X_m(T_1) = X - X_m(t_1), \tag{22}$$

from equations (18) and (9c). Differentiation of equation (21) with respect to T gives

$$\mathscr{I}\,\psi'(x,\,T) = \mathscr{I}\,\psi'(x,\,t) = -\frac{\dot{b}_0}{2}-\sum_{n=1}^{\infty}\dot{b}_n\cos n\mu - U\frac{\partial}{\partial x}\left(\frac{b_0}{2}+\sum_{n=1}^{\infty} b_n\cos n\mu\right), \tag{23}$$

from equations (10a), (10c) and (9c). Now, from equations (11) and (12),

$$\psi'(z,\,t) = \frac{\partial\psi/\partial\zeta}{\partial z/\partial\zeta} = \frac{-i\left[\dfrac{a_0}{(\zeta+1)^2}+\displaystyle\sum_{n=1}^{\infty}\frac{na_n}{\zeta^{(n+1)}}\right]}{\dfrac{c}{4}\left(1-\dfrac{1}{\zeta^2}\right)}; \tag{24}$$

hence, on the aerofoil,

$$\begin{aligned}\psi'(x,\,t) &= -\frac{4i}{c}\,\frac{\left[\dfrac{a_0e^{i\mu}}{(e^{i\mu}+1)^2}+\displaystyle\sum_{n=1}^{\infty} na_ne^{-in\mu}\right]}{(e^{i\mu}-e^{-i\mu})}\\ &= -\frac{2}{c\sin\mu}\left[\frac{a_0}{2(\cos\mu+1)}+\sum_{n=1}^{\infty} na_n(\cos n\mu - i\sin n\mu)\right].\end{aligned} \tag{25}$$

From equation (15b),

$$\frac{\partial}{\partial x} = -\frac{2}{c\sin\mu}\frac{\partial}{\partial\mu} \tag{26}$$

on the aerofoil. From equations (23), (25) and (26),

$$\frac{2}{c\sin\mu}\sum_{n=1}^{\infty} na_n\sin n\mu = -\frac{\dot{b}_0}{2}-\sum_{n=1}^{\infty}\dot{b}_n\cos n\mu - \frac{2U}{c\sin\mu}\sum_{n=1}^{\infty} nb_n\sin n\mu\,;$$

it follows that

$$a_n = -\frac{c}{4n}\dot{b}_{n-1}+\frac{c}{4n}\dot{b}_{n+1}-Ub_n \qquad (n\geqslant 1). \tag{27}$$

The flow field is completely determined once ψ is known, and the only unknown quantity in the expression for ψ given by equation (12) is a_0. Only the differentiated form of equation (21) has so far been used; substitution of $\psi'(z_1,\,T_1)$ into equation (21) provides the desired relation for a_0. From equations (24) and (27),

$$\begin{aligned}\psi'(z,\,T) = \psi'(z,\,t) &= -\frac{4i\zeta^2}{c(\zeta^2-1)}\left\{\frac{a_0}{(\zeta+1)^2}-\sum_{n=1}^{\infty}\left(\frac{c}{4}\dot{b}_{n-1}-\frac{c}{4}\dot{b}_{n+1}+Unb_n\right)\frac{1}{\zeta^{(n+1)}}\right\}\\ &= \frac{i}{(\zeta^2-1)}\left\{-\frac{4a_0\zeta^2}{c(\zeta+1)^2}+(\dot{b}_0+\zeta\dot{b}_1)\right\}-i\sum_{n=1}^{\infty}\left\{\frac{\dot{b}_n}{\zeta^n}-\frac{4U}{c}\frac{nb_n}{(\zeta^2-1)\zeta^{n-1}}\right\}\\ &= \frac{i}{(\zeta^2-1)}\left\{-\frac{4a_0\zeta^2}{c(\zeta+1)^2}+(\dot{b}_0+\zeta\dot{b}_1)\right\}-iU\left(\frac{\partial}{\partial z}+\frac{1}{U}\frac{\partial}{\partial T}\right)\sum_{n=1}^{\infty}\frac{b_n}{\zeta^n}.\end{aligned}$$

When X and Y are held constant,

$$\frac{d}{dz_1} = \frac{\partial}{\partial z_1} + \frac{dT_1}{dz_1}\frac{\partial}{\partial T_1} = \frac{\partial}{\partial z_1} + \frac{1}{U(T_1)}\frac{\partial}{\partial T_1},$$

from equation (18); in the integral in equation (21), therefore, $\psi'(z_1, T_1)$ is given by

$$\psi'(z_1, T_1) = \frac{i}{(\zeta_1^2-1)}\left\{-\frac{4a_0\zeta_1^2}{c(\zeta_1+1)^2} + (\dot{b}_0 + \zeta_1\dot{b}_1)\right\} - iU\frac{d}{dz_1}\sum_{n=1}^{\infty}\frac{b_n}{\zeta_1^n},$$

where ζ_1 is defined by

$$z_1 = \frac{c}{4}\left(\zeta_1 + \frac{1}{\zeta_1}\right). \tag{28}$$

Hence,

$$\mathscr{I}\int_{-\infty}^{x}\psi'(z_1, T_1)\frac{dz_1}{U(T_1)} = -\mathscr{R}\left[\int_{-\infty}^{x}\frac{1}{(\zeta_1^2-1)}\left\{\frac{4a_0\zeta_1^2}{c(\zeta_1+1)^2} - (\dot{b}_0 + \zeta_1\dot{b}_1)\right\}\frac{dz_1}{U(T_1)} + \sum_{n=1}^{\infty}\frac{b_n}{\zeta^n}\right].$$

From equation (21),

$$\mathscr{R}\int_{-\infty}^{x}\frac{1}{(\zeta_1^2-1)}\left\{\frac{4a_0\zeta_1^2}{c(\zeta_1+1)^2} - (\dot{b}_0 + \zeta_1\dot{b}_1)\right\}\frac{dz_1}{U(T_1)} = \frac{b_0}{2}. \tag{29}$$

The path of integration may be taken as the part of the x axis from $-\infty$ to x; but x lies in the range $-c/2 \leqslant x \leqslant c/2$, so that the path includes the point $x = -c/2$; the integral becomes singular there, since, from equation (11), the point $x = -c/2$ corresponds to $\zeta = -1$; hence, the path of integration must be indented at the point $x = -c/2$.

In terms of ζ_1 equation (29) becomes

$$\mathscr{R}\int_{-\infty}^{i\mu}\left\{\frac{4a_0}{c(\zeta_1+1)^2} - \frac{1}{\zeta_1^2}(\dot{b}_0 + \zeta_1\dot{b}_1)\right\}\frac{d\zeta_1}{U(T_1)} = \frac{2b_0}{c},$$

from equations (28) and (15a); the path of integration runs from $-\infty$ to -1 along the real axis of ζ_1, is indented at $\zeta_1 = -1$, and then runs from $\zeta_1 = -1$ to $\zeta_1 = e^{i\mu}$ around the unit circle. The last equation may be written

$$\mathscr{R}\int_{-\infty}^{e^{i\mu}}\frac{1}{(\zeta_1+1)^2}\left\{\frac{4a_0(T_1)}{cU(T_1)} - \frac{4a_0(T)}{cU(T)}\right\}d\zeta_1 = \frac{2b_0}{c} +$$

$$+\mathscr{R}\int_{-\infty}^{e^{i\mu}}\frac{(\dot{b}_0+\zeta_1\dot{b}_1)}{\zeta_1^2}\frac{d\zeta_1}{U(T_1)} - \mathscr{R}\frac{4a_0(T)}{cU(T)}\int_{-\infty}^{e^{i\mu}}\frac{1}{(\zeta_1+1)^2}d\zeta_1. \tag{30}$$

But

$$\mathscr{R}\int_{-\infty}^{e^{i\mu}}\frac{1}{(\zeta_1+1)^2}d\zeta_1 = -\mathscr{R}\frac{1}{(e^{i\mu}+1)} = \mathscr{R}\frac{-\left(\cos\frac{\mu}{2} + i\sin\frac{\mu}{2}\right)}{2\cos\frac{\mu}{2}} = -\frac{1}{2};$$

hence, equation (30) becomes

$$\mathscr{R}\int_{-\infty}^{e^{i\mu}}\frac{1}{(\zeta_1+1)^2}\left\{\frac{4a_0(T_1)}{cU(T_1)} - \frac{4a_0(T)}{cU(T)}\right\}d\zeta_1 - \frac{2a_0(T)}{cU(T)} = \frac{2b_0}{c} + \mathscr{R}\int_{-\infty}^{e^{i\mu}}\frac{(\dot{b}_0+\zeta_1\dot{b}_1)}{\zeta_1^2}\frac{d\zeta_1}{U(T_1)}.$$

This equation has to be satisfied for one value only of μ, since the differentiated form is known to be true; when $\mu = \pi$ it becomes

$$\mathscr{R}\int_{-\infty}^{-1}\frac{1}{(\zeta_1+1)^2}\left\{\frac{4a_0(T_1)}{cU(T_1)} - \frac{4a_0(T)}{cU(T)}\right\}d\zeta_1 - \frac{2a_0(T)}{cU(T)} = \frac{2b_0}{c} + \int_{-\infty}^{-1}\frac{(\dot{b}_0+\zeta_1\dot{b}_1)}{\zeta_1^2}\frac{d\zeta_1}{U(T_1)}. \tag{31}$$

From equations (28) and (22),

$$\frac{c}{4}\left(\zeta_1+\frac{1}{\zeta_1}\right) = X - X_m(T_1)\,; \tag{32}$$

at the chosen instant of time, T, $\mu = \pi$, and the last equation becomes

$$-\frac{c}{2} = X - X_m(T), \tag{33}$$

from equations (15b) and (9a); subtraction of equation (33) from equation (32) gives a relation between ζ_1 and T_1 with T as a parameter,

$$\frac{c}{4}\frac{(1+\zeta_1)^2}{\zeta_1} = X_m(T) - X_m(T_1)\,. \tag{34}$$

Provided that a_0 and U have a Taylor expansion about $T_1 = T$, it follows that the integrand in the integral on the left-hand side of equation (31) is not singular at $\zeta_1 = -1$, so that the path of integration may be taken as the part of the real axis of ζ_1 from $-\infty$ to -1.

Equations (6) and (7) give

$$p - p_\infty = -\rho\Omega\,;$$

hence, the loading at a point on the aerofoil is $\rho(\Omega_u - \Omega_l)$, where Ω_u and Ω_l are respectively the values of Ω on the upper and lower surfaces of the aerofoil; from equations (12) and (15a),

$$\rho(\Omega_u - \Omega_l) = \rho\left(a_0 \tan\frac{\mu}{2} + 2\sum_{n=1}^{\infty} a_n \sin n\mu\right).$$

The lift on the aerofoil is L, where

$$\begin{aligned} L &= \rho\int_{-c/2}^{c/2}(\Omega_u - \Omega_l)dx \\ &= \frac{1}{2}\rho c\int_0^{\pi}\left(a_0\tan\frac{\mu}{2} + 2\sum_{n=1}^{\infty} a_n \sin n\mu\right)\sin\mu\, d\mu = \frac{\pi}{2}\rho c(a_0 + a_1)\,, \end{aligned}$$

from equation (15b); the lift coefficient is C_L, where

$$C_L = \frac{L}{\frac{1}{2}\rho c U^2} = \frac{\pi(a_0+a_1)}{U^2}\,. \tag{35}$$

The moment about the leading edge is M, where

$$\begin{aligned} M &= \rho\int_{-c/2}^{c/2}(\Omega_u - \Omega_l)\left(x+\frac{1}{2}c\right)dx = \rho\int_{-c/2}^{c/2}(\Omega_u - \Omega_l)x\, dx + \frac{1}{2}cL \\ &= \rho\frac{c^2}{4}\int_0^{\pi}\left(a_0\tan\frac{\mu}{2} + 2\sum_{n=1}^{\infty} a_n \sin n\mu\right)\sin\mu\cos\mu\, d\mu + \frac{1}{2}cL \\ &= \frac{\pi\rho c^2}{8}(a_0 + 2a_1 + a_2)\,; \end{aligned}$$

the moment coefficient is C_m, where

$$C_m = \frac{M}{\frac{1}{2}\rho c^2 U^2} = \frac{\pi(a_0+2a_1+a_2)}{4U^2}. \tag{36}$$

2.2. *Flow Past a Particular Aerofoil.*

Now consider the aerofoil defined in Section 1. From equation (1),

$$U(T) = U_0(1+\Upsilon\cos\omega T). \tag{37}$$

If the mid-point of the aerofoil is at the origin of the (X, Y) coordinates when $T = 0$, then, from equations (3), (37) and (9c),

$$X_m(T) = -U_0\left(T+\frac{\Upsilon}{\omega}\sin\omega T\right) = -U_0\left(t+\frac{\Upsilon}{\omega}\sin\omega t\right). \tag{38}$$

From equations (13) and (2), if the aerofoil is assumed to be pitching about the leading edge,

$$F(x, t) = -\alpha_0\{1 + a\cos(\omega t+\epsilon)\}\left(x+\frac{1}{2}c\right);$$

hence, from equation (37),

$$U\frac{\partial F}{\partial x}+\frac{\partial F}{\partial t} = -\alpha_0 U_0(1+\Upsilon\cos\omega t)\{1 + a\cos(\omega t+\epsilon)\} + \alpha_0 a\omega\sin(\omega t+\epsilon)\left(x+\frac{1}{2}c\right)$$

$$= -\alpha_0\left[U_0(1+\Upsilon\cos\omega t)\{1 + a\cos(\omega t+\epsilon)\} - \frac{a\omega\epsilon}{2}\sin(\omega t+\epsilon)\right] +$$

$$+\alpha_0 a\omega x\sin(\omega t+\epsilon),$$

so that, from equations (14), (16) and (15b)

$$b_0 = -2\alpha_0\left[U_0(1+\Upsilon\cos\omega t)\{1 + a\cos(\omega t+\epsilon)\} - \frac{ac\omega}{2}\sin(\omega t+\epsilon)\right], \tag{39}$$

$$b_1 = \frac{\alpha_0 c\omega a}{2}\sin(\omega t+\epsilon), \tag{40}$$

$$b_n = 0 \qquad (n > 1). \tag{41}$$

From equation (39),

$$\dot{b}_0 = 2\alpha_0 U_0\omega\left[\Upsilon\sin\omega t\{1 + \alpha\cos(\omega t+\epsilon)\} + a(1+\Upsilon\cos\omega t)\sin(\omega t+\epsilon) + \right.$$

$$\left. + \frac{a\nu}{2}\cos(\omega t+\epsilon)\right], \tag{42}$$

where ν is the reduced frequency, $c\omega/U_0$; from equation (40),

$$\dot{b}_1 = \frac{1}{2}\alpha_0 U_0\omega\nu\alpha\cos(\omega t+\epsilon); \tag{43}$$

from equation (41),

$$\dot{b}_n = 0 \qquad (n > 1). \tag{44}$$

From equations (27), (37) and (39) to (44) inclusive,

$$a_1 = -\frac{1}{2}\alpha_0 U_0{}^2\nu\left[\Upsilon \sin \omega t\,\{1 + a\cos(\omega t+\epsilon)\} + a(1+\Upsilon\cos\omega t)\sin(\omega t+\epsilon) + \right.$$

$$\left. + \frac{a\nu}{2}\cos(\omega t+\epsilon)\right] - \frac{1}{2}\alpha_0 U_0{}^2\nu a(1+\Upsilon\cos\omega t)\sin(\omega t+\epsilon)$$

$$= -\frac{1}{2}\alpha_0 U_0{}^2\nu\left[\Upsilon\sin\omega t\,\{1 + a\cos(\omega t+\epsilon)\} + \right.$$

$$\left. + 2a(1+\Upsilon\cos\omega t)\sin(\omega t+\epsilon) + \frac{a\nu}{2}\cos(\omega t+\epsilon)\right], \tag{45}$$

$$a_2 = -\frac{\alpha_0 c^2\omega^2 a}{16}\cos(\omega t+\epsilon) = -\frac{\alpha_0 U_0{}^2\nu^2 a}{16}\cos(\omega t+\epsilon), \tag{46}$$

$$a_n = 0 \qquad (n > 2).$$

To obtain a_0 equation (31) must be solved. It is convenient to make the following substitutions:

$$\theta = -U_0 T/c, \tag{47a}$$

$$\theta_1 = -U_0 T_1/c, \tag{47b}$$

$$A_0(\theta) = \frac{a_0(T)}{U_0{}^2\alpha_0(1+\Upsilon\cos\omega T)}, \tag{47c}$$

$$\eta = -\zeta_1. \tag{47d}$$

From equations (31), (37), (39), (42), (43) and (47),

$$\int_1^\infty \{A_0(\theta_1) - A_0(\theta)\}\frac{d\eta}{(\eta-1)^2} - \frac{1}{2}A_0(\theta)$$

$$= -\left[(1+\Upsilon\cos\nu\theta)\{1 + a\cos(\nu\theta-\epsilon)\} + \frac{a\nu}{2}\sin(\nu\theta-\epsilon)\right] -$$

$$-\int_1^\infty\left\{\frac{\nu\left[\Upsilon\sin\nu\theta_1\{1 + a\cos(\nu\theta_1-\epsilon)\} + a(1+\Upsilon\cos\nu\theta_1)\sin(\nu\theta_1-\epsilon) - \frac{a\nu}{2}\cos(\nu\theta_1-\epsilon)\right]}{2\eta_1{}^2} - \right.$$

$$\left. - \frac{a\nu^2}{8\eta_1}\cos(\nu\theta_1-\epsilon)\right\}\frac{d\eta_1}{(1+\Upsilon\cos\nu\theta_1)}; \tag{48}$$

here, from equations (34), (38) and (47),

$$(\theta_1-\theta) + \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta) = \frac{1}{4}\frac{(\eta_1-1)^2}{\eta_1}. \tag{49}$$

3. *Approximate Form of the Integral Equation.*

Equation (48) may be written in the form

$$A_0(\theta) = 2(1+\Upsilon \cos \nu\theta)\{1 + a \cos (\nu\theta - \epsilon)\} + a\nu \sin (\nu\theta - \epsilon) +$$

$$+ \nu \int_1^\infty \left[\frac{\Upsilon \sin \nu\theta_1\{1 + a \cos (\nu\theta_1 - \epsilon)\}}{(1+\Upsilon \cos \nu\theta_1)} + a \sin (\nu\theta_1 - \epsilon)\right] \frac{d\eta_1}{\eta_1^2} -$$

$$- \frac{a\nu^2}{4} \int_1^\infty \left(\frac{2}{\eta_1^2} + \frac{1}{\eta_1}\right) \frac{\cos (\nu\theta_1 - \epsilon) d\eta_1}{(1+\Upsilon \cos \nu\theta_1)} + 2 \int_1^\infty \{A_0(\theta_1) - A_0(\theta)\} \frac{d\eta_1}{(\eta_1 - 1)^2}. \tag{50}$$

First, it is shown that the second integral is O(log ν). Clearly,

$$\left| \int_1^\infty \frac{2}{\eta_1^2} \frac{\cos (\nu\theta_1 - \epsilon) d\eta_1}{(1+\Upsilon \cos \nu\theta_1)} \right| < \frac{2}{(1-\Upsilon)} \int_1^\infty \frac{d\eta_1}{\eta_1^2} = \mathrm{O}(1);$$

so it suffices to show that

$$\int_1^\infty \frac{1}{\eta_1} \frac{\cos (\nu\theta_1 - \epsilon) d\eta_1}{(1+\Upsilon \cos \nu\theta_1)} = \mathrm{O}(\log \nu). \tag{51}$$

Write

$$Q_1 = (\theta_1 - \theta) + \frac{\Upsilon}{\nu} (\sin \nu\theta_1 - \sin \nu\theta). \tag{52}$$

From equation (49),

$$\frac{1}{4} \frac{(\eta_1 - 1)^2}{\eta_1} = Q_1.$$

It follows that

$$\eta_1 = (1+2Q_1) + 2(Q_1 + Q_1^2)^{1/2}; \tag{53}$$

the positive sign of the square root is required, since, as $\theta_1 \to \infty$, $Q_1 \to \infty$ and $\eta_1 \to \infty$. From equations (52) and (53),

$$\frac{d\eta_1}{\eta_1} = \frac{dQ_1}{(Q_1 + Q_1^2)^{1/2}} = \frac{(1+\Upsilon \cos \nu\theta_1) d\theta_1}{(Q_1 + Q_1^2)^{1/2}}.$$

Therefore, the equation to be verified, equation (51), may be written

$$\int_\theta^\infty \frac{\cos (\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2}} = \mathrm{O}(\log \nu). \tag{54}$$

It is convenient to split the range of integration into two parts by writing

$$\int_\theta^\infty \frac{\cos (\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2}} = \int_\theta^{\theta + 2\pi/\nu} \frac{\cos (\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2}} + \int_{\theta + 2\pi/\nu}^\infty \frac{\cos (\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2}}; \tag{55}$$

the only significance in choosing $\theta + 2\pi/\nu$ as the intermediate stage is that it makes the analysis simpler; the essential points are that the integrand in the first integral is singular at the lower limit

($Q_1 = 0$ when $\theta_1 = \theta$) but not at the upper one, whereas in the second integral only the upper limit requires care. From equations (52) and (53),

$$\theta_1 = \theta + \frac{2\pi}{\nu} \rightarrow Q_1 = \frac{2\pi}{\nu}, \tag{56a}$$

$$\theta_1 = \theta + \frac{2\pi}{\nu} \rightarrow \eta_1 = \left(1 + \frac{4\pi}{\nu}\right) + 2\left(\frac{2\pi}{\nu} + \frac{4\pi^2}{\nu^2}\right)^{1/2} = \eta_A ; \tag{56b}$$

the last equation defines η_A.

Consider the second integral on the right-hand side of equation (55). From equation (52),

$$\begin{aligned}\left(\frac{1}{\theta_1} - \frac{1}{Q_1}\right) + \left[\frac{1}{Q_1} - \frac{1}{(Q_1 + Q_1^2)^{1/2}}\right] &= - \frac{\theta - \frac{\Upsilon}{\nu}(\sin \nu\theta_1 - \sin \nu\theta)}{\theta_1 Q_1} + \frac{(Q_1 + Q_1^2)^{1/2} - Q_1}{Q_1(Q_1 + Q_1^2)^{1/2}} \\ &= - \frac{\theta - \frac{\Upsilon}{\nu}(\sin \nu\theta_1 - \sin \nu\theta)}{\theta_1 Q_1} + \\ &\quad + \frac{1}{(Q_1 + Q_1^2)^{1/2}\,[(Q_1 + Q_1^2)^{1/2} + Q_1]} ;\end{aligned}$$

hence,

$$\begin{aligned}\int_{\theta+2\pi/\nu}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon)d\theta_1}{(Q_1 + Q_1^2)^{1/2}} &= \int_{\theta+2\pi/\nu}^{\infty} \frac{\cos(\nu\theta - \epsilon)d\theta_1}{\theta_1} + \\ &\quad + \int_{\theta+2\pi/\nu}^{\infty} \frac{\left[\theta - \frac{\Upsilon}{\nu}(\sin \nu\theta_1 - \sin \nu\theta)\right]\cos(\nu\theta_1 - \epsilon)}{\theta_1 Q_1} d\theta_1 - \\ &\quad - \int_{\theta+2\pi/\nu}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon)d\theta_1}{(Q_1 + Q_1^2)^{1/2}\,[(Q_1 + Q_1^2)^{1/2} + Q_1]} .\end{aligned} \tag{57}$$

The first of the integrals on the right-hand side can be written in terms of tabulated functions, as follows:

$$\int_{\theta}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon)d\theta_1}{+2\pi/\nu\theta_1} = -\cos\epsilon\ \mathrm{Ci}(2\pi + \nu\theta) + \sin\epsilon\left[\frac{\pi}{2} - \mathrm{Si}(2\pi + \nu\theta)\right] ; \tag{58}$$

here, Ci and Si are respectively the cosine and sine integrals (Ref. 2, page 3), defined by

$$\mathrm{Ci}(\vartheta) = -\int_{\vartheta}^{\infty} \frac{\cos\vartheta_1}{\vartheta_1} d\vartheta_1,$$

and

$$\mathrm{Si}(\vartheta) = \int_0^{\vartheta} \frac{\sin\vartheta_1}{\vartheta_1} d\vartheta_1 ;$$

it is known that[2]

$$\mathrm{Si}(\infty) = \frac{\pi}{2}.$$

Since[2] $\mathrm{Ci}(2\pi + \nu\theta)$ and $\mathrm{Si}(2\pi + \nu\theta)$ are both O(1), it follows from equation (58) that

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon)d\theta_1}{\theta_1} = O(1). \tag{59}$$

For the second of the integrals, equation (52) gives

$$\left|\int_{\theta+2\pi/\nu}^{\infty} \frac{\left[\theta - \frac{\Upsilon}{\nu}(\sin \nu\theta_1 - \sin \nu\theta)\right] \cos(\nu\theta_1 - \epsilon)}{\theta_1 Q_1} d\theta_1\right|$$

$$= \left|\int_{\theta+2\pi/\nu}^{\infty} \frac{\left[\theta - \frac{\Upsilon}{\nu}(\sin \nu\theta_1 - \sin \nu\theta)\right] \cos(\nu\theta_1 - \epsilon)}{\theta_1 \left[(\theta_1 - \theta) + \frac{\Upsilon}{\nu}(\sin \nu\theta_1 - \sin \nu\theta)\right]} d\theta_1\right|$$

$$< \left(\theta + \frac{2\Upsilon}{\nu}\right) \int_{\theta+2\pi/\nu}^{\infty} \frac{d\theta_1}{\theta_1 \left(\theta_1 - \theta - \frac{2\Upsilon}{\nu}\right)} = \log \frac{2(\pi - \Upsilon)}{(2\pi + \nu\theta)} = \mathrm{O}(1). \tag{60}$$

For the third of the integrals, equations (52) and (56a) give

$$\left|\int_{\theta+2\pi/\nu}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2} [(Q_1 + Q_1^2)^{1/2} + Q_1]}\right|$$

$$= \left|\int_{2\pi/\nu}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon) dQ_1}{(1 + \Upsilon \cos \nu\theta_1)(Q_1 + Q_1^2)^{1/2} [(Q_1 + Q_1^2)^{1/2} + Q_1]}\right|$$

$$< \frac{1}{2(1 - \Upsilon)} \int_{2\pi/\nu}^{\infty} \frac{dQ_1}{Q_1^2} = \frac{\nu}{4\pi(1 - \Upsilon)} = \mathrm{O}(\nu). \tag{61}$$

From equations (57), (59), (60) and (61), it follows that

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\cos(\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2}} = \mathrm{O}(1). \tag{62}$$

Now consider the first integral on the right-hand side of equation (55). From equations (52) and (56a),

$$\left|\int_0^{\theta+2\pi/\nu} \frac{\cos(\nu\theta_1 - \epsilon) d\theta_1}{(Q_1 + Q_1^2)^{1/2}}\right| = \left|\int_0^{2\pi/\nu} \frac{\cos(\nu\theta_1 - \epsilon) dQ_1}{(1 + \Upsilon \cos \nu\theta_1)(Q_1 + Q_1^2)^{1/2}}\right|$$

$$= \left|\left[2 \sinh^{-1} Q_1^{1/2} \frac{\cos(\nu\theta_1 - \epsilon)}{(1 + \Upsilon \cos \nu\theta_1)}\right]_0^{2\pi/\nu} + \right.$$

$$\left. + 2\nu \int_0^{2\pi/\nu} \left[\frac{\sin(\nu\theta_1 - \epsilon)}{(1 + \Upsilon \cos \nu\theta_1)^2} - \frac{\Upsilon \sin \nu\theta_1 \cos(\nu\theta_1 - \epsilon)}{(1 + \Upsilon \cos \nu\theta_1)^3}\right] \sinh^{-1} Q_1^{1/2} dQ_1\right|$$

$$< \frac{2}{(1 - \Upsilon)} \sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2} + 2\nu \left[\frac{1}{(1 - \Upsilon)^2} + \frac{\Upsilon}{(1 - \Upsilon)^3}\right] \int_0^{2\pi/\nu} \sinh^{-1} Q_1^{1/2} dQ_1$$

$$= \frac{2}{(1 - \Upsilon)} \sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2} + 2\nu \left[\frac{1}{(1 - \Upsilon)^2} + \frac{\Upsilon}{(1 - \Upsilon)^3}\right] \times$$

$$\times \left[\frac{\sinh^{-1} Q_1^{1/2}}{2}(1 + 2Q_1) - \frac{1}{2}(Q_1 + Q_1^2)^{1/2}\right]_0^{2\pi/\nu}$$

$$= \frac{2}{(1 - \Upsilon)} \sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2} + \frac{2}{(1 - \Upsilon)^3} \left[\frac{\sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2}}{2}(4\pi + \nu) - \frac{1}{2}(4\pi^2 + 2\pi\nu)^{1/2}\right]$$

$$= \mathrm{O}(\log \nu). \tag{63}$$

From equations (55), (62) and (63), it follows that equation (54) is correct; hence, equation (51) has been proved. Therefore, equation (50) may be written

$$\begin{aligned} A_0(\theta) &= 2(1+\Upsilon\cos\nu\theta)\{1+a\cos(\nu\theta-\epsilon)\}+a\nu\sin(\nu\theta-\epsilon)+ \\ &\quad +\nu\int_1^\infty\left[\frac{\Upsilon\sin\nu\theta_1\{1+a\cos(\nu\theta_1-\epsilon)\}}{(1+\Upsilon\cos\nu\theta_1)}+a\sin(\nu\theta_1-\epsilon)\right]\frac{d\eta_1}{\eta_1^2}+ \\ &\quad +2\int_1^\infty\{A_0(\theta_1)-A_0(\theta)\}\frac{d\eta_1}{(\eta_1-1)^2}+\mathrm{O}(\nu^2\log\nu) \\ &= 2(1+\Upsilon\cos\nu\theta)\{1+a\cos(\nu\theta-\epsilon)\}+a\sin(\nu\theta-\epsilon)+ \\ &\quad +\nu\left[\frac{\Upsilon\sin\nu\theta\{1+a\cos(\nu\theta-\epsilon)\}}{(1+\Upsilon\cos\nu\theta)}+a\sin(\nu\theta-\epsilon)\right]+ \\ &\quad +\nu^2\int_\theta^\infty\frac{Ad\theta_1}{\eta_1}+2\int_1^\infty\{A_0(\theta_1)-A_0(\theta)\}\frac{d\eta_1}{(\eta_1-1)^2}+\mathrm{O}(\nu^2\log\nu), \end{aligned} \tag{64}$$

where

$$\begin{aligned} A &= a\cos(\nu\theta_1-\epsilon)+\frac{\Upsilon\cos\nu\theta_1[1+a\cos(\nu\theta_1-\epsilon)-a\Upsilon\sin\nu\theta_1\sin(\nu\theta_1-\epsilon)]}{(1+\Upsilon\cos\nu\theta_1)}+ \\ &\quad +\frac{\Upsilon^2\sin^2\nu\theta_1[1+a\cos(\nu\theta_1-\epsilon)]}{(1+\Upsilon\cos\nu\theta_1)^2} \\ &= a\cos(\nu\theta_1-\epsilon)+\frac{\Upsilon[\cos\nu\theta_1+a\cos(2\nu\theta_1-\epsilon)]}{(1+\Upsilon\cos\nu\theta_1)}+\frac{\Upsilon^2\sin^2\nu\theta_1[1+a\cos(\nu\theta_1-\epsilon)]}{(1+\Upsilon\cos\nu\theta_1)^2}. \end{aligned} \tag{65}$$

Clearly,

$$|A| < a+\frac{\Upsilon(1+a)}{(1-\Upsilon)}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^2}. \tag{66}$$

It is now shown that the last but one integral in equation (64) is $\mathrm{O}(\log\nu)$. The range of the integral is split up into two parts by writing

$$\int_\theta^\infty\frac{A\,d\theta_1}{\eta_1}=\int_\theta^{\theta+2\pi/\nu}\frac{A\,d\theta_1}{\eta_1}+\int_{\theta+2\pi/\nu}^\infty\frac{A\,d\theta_1}{\eta_1}. \tag{67}$$

First, consider the second integral. From equations (52) and (53),

$$\begin{aligned} \left(\frac{1}{4\theta}-\frac{1}{4Q_1}\right)+\left(\frac{1}{4Q_1}-\frac{1}{\eta_1}\right) &= -\frac{\theta-\frac{\Upsilon}{\nu}(\sin\nu\theta_1-\sin\nu\theta)}{4\theta_1Q_1}+\frac{2(Q_1+Q_1^2)^{1/2}-(2Q_1-1)}{4Q_1\eta_1} \\ &= -\frac{\theta-\frac{\Upsilon}{\nu}(\sin\nu\theta_1-\sin\nu\theta)}{4\theta_1Q_1}+\frac{(8Q_1-1)}{4Q_1\eta_1[2(Q_1+Q_1^2)^{1/2}+(2Q_1-1)]}; \end{aligned}$$

hence,

$$\begin{aligned} \int_{\theta+2\pi/\nu}^\infty\frac{A\,d\theta_1}{\eta_1} &= \int_{\theta+2\pi/\nu}^\infty\frac{A\,d\theta_1}{4\theta_1}+\frac{1}{4}\int_{\theta+2\pi/\nu}^\infty\frac{A\left[\theta-\frac{\Upsilon}{\nu}(\sin\nu\theta_1-\sin\nu\theta)\right]}{\theta_1Q_1}d\theta_1- \\ &\quad -\frac{1}{4}\int_{\theta+2\pi/\nu}^\infty\frac{A(8Q_1-1)d\theta_1}{Q_1\eta_1[2(Q_1+Q_1^2)^{1/2}+(2Q_1-1)]}. \end{aligned} \tag{68}$$

For the second of the integrals on the right-hand side, equations (52) and (66) give

$$\left|\int_{\theta+2\pi/\nu}^{\infty} \frac{A\left[\theta-\frac{\Upsilon}{\nu}(\sin\nu\theta_1-\sin\nu\theta)\right]}{\theta_1 Q_1}\,d\theta_1\right|$$

$$< \left[a+\frac{\Upsilon(1+a)}{(1-\Upsilon)}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^2}\right]\left(\theta+\frac{2\Upsilon}{\nu}\right)\int_{\theta+2\pi/\nu}^{\infty}\frac{d\theta_1}{\theta_1\left(\theta_1-\theta-\frac{2\Upsilon}{\nu}\right)}$$

$$= \left[a+\frac{\Upsilon(1+a)}{(1-\Upsilon)}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^2}\right]\log\frac{(2\pi+\nu\theta)}{2(\pi-\Upsilon)} = O(1). \tag{69}$$

For the third of these integrals, equations (52), (53), (56a) and (66) give

$$\left|\int_{\theta+2\pi/\nu}^{\infty}\frac{A(8\,Q_1-1)d\theta_1}{Q_1\eta_1[2(Q_1+Q_1^2)^{1/2}+(2Q_1-1)]}\right|$$

$$= \left|\int_{\theta+2\pi/\nu}^{\infty}\frac{A(8\,Q_1-1)dQ_1}{(1+\Upsilon\cos\nu\theta_1)Q_1\,[2(Q_1+Q_1^2)^{1/2}+(2Q_1+1)]\,[2(Q_1+Q_1^2)^{1/2}+(2Q_1-1)]}\right|$$

$$< \frac{8}{(1-\Upsilon)}\left[a+\frac{\Upsilon(1+a)}{(1-\Upsilon)}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^2}\right]\int_{2\pi/\nu}^{\infty}\frac{dQ_1}{4Q_1(4Q_1-1)}$$

$$= \frac{2}{(1-\Upsilon)}\left[a+\frac{\Upsilon(1+a)}{(1-\Upsilon)}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^2}\right]\log\frac{8\pi}{(8\pi-\nu)} = O(\nu). \tag{70}$$

From equations (68), (69) and (70)

$$\int_{\theta+2\pi/\nu}^{\infty}\frac{A\,d\theta_1}{\eta_1} = \frac{1}{4}\int_{\theta+2\pi/\nu}^{\infty}\frac{A\,d\theta_1}{\theta_1} + O(1). \tag{71}$$

Equation (65) may be written as

$$A = 2a\cos(\nu\theta_1-\epsilon)+\frac{(\Upsilon-a\cos\epsilon)}{\Upsilon(1+\Upsilon\cos\nu\theta_1)}-\frac{(1-\Upsilon^2)(\Upsilon-a\cos\epsilon)}{\Upsilon(1+\Upsilon\cos\nu\theta_1)^2}-\frac{a(1-\Upsilon^2)\sin\epsilon\sin\nu\theta_1}{(1+\Upsilon\cos\nu\theta_1)^2}.$$

Now,

$$\int\left[\frac{1}{(1+\Upsilon\cos\nu\theta_1)}-\frac{(1-\Upsilon^2)}{(1+\Upsilon\cos\nu\theta_1)^2}\right]d\theta_1 = \frac{\Upsilon\sin\nu\theta_1}{\nu(1+\Upsilon\cos\nu\theta_1)},$$

and

$$\int\frac{\sin\nu\theta_1\,d\theta_1}{(1+\Upsilon\cos\nu\theta_1)^2} = \frac{1}{\nu\Upsilon(1+\Upsilon\cos\nu\theta_1)}.$$

From the last three equations and equation (58),

$$\left|\int_{\theta+2\pi/\nu}^{\infty}\frac{A\,d\theta_1}{\theta_1}\right| < 2a\left|\mathrm{Ci}\,(2\pi+\nu\theta)\right| + 2a\left|\frac{\pi}{2}-\mathrm{Si}\,(2\pi+\nu\theta)\right| + \left|\left[\frac{(\Upsilon-a\cos\epsilon)\sin\nu\theta_1}{\nu\theta_1(1+\Upsilon\cos\nu\theta_1)}-\right.\right.$$

$$\left.\left.-\frac{a(1-\Upsilon^2)\sin\epsilon}{\nu\Upsilon\theta_1(1+\Upsilon\cos\nu\theta_1)}\right]_{\theta+2\pi/\nu}^{\infty}\right| + \frac{1}{\nu}\left|\int_{\theta+2\pi/\nu}^{\infty}\left[\frac{(\Upsilon-a\cos\epsilon)\sin\nu\theta_1}{(1+\Upsilon\cos\nu\theta_1)}-\frac{a(1-\Upsilon^2)\sin\epsilon}{\Upsilon(1+\Upsilon\cos\nu\theta_1)}\right]\frac{d\theta_1}{\theta_1^2}\right|$$

$$< 2a\left|\mathrm{Ci}\,(2\pi+\nu\theta)\right| + 2a\left|\frac{\pi}{2}-\mathrm{Si}\,(2\pi+\nu\theta)\right| + \frac{(\Upsilon+a)}{(1-\Upsilon)(2\pi+\nu\theta)}+\frac{a(1-\Upsilon^2)}{\Upsilon(1-\Upsilon)(2\pi+\nu\theta)}+$$

$$+\frac{1}{\nu}\left[\frac{(\Upsilon+a)}{(1-\Upsilon)}+\frac{a(1-\Upsilon^2)}{\Upsilon(1-\Upsilon)}\right]\int_{\theta+2\pi/\nu}^{\infty}\frac{d\theta_1}{\theta_1^2}$$

$$= 2a\left|\mathrm{Ci}\,(2\pi+\nu\theta)\right| + 2a\left|\frac{\pi}{2}-\mathrm{Si}\,(2\pi+\nu\theta)\right| + \frac{2}{(2\pi+\nu\theta)}\left[\frac{(\Upsilon+a)}{(1-\Upsilon)}+\frac{a(1+\Upsilon)}{\Upsilon}\right]$$

$$= O(1).$$

From this equation and equation (71) it follows that

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{A\, d\theta_1}{\eta_1} = O(1). \tag{72}$$

Now consider the first integral on the right-hand side of equation (67). From equations (52), (53) and (56a),

$$\begin{aligned}\left|\int_{\theta}^{\theta+2\pi/\nu} \frac{A\, d\theta_1}{\eta_1}\right| &= \left|\int_0^{2\pi/\nu} \frac{A\{(1+2Q_1)-2(Q_1+Q_1^2)^{1/2}\}dQ_1}{(1+\Upsilon\cos\nu\theta_1)}\right| \\ &\leqslant \left|\left[\frac{A}{(1+\Upsilon\cos\nu\theta_1)}\left\{Q_1+Q_1^2-\frac{1}{2}(Q_1+Q_1^2)^{1/2}(1+2Q_1)+\frac{1}{2}\sinh^{-1}Q_1^{1/2}\right\}\right]_0^{2\pi/\nu}\right| + \\ &+ \left|\int_{\theta}^{\theta+2\pi/\nu} \frac{d}{d\theta_1}\left\{\frac{A}{(1+\Upsilon\cos\nu\theta_1)}\right\}\left\{Q_1+Q_1^2-\frac{1}{2}(Q_1+Q_1^2)^{1/2}(1+2Q_1)+\right.\right. \\ &\left.\left.+\frac{1}{2}\sinh^{-1}Q_1^{1/2}\right\} d\theta_1\right|.\end{aligned} \tag{73}$$

From equation (66),

$$|A| = O(1); \tag{74}$$

from equations (65) and (66),

$$\begin{aligned}\left|\frac{d}{d\theta_1}\frac{A}{(1+\Upsilon\cos\nu\theta_1)}\right| &= \nu\left|\frac{\Upsilon\sin\nu\theta_1}{(1+\Upsilon\cos\nu\theta_1^2)} - \frac{a\sin(\nu\theta_1-\epsilon)}{(1+\Upsilon\cos\nu\theta_1)}\right. \\ &-\frac{\Upsilon\{\sin\nu\theta_1+2a\sin(2\nu\theta_1-\epsilon)\}}{(1+\Upsilon\cos\nu\theta_1)^2}+\frac{\Upsilon^2\sin\nu\theta_1\{\cos\nu\theta_1+a\cos(2\nu\theta_1-\epsilon)\}}{(1+\Upsilon\cos\nu\theta_1)^3}+ \\ &+\frac{\Upsilon^2[2\sin\nu\theta_1\cos\nu\theta_1\{1+a\cos(\nu\theta_1-\epsilon)\}-a\sin^2\nu\theta_1\sin(\nu\theta_1-\epsilon)]}{(1+\Upsilon\cos\nu\theta_1)^3}+ \\ &\left.+\frac{2\Upsilon^3\sin^3\nu\theta_1\{1+a\cos(\nu\theta_1-\epsilon)\}}{(1+\Upsilon\cos\nu\theta_1)^4}\right| \\ &< \nu\left[\frac{\Upsilon}{(1-\Upsilon)^2}\left\{a+\frac{\Upsilon(1+a)}{(1-\Upsilon)}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^2}\right\}+\frac{a}{(1-\Upsilon)}+\frac{\Upsilon(1+2a)}{(1-\Upsilon)^2}+\frac{\Upsilon^2(1+a)}{(1-\Upsilon)^3}+\right. \\ &\left.+\frac{\Upsilon^2(2+3a)}{(1-\Upsilon)^3}+\frac{2\Upsilon^3(1+a)}{(1-\Upsilon)^4}\right] = O(\nu).\end{aligned} \tag{75}$$

Now,

$$\begin{aligned}\frac{d}{dQ_1}\left[Q_1+Q_1^2-\frac{1}{2}(Q_1+Q_1^2)^{1/2}(1+2Q_1)+\frac{1}{2}\sinh^{-1}Q_1^{1/2}\right] &= \\ &= (1+2Q_1)-2(Q_1+Q_1^2)^{1/2} \\ &= [(1+2Q_1)+2(Q_1+Q_1^2)^{1/2}]^{-1}.\end{aligned}$$

This is always positive in the range of integration, so the maximum of $[Q_1+Q_1^2-\frac{1}{2}(Q_1+Q_1^2)^{1/2}(1+2Q_1)+\frac{1}{2}\sinh^{-1}Q_1^{1/2}]$ in the range occurs at $Q_1 = 2\pi/\nu$. Hence,

$$\begin{aligned}&Q_1+Q_1^2-\frac{1}{2}(Q_1+Q_1^2)^{1/2}(1+2Q_1)+\frac{1}{2}\sinh^{-1}Q_1^{1/2} \\ &\qquad\leqslant \frac{2\pi}{\nu}+\frac{4\pi^2}{\nu^2}-\frac{1}{2}\left(\frac{2\pi}{\nu}+\frac{4\pi^2}{\nu^2}\right)^{1/2}\left(1+\frac{4\pi}{\nu}\right)+\frac{1}{2}\sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2} = O(\log\nu).\end{aligned} \tag{76}$$

From equations (73), (74), (75) and (76),

$$\left| \int_{\theta}^{\theta+2\pi/\nu} \frac{A \, d\theta_1}{\eta_1} \right| = \mathrm{O}(\log \nu) . \tag{77}$$

From this equation and equations (67) and (72),

$$\int_{\theta}^{\infty} \frac{A \, d\theta_1}{\eta_1} = \mathrm{O}(\log \nu) . \tag{78}$$

Hence, equation (64) becomes

$$\begin{aligned} A_0(\theta) = {} & 2(1+\Upsilon \cos \nu\theta)\{1 + a \cos(\nu\theta - \epsilon)\} + \\ & + \nu \left[2a \sin(\nu\theta - \epsilon) + \frac{\Upsilon \sin \nu\theta \{1 + a\cos(\nu\theta - \epsilon)\}}{(1+\Upsilon \cos \nu\theta)} \right] + \\ & + 2 \int_1^{\infty} \{A_0(\theta_1) - A_0(\theta)\} \frac{d\eta_1}{(\eta_1 - 1)^2} + \mathrm{O}(\nu^2 \log \nu) . \end{aligned} \tag{79}$$

This is the required approximate form of the integral equation, equation (48).

4. *Solution of the Approximate Integral Equation.*

4.1. *A Preliminary Result.*

The problem considered in this section is the expansion of the integral

$$\int_1^{\infty} \frac{(1+\Upsilon \cos \nu\theta_1)\{1 + a\cos(\nu\theta_1 - \epsilon)\} - (1+\Upsilon\cos\nu\theta)\{1 + a\cos(\nu\theta - \epsilon)\}}{(\eta_1 - 1)^2} \, d\eta_1$$

as a term in $\nu \log \nu$ plus a term in ν plus higher-order terms. It is convenient to write

$$\begin{aligned} & \int_1^{\infty} \frac{(1+\Upsilon \cos \nu\theta_1)\{1 + a\cos(\nu\theta_1 - \epsilon)\} - (1+\Upsilon\cos\nu\theta)\{1 + a\cos(\nu\theta - \epsilon)\}}{(\eta_1 - 1)^2} \, d\eta_1 \\ & = \int_1^{\infty} \frac{\left[\Upsilon(\cos \nu\theta_1 - \cos\nu\theta) + a\{\cos(\nu\theta_1 - \epsilon) - \cos(\nu\theta - \epsilon)\} + \frac{a\Upsilon}{2}\{\cos(2\nu\theta_1 - \epsilon) - \cos(2\nu\theta - \epsilon)\} \right]}{(\eta_1 - 1)^2} \, d\eta_1 \\ & = \Upsilon J(1, 0) + aJ(1, \epsilon) + \frac{a\Upsilon}{2} J(2, \epsilon) , \end{aligned} \tag{80}$$

where

$$J(n, \delta) = \int_1^{\infty} \frac{\cos(n\nu\theta_1 - \delta) - \cos(n\nu\theta - \delta)}{(\eta_1 - 1)^2} \, d\eta_1 .$$

After an integration by parts,

$$J(n, \delta) = \left[- \frac{\cos(n\nu\theta_1 - \delta) - \cos(n\nu\theta - \delta)}{(\eta_1 - 1)} \right]_1^{\infty} - n\nu \int_{\theta}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{(\eta_1 - 1)} \, d\theta_1 . \tag{81}$$

From equations (52) and (53),

$$\begin{aligned} \lim_{\eta_1 \to 0} \left[- \frac{\cos(n\nu\theta_1 - \delta) - \cos(n\nu\theta - \delta)}{\eta_1 - 1} \right] & = \lim_{Q_1 \to 0} \left[-\frac{1}{2} \left\{ \frac{\cos(n\nu\theta_1 - \delta) - \cos(n\nu\theta - \delta)}{Q_1 + (Q_1 + Q_1^2)^{1/2}} \right\} \right] \\ & = \lim_{Q_1 \to 0} \left[-\frac{1}{2} \left\{ \frac{-n\nu \sin(n\nu\theta_1 - \delta) \dfrac{1}{(1+\Upsilon\cos\nu\theta_1)}}{1 + \dfrac{(1+2Q_1)}{2(Q_1 + Q_1^2)^{1/2}}} \right\} \right] = 0 , \end{aligned} \tag{82}$$

by l'Hôpital's rule; so equation (81) becomes

$$J(n, \delta) = -n\nu \int_{\theta}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{(\eta_1 - 1)} \, d\theta_1 = -\frac{n\nu}{2} \int_{\theta}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]} \, d\theta_1 , \tag{83}$$

from equation (53). Split the range of integration into two parts and write

$$\int_{\theta}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]} d\theta_1 = \int_{\theta}^{\theta+2\pi/\nu} \frac{\sin(n\nu\theta_1 - \delta)d\theta_1}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]} + \int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)d\theta_1}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]}. \tag{84}$$

Consider the second integral on the right-hand side. From equation (52).

$$\left(\frac{1}{2\theta_1} - \frac{1}{2Q_1}\right) + \left[\frac{1}{2Q_1} - \frac{1}{Q_1 + (Q_1 + Q_1^2)^{1/2}}\right] =$$

$$= -\frac{\theta - \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta)}{2\theta_1 Q_1} + \frac{(Q_1 + Q_1^2)^{1/2} - Q_1}{2Q_1[Q_1 + (Q_1 + Q_1^2)^{1/2}]}$$

$$= -\frac{\theta - \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta)}{2\theta_1 Q_1} + \frac{1}{2[Q_1 + (Q_1 + Q_1^2)^{1/2}]^2};$$

hence,

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)d\theta_1}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]} = \frac{1}{2}\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{\theta_1} d\theta_1 +$$

$$+ \frac{1}{2}\int_{\theta+2\pi/\nu}^{\infty} \frac{\left[\theta - \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta)\right]\sin(n\nu\theta_1 - \delta)d\theta_1}{\theta_1 Q_1} - \frac{1}{2}\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)d\theta_1}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]^2}. \tag{85}$$

The first of the integrals on the right-hand side of this equation can be written as[2]

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{\theta_1} d\theta_1 = \cos\delta\left[\frac{\pi}{2} - \text{Si}(2\pi n + n\nu\theta)\right] + \sin\delta\,\text{Ci}(2\pi n + n\nu\theta);$$

Si $(2\pi n + n\nu\theta)$ and Ci $(2\pi n + n\nu\theta)$ are both[2] O(1), so that

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)}{\theta_1} d\theta_1 = O(1). \tag{86}$$

For the second of these integrals, equation (52) gives

$$\left|\int_{\theta+2\pi/\nu}^{\infty} \frac{\left[\theta - \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta)\right]\sin(n\nu\theta_1 - \delta)d\theta_1}{\theta_1 Q_1}\right|$$

$$= \left|\int_{\theta+2\pi/\nu}^{\infty} \frac{\left[\theta - \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta)\right]\sin(n\nu\theta_1 - \delta)d\theta_1}{\theta_1\left[(\theta_1 - \theta) + \frac{\Upsilon}{\nu}(\sin\nu\theta_1 - \sin\nu\theta)\right]}\right|$$

$$< \left(\theta + \frac{2\Upsilon}{\nu}\right)\int_{\theta+2\pi/\nu}^{\infty} \frac{d\theta_1}{\theta_1\left(\theta_1 - \theta - \frac{2\Upsilon}{\nu}\right)} = \log\frac{(2\pi + \nu\theta)}{2(\pi - \Upsilon)} = O(1). \tag{87}$$

For the third of these integrals, equations (52) and (56a) give

$$\left|\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)d\theta_1}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]^2}\right| = \left|\int_{2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1 - \delta)dQ_1}{[Q_1 + (Q_1 + Q_1^2)^{1/2}]^2(1 + \Upsilon\cos\nu\theta_1)}\right|$$

$$< \frac{1}{4(1-\Upsilon)}\int_{2\pi/\nu}^{\infty} \frac{dQ_1}{Q_1^2} = \frac{\nu}{8\pi(1-\Upsilon)} = O(\nu). \tag{88}$$

From equations (85), (86), (87) and (88), it follows that

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1-\delta)}{[Q_1+(Q_1+Q_1^2)^{1/2}]} = d\theta_1 = \mathrm{O}(1). \tag{89}$$

From equations (85), (88), and (52), this integral can be approximated by

$$\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1-\delta)d\theta_1}{[Q_1+(Q_1+Q_1^2)^{1/2}]} = \frac{1}{2}\int_{\theta+2\pi/\nu}^{\infty} \frac{\sin(n\nu\theta_1-\delta)}{Q_1}\, d\theta_1 + \mathrm{O}(\nu). \tag{90}$$

Now consider the first integral on the right-hand side of equation (84). From equations (52) and (56a),

$$\begin{aligned}
\int_{\theta}^{\theta+2\pi/\nu} \frac{\sin(n\nu\theta_1-\delta)d\theta_1}{[Q_1+(Q_1+Q_1^2)^{1/2}]} &= \int_0^{2\pi/\nu} \frac{\sin(n\nu\theta_1-\delta)dQ_1}{(1+\Upsilon\cos\nu\theta_1)\,[Q_1+(Q_1+Q_1^2)^{1/2}]} \\
&= \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta_1)}\left[\sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2} + \left(\frac{2\pi}{\nu}+\frac{4\pi^2}{\nu^2}\right)^{1/2} - \frac{2\pi}{\nu}\right] - \\
&\quad - \nu\int_{\theta}^{\theta+2\pi/\nu}\left[\frac{n\cos(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} + \frac{\Upsilon\sin\nu\theta_1\sin(n\nu\theta_1-\xi)}{(1+\Upsilon\cos\nu\theta_1)^2}\right]\times \\
&\quad \times [\sinh^{-1}Q_1^{1/2} + (Q_1+Q_1^2)^{1/2} - Q_1]d\theta_1.
\end{aligned} \tag{91}$$

Now,

$$\sinh^{-1}\left(\frac{2\pi}{\nu}\right)^{1/2} + \left(\frac{2\pi}{\nu}+\frac{4\pi^2}{\nu^2}\right)^{1/2} - \frac{2\pi}{\nu} = \frac{1}{2}\log\frac{8\pi}{\nu} + \frac{1}{2} + \mathrm{O}(\nu);$$

and, from equation (52),

$$\begin{aligned}
&\left|\int_{\theta}^{\theta+2\pi/\nu}\left[\frac{n\cos(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} + \frac{\Upsilon\sin\nu\theta_1\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)^2}\right]\left[\frac{1}{2}+Q_1-(Q_1+Q_1^2)^{1/2}\right]dQ_1\right| \\
&< \left[\frac{n}{(1-\Upsilon)^2}+\frac{\Upsilon}{(1-\Upsilon)^3}\right]\int_0^{2\pi/\nu}\left[\frac{1}{2}+Q_1-(Q_1+Q_1^2)^{1/2}\right]dQ_1 \\
&= \left[\frac{n}{(1-\Upsilon)^2}+\frac{\Upsilon}{(1-\Upsilon)^3}\right]\left[-\frac{1}{4}(1+2Q_1)(Q_1+Q_1^2)^{1/2} + \frac{1}{4}\sinh^{-1}Q_1^{1/2} + \frac{Q_1^2}{2} + \frac{Q_1}{2}\right]_0^{2\pi/\nu} \\
&= \mathrm{O}(\log\nu);
\end{aligned}$$

here, the fact that $\frac{1}{2}+Q_1-(Q_1+Q_1^2)^{1/2} > 0$ for $0 \leqslant Q_1 < \infty$ has been used. So equation (91) may be written

$$\begin{aligned}
\int_{\theta}^{\theta+2\pi/\nu} \frac{\sin(n\nu\theta_1-\delta)d\theta_1}{[Q_1+(Q_1+Q_1^2)^{1/2}]} &= \frac{1}{2}\left(\log\frac{8\pi}{\nu}+1\right)\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)} - \\
&\quad - \nu\int_{\theta}^{\theta+2\pi/\nu}\left[\frac{n\cos(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} + \frac{\Upsilon\sin\nu\theta_1\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)^2}\right]\left(\sinh^{-1}Q_1^{1/2}+\frac{1}{2}\right)d\theta_1 + \mathrm{O}(\nu\log\nu) \\
&= \frac{1}{2}\left(\log\frac{8\pi}{\nu}+1\right)\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)} - \\
&\quad - \nu\int_{\theta}^{\theta+2\pi/\nu} d\theta_1\left[\frac{n\cos(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} + \frac{\Upsilon\sin\nu\theta_1\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)^2}\right]\sinh^{-1}Q_1^{1/2} + \mathrm{O}(\nu\log\nu).
\end{aligned} \tag{92}$$

From equation (56a),

$$\nu \int_{\theta}^{\theta+2\pi/\nu} \left[\frac{n \cos (n\nu\theta_1 - \delta)}{(1-\Upsilon \cos \nu\theta_1)} + \frac{\Upsilon \sin \nu\theta_1 \sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)^2}\right] \sinh^{-1} Q_1^{1/2} d\theta_1$$

$$= \left[\left\{\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right\} \sinh^{-1} Q_1^{1/2}\right]_{\theta}^{\theta+2\pi/\nu} -$$

$$- \frac{1}{2}\int_0^{2\pi/\nu} \left[\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right] \frac{dQ_1}{(Q_1 + Q_1^2)^{1/2}}$$

$$= -\frac{1}{2}\int_0^{2\pi/\nu} \left[\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right] \frac{dQ_1}{(Q_1 + Q_1^2)^{1/2}}. \tag{93}$$

Now,

$$\left|\int_0^{2\pi/\nu} \left[\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right] \left[\frac{1}{Q_1} - \frac{1}{(Q_1 + Q_1^2)^{1/2}}\right] dQ_1\right|$$

$$< \left|\left[\left\{\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right\} \{\log Q_1 + 2 \log 2 - 2 \sinh^{-1} Q_1^{1/2}\}\right]_0^{2\pi/\nu}\right| +$$

$$+ \nu \left|\int_{\theta}^{\theta+2\pi/\nu} \left[\frac{n \cos (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} + \frac{\Upsilon \sin \nu\theta_1 \sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)^2}\right] \times\right.$$

$$\left.\times [\log Q_1 + 2 \log 2 - 2 \sinh^{-1} Q_1^{1/2}]\, d\theta_1\right|$$

$$= \nu \left|\int_0^{2\pi/\nu} \left[\frac{n \cos (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} + \frac{\Upsilon \sin \nu\theta_1 \sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)^2}\right] \times\right.$$

$$\left.\times [\log Q_1 + 2 \log 2 - 2 \sinh^{-1} Q_1^{1/2} \frac{dQ_1}{(1+\Upsilon \cos \nu\theta_1)}\right|. \tag{94}$$

from equations (52) and (56a). Hence,

$$\left|\int_0^{2\pi/\nu} \left[\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right] \left[\frac{1}{Q_1} - \frac{1}{(Q_1 + Q_1^2)^{1/2}}\right] dQ_1\right|$$

$$< \nu \left[\frac{n}{(1-\Upsilon)^2} + \frac{\Upsilon}{(1-\Upsilon)^3}\right] \int_0^{2\pi/\nu} [2 \sinh^{-1} Q_1^{1/2} - \log Q_1 - 2 \log 2]\, dQ_1$$

$$= \nu \left[\frac{n}{(1-\Upsilon)^2} + \frac{\Upsilon}{(1-\Upsilon)^3}\right] \Big[- Q_1 \log Q_1 + Q_1 + (1+2Q_1) \sinh^{-1} Q_1^{1/2} - (Q_1 + Q_1^2)^{1/2} -$$

$$\Big. - 2Q_1 \log 2\Big]_0^{2\pi/\nu}$$

$$= O(\nu \log \nu)\,; \tag{95}$$

here, the fact that $[2 \sinh^{-1} Q_1^{1/2} - \log Q_1 - 2 \log 2] > 0$ for $0 \leqslant Q_1 < \infty$ has been used. From equations (92, (93) and (95),

$$\int_{\theta}^{\theta+2\pi/\nu} \frac{\sin (n\nu\theta_1 - \delta) d\theta_1}{Q_1 + (Q_1 + Q_1^2)^{1/2}]}$$

$$= \frac{1}{2}\left(\log \frac{8\pi}{\nu} + 1\right) \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)} + \frac{1}{2}\int_0^{2\pi/\nu} \left[\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right] \frac{dQ_1}{Q_1} + O(\nu \log \nu)$$

$$= \frac{1}{2}\left(\log \frac{8\pi}{\nu} + 1\right) \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)} +$$

$$+ \frac{1}{2}\int_{\theta}^{\theta+2\pi/\nu} \left[\frac{\sin (n\nu\theta_1 - \delta)}{(1+\Upsilon \cos \nu\theta_1)} - \frac{\sin (n\nu\theta - \delta)}{(1+\Upsilon \cos \nu\theta)}\right] \frac{(1+\Upsilon \cos \theta_1) d\theta_1}{Q_1} + O(\nu \log \nu), \tag{96}$$

from equations (52) and (56a). From equations (83), (84), (90) and (96),

$$J(n, \delta) = -\frac{n\nu}{4}\left(\log\frac{8\pi}{\nu}+1\right)\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)} -$$

$$-\frac{n\nu}{4}\int_{\theta}^{\theta+2\pi/\nu}\left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)}-\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right]\frac{(1+\Upsilon\cos\theta_1)d\theta_1}{Q_1} -$$

$$-\frac{n\nu}{4}\int_{\theta+2\pi/\nu}^{\infty}\sin(n\nu\theta_1-\delta)\frac{d\theta_1}{Q_1} + O(\nu^2\log\nu) = O(\nu\log\nu), \tag{97}$$

from equations (89) and (90). Change the variable of integration from θ_1 to χ where

$$\chi = \nu\theta_1.$$

Equation (97) then becomes

$$J(n, \delta) = \frac{n}{4}\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\nu\log\nu - \frac{n\nu}{4}K(n, \delta) + O(\nu^2\log\nu),$$

where

$$K(n, \delta) = (\log 8\pi+1)\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)} +$$

$$+\int_{\nu\theta}^{2\pi+\nu\theta}\left[\frac{\sin(n\chi-\delta)}{(1+\Upsilon\cos\chi)}-\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right]\frac{(1+\Upsilon\cos\chi)d\chi}{R} + \int_{2\pi+\nu\theta}^{\infty}\sin(n\chi-\delta)\frac{d\chi}{R}; \tag{98}$$

here, from equation (52),

$$R = \nu Q_1 = (\chi-\nu\theta) + \Upsilon(\sin\chi-\sin\nu\theta). \tag{99}$$

From equation (80),

$$\int_1^{\infty}\frac{(1+\Upsilon\cos\nu\theta_1)[1+a\cos(\nu\theta_1-\epsilon)]-(1+\Upsilon\cos\nu\theta)[1+a\cos(\nu\theta-\epsilon)]}{(\eta_1-1)^2}d\eta_1$$

$$= \frac{1}{4}\left[\Upsilon\frac{\sin\nu\theta}{(1+\Upsilon\cos\nu\theta)} + a\frac{\sin(\nu\theta-\epsilon)}{(1+\Upsilon\cos\nu\theta)} + a\Upsilon\frac{\sin(2\nu\theta-\epsilon)}{(1+\Upsilon\cos\nu\theta)}\right]\nu\log\nu -$$

$$-\frac{1}{4}[\Upsilon K(1, 0) + aK(1, \epsilon) + a\Upsilon K(2, \epsilon)]\nu + O(\nu^2\log\nu). \tag{100}$$

4.2. *The Approximate Solution.*

It is now shown that the approximate solution of equation (77) is

$$A_0(\theta) = 2(1+\Upsilon\cos\nu\theta)[1+a\cos(\nu\theta-\epsilon)] + \left[2a\sin(\nu\theta-\epsilon)+\frac{\Upsilon\sin\nu\theta\{1+a\cos(\nu\theta-\epsilon)\}}{(1+\Upsilon\cos\nu\theta)}\right]\nu +$$

$$+\frac{[\Upsilon\sin\nu\theta + a\sin(\nu\theta-\epsilon) + a\Upsilon\sin(2\nu\theta-\epsilon)]}{(1+\Upsilon\cos\nu\theta)}\nu\log\nu -$$

$$-[\nu K(1, 0) + aK(1, \epsilon) + a\Upsilon K(2, \epsilon)]\nu + O[(\nu\log\nu)^2]. \tag{101}$$

Write equation (77) as

$$A_0(\theta) - 2(1+\Upsilon\cos\nu\theta)[1+a\cos(\nu\theta-\epsilon)] -$$

$$-\nu\left[2a\sin(\nu\theta-\epsilon)+\frac{\Upsilon\sin\nu\theta\{1+a\cos(\nu\theta-\epsilon)\}}{(1+\Upsilon\cos\nu\theta)}\right] - 2G[A_0(\theta)] + O(\nu^2\log\nu) = 0, \tag{102}$$

where G is an operator defined by

$$G[f(\theta)] = \int_1^{\infty}[f(\theta_1)-f(\theta)]\frac{d\eta_1}{(\eta_1-1)^2}.$$

Clearly

$$G[kf(\theta)] = kG[f(\theta)], \tag{103}$$

when k is a constant, and

$$G[f_1(\theta) + f_2(\theta)] = G[f_1(\theta)] + G[f_2(\theta)]. \tag{104}$$

If $A_0(\theta)$ as given by equation (101) is to satisfy equation (102), then, from equations (100), (103) and (104), the following equation must be true:

$$2a\nu G[\sin(\nu\theta-\epsilon)] + \nu\Upsilon G\left[\frac{\sin\nu\theta\{1+\cos(\nu\theta-\epsilon)\}}{(1+\Upsilon\cos\nu\theta)}\right] +$$

$$+ \nu\log\nu\left[\Upsilon G\left\{\frac{\sin\nu\theta}{(1+\Upsilon\cos\nu\theta)}\right\} + aG\left\{\frac{\sin(\nu\theta-\epsilon)}{(1+\Upsilon\cos\nu\theta)}\right\} + a\Upsilon G\left\{\frac{\sin(2\nu\theta-\epsilon)}{(1+\Upsilon\cos\nu\theta)}\right\}\right] -$$

$$- \nu\Upsilon G[K(1,0)] - \nu aG[K(1,\epsilon)] - \nu a\Upsilon G[K(2,\epsilon)] = \mathrm{O}[(\nu\log\nu)^2].$$

To prove this it is sufficient to show that

$$G[\sin(\nu\theta-\epsilon)] = \mathrm{O}(\nu\log\nu), \tag{105}$$

$$G\left[\frac{\sin\nu\theta\{1+a\cos(\nu\theta-\epsilon)\}}{(1+\Upsilon\cos\nu\theta)}\right] = \mathrm{O}(\nu\log\nu), \tag{106}$$

$$G\left[\frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] = \mathrm{O}(\nu\log\nu), \tag{107}$$

and

$$G[K(n,\delta)] = \mathrm{O}(\nu\log\nu), \tag{108}$$

where n is equal to 1 or 2 and δ is equal to 0 or ϵ. Equation (106) can be written

$$G\left[\frac{\sin\nu\theta + \frac{a}{2}\sin(2\nu\theta-\epsilon) + \frac{a}{2}\sin\epsilon}{(1+\Upsilon\cos\nu\theta)}\right] = \mathrm{O}(\nu\log\nu);$$

from equations (103) and (104), the proof of this is dependent only on the proof of equation (107), with n now equal to 0, 1 or 2.

Equation (105) is

$$\int_1^\infty \frac{[\sin(\nu\theta_1-\epsilon) - \sin(\nu\theta-\epsilon)]d\eta_1}{(\eta_1-1)^2} = \mathrm{O}(\nu\log\nu).$$

Now,

$$\int_1^\infty \frac{[\sin(\nu\theta_1-\epsilon) - \sin(\nu\theta-\epsilon)]d\eta_1}{(\eta_1-1)^2}$$

$$= \int_1^\infty \frac{\left[\cos\left(\nu\theta_1 - \frac{\pi}{2} - \epsilon\right) - \cos\left(\nu\theta - \frac{\pi}{2} - \epsilon\right)\right]}{(\eta_1-1)^2}\, d\eta_1$$

$$= J\left(1, \frac{\pi}{2}+\epsilon\right) = \mathrm{O}(\nu\log\nu),$$

from equation (97); this proves equation (105).

Equation (107) is

$$\int_1^\infty \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right]\frac{d\eta_1}{(\eta_1-1)^2} = \mathrm{O}(\nu\log\nu).$$

Split the range of integration into two parts by writing

$$\int_1^\infty \frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\Bigg] \frac{d\eta_1}{(\eta_1-1)^2}$$

$$= \int_1^{\eta_A} \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] \frac{d\eta_1}{(\eta_1-1)^2} +$$

$$+ \int_{\eta_A}^\infty \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] \frac{d\eta_1}{(\eta_1-1)^2}, \tag{109}$$

where η_A is given by equation (56b). Now,

$$\left|\int_{\eta_A}^\infty \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] \frac{d\eta_1}{(\eta_1-1)^2}\right| < \frac{2}{(1-\Upsilon)} \int_{\eta_A}^\infty \frac{d\eta_1}{(\eta_1-1)^2}$$

$$= \frac{2}{(1-\Upsilon)(\eta_A-1)} = \frac{\nu}{(1-\Upsilon)\,[2\pi+(4\pi^2+2\pi\nu)^{1/2}]} = \mathrm{O}(\nu);$$

hence, from equations (109) and (56b),

$$\int_1^\infty \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] \frac{d\eta_1}{(\eta_1-1)^2}$$

$$= \int_1^{\eta_A} \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] \frac{d\eta_1}{(\eta_1-1)^2} + \mathrm{O}(\nu)$$

$$= \left[-\frac{1}{(\eta_1-1)}\left\{\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right\}\right]_1^{\eta_A} +$$

$$+ \nu\int_\theta^{\theta+2\pi/\nu} \left[\frac{n\cos(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} + \frac{\Upsilon\sin\nu\theta_1\sin(\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)^2}\right] \frac{d\theta_1}{(\eta_1-1)} + \mathrm{O}(\nu). \tag{110}$$

From equations (56), the upper limit in the first term vanishes; the argument used to prove equation (82) shows that the lower limit also vanishes. Hence, from equations (52), (53), (56a) and (110),

$$\left|\int_1^\infty \left[\frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right] \frac{d\eta_1}{(\eta_1-1)^2}\right|$$

$$= \frac{\nu}{2}\left|\int_0^{2\pi/\nu} \left[\frac{n\cos(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)^2} + \frac{\Upsilon\sin\nu\theta_1\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)^3}\right] \frac{dQ_1}{[Q_1+(Q_1+Q_1^2)^{1/2}]}\right| + \mathrm{O}(\nu)$$

$$< \frac{\nu}{2}\left[\frac{n}{(1-\Upsilon)^2} + \frac{\Upsilon}{(1-\Upsilon)^3}\right] \int_0^{2\pi/\nu} \frac{dQ_1}{[Q_1+(Q_1+Q_1^2)^{1/2}]} + \mathrm{O}(\nu)$$

$$= \frac{\nu}{2}\left[\frac{n}{(1-\Upsilon)^2} + \frac{\Upsilon}{(1-\Upsilon)^3}\right] \left[\sinh^{-1} Q_1^{1/2} + (Q_1+Q_1^2)^{1/2} - Q_1\right]_0^{2\pi/\nu} + \mathrm{O}(\nu)$$

$$= \mathrm{O}(\nu\log\nu); \tag{111}$$

this proves equation (107).

To prove equation (108) it is sufficient to demonstrate the truth of both

$$\int_1^\infty \left[\int_{\nu\theta_1}^{(2\pi+\nu\theta_1)} \left\{\frac{\sin(n\chi-\delta)}{(1+\Upsilon\cos\chi)} - \frac{\sin(n\nu\theta_1-\delta)}{(1+\Upsilon\cos\nu\theta_1)}\right\} \frac{(1+\Upsilon\cos\chi)d\chi}{R_1} - \right.$$

$$\left. - \int_{\nu\theta}^{(2\pi+\nu\theta)} \left\{\frac{\sin(n\chi-\delta)}{(1+\Upsilon\cos\chi)} - \frac{\sin(n\nu\theta-\delta)}{(1+\Upsilon\cos\nu\theta)}\right\} \frac{(1+\Upsilon\cos\chi)d\chi}{R}\right] \frac{d\eta_1}{(\eta_1-1)^2} = O(\nu\log\nu), \tag{112}$$

and

$$\int_1^\infty \left\{\int_{(2\pi+\nu\theta_1)}^\infty \sin(n\chi-\delta)\frac{d\chi}{R_1} - \int_{(2\pi+\nu\theta)}^\infty \sin(n\chi-\delta)\frac{d\chi}{R}\right\} \frac{d\eta_1}{(\eta_1-1)^2} = O(\nu\log\nu), \tag{113}$$

where equations (98), (111), (103) and (104) have been used; here,

$$R_1 = (\chi-\nu\theta_1) + \Upsilon(\sin\chi-\sin\nu\theta_1),$$

and

$$R = (\chi-\nu\theta) + \Upsilon(\sin\chi-\sin\nu\theta),$$

from equation (99). Equations (112) and (113) are proved by splitting the range of integration into two, as before; the expressions are now fairly cumbersome, and it is hoped that the reader will be satisfied with a brief sketch of the procedure; the arguments are very similar to those used before.

Consider equation (112) first. Split the range of the outer integral into two, 1 to η_A, and η_A to ∞, where η_A is given by equation (56b). The inner integrals are both O(1), and so are less than C, where $C = O(1)$; hence, the outer integral from η_A to ∞ is less than $C\int_{\eta_A}^\infty d\eta_1/(\eta_1-1)^2$, which is $O(\nu)$. The outer integral from 1 to η_A is integrated by parts: $1/(\eta_1-1)$ times the inner integral at η_1 equal to η_A is $O(\nu)$; $1/(\eta_1-1)$ times the inner integral vanishes as $\eta_1 \to 1$, exactly as in the proof of equation (82). The remaining term is ν times a double integral. This double integral is less than $C\int_\theta^{2\pi+\theta} d\theta_1/(\eta_1-1)$, where C is O(1); as in the proof of equation (111), this can be shown to be $O(\log\nu)$. Hence, equation (112) is true. A similar argument holds for equation (113); here, the inner integrals are known to be O(1) from equations (89), (90) and (52). Since equations (112) and (113) are both true, it follows that equation (108) is also true.

All four relations of equations (105) to (108) have now been verified, so that $A_0(\theta)$ is indeed given by equation (101), the error being $O[(\nu\log\nu)^2]$.

5. *Results and Discussion.*

From the analysis in the preceding three sections, approximate expressions can be obtained for C_L, the lift coefficient, and C_m, the moment coefficient referred to the leading edge. From equations (35) and (37),

$$C_L = \frac{\pi(a_0+a_1)}{U_0^2(1+\Upsilon\cos\omega t)^2}; \tag{114}$$

from equations (36) and (37)

$$C_m = \frac{\pi(a_0+2a_1+a_2)}{4U_0^2(1+\Upsilon\cos\omega t)^2}. \tag{115}$$

Here, from equation (45),

$$\frac{a_1}{\alpha U_0^2} = -\frac{\nu}{2}\{\Upsilon \sin \omega t\,[1 + a \cos(\omega t+\epsilon)] + 2a(1+\Upsilon \cos \omega t) \sin(\omega t+\epsilon)\} + \mathrm{O}(\nu^2)\,; \tag{116a}$$

from equation (46),

$$\frac{a_2}{\alpha U_0^2} = \mathrm{O}(\nu^2)\,; \tag{116b}$$

and, from equation (47c),

$$\frac{a_0}{\alpha U_0^2} = (1+\Upsilon \cos \omega t)A_0\,. \tag{117a}$$

From equations (101) and (47a),

$$\begin{aligned} A_0 = {} & 2(1+\Upsilon \cos \omega t)\,\{1 + a \cos(\omega t+\epsilon)\} - \\ & - \nu \log \nu \frac{\{\Upsilon \sin \omega t + a \sin(\omega t+\epsilon) + a\Upsilon \sin(2\omega t+\epsilon)\}}{(1+\Upsilon \cos \omega t)} - \\ & - \nu \left[2a \sin(\omega t+\epsilon) + \frac{\Upsilon \sin \omega t\{1 + a \cos(\omega t+\epsilon)\}}{(1+\Upsilon \cos \omega t)} + \Upsilon K(1, 0) + aK(1, \epsilon) + \right. \\ & \left. + a\Upsilon K(2, \epsilon)\right] + \mathrm{O}[(\nu \log \nu)^2]\,; \end{aligned} \tag{117b}$$

here, from equations (47a) and (98) (with γ written for $-\chi$),

$$\begin{aligned} K(n, \delta) = {} & -(\log 8\pi+1)\frac{\sin(n\omega t+\delta)}{(1+\Upsilon \cos \omega t)} + \\ & + \int_{(\omega t-2\pi)}^{\omega t} \left[\frac{\sin(n\omega t+\delta)}{(1+\Upsilon \cos \omega t)} - \frac{\sin(n\gamma+\delta)}{(1+\Upsilon \cos \gamma)}\right] \frac{(1+\Upsilon \cos \gamma)d\gamma}{R} - \int_{-\infty}^{(\omega t-2\pi)} \sin(n\gamma+\delta)\frac{d\gamma}{R}\,; \end{aligned}$$

where, from equations (99) and (47a),

$$R = (\omega t-\gamma) + \Upsilon(\sin \omega t - \sin \gamma)\,.$$

These formulas give C_L and C_m as terms O(1) plus terms $\mathrm{O}(\nu \log \nu)$ and $\mathrm{O}(\nu)$; terms $\mathrm{O}[(\nu \log \nu)^2]$ and higher-order terms are neglected. When ν is put equal to zero, the formulas give the 'quasi-steady' results, $C_L = 2\pi\alpha_0(1 + a \cos(\omega t+\epsilon)]$, $C_m = (\pi/2)\,\alpha_0[1 + a \cos(\omega t+\epsilon)]$; here, from equation (2), $\alpha_0[1 + a \cos(\omega t+\epsilon)]$ is the instantaneous incidence. Consequently, an idea of the error incurred by the use of quasi-steady theory can be obtained by comparing the terms $\mathrm{O}(\nu \log \nu)$ and $\mathrm{O}(\nu)$ in C_L and C_m with the terms O(1).

Now, the formulas for C_L and C_m may be written in the following form:

$$\begin{aligned} \frac{C_L}{2\pi\alpha_0} = {} & [1 + a \cos(\omega t+\epsilon)] + \nu[l_1+l_2 \log \nu) + (m_1+m_2 \log \nu)\, a \sin \epsilon + \\ & + (n_1+n_2 \log \nu)a \cos \epsilon] + \mathrm{O}[(\nu \log \nu)^2]\,; \end{aligned} \tag{118}$$

$$\begin{aligned} \frac{C_m}{\pi/2\alpha_0} = {} & [1 + a \cos(\omega t+\epsilon)] + \nu[(l_3+l_2 \log \nu) + (m_3+m_2 \log \nu)a \sin \epsilon + \\ & + (n_3+n_2 \log \nu)a \cos \epsilon] + \mathrm{O}[(\nu \log \nu)^2]\,. \end{aligned} \tag{119}$$

These follow from equations (114) to (117) inclusive. Table 1 gives values for l_1, l_2, etc: for each of five values of $\Upsilon(0, 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8)$ the quantities are given for twelve values of ωt (0 to $11\pi/6$ at intervals of $\pi/6$).

In Fig. 1; L (non-dimensionalised by dividing by $2\pi\alpha_0\frac{1}{2}\rho U_0^2c$) is plotted against ωt for representative values (for a helicopter blade) of Υ, a, ν and ϵ; these are $\Upsilon = 0\cdot6$, $a = 0\cdot8$, $\nu = 0\cdot1$, $\epsilon = \pi$. The full line is the quasi-steady result,

$$\frac{L}{2\pi\alpha_0\frac{1}{2}\rho U_0^2c} = (1+\Upsilon\cos\omega t)^2[1 + a\cos(\omega t+\epsilon)]\,;$$

the crosses are the values obtained by using Table 1. It is seen that in this example retention of terms $O(\nu\log\nu)$ and $O(\nu)$ makes little change to the quasi-steady curve.

The special case where a is unity and Υ is zero corresponds to constant forward speed and harmonic variation of incidence. Since $\Upsilon = 0$, ϵ may be chosen arbitrarily; it is taken to be zero. This case is treated in Ref. 1 (page 503), where an exact solution is obtained. For a reduced frequency of $0\cdot1$, the following results are obtained for the non-dimensionalised L.

ωt	Quasi-steady	Present theory	Exact theory[1]
0	2	1·922	1·916
$\pi/2$	1	1·056	1·038
π	0	0·078	0·084
$3\pi/2$	1	0·944	0·962
2π	2	1·922	1·916

The terms $O(\nu\log\nu)$ and $O(\nu)$ produce the differences between the second and third columns; the terms $O[(\nu\log\nu)^2]$ and higher-order terms produce the differences between the third and fourth columns.

After the present work had been completed, the author's attention was drawn to two papers by Issacs[3, 4]. Isaacs considers the same problem as the one treated in this report. His method is to solve equation (48) by expanding it as a Fourier series in $\nu\theta$ (that is, in ωt). The Fourier coefficients are complicated functions of Bessel functions; but, in principle, any number of them can be calculated. Isaacs gives results only for an example where there is constant incidence and harmonic variation of forward speed (*see* Ref. 3). He truncates the Fourier series at the terms in $\cos 4\omega t$ and $\sin 4\omega t$, because the coefficients of these are already small enough to be neglected. It seems likely that the same will be true for the general case (incidence and forward speed both varying); if so, Isaacs's method will be more satisfactory than the present one, since it effectively provides the exact solution. However, it is felt that the present analysis is of sufficient interest to warrant publication.

In Isaacs's example $a = 0$, $\Upsilon = 0\cdot4$, and $\nu = 0\cdot0848$; since $a = 0$, there is no need to specify ϵ. The following results are obtained for the non-dimensionalised L.

ωt	Quasi-steady	Present theory	Issacs's theory[3]
0	1·96	1·947	1·947
$\pi/2$	1	1·047	1·039
π	0·36	0·430	0·427
$3\pi/2$	1	0·954	0·963
2π	1·96	1·947	1·947

The terms $O(\nu\log\nu)$ and $O(\nu)$ produce the differences between the second and third columns; the terms $O[(\nu\log\nu)^2]$ and higher-order terms effectively produce the differences between the third and fourth columns.

SYMBOLS

A		Defined by equation (65)
A_0		Defined by equation (47c)
a		See equation (2)
a_s		See equation (12)
b_s		Defined after equation (15b)
Ci		Cosine integral (see page 3 of Ref. 2)
C_L		Lift coefficient
C_m		Coefficient of pitching moment about leading edge
c		Aerofoil chord
F		$y = F(x, t)$ is aerofoil equation
G		Operator defined after equation (102)
J		Defined after equation (80)
K		Defined by equation (98)
L		Lift acting on aerofoil
l_s, m_s, n_s		See equations (118) and (119)
M		Pitching moment about the leading edge
p		Pressure
p_∞		Free-stream pressure
Q_1		Defined by equation (52)
R	=	$(\chi - \nu\theta) + \Upsilon(\sin\chi + \sin\nu\theta)$
R_1	=	$(\chi - \nu\theta_1) + \Upsilon(\sin\chi - \sin\nu\theta_1)$
r		Length of blade
Si		Sine integral (see page 3 of Ref. 2)
T, t		Time (T when space coordinates are X and Y, t when they are x and y)
T_1, t_1		Variable of integration
U		Aerofoil speed
U_0		See equation (1)
X		Coordinate: origin fixed in space and lying on flight path, but otherwise immaterial; direction opposed to that of aerofoil motion
X_m		X coordinate of mid-point of aerofoil
x		Coordinate: origin at $X = X_m$; direction the same as that of X
Y		Coordinate: origin same as that of X; direction perpendicular to that of X
y		Coordinate: origin same as that of x; direction perpendicular to that of x

z	=	$x + iy$
z_1	=	$X - X_m(t_1) + iY$; on the aerofoil, $X - X_m(t_1)$
α		Aerofoil incidence
α_0		See equation (2)
γ		Variable of integration
ϵ		See equation (2)
ζ		$z = \frac{c}{4}\left(\zeta + \frac{1}{\zeta}\right)$
ζ_1		$z_1 = \frac{c}{4}\left(\zeta_1 + \frac{1}{\zeta_1}\right)$
η	=	$-\zeta_1$
η_A	=	$\left(1 + \frac{4\pi}{\nu}\right) + 2\left(\frac{2\pi}{\nu} + \frac{4\pi^2}{\nu^2}\right)^{1/2}$
θ	=	$-U_0 t/c$
θ_1	=	$-U_0 t_1/c$
μ		$\zeta = e^{i\mu}$
ν		Reduced frequency
ρ		Density
Υ		See equation (1)
ϕ		Velocity potential
χ	=	$\nu\theta_1$
ψ		$\Omega = \mathscr{R}\psi(z, t)$
Ω		Acceleration potential
ω		Circular frequency

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TABLE 1

Coefficients for Determining Lift and Pitching Moment

	Υ = 0			Υ = 0·2			Υ = 0·4			Υ = 0·6			Υ = 0·8		
ωt	l_1	l_2	l_3	l_1	l_2	l_3	l_1	l_2	l_3	l_1	l_2	l_3	l_1	l_2	l_3
0	0	0	0	−0·10833	0	−0·10833	−0·15479	0	−0·15479	−0·17010	0	−0·17010	−0·16827	0	−0·16827
$\pi/6$	0	0	0	−0·07917	−0·03633	−0·09733	−0·10704	−0·05516	−0·13462	−0·11046	−0·06496	−0·14294	−0·10168	−0·06979	−0·13658
$\pi/3$	0	0	0	−0·03355	−0·07157	−0·06933	−0·04087	−0·12028	−0·10101	−0·03325	−0·15373	−0·11011	−0·01730	−0·17674	−0·10566
$\pi/2$	0	0	0	+0·02988	−0·10000	−0·02012	+0·05791	−0·20000	−0·04209	+0·08168	−0·30000	−0·06832	+0·10051	−0·40000	−0·09949
$2\pi/3$	0	0	0	+0·11237	−0·10692	+0·05891	+0·22879	−0·27063	+0·09347	+0·31415	−0·53022	+0·04904	+0·27305	−0·96225	−0·20809
$5\pi/6$	0	0	0	+0·19924	−0·07314	+0·16268	+0·53351	−0·23409	+0·41646	+1·06083	−0·65000	+0·73583	+1·32581	−2·11956	+0·26603
π	0	0	0	+0·24241	0	+0·24241	+0·83033	0	+0·83033	+2·57775	0	+2·57775	+11·33460	0	+11·33460
$7\pi/6$	0	0	0	+0·19385	+0·07314	+0·23042	+0·68077	+0·23409	+0·79782	+2·08138	+0·65000	+2·40638	+7·60139	+2·11956	+8·66117
$4\pi/3$	0	0	0	+0·07890	+0·10692	+0·13236	+0·24354	+0·27063	+0·37886	+0·57951	+0·53022	+0·84462	+1·31715	+0·96225	+1·79828
$3\pi/2$	0	0	0	+0·03032	+0·10000	+0·01968	−0·05499	+0·20000	+0·04501	−0·06200	+0·30000	+0·08802	−0·02243	+0·40000	+0·17757
$5\pi/3$	0	0	0	−0·09563	+0·07157	−0·05984	−0·16822	+0·12028	−0·10808	−0·21487	+0·15373	−0·13801	−0·23189	+0·17674	−0·14352
$11\pi/6$	0	0	0	−0·11706	+0·03633	−0·09890	−0·18346	+0·05516	−0·15588	−0·21396	+0·06496	−0·18148	−0·22184	+0·06979	−0·18694
ωt	m_1	m_2	m_3	m_1	m_2	m_3	m_1	m_2	m_3	m_1	m_2	m_3	m_1	m_2	m_3
0	−0·59546	−0·50000	−1·09546	−0·49061	−0·41667	−0·90728	−0·40943	−0·35714	−0·76658	−0·34603	−0·31250	−0·65853	−0·29584	−0·27778	−0·57362
$\pi/6$	−0·12299	−0·43301	−0·55600	−0·05826	−0·35092	−0·41827	−0·01708	−0·29402	−0·32490	+0·01058	−0·25247	−0·25813	+0·02930	−0·22090	−0·20904
$\pi/3$	+0·38244	−0·25000	+0·13244	+0·38354	−0·16529	+0·18726	+0·37648	−0·10417	+0·22023	+0·36290	−0·05917	+0·23715	+0·34694	−0·02551	+0·24489
$\pi/2$	+0·78540	0	+0·78540	+0·76178	+0·10000	+0·81178	+0·75106	+0·20000	+0·85106	+0·75120	+0·30000	+0·90120	+0·76109	+0·40000	+0·96109
$2\pi/3$	+0·97791	+0·25000	+1·22791	+0·99374	+0·37037	+1·31782	+1·03939	+0·54688	+1·46908	+1·16098	+0·81633	+1·74772	+1·45578	+1·25000	+2·28910
$5\pi/6$	+0·90839	+0·43301	+1·34140	+0·99497	+0·56030	+1·53679	+1·09597	+0·77956	+1·81701	+1·22566	+1·22639	+2·28955	+1·69904	+2·46942	+3·63857
π	+0·59546	+0·50000	+1·09546	+0·73126	+0·62500	+1·35626	+0·89936	+0·83333	+1·73270	+1·03224	+1·25000	+2·28224	+0·16022	+2·50000	+2·66022
$7\pi/6$	+0·12299	+0·43301	+0·55600	+0·23106	+0·56030	+0·77307	+0·42924	+0·77956	+1·15027	+0·87485	+1·22639	+1·93873	+2·23415	+2·46942	+4·17367
$4\pi/3$	−0·38244	+0·25000	−0·13244	−0·36892	+0·37037	−0·04485	−0·33471	+0·54688	+0·09497	−0·28647	+0·81633	+0·30026	−0·27757	+1·25000	+0·55576
$3\pi/2$	−0·78540	0	−0·78540	−0·82487	+0·10000	−0·77487	−0·88515	+0·20000	−0·78515	−0·97503	+0·30000	−0·82503	−1·11418	+0·40000	−0·91418
$5\pi/3$	−0·97791	−0·25000	−1·27791	−0·97598	−0·16529	−1·17226	−0·98233	−0·10417	−1·13858	−0·99016	−0·05917	−1·11589	−1·00458	−0·02551	−1·10662
$11\pi/6$	−0·90839	−0·43301	−1·34140	−0·83166	−0·35092	−1·19166	−0·76614	−0·29402	−1·07395	−0·70649	−0·25247	−0·97520	−0·67511	−0·22090	−0·89546
ωt	n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3
0	−0·78540	0	−0·78540	−0·65080	0	−0·65080	−0·54763	0	−0·54763	−0·46864	0	−0·46864	−0·40662	0	−0·40662
$\pi/6$	−0·97791	−0·25000	−1·27791	−0·78631	−0·24455	−1·01514	−0·64038	−0·23345	−0·84995	−0·52849	−0·22077	−0·72113	−0·44096	−0·20813	−0·61886
$\pi/3$	−0·90839	−0·43301	−1·34140	−0·71605	−0·42944	−1·12760	−0·56536	−0·42099	−0·95628	−0·44661	−0·40996	−0·81813	−0·35164	−0·39766	−0·70513
$\pi/2$	−0·59546	−0·50000	−1·09546	−0·44875	−0·50000	−0·94875	−0·31149	−0·50000	−0·81149	−0·18753	−0·50000	−0·68753	−0·07485	−0·50000	−0·57485
$2\pi/3$	−0·12299	−0·43301	−0·55600	−0·01389	−0·42766	−0·46829	+0·15299	−0·40595	−0·32061	+0·42306	−0·35348	−0·06298	+0·89308	−0·24056	+0·41194
$5\pi/6$	+0·38244	−0·25000	+0·13244	+0·50185	−0·23903	+0·23114	+0·73537	−0·17978	+0·45424	+1·33522	+0·04250	+1·09626	+3·76832	+1·02174	+3·87227
π	+0·78540	0	+0·78540	+0·96671	0	+0·96671	+1·20697	0	+1·20697	+1·41378	0	+1·41378	−0·16515	0	−0·16515
$7\pi/6$	+0·97791	+0·25000	+1·22791	+1·23391	+0·23903	+1·50462	+1·56939	+0·17978	+1·85052	+1·91347	−0·04250	+2·15243	+1·28941	−1·02174	+1·18547
$4\pi/3$	+0·90839	+0·43301	+1·34140	+1·16420	+0·42766	+1·61859	+1·51910	+0·40595	+1·99271	+2·02717	+0·35348	+2·51320	+2·78648	+0·24056	+3·26760
$3\pi/2$	+0·59546	+0·50000	+1·09546	+0·76064	+0·50000	+1·26064	+0·94629	+0·50000	+1·44629	+1·15609	+0·50000	+1·65609	+1·39221	+0·50000	+1·89221
$5\pi/3$	+0·12299	+0·43301	+0·55600	+0·19991	+0·42944	+0·61144	+0·25993	+0·42099	+0·65085	+0·30738	+0·40996	+0·67890	+0·34685	+0·39766	+0·70033
$11\pi/6$	−0·38244	+0·25000	−0·13244	−0·30901	+0·24455	−0·08019	−0·25987	+0·23345	−0·05031	−0·21890	+0·22077	−0·02626	−0·18919	+0·20813	−0·01129

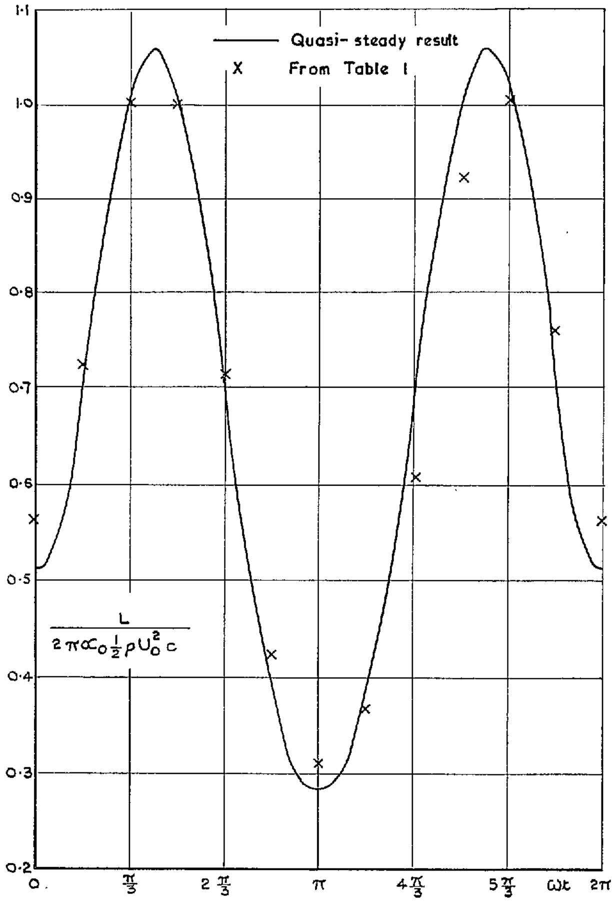

FIG. 1. Quasi-steady and corrected lift for a representative aerofoil.

(91640) Wt. 66/2301 K 5 6/65 Hw.

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