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Part I.—The Equations of Motion

Part II.—A Study of the Trim State and Longitudinal Stability of the Slender Integrated Aeroplane Configuration

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Part I.—The Equations of Motion

Summary.

An integrated analytical treatment is presented which deals with the equilibrium and stability of the flexible aeroplane in flight. The analysis embodies those methods currently employed to investigate the behaviour of the flexible aeroplane stemming on the one hand from the stability theory of the rigid aeroplane and on the other from conventional aeroelastic studies. The integrated treatment serves to clarify the regions of application of these restricted methods.

In Part I the equations of motion for a flexible aeroplane are developed in as general a manner as possible.

In Part II the general analysis is applied to a detailed study of the equilibrium and stability of the slender, integrated aeroplane configuration.

1. *Introduction.*

The effect of flexibility on the stability and control of aeroplanes is recognised as being of paramount importance. Yet this problem tends to be treated either as a modification of rigid-aeroplane stability theory or as an extension of the methods common to flutter analysis. In the first case the rigid-aeroplane equations of motion are modified by the use of so-called 'modified derivatives' which include an allowance only for the steady or equilibrium deformation of the aeroplane structure. The flutter equations are extended to include small translation and rotation of the aeroplane as a whole about a zero position: but the zero position can not, with the modification adopted, be a true equilibrium state for the aeroplane in flight. Both these approaches are, to some extent, deficient in dealing with the general problem of the stability and control of the flexible aeroplane.

The advent of the slender, integrated aeroplane configuration which is currently thought to be suitable for a supersonic transport demands the development of an analysis dealing with the dynamics of the deformable aeroplane in as fundamental a manner as possible. Part I of this paper

* Replaces A.R.C. 24,060.

presents such an analysis in general terms: it is natural that the choice of an axis system for an aeroplane in flight should be, in a generalised sense, body axes and a central consideration of the analysis is the definition of body axes for a deformable aeroplane. Part II applies the general analysis to the investigation of the trim states and the stability of these trim states for the slender, integrated aeroplane configuration. This type of aeroplane configuration is very different from the classical layout and illustrates well the extent to which overall aeroplane stability is inseparable from aeroplane flexibility.

2. *The Equations of Motion.*

2.1. *The Equations of Motion of a Deformable Body in the Absence of Kinematic Constraints.*

2.1.1. *Introduction.*—The equations of motion are to be set up for a body which possesses, in addition to an overall spatial motion, a local deformation motion due to its inherent flexibility, the body as a whole being subjected to gravitational (body) forces and such external forces as are caused entirely by the relative motion of the body surface through a fluid medium. In particular, the body suffers no external kinematic constraints.

It is assumed in all that follows that the relative displacement of any point of the body from the position it occupies in some assigned reference configuration is small in comparison with a typical overall linear dimension of the body: thus second and higher powers of the displacement are neglected. This assumption is that usually made in the Classical Theory of Elasticity: it implies that the strain at any point is small and, in addition, that the relative rotation of any element is small. As a consequence of these restrictions a set of linear relations connects the strain and displacement components at a point. It is not necessarily assumed that the relation between stress and strain is a linear one.

The lack of kinematic boundary conditions means that the Elastic Boundary Value Problem is the Neumann Problem[1], any solution of which is arbitrary to the extent of a small rigid-body displacement and rotation. The resolution of this arbitrariness will be discussed at length in connection with the choice of reference axes moving in a generalised sense with the body. However, it may be emphasised at this point that the arbitrary nature of the Neumann Solution is quite inadequate to describe the overall motion of the body because of its necessary smallness: indeed, any interpretation in this light is essentially misleading.

The equations of motion must be referred to inertial or space axes and for the purpose of aeroplane stability and control the motion of the earth may be neglected and 'earth' axes adopted. However, as in the case of the motion of rigid bodies it is advantageous to interpose a set of axes moving with the body and in a conventional sense the motion is then referred to body axes. In the case of a deformable body the specification of such an axis system is not obvious or indeed unique; the resolution of this question is postponed for reasons which will become clear.

Accordingly we shall refer to body axes (origin O) whose position and orientation are not specified except in so far as they lie always in the region of a set of axes positioned at a definite point and along definite directions in the body in a reference configuration.

The specification of this reference configuration is not unique but, once chosen, it remains unchanged. It may coincide, for example, with a particular equilibrium configuration of the body but more naturally it will be taken to coincide with the body configuration when completely free from external or body forces. In the latter case it is then essentially an idealised assembly of material points in a purely geometric sense.

Let the position vector of a general point of the body be $\mathbf{r}$: let the position vector of the same point in the reference configuration be $\mathbf{r}_0$ then a displacement vector $\mathbf{r}'$ is defined by

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_0 \tag{2.1, 1}$$

where, in accordance with the condition of smallness of the displacement,

$$\mathbf{r}'^2 \ll l^2 .$$

2.1.2. *The linear and angular momenta.*—Let σ be the mass per unit volume at any point and dV an element of volume. The linear momentum of the body is

$$\mathbf{M} = \int_V \sigma \left(\mathbf{v} + \frac{d\mathbf{r}}{dt} \right) dV \tag{2.1, 2}$$

where $\mathbf{v}$ is the velocity of the origin of the body axes relative to inertial axes and d/dt represents time rate of change with respect to inertial axes.

The corresponding angular momentum about the origin of the body axes is

$$\mathbf{H} = \int_V \sigma \mathbf{r} \times \left(\mathbf{v} + \frac{d\mathbf{r}}{dt} \right) dV . \tag{2.1, 3}$$

Let the angular velocity of the body axes at any instant be $\mathbf{\Omega}$ and let the operator $\partial/\partial t$ represent time rate of change with respect to an observer stationed in the body axes; then the operators

$$\frac{d}{dt}, \left\{ \frac{\partial}{\partial t} + \mathbf{\Omega} \times \right\}$$

are commutable.

The linear momentum {equation (2.1, 2)} may be written

$$\mathbf{M} = M\mathbf{v} + M\mathbf{\Omega} \times \mathbf{r}_{0g} + \mathbf{\Omega} \times \int_V \sigma \mathbf{r}' dV + \int_V \sigma \frac{\partial \mathbf{r}'}{\partial t} dV \tag{2.1, 4}$$

where

$$M = \int_V \sigma \, dV$$

is the mass of the body and

$$\mathbf{r}_{0g} = \frac{1}{M} \int_V \sigma \mathbf{r}_0 \, dV$$

is the position vector of the centre of mass of the reference configuration.

The angular momentum {equation (2.1, 3)} may be written

$$\begin{aligned} \mathbf{H} = {} & M\mathbf{r}_{0g} \times \dot{\mathbf{v}} + (\mathbf{\Phi}_0 + \mathbf{\Phi}') \cdot \mathbf{\Omega} + \\ & + \int_V \sigma \mathbf{r}' dV \times \mathbf{v} + \int_V \sigma \mathbf{r}_0 \times \frac{\partial \mathbf{r}'}{\partial t} dV \end{aligned} \tag{2.1, 5}$$

where

$$\mathbf{\Phi}_0 = \int_V \sigma [\mathbf{r}_0^2 \mathbf{I} - \mathbf{r}_0 \mathbf{r}_0] \, dV$$

is the inertia tensor (or dyadic) for the reference configuration and

$$\mathbf{\Phi}' = \int_V \sigma [2\mathbf{r}_0 \cdot \mathbf{r}' \mathbf{I} - (\mathbf{r}' \mathbf{r}_0 + \mathbf{r}_0 \mathbf{r}')] \, dV$$

represents (to first order in $\mathbf{r}'$) the addition to $\mathbf{\Phi}_0$ due to the relative deformation.

2.1.3. *Overall and elemental equations of motion.*—The equations of motion for the body take the form of two equations which relate the overall force and moment on the body to its motion and in which the internal reactions do not explicitly appear and an equation which embodies the conditions of equilibrium for the elements of the body.

Thus if $\mathbf{F}$ is the resultant force and $\mathbf{L}$ the resultant moment about the origin of the body axes of the external (surface) tractions and $\mathbf{g}$ is the acceleration due to gravity then

$$\frac{d\mathbf{M}}{dt} = \mathbf{F} + M\mathbf{g}, \tag{2.1, 6}$$

$$\frac{d\mathbf{H}}{dt} = \mathbf{L} - \mathbf{v} \times \mathbf{M} + \left\{ M\mathbf{r}_{0g} + \int_V \sigma\mathbf{r}'\, dV \right\} \times \mathbf{g} \tag{2.1, 7}$$

are the overall equations of motion.

The equation of motion for an element of the body is

$$\sigma \left\{ \frac{d}{dt}\left(\mathbf{v} + \frac{d\mathbf{r}}{dt}\right) - \mathbf{g} \right\} = \nabla \cdot \mathbf{\Sigma}, \tag{2.1, 8}$$

within $\mathbf{V}$, where $\mathbf{\Sigma}$ is the stress tensor. On the surface of the body the statical boundary condition is that the surface stress components must be equivalent to the external surface traction $\boldsymbol{\psi}$; thus on the body surface S,

$$\mathbf{n} \cdot \mathbf{\Sigma} = \boldsymbol{\psi} \tag{2.1, 9}$$

where $\mathbf{n}$ is the outward normal. The overall equations of motion (2.1, 6) and (2.1, 7) may be considered as necessary conditions for the consistency of equation (2.1, 8) and the statical boundary condition (2.1, 9).

2.1.4. *Specification of the body axes.*—The detailed specification of the body axes may now profitably be discussed. Let $\mathbf{r}'_1$ be a solution of equation (2.1, 8) satisfying (2.1, 9) then

$$\mathbf{r}'_2 = \Delta\mathbf{R} + \Delta\boldsymbol{\theta} \times \mathbf{r}_0 + \mathbf{r}'_1$$

where $\Delta\mathbf{R}(t)$, $\Delta\boldsymbol{\theta}(t) = 0(\mathbf{r}')$ is also a solution where the rotation may be represented as a vector $\Delta\boldsymbol{\theta}$ by virtue of its smallness.

Let A_0 be an axis system set up in the reference configuration by choosing some (material) point as origin and a line of (material) points as an axis of orientation. Then if motion ensues at time t_0 the specification of the body axes (of a similar nature to the original) may formally be said to be specified by a knowledge of $\Delta\mathbf{R}$ and $\Delta\boldsymbol{\theta}$ at any subsequent time t. For it must be noted that the origin of the body axes will no longer necessarily be invested in a material point of the body nor will the axis of orientation contain the initial material points. It need only be demonstrated that $\Delta\mathbf{R}$ and $\Delta\boldsymbol{\theta}$ may be consistently specified in terms of a solution of (2.1, 8): in practice a knowledge of $\Delta\mathbf{R}$ and $\Delta\boldsymbol{\theta}$ is not required directly as will be seen in the sequel.

Any number of ways of choosing the body axes exist but in practice three particular choices would seem to be worthy of discussion.

(*a*) Attached Axes.

These axes are specified by the simple conditions that

$$\Delta\mathbf{R} = \Delta\boldsymbol{\theta} \equiv 0.$$

In this case the origin of the body axes remains invested in one material point of the body while an axis of orientation is tangent to the curve formed by the material points originally defining the axis of orientation. For example, in the case of Cartesian Axes the axis directions may be the tangent, normal and binormal to a curve of material points. Further, any set of axes which have a fixed orientation to such a set of axes, are also Attached axes.

(*b*) Mean Axes.[2, 3]

These axes are chosen in such a way that, at every instant, the linear and angular momenta of the relative motion with respect to the body axes are identically zero. Thus,

$$\int_V \sigma \frac{\partial \mathbf{r}'_2}{\partial t} dV = \int_V \sigma \mathbf{r}_0 \times \frac{\partial \mathbf{r}'_2}{\partial t} dV \equiv 0$$

or

$$\int_V \sigma\{\Delta\mathbf{R} + \Delta\boldsymbol{\theta} \times \mathbf{r}_0 + \mathbf{r}'_1\} dV = \text{const.}\Big|_{t=t_0},$$

$$\int_V \sigma\mathbf{r}_0 \times \{\Delta\mathbf{R} + \Delta\boldsymbol{\theta} \times \mathbf{r}_0 + \mathbf{r}'_1\} dV = \text{const.}\Big|_{t=t_0},$$

where, for coincidence of the body axes and reference axis system A_0 at time t_0 the constants should be taken to be zero. The latter equations are sufficient to determine $\Delta\mathbf{R}$, $\Delta\boldsymbol{\theta}$, thus,

$$M\Delta\mathbf{R} + \Delta\boldsymbol{\theta} \times M\mathbf{r}_{0g} = \int_V \sigma\mathbf{r}'_1 dV \tag{2.1, 10a}$$

$$M\mathbf{r}_{0g} \times \Delta\mathbf{R} + \Delta\boldsymbol{\theta} \cdot \boldsymbol{\Phi}_0 = \int_V \sigma\mathbf{r}_0 \times \mathbf{r}'_1 dV. \tag{2.1, 10b}$$

In practice the specification that the deformation motion shall satisfy the conditions

$$\int_V \sigma\mathbf{r}' dV = 0 \tag{2.1, 11a}$$

$$\int \sigma\mathbf{r}_0 \times \mathbf{r}' dV = 0 \tag{2.1, 11b}$$

is equivalent to reference of the motion to Mean Axes. Then equations (2.1, 4) and (2.1, 5) respectively take the forms

$$\mathbf{M} = M\mathbf{v} + M\boldsymbol{\Omega} \times \mathbf{r}_{0g}$$

$$\mathbf{H} = M\mathbf{r}_{0g} \times \mathbf{v} + (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}') \cdot \boldsymbol{\Omega}.$$

The use of Mean Axes effectively reduces the inertial coupling between the overall and relative deformation motions.

It may be noted that if the origin of the reference axis system A_0 is chosen to be the centre of mass of the reference configuration then because of condition (2.1, 11a) the origin is always at the centre of mass.

(*c*) Principal Axes.

The basic requirement in this case is that the tensor $\boldsymbol{\Phi}'$ should be diagonal and this is most conveniently coupled with the condition (2.1, 11a) which ensures that $\mathbf{r}'_g$ is zero. The equations determining $\Delta\mathbf{R}$ and $\Delta\boldsymbol{\theta}$ are complicated but in the case when the origin of the reference axis system A_0 is chosen so that $\mathbf{r}_{0g} = 0$ they simplify to

$$M\Delta\mathbf{R} = \int_V \sigma\mathbf{r}'_1\, dV \tag{2.1, 10a}$$

and the three scalar equations,

$$\mathbf{j}\cdot\boldsymbol{\Phi}'_2\cdot\mathbf{k} = \mathbf{k}\cdot\boldsymbol{\Phi}'_2\cdot\mathbf{i} = \mathbf{i}\cdot\boldsymbol{\Phi}'_2\cdot\mathbf{j} = 0 \tag{2.1, 12}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are three orthogonal unit vectors parallel to the body axes and

$$\begin{aligned}\boldsymbol{\Phi}'_2 &= \int_V \sigma[2\mathbf{r}_0\cdot\mathbf{r}'_2\mathbf{I} - (\mathbf{r}'_2\mathbf{r}_0 + \mathbf{r}_0\mathbf{r}'_2)]\, dV \\ &= \int_V \sigma[2\mathbf{r}_0\cdot\mathbf{r}'_1\mathbf{I} - (\mathbf{r}'_1\mathbf{r}_0 + \mathbf{r}_0\mathbf{r}'_1)]\, dV - \\ &\quad - \Delta\boldsymbol{\theta}\times\int_V \sigma[\mathbf{r}_0{}^2\mathbf{I} - \mathbf{r}_0\mathbf{r}_0]\, dV - \int_V \sigma[\mathbf{r}_0{}^2\mathbf{I} + \mathbf{r}_0\mathbf{r}_0]\, dV \times \Delta\boldsymbol{\theta}.\end{aligned}$$

The three scalar equations (2.1, 12) are sufficient to determine the components of $\Delta\boldsymbol{\theta}$.

Principal Axes in this sense will most often be combined with the choice of Principal Axes in the usual geometric sense situated at the centre of mass for the reference axis system A_0. The body axes are then always Principal Axes situated at the centre of mass of the deformed body. The conditions (2.1, 11b) and (2.1, 12) imposed on $\mathbf{r}'$ by the choice respectively of Mean Axes or Principal Axes are more clearly illustrated by writing these conditions in terms of Cartesian components. Thus, with

$$\begin{aligned}\mathbf{r}_0 &= x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}, \\ \mathbf{r}' &= x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}\end{aligned}$$

where, as before, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are a unit (body) triad, conditions (2.1, 11b) are

$$\int_V \sigma(y_0z' - z_0y')\, dV = \int_V \sigma(z_0x' - x_0z')\, dV = \int_V \sigma(x_0y' - y_0x')\, dV = 0 \tag{2.1, 13}$$

while conditions (2.1, 12) become

$$\int_V \sigma(y_0z' + z_0y')\, dV = \int_V \sigma(z_0x' + x_0z')\, dV = \int_V \sigma(x_0y' + y_0x')\, dV = 0. \tag{2.1, 14}$$

For shapes which are typical of aeroplanes in which transverse displacement relative to a plane or line contributes the main deformation the conditions (2.1, 13) and (2.1, 14) may be identical. For example, let the median plane of a plate-like structure be flat in the reference configuration and let a Cartesian axis system O, x, y, z be chosen to have the (x, y) plane as the median plane. Then if $z'(x_0, y_0)$ is the transverse displacement component and it is assumed that terms of $O(z_0x', z_0y')$ are much smaller than terms of $O(z'x_0, z'y_0)$ then since the last integral vanishes identically conditions (2.1, 13) and (2.1, 14) are identical. This latter assumption is effectively the neglect of rotatory inertia.

2.1.5. *Variational form of the elemental equation of motion.*—Having discussed the question of the specification and choice of the body axes we may return to further consideration of the equations of motion, in particular the differential equation (2.1, 8) and boundary condition (2.1, 9). These may conveniently be combined in a single variational equation of motion. Furthermore, the variational form of the elemental equation of motion is by far the most fertile for the deduction of approximate representations of the flexibility of the body.

Let the path of the motion over a fixed, arbitrary time interval $t_1 < t < t_2$ be varied from the actual path by the virtual displacement $\delta\mathbf{r}'$, then since the forces on an element of the body are, at every instant, in equilibrium (over the actual path) then to a first-order variation in the path no work is performed by these forces through the virtual displacement. Thus, integrating over every element and over the time interval $t_1 < t < t_2$,

$$\int_{t_1}^{t_2}\int_V \left[\sigma\left\{\frac{d}{dt}\left(\mathbf{v}+\frac{d\mathbf{r}}{dt}\right)-\mathbf{g}\right\}-\nabla\cdot\boldsymbol{\Sigma}\right]\cdot\delta\mathbf{r}'\,dV\,dt = 0\,. \qquad (2.1,\ 15)$$

Transforming the third term by the Divergence Theorem, using the boundary condition (2.1, 9) and noting that

$$\boldsymbol{\Sigma}:\nabla\delta\mathbf{r}' = \boldsymbol{\Sigma}:\tfrac{1}{2}(\nabla\delta\mathbf{r}'+\delta\mathbf{r}'\nabla) = \boldsymbol{\Sigma}:\delta\boldsymbol{\Psi}$$

where $\boldsymbol{\Psi}$ is the strain tensor then, finally, the variational equation of motion is

$$\int_{t_1}^{t_2}\left\{\int_V \left[\sigma\left\{\frac{d}{dt}\left(\mathbf{v}+\frac{d\mathbf{r}}{dt}\right)-\mathbf{g}\right\}\cdot\delta\mathbf{r}'+\boldsymbol{\Sigma}:\delta\boldsymbol{\Psi}\right]dV-\int_S \boldsymbol{\psi}\cdot\delta\mathbf{r}'dS\right\}dt = 0\,. \qquad (2.1,\ 16)$$

The variation $\delta\mathbf{r}'$ is arbitrary except that it must satisfy the same (quasi-kinematic) conditions as are satisfied by $\mathbf{r}'$ consequent upon the choice of a particular type of body axes. Thus, in particular, the variational modes $\delta\mathbf{r}' = \text{const.}$ and $\delta\mathbf{r}' = \text{const.}\times\mathbf{r}_0$ are not admissible under any choice of body axes so that equation (2.1, 16) does not contain equations (2.1, 6) and (2.1, 7) as special cases. Similarly equation (2.1, 8) and any differential equation (relating to some approximate type of analysis) deduced from (2.1, 16) may not have as a solution $\text{const.}_1 + \text{const.}_2 \times \mathbf{r}_0$.

To the equations of motion for the aeroplane may be added equations representing control systems incorporating servo-mechanisms. With large controls it may be important to include the inertia of the control and in that case a part of $\mathbf{r}'$ may be allotted to control deflection; a part of the surface loading $\boldsymbol{\psi}$ will of course be associated with control deflection. These additional equations of motion will embody {in place of the variation in strain energy integral of equation (2.1, 16)} the Transfer Function of the control as related to the demand and to the overall and deformation motions of the aeroplane.

2.1.6. *Attitude of the body axes in space.*—The presence of the gravitational force in the equations of motion requires that reference be made to the attitude of the body axes in space since this force is fixed in direction relative to 'earth' axes.

It is necessary to adopt a scheme whereby a sequence of rotations will, from a reference attitude, lead uniquely to a final attitude: the following scheme[4] is usually adopted. In the reference position, axis 0, 3 of the (inertial) triad 0, 1, 2, 3 is vertically downward; taking all rotations to be right-handed the final attitude is obtained from the reference by the sequence of rotations ϕ_3, ϕ_2, ϕ_1 each rotation being about the carried position of the relevant axis. Thus, using the abbreviations $\cos\phi_i = c_i$ $\sin\phi_i = s_i$ the orthogonal matrix of direction cosines[5] for the final attitude is

$$[l] = \begin{bmatrix} c_2c_3\,, & c_2s_3\,, & -s_2 \\ -c_1s_3+s_1s_2c_3\,, & c_1c_3+s_1s_2s_3\,, & s_1c_2 \\ s_1s_3+c_1s_2c_3\,, & -s_1c_3+c_1s_2s_3\,, & c_1c_2 \end{bmatrix}. \qquad (2.1,\ 17)$$

Then if the column $\{v_{iF}\}$ represents the components of a vector **v** in the fixed (vertical) axis system and $\{v_{iM}\}$ its components in the moving (body) axis system,

$$\{v_{iM}\} = [l]\{v_{iF}\}. \tag{2.1, 18}$$

A kinematic relation is also required between the components of $\boldsymbol{\Omega}$ referred to the body axes, say (p, q, r), and the ϕ_i and their time rates of change, $\dot{\phi}_i$. The required relation is

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s_2 \\ 0 & c_1 & s_1c_2 \\ 0 & -s_1 & c_1c_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} \tag{2.1, 19}$$

2.2. *The Deviant Equations of Motion.*

2.2.1. *Introduction.*—A consideration of the behaviour of an aeroplane in flight will deal essentially with three distinct problems:

(1) equilibrium of a steady-flight state,

(2) the stability of such steady-flight states,

(3) the response of the aeroplane to controls or gusts and behaviour in unsteady manoeuvres (rapidly rolling flight, rapid pull-outs, etc.).

Of these three problems the last is very considerably more difficult than the first two. The problem of equilibrium by virtue of its definition is independent of time but it may often be non-linear in character. The stability of such equilibrium may, by virtue of the stability theory due to Liapunov[6,7], be tested by considering the stability of a linearised system having a small disturbed motion about the position of equilibrium. If the system returns to its equilibrium position under perturbations of sufficiently small magnitude, the equilibrium position is said to be stable. If it does so under all possible perturbations of arbitrary magnitude, the equilibrium position is said to be totally stable. The linear approximation is not a test for total stability.

The third problem will generally be non-linear except when the control forces or external disturbances are restricted to be small enough to permit linearisation of the equations of motion as for stability: in this case the stability and response problems are solutions of the homogeneous and inhomogeneous forms of the same set of equations.

2.2.2. *The steady state.*—Without discussing in detail the problem of equilibrium (*see* Part II) we may consider the nature of possible steady-flight states. To do this it need only be recalled that the aerodynamic forces are not dependent on the position or attitude of the aeroplane in space while the gravitational force is of constant magnitude and direction with respect to 'earth' axes.

The most general steady state in a homogeneous atmosphere clearly consists in $\mathbf{v} =$ const. and $\mathbf{r}' \neq f(t)$ while $\boldsymbol{\Omega}$ may be a vertically directed vector of constant magnitude; that is, a spiralling motion at constant speed. When the atmosphere is recognised to be vertically inhomogeneous then **v** must be a horizontally directed vector.

The most usual steady-flight case is that of rectilinear flight ($\mathbf{\Omega} = 0$) for which the equations of equilibrium take the forms

$$\mathbf{F}_1 + M\mathbf{g}_1 = 0$$

$$\mathbf{L}_1 + M\mathbf{r}_{01g} \times \mathbf{g}_1 = 0$$

$$\int_V \mathbf{\Sigma}_1 : \delta\mathbf{\Psi}\, dV - \mathbf{g}_1 \cdot \int_V \sigma\delta\mathbf{r}'\, dV - \int_S \boldsymbol{\psi}_1 \cdot \delta\mathbf{r}'\, dS = 0 \qquad (2.2, 1)$$

where the suffix 1 refers to the steady state.

These equations determine, for given control forces or settings, the speed of flight, the attitude of the aeroplane and the form of the deformation: alternatively, when the speed (and altitude) is specified, the required control forces and the resulting attitude and deformation may be determined (*see* Part II).

Upon solving the equilibrium problem any equilibrium state $\mathbf{r}_{01} = \mathbf{r}_0 + \mathbf{r}'_1$ may be chosen as a new reference configuration in the sense of Section 2.1.1.

For some purposes it may be possible to neglect gravitational forces. This arises when the (integrated) inertial forces in the steady state are large such as in a rapid pull-out or rapidly rolling motions. In this case the attitude of the aeroplane in space is immaterial and the most general steady state is $\mathbf{v}$ = const., $\mathbf{\Omega}$ = const. and $\mathbf{r}' \neq f(t)$.

2.2.3. *The form of the deviant equations of motion.*—The deviant equations of motion relate to the disturbed motion of the aeroplane relative to a specified steady or equilibrium state and can only be constructed once the relevant equilibrium state has been solved. The variables in the deviant equations of motion are so defined that when they are all identically zero the equilibrium state is recovered.

Using the suffix 1 as in Section 2.2.2 to mean an equilibrium state then we define the deviant variables (without suffix) by the relations

$$\begin{aligned} \mathbf{v}_t &= \mathbf{v}_1 + \mathbf{v} \\ \mathbf{\Omega}_t &= \mathbf{\Omega} \\ \mathbf{r}_t &= (\mathbf{r}_0 + \mathbf{r}'_1) + \mathbf{r}' \end{aligned} \qquad (2.2, 2)$$

where the suffix t indicates that the variables refer to the total motion. Similarly, the forces are given by the relations

$$\begin{aligned} \mathbf{F}_t &= \mathbf{F}_1(\mathbf{v}_1, \mathbf{r}'_1) + \mathbf{F}(\mathbf{v}_1, \mathbf{r}'_1, \mathbf{v}, \mathbf{\Omega}, \mathbf{r}') \\ \mathbf{L}_t &= \mathbf{L}_1(\mathbf{v}_1, \mathbf{r}'_1) + \mathbf{L}(\mathbf{v}_1, \mathbf{r}'_1, \mathbf{v}, \mathbf{\Omega}, \mathbf{r}') \\ \boldsymbol{\psi}_t &= \boldsymbol{\psi}_1(\mathbf{v}_1, \mathbf{r}'_1) + \boldsymbol{\psi}(\mathbf{v}_1, \mathbf{r}'_1, \mathbf{v}, \mathbf{\Omega}, \mathbf{r}') \end{aligned} \qquad (2.2, 3)$$

and

$$\mathbf{g}_t = \mathbf{g}_1 + \mathbf{g}. \qquad (2.2, 4)$$

Also, the total attitude of the body axes is given by the rotations ϕ_{i1} followed by the rotations ϕ_i (the deviant rotations). The deviations ϕ_i do not have the same meaning as the rotations ϕ_{i1} for the rotations ϕ_i are carried out about the axis directions of the equilibrium state 1 whereas the rotations ϕ_{i1} were carried out about the 'vertical' axis system fixed in spatial orientation. Thus if $\{v_{iF}\}$

and $\{v_{iM}\}_1$ are the components of a vector in the 'earth' and equilibrium axes respectively and $\{v_{iM}\}$ its components in the moving axes then

$$\{v_{iM}\} = [l]\{v_{iM}\}_1 = [l][l]_1\{v_{iF}\}$$

and, in particular,

$$\{v_{iM}\} - \{v_{iM}\}_1 = ([l]-I)[l]_1\{v_{iF}\}. \tag{2.2, 5}$$

The deviant equations of motion are obtained by substituting (2.2, 2), (2.2, 3) and (2.2, 4) in the equations of motion (2.1, 6), (2.1, 7) and (2.1, 16) and using the equations of equilibrium (2.2, 1). The deviant equations of motion are written out in full in Appendix I.

2.2.4. *The deviant equations to first order in the velocities.*—The main step in the linearisation of the deviant equations is to retain only those terms which are of the first order when $\mathbf{v}$, $\mathbf{\Omega}$ (and, of course, $\mathbf{r}'$) are treated as small quantities. It is shown in Ref. 8 that when $\mathbf{v}$, $\mathbf{\Omega}$ and $\mathbf{r}'$ are all small then the aerodynamic forces are linear (integral or differential) functions of $\mathbf{v}$, $\mathbf{\Omega}$ and $\mathbf{r}'$ (the functional forms are dependent on the actual equilibrium configuration under consideration). This degree of linearisation is thus sufficient to make the deviant equations linear except for those terms which involve the gravitational force and are dependent on the attitude of the aeroplane in space.

Thus for those problems in which gravity may be neglected the equations are already linear. For those in which gravity cannot be ignored a further linearisation is required in rotational attitude: no restriction is required on the displacement of the origin unless the atmosphere is inhomogeneous.

2.2.5. *Non-dimensional form of the deviant equations to first order in* $\mathbf{v}$ *and* $\mathbf{\Omega}$.—The deviant equations are rendered non-dimensional by choosing

(*a*) $\rho V_1^2 l^2$ as the unit of force,

(*b*) l, a typical overall dimension of the aeroplane, as the unit of length, and

(*c*) l/V_1 as the unit of time,

where

$$V_1 = |\mathbf{v}_1|.$$

Then the non-dimensional deviant equations of motion, to first order in $\mathbf{v}$, $\mathbf{\Omega}$ and $\mathbf{r}'$ are (*see* Appendix I)

$$M^*\left[\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{\Omega}^* \times \mathbf{v}_1^*\right] + \frac{\partial \mathbf{\Omega}^*}{\partial t^*} \times M^*\mathbf{r}_{0g}^* + \int_{V^*} \sigma^* \frac{\partial^2 \mathbf{r}'^*}{\partial t^{*2}}\, dV^* = \mathbf{F}^* + M^*\mathbf{g}^* \tag{2.2, 6a}$$

$$M^*\mathbf{r}_{0g}^* \times \frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{\Phi}_0^* \cdot \frac{\partial \mathbf{\Omega}^*}{\partial t^*} + M^*\mathbf{r}_{0g}^* \times (\mathbf{\Omega}^* \times \mathbf{v}_1^*) +$$

$$+ \int_{V^*} \sigma^* \mathbf{r}_0^* \times \frac{\partial^2 \mathbf{r}'^*}{\partial t^{*2}}\, dV^* = \mathbf{L}^* + M\mathbf{r}_{01g}^* \times \mathbf{g}^* + \int_{V^*} \sigma^* \mathbf{r}'^* dV^* \times (\mathbf{g}_1^* + \mathbf{g}^*) \tag{2.2, 6b}$$

$$\int_{t_1^*}^{t_2^*} \left\{ \left(\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{\Omega}^* \times \mathbf{v}_1^* \right) \cdot \int_{V^*} \sigma^* \delta \mathbf{r}^* dV^* + \int_{V^*} \sigma^* \frac{\partial^2 \mathbf{r}'^*}{\partial t^{*2}} \cdot \delta \mathbf{r}'^* dV^* + \right.$$

$$+ \frac{\partial \mathbf{\Omega}^*}{\partial t^*} \cdot \int_{V^*} \sigma^* \mathbf{r}_0^* \times \delta \mathbf{r}'^* dV^* - \mathbf{g}^* \cdot \int_{V^*} \sigma^* \delta \mathbf{r}'^* dV^* +$$

$$\left. + \int_{V^*} \mathbf{\Sigma}^* : \delta \mathbf{\Psi}^* dV^* - \int_{S^*} \boldsymbol{\psi}^* \cdot \delta \mathbf{r}'^* dS^* \right\} dt^* = 0 \tag{2.2, 6c}$$

where

$$\mathbf{r}_{0g}^* = \frac{\mathbf{r}_{0g}}{l} \quad , \mathbf{r}'^* = \frac{\mathbf{r}'}{l} \quad , dS^* = \frac{dS}{l^2} \quad , dV^* = \frac{dV}{l^3} \quad ,$$

$$t^* = \frac{t}{(l/V_1)}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{V_1} \quad , \mathbf{\Omega}^* = \frac{\mathbf{\Omega} l}{V_1} \quad , \mathbf{g}^* = \frac{\mathbf{g} l}{V_1^2} \quad ,$$

$$\sigma^* = \frac{\sigma}{\rho} \quad , M^* = \frac{M}{\rho l^3} \quad , \mathbf{\Phi}_0^* = \frac{\mathbf{\Phi}_0}{\rho l^5} \quad ,$$

$$\boldsymbol{\psi}^* = \frac{\boldsymbol{\psi}}{\rho V_1^2} \quad , \mathbf{\Sigma}^* = \frac{\mathbf{\Sigma}}{\rho V_1^2}, \quad \mathbf{F}^* = \frac{\mathbf{F}}{\rho V_1^2 l^2}, \quad \mathbf{L}^* = \frac{\mathbf{L}}{\rho V_1^2 l^3}. \tag{2.2, 7}$$

In the above equations $\mathbf{r}_{01}$ has been replaced by $\mathbf{r}_0$ in those terms which would otherwise involve products of $O(\mathbf{r}'^2)$, $O(\mathbf{v}^2)$, etc.

The kinematic relations (2.1, 19) apply with (p, q, r) the deviant angular velocities, the $\dot{\phi}_i$ the 'carried axis' angular velocities about the equilibrium axes and the ϕ_i the rotations from the steady-state orientation.

2.2.6. *The deviant equations to first order in attitude.*—The deviant equations are fully linearised by taking the deviant rotations ϕ_i to be small. The form of the equations (2.2, 6) is unaltered except for those terms involving $\mathbf{g}$. The relations (2.1, 19) and (2.2, 5) are linearised, the rotations ϕ_i becoming the components of a vector $\boldsymbol{\phi}$; thus,

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} \tag{2.2, 8}$$

and

$$\{v_{iM}\} - \{v_{iM}\}_1 = \begin{bmatrix} 0 & \phi_3 & -\phi_2 \\ -\phi_3 & 0 & \phi_1 \\ \phi_2 & -\phi_1 & 0 \end{bmatrix} [l]_1 \{v_{iF}\}. \tag{2.2, 9}$$

In (2.2, 9) the antisymmetric matrix is equivalent to a vector multiplication by $\boldsymbol{\phi}$.

2.3. *The Forces on the Aeroplane.*

2.3.1. *The gravitational force.*—In the deviant equations of motion the components of the vector $\mathbf{g}$ {equation (2.2, 4)} are given by an application of equation (2.2, 5), thus,

$$\mathbf{g} = ([l] - I)\mathbf{g}_1 \tag{2.3, 1}$$

so that, to first order in the deviant rotations ϕ_i,

$$\mathbf{g} = \boldsymbol{\phi} \times \mathbf{g}_1. \tag{2.3, 2}$$

2.3.2. *The propulsive force.*—The propulsive force will generally have a fixed direction relative to the power unit but its direction may vary relative to the body axes by an angle which will be of the same order of smallness as $\mathbf{r}'$. The magnitude of the force while being controllable will also change with the motion of the aeroplane and in particular with changes in forward speed.

2.3.3. *The aerodynamic forces.*—The surface traction $\boldsymbol{\psi}$ due to the motion of the aeroplane through the air is obviously extremely difficult to specify for a general motion. It will depend upon the whole history of the motion (due to wake effects): the pressure and shears at any point on the surface will depend on the integrated effect of the whole motion of every part of the aeroplane.

In addition, the type of flow regime encountered will depend on the variation, throughout the motion, in the values of a typical Reynolds number and Mach number for the aeroplane.

For the deviant motion relative to a steady-flight state the appropriate Reynolds number may be taken to be that of the steady state but the variation in the Mach number may still require to be taken into account particularly in the transonic regime.

The problem becomes tractable when the deviant motion is linearised in the velocities $\mathbf{v}$ and $\mathbf{\Omega}$. The first-order deviant aerodynamic forces may then be said to be given by a sum of the following contributions:

(1) the (first-order) change in the (unit-order) equilibrium forces due to (first-order) change in speed treating the equilibrium stress coefficients as constant,

(2) the (first-order) change in the directions of the (unit-order) overall equilibrium force coefficients due to (first-order) rotation of the resultant velocity vector relative to the body axes,

(3) the (first-order) change in the (unit-order) equilibrium stress coefficients due to (first-order) change in the Mach number of the equilibrium state and

(4) the (first-order) unsteady pressure field generated by the (first-order) motion of the aeroplane when changes in Mach number are ignored: this component will generally be treated on an inviscid-flow basis. It is shown in Ref. 8 that this pressure field may be derived from the standard linearised potential unsteady-flow theory when due allowance is made for the difference between body axes as used here and the steadily translating axes employed in the standard theory.

2.4. *Representation of the Aeroplane Structure.*

2.4.1. *Introduction.*—The equations of motion (2.1, 6), (2.1, 7) and (2.1, 16) are not, in themselves, sufficient for the solution of the aeroplane motion even when the surface tractions are completely specified as functions of the surface motion. The additional equations required are:

(1) the stress-strain relation,

(2) the equations of strain compatibility.

In effect, in order to proceed with a solution of the motion it is necessary to solve the Elastic Boundary Value problem for the aeroplane structure in terms of a general surface loading. When it is assumed that the stress instantaneously attains its equilibrium value consequent upon a rapid change in strain then the elastic problem is effectively reduced to the solution of the aeroplane structure under general steady surface and body forces when the inertia forces are represented by their instantaneous values (d'Alembert's Principle). However, the assumption of an instantaneous (conservative) stress-strain relation may not be justified in application to unsteady aeroelastic problems since it cannot allow for internal damping: the solution of the elastic problem if this assumption is abandoned becomes difficult and involves the history of the motion. A theoretical treatment of internal damping in elasticity is given in Ref. 9.

In what follows here it will be assumed that the stress-strain law is the Generalised Hooke's Law: the modification of the equations of motion consequent upon the presence of structural damping may then be made for those cases covered in Ref. 9: as a consequence of assuming an instantaneous stress-strain relation there is no need to retain the integration with respect to time in the variational equation of motion, (2.1, 16).

2.4.2. *Suitable forms for the displacement field.*—On the aeroelastic scale,* the classical aeroplane consists of an assembly of beam-like and plate-like structures, while the modern integrated aeroplane may consist largely of a single plate-like structure. More particularily, from an aeroelastic point of view the deformations of interest are solely those at the surface, the internal displacement field being of secondary importance.

As a consequence wide use is made of the simple bending theory of plates and the simple bending and torsion theories of beams sometimes with approximate corrections for shear deformation. When the simple theories of bending are inapplicable then methods of structural analysis[13, 14] are used based on the consistent assembly (by displacement or force compatibility) of all the internal elements of the structure and the external (point) force system. Nevertheless, in this case also the part of the solution of interest to the aeroelastician is that which relates the 'transverse surface displacements' of the structure at a finite number of points to the loads at these points.

Having synthesised the structure in some way then two main methods are available for representing the characteristics of the structure in the equations of motion:

(*a*) in the case when beam or plate theory is applicable resort may be made to a Rayleigh-Ritz analysis thereby expressing the surface displacement in terms of a series of weighted co-ordinate functions; this approach stems directly from the variational equation of motion (2.1, 16);

(*b*) the Green's or Influence Function for beam or plate may be calculated or for more general structures a set of influence coefficients and solution of the equations of motion obtained by numerical integration (collocation); the variational equation (2.1, 16) yields the integral equation of motion directly by the simple device of taking the virtual displacement to be a (small) arbitrary constant times the appropriate influence function when the variation in strain-energy integral becomes, by definition, the displacement at the general point.

In both instances the result is to replace the variational equation by a finite set of ordinary differential equations with time as the independent variable.

2.4.3. *Application of the Rayleigh-Ritz procedure.*—The method is extremely well known and the only point of interest here refers to the choice of body axes. Thus whatever axes are used each co-ordinate function should satisfy the appropriate axes conditions {e.g. equations (2.1, 11a), (2.1, 11b) for Mean Axes}. When *in-vacuo* vibration modes (normal modes) are used as co-ordinate functions they will already satisfy the mean-axes conditions. It is commonly asserted, for example, that normal modes are 'orthogonal to rigid-body modes' as if this were a unique property of normal modes whereas in fact it is a consequence of referring the vibration modes to mean body axes: so-called arbitrary modes can always be chosen to be 'orthogonal to rigid-body modes' simply by applying the conditions (2.1, 11a), (2.1, 11b). The role of the overall body motions in vibration studies is discussed in Appendix II.

2.4.4. *Application of the Influence Function.*—In like manner our main interest in discussing the application of the method (*b*) above is in defining the Influence Function for a structure which is not subject to kinematic constraints, in conjunction with the choice of body axes.

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* Omitting local aeroelastic effects such as panel flutter.

The following discussion will naturally also have reference to those cases where a set of influence coefficients rather than an Influence Function is available but some remarks are added at the end which refer more particularily to these cases.

The existence and nature of Influence Functions[10, 11] for plates and beams is well known so that it will be convenient here to pursue the discussion with reference to the simple bending of beams; corresponding results for other cases are obvious.

At the outset, in defining the Influence Function for a beam, it is necessary to consider the beam to have sufficient kinematic constraint to prevent bodily motion and, for our purposes, it is convenient but not essential to consider a cantilever beam since conditions at the free end already satisfy the requirements regarding lack of kinematic constraint.

Thus, let

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = p(x), \quad 0 < x < 1 \tag{2.4, 1}$$

be the equation of the loaded beam subject to the boundary conditions

$$w = \frac{dw}{dx} = 0 \text{ at } x = 0,$$

$$EI\frac{d^2w}{dx^2} = \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = 0 \text{ at } x = l. \tag{2.4, 2}$$

A formal solution of the differential equation and boundary conditions is given by a Fredholm Integral Equation[12], thus,

$$w(x) = \int_0^l G(x, \xi)p(\xi)d\xi \tag{2.4, 3}$$

where the Influence Function $G(x, \xi)$ satisfies the differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2G}{dx^2}\right) = \delta(x-\xi), \tag{2.4, 4}$$

δ being the Dirac Function, and the boundary conditions

$$G = \frac{dG}{dx} = 0 \quad \text{at } x = 0,$$

$$EI\frac{d^2G}{dx^2} = \frac{d}{dx}\left(EI\frac{d^2G}{dx^2}\right) = 0 \text{ at } x = l. \tag{2.4, 5}$$

As a consequence of the fact that $(d^2/dx^2)\{EI(d^2/dx^2)\}$ is a self-adjoint differential operator, the function $G(x, \xi)$ is symmetrical.

In the case of a beam without kinematic constraint and in which the ends are unloaded it is a necessary condition for the consistency of the differential equation (2.4, 1) and the boundary conditions

$$EI\frac{d^2w}{dx^2} = \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = 0 \text{ at } x = 0, l \tag{2.4, 6}$$

that

$$\int_0^l p(x)dx = \int_0^l xp(x)dx = 0. \tag{2.4, 7}$$

Since these conditions will be satisfied by any real motion involving the beam (by virtue of the application of the overall equations of motion) we may, for the purpose of defining an Influence Function, equilibrate the unit load $\delta(x-\xi)$ by any convenient loading system provided only that it alone cannot satisfy (2.4, 7). This arbitrary balancing system will clearly vanish from any real solution by virtue of the satisfaction of the overall equations of motion (2.1, 6), (2.1, 7). A convenient balancing system is the loading $a + bx$ where a, b satisfy the equations

$$\int_0^l \{\delta(x-\xi) - (a+bx)\}\, dx = \int_0^l x\{\delta(x-\xi) - (a+bx)\}\, dx = 0 \tag{2.4, 8}$$

giving

$$a(\xi) = \frac{2}{l^2}(2l-3\xi), \quad b(\xi) = \frac{6}{l^3}(2\xi - l).$$

It is easily verified that $a + bx$ by itself cannot satisfy equations (2.4, 7) unless $a = b \equiv 0$.

The Influence Function $G'(x, \xi)$ for the beam without kinematic constraint is then

$$G'(x, \xi) = G(x, \xi) - \int_0^l G(x, \xi')\,[a(\xi) + b(\xi)\xi']\, d\xi'. \tag{2.4, 9}$$

The function $G'(x, \xi)$ is not symmetrical. The function $G'(x, \xi)$ obviously satisfies the differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2G'}{dx^2}\right) = \delta(x-\xi) - (a+bx) \tag{2.4, 10}$$

and the boundary conditions (2.4, 6) with G' written for w.

As it happens the function G' will still satisfy the conditions $G' = dG'/dx = 0$ at $x = 0$ but these conditions are no longer necessary. In fact, G' is, for fixed ξ, arbitrary up to a small rigid-body displacement so that, in general,

$$G'(x, \xi) = G(x, \xi) - \int_0^l G(x, \xi')\,[a + b\xi']\, d\xi' + A(\xi) + B(\xi)x. \tag{2.4, 11}$$

In the context of the equations of motion of this beam the functions A and B are determined by the choice of body axes. Thus, for attached axes $A = B \equiv 0$ while for mean axes

$$\int_0^l m(x)G'(x, \xi)dx = \int_0^l m(x)xG'(x, \xi)dx = 0 \tag{2.4, 12}$$

where $m(x)$ is the mass per unit length of the beam: these two conditions yield simultaneous equations for $A(\xi)$ and $B(\xi)$ which are always consistent. There is, of course, no need to choose as origin of co-ordinates one end of the beam but should an intermediate point be chosen then G' will be an amalgam of two abutting cantilever influence functions: the application of the conditions (2.4, 8) (embodied in the balancing load system) ensures continuity of shear and moment between the parts of the beam meeting at the origin. For example, in a conventional aeroplane the origin will generally be in the region where the fore and aft fuselage beams and port and starboard wing beams meet.

By choosing the centre of mass as origin an alternative balancing system may be employed which is particularily convenient when also mean axes are used. This system is

$$a(\xi)m(x) + b(\xi)xm(x)$$

and, as before, it is easily verified that this system alone cannot satisfy (2.4, 7). The equations (2.4, 7) lead to

$$a = \frac{1}{M}, \quad b = \frac{\xi}{Mk_g^2}$$

upon using the fact that

$$\int_0^l m(x)x\,dx = 0$$

where M is the mass of the beam and k_g the radius of gyration about the centre of mass. When the mean-axes conditions are used to determine $A(\xi)$ and $B(\xi)$ the resulting influence function is, conveniently, symmetrical. Other forms of balancing systems may be advantageous in specific cases: the extension to two and three dimensions is obvious.

In those cases where a matrix of influence coefficients represents the structure then the structure will have been assumed to have sufficient kinematic constraints to prevent bodily motion: the foregoing integral operations for deriving the 'unrestrained' influence function may then be interpreted suitably as matrix multiplications preferably with the addition of a matrix which represents a consistent set of integrating weighting numbers.

The matrix of influence coefficients for an unrestrained structure is necessarily singular. In fact, if this matrix is of order m then its rank is $(m-n)$ where n is the number of necessary external equilibrium relations to be satisfied. As an illustration consider a beam deflecting in a principal plane: in this case there are two necessary external equilibrium relations, namely that overall force and moment in the principal plane should be zero.

Let G be the matrix of influence coefficients for the beam under $(m-1)$ point loads $\{p\}$ when the beam is suitably constrained. Again, the manner of constraint is arbitrary but we choose the cantilever as being most convenient. Then if the $(m-1)$ deflections at the load stations are $\{w\}$,

$$\{w\} = G\{p\}: \tag{2.4, 13}$$

the built-in end is not included as a point-direction. To construct the influence matrix for the unrestrained beam we proceed as for the influence function but first include the root as a station by writing

$$\begin{bmatrix} w_0 \\ \{w\} \end{bmatrix} = \begin{bmatrix} 0 & \{0\}' \\ \{0\} & G \end{bmatrix} \begin{bmatrix} p_0 \\ \{p\} \end{bmatrix} \tag{2.4, 14}$$

where w_0 and p_0 are the displacement and (point) load at the root station. The balancing load is again taken as $A_j\{1\} + B_j\{x\}$ where A_j and B_j are given by the overall equilibrium equations

$$1 - \{1\}'\{1\}A_j - \{1\}'\{x\}B_j = 0$$

$$x_j - \{x\}'\{1\}A_j - \{x\}'\{x\}B_j = 0, \quad j = 0 \text{ to } m. \tag{2.4, 15}$$

The matrix G' is given by {cf. equation (2.4, 9)}

$$G' = \begin{bmatrix} 0 & \{0\}' \\ \{0\} & G \end{bmatrix} - \begin{bmatrix} 0 & \{0\}' \\ \{0\} & G \end{bmatrix} \{A_j\{1\} + B_j\{x\}\}' \tag{2.4, 16}$$

and clearly takes the form

$$G' = \begin{bmatrix} 0 & \{0\}' \\ \{G_{21}\} & G_{22} \end{bmatrix} \tag{2.4, 17a}$$

finally,

$$\begin{bmatrix} w_0 \\ \{w\} \end{bmatrix} = G' \begin{bmatrix} p_0 \\ \{p\} \end{bmatrix}. \tag{2.4, 17b}$$

That G' (of order m) is of rank $(m-2)$ may be demonstrated by noting that for the loading systems

$$\begin{bmatrix} p_0 \\ \{p\} \end{bmatrix} = \begin{bmatrix} 1 \\ \{1\} \end{bmatrix} \text{ and } \begin{bmatrix} p_0 \\ \{p\} \end{bmatrix} = \begin{bmatrix} 0 \\ \{x\} \end{bmatrix}$$

then

$$\begin{bmatrix} w_0 \\ \{w\} \end{bmatrix} \equiv 0.$$

Thus the columns of G' are connected by two linear relations, that is, the rank of G' is $(m-2)$. In addition since

$$\alpha\{1\}' \begin{bmatrix} p_0 \\ \{p\} \end{bmatrix} = 0 \text{ and } \beta\{x\}'\{p\} = 0$$

for all α, β when the loading system p_j is self-equilibrating we may add to G' the arbitrary columns $\alpha\{1\}$ and $\beta\{x\}$. Similar results follow for other balancing load systems.

3. *A Discussion of the Equations of Motion with Reference to Current Methods of Investigating Aeroplane Stability.*

3.1. *Introduction.*

This discussion relates the foregoing general analysis to the methods currently used to estimate the static and dynamic stability of flexible aeroplanes. Emphasis is laid on the estimation of the stability of the trimmed, level-flight state.

Broadly speaking, current methods for dealing with these problems fall into two types, one an extension of the classical flutter analysis, the other an extension of classical, rigid-aeroplane stability.

The slender integrated configuration differs considerably in layout from the classical aeroplane and it is by no means obvious that behaviour known to be typical of classical aircraft will apply to this configuration. Here, attention is drawn to some of the points over which some doubt may arise, while in Part II this type of configuration is dealt with in some detail.

3.2. *A Discussion of Current Methods.*

3.2.1. *Inclusion of the 'rigid-body modes' in flutter analyses*[17].—When, in addition to the assumption of small change in attitude of the aeroplane, it is also assumed that the displacement of any point of the aeroplane from a rectilinear flight path is small then the equations of motion may be constructed so as to refer the motion to steadily translating (i.e. Newtonian) axes. A procedure of this type is usually followed in investigations of the flutter of aeroplanes including the so-called 'rigid-body modes' (of pitch and vertical translation, for example, in the symmetrical case).

The chief drawback of this approach is that changes in forward speed must be excluded from the deviant equations of motion in order to eliminate aeroplane motions which imply deviations from a rectilinear flight path which are many times larger than a typical aeroplane reference length (motions of phugoid type). The result of this is to suppress any reference in the deviant equations of motion to the actual equilibrium configuration under consideration: the trim speed is irrelevant except in so far as it is implicitly present in the ratio of a typical structural stiffness to a typical dynamic pressure. In practice, in these cases it is usual to imagine an 'equilibrium state' in which all forces both elastic and aerodynamic are zero: weight is necessarily ignored.

With the advent of the integrated configuration it is felt that stability investigations should properly include the full overall motion of the aeroplane. The deviant equations of motion then yield information relating to the static stability of the aeroplane whereas the roots of lowest frequency for the abbreviated equations yield information about a mode which often resembles the short-period motion of a rigid aeroplane: whenever this mode shows a 'static' instability the neglect of change in forward speed is not justifiable.

3.2.2. *The method of modified derivatives*[16, 4].—Until fairly recently the approach used in aeroplane stability and response calculations which take account of flexibility has been based on the idea of frequency-separated systems. The method is essentially a modification of the rigid-aeroplane equations of motion and quasi-steady aerodynamic forces are used based on the assumption that, for the modes of interest, the frequency parameter will be low. The number of equations of motion remains unaltered but the lowest-order coefficients are modified by an allowance for flexibility, such allowance being based on an equilibrium or steady-deformation analysis of the aeroplane structure (e.g. interia forces are neglected). Practically speaking, this approach is applicable whenever the typical overall-motion frequencies are much smaller than the lower typical vibration natural frequencies of the structure. But the vibration frequencies of interest are those of the aeroplane in flight and these frequencies may depart considerably from their 'still-air' values: under such conditions the principle of frequency separation may often fail and the number of equations of motion should be increased.

Further, in calculating modified derivatives it is usual to imagine the major parts of the aeroplane to be kinematically constrained (i.e. built-in) at various points. For the classical layout this procedure leads to modified derivatives which are physically meaningful but it would not be an exaggeration to say that the concept of the modified derivative as applied to the integrated configuration is vitiated by the lack of obvious physical meaning to be attached to such derivatives.

The pitfalls associated with the application of kinematic constraint of any kind to the slender configuration are discussed in Ref. 15.

APPENDIX I

The Deviant Equations of Motion

$$M\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{\Omega} \times (\mathbf{v}_1 + \mathbf{v})\right] + \frac{\partial \mathbf{\Omega}}{\partial t} \times \left[M\mathbf{r}_{01g} + \int_V \sigma \mathbf{r}'\, dV\right] +$$

$$+ \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \left[M\mathbf{r}_{01g} + \int_V \sigma \mathbf{r}'\, dV\right]\right) + 2\mathbf{\Omega} \times \int_V \sigma \frac{\partial \mathbf{r}'}{\partial t}\, dV +$$

$$+ \int_V \sigma \frac{\partial^2 \mathbf{r}'}{\partial t^2}\, dV = \mathbf{F} + M\mathbf{g}.$$

$$\left[M\mathbf{r}_{01g} + \int_V \sigma \mathbf{r}'\, dV\right] \times \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{\Phi}_{01} + \mathbf{\Phi}') \cdot \frac{\partial \mathbf{\Omega}}{\partial t} +$$

$$+ \frac{\partial \mathbf{\Phi}'}{\partial t} \cdot \mathbf{\Omega} + \mathbf{\Omega} \times [(\mathbf{\Phi}_{01} + \mathbf{\Phi}') \cdot \mathbf{\Omega}] +$$

$$+ \left[M\mathbf{r}_{01g} + \int_V \sigma \mathbf{r}'\, dV\right] \times [\mathbf{\Omega} \times (\mathbf{v}_1 + \mathbf{v})]$$

$$+ \mathbf{\Omega} \times \left(\int_V \sigma \mathbf{r}_0 \times \frac{\partial \mathbf{r}'}{\partial t}\, dV\right) + \int_V \sigma \mathbf{r}_0 \times \frac{\partial^2 \mathbf{r}'}{\partial t^2}\, dV$$

$$= \mathbf{L} + M\mathbf{r}_{01g} \times \mathbf{g} + \int_V \sigma \mathbf{r}'\, dV \times (\mathbf{g}_1 + \mathbf{g})$$

$$\int_{t_1}^{t_2} \left\{ \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{\Omega} \times (\mathbf{v}_1 + \mathbf{v})\right] \cdot \int_V \sigma \delta \mathbf{r}'\, dV + \int_V \sigma \frac{\partial^2 \mathbf{r}'}{\partial t^2} \cdot \delta \mathbf{r}'\, dV + \right.$$

$$+ 2\mathbf{\Omega} \cdot \int_V \sigma \frac{\partial \mathbf{r}'}{\partial t} \times \delta \mathbf{r}'\, dV + \frac{\partial \mathbf{\Omega}}{\partial t} \cdot \int_V \sigma (\mathbf{r}_{01} + \mathbf{r}') \times \delta \mathbf{r}'\, dV -$$

$$- \mathbf{\Omega} \cdot \int_V \sigma\, [(\mathbf{r}_{01} + \mathbf{r}') \cdot \delta \mathbf{r}' \mathbf{I} - (\mathbf{r}_{01} + \mathbf{r}') \delta \mathbf{r}']\, dV \cdot \mathbf{\Omega} -$$

$$\left. - \mathbf{g} \cdot \int_V \sigma \delta \mathbf{r}'\, dV + \int_V \mathbf{\Sigma} : \delta \mathbf{\Psi}\, dV - \int_S \boldsymbol{\psi} \cdot \delta \mathbf{r}'\, dS \right\} dt = 0$$

APPENDIX II

Motion under no Forces—Free Vibration

In the usual approach to the free vibration of unrestrained structures the motion is referred directly to Newtonian Axes with the result that bodily motion may only be allowed for within the restrictions of small overall displacement and rotation. Overall equations of equilibrium are then applied which lead to conditions on the resulting motion which are identical to the mean-axes conditions (2.1, 11a), (2.1, 11b). The result is to refer the motion to mean axes which are at rest and are therefore, *ipso facto*, Newtonian Axes.

However, there is no need to assume that the mean axes are at rest and more general motions exist which satisfy the equations of motion when no external forces act on the system. Of all these general motions only that involving steady, non-rotating translation ($\mathbf{v}$ = const., $\mathbf{\Omega} = 0$) of the mean axes will yield what is normally referred to as free-vibration modes.

But above all it should be noted that the equations of motion when no external forces act contain no reference to position or orientation in space so that these are, at all times, arbitrary and without limit. This conclusion is quite outside the scope of the solution, $\text{const.}_1 + \text{const.}_2 \times \mathbf{r}_0$ associated with the Neumann Problem in elasticity.

Finally, while it is certainly convenient to refer vibration motion to mean axes it is not essential: the contributions of overall and deformation motions will merely be altered to yield the same total motion.

Part II.—A Study of the Trim State and Longitudinal Stability of the Slender Integrated Aeroplane Configuration

Summary.

The general analysis developed in Part I is applied to the calculation of the equilibrium states and the longitudinal stability of such equilibrium states for the slender, integrated aeroplane configuration.

The slender configuration is treated essentially as having only longitudinal flexibility but an extension to include spanwise flexibility is included.

Slender-wing theory is employed both in the trim state and in the deviant equations of motion to give the aerodynamic loading.

The method of solution of the equations of equilibrium and the deviant equations of motion is by a collocation procedure well suited to digital computation.

A simple numerical example is presented to illustrate the application of the analysis.

1. *Introduction.*

In this Part the general analysis of Part I is applied to the estimation of the stability of the symmetric motion of a slender flexible flying wing this being a model of the slender integrated type of aeroplane configuration which may prove suitable as a supersonic transport cruising in the Mach number range $1 \cdot 8$ to $2 \cdot 2$ or thereabouts.

Before the stability of the motion relative to a specified trimmed state can be studied the trimmed state itself must be determined at all airspeeds so that the calculation of this steady state forms an integral part of the following analysis.

The trimmed state is taken as level trimmed flight and the atmosphere is treated as being homogeneous from the point of view of the deviant motion.

The main interest is in the stability of the aeroplane as a whole and not in flutter as such. Thus only those modes of motion having significant contributions from overall body motion are of direct interest. Hence the slender wing is treated essentially as a flying beam bending longitudinally and having rigid spanwise sections but the extension of the analysis to include spanwise flexibility is discussed.

Linearised slender-wing theory is used in setting up the deviant equations of motion and the equations for the trim state. However, the aerodynamic theory used in determining the trim state need not be identical to that used to obtain the deviant forces and may allow for non-linearity. But it should be borne in mind that since the relative deformation is assumed to be small the change in the local angle of incidence over the wing surface due to flexibility will also be small: hence if a non-linear aerodynamic theory is to be used it should take the form of a suitable Taylor Expansion in the relative deformation about a mean overall incidence.

The actual method of solution of both the deviant equations of motion and the equations of equilibrium is by collocation. That is, the variational equation of motion is satisfied at only a finite number of points, in this case distributed along the wing root chord. By this means the continuous system is reduced to one having a finite number of degrees of freedom and the usual methods of solution are available. In deriving the deviant equations of motion and equations of equilibrium for this equivalent dynamical system it will be seen that the only numerical technique required throughout is that of numerical integration.

An idealised point force is supposed to act at the trailing edge of the wing in order to be able to trim the aeroplane. This control force is assumed to be infinitely disposable and no attempt is made to elucidate its origin but it is a close representation of a flap-type control situated at the trailing edge of the wing.

Finally, the analysis is applied to the simple example of a delta wing having a given mass distribution and whose overall characteristics are probably typical of an aeroplane suitable as a supersonic transport.

2. *The Integrated Slender Configuration.*

2.1. *General Specification.*

The general layout of an idealised, slender configuration is shown in Figs. 1 and 2: the cross-section could be more generally a wing-body shape. Fig. 1 shows the main geometric parameters of the aeroplane while Fig. 2 shows the sense of the linear and angular velocities, forces and moments and loading per unit length.

The reference length is taken as the root chord l and the origin of the axis system is at the mid-point of the trailing edge.

The control force P represents an idealised aerodynamic control; in practice P would be supplied by elevator-type controls giving a short region of distributed pressure loading. The force P being aerodynamic in origin will have the form

$$P = \rho V^2 l^2 (\text{control coefficient})$$

for fixed control angle relative to the trailing edge of the wing. On the basis of Slender-Wing Theory the control force may be altered by varying the control coefficient without affecting the pressure distribution on the rest of the wing: the control coefficient (symbol P^*) may be loosely referred to as elevator angle. It is assumed that the control is irreversible so that in a perturbed motion the control coefficient is constant. Thus the control force P varies in proportion to the deviation in forward speed (Section 2.3, 3, Part I).

2.2. *Numerical Integration.*

As pointed out in the Introduction all the numerical operations required for solution of the trim equations and the deviant equations of motion are based on the evaluation of definite integrals.

The reduction from a continuous system to a dynamical system is made by representing the continuous (longitudinal) displacement curve $\zeta(x)$ by its values ζ_i at a chosen set of collocation points: thus every numerical integration will be based on this set of points throughout the calculation. The points will be associated with a preferred numerical integration formula and may not, in consequence, be equally spaced.

In the general form of the deviant equations of motion and the trim equations the set of collocation points is not specified beyond an indication of their total number. The integration formula is represented as a set of numbers assembled into a diagonal, weighting matrix indicated by the symbol

$$\left[\int\right]_D .$$

2.3. *Structural Influence Coefficients.*

Since spanwise sections of the wing are treated as being rigid the wing behaves essentially as a non-uniform beam in bending. The simple theory of bending is assumed to hold but no difficulty is presented if an approximate allowance for shear deflection is made based on the usual simple theory of shear in slender beams. A discussion of the calculation of influence coefficients when spanwise flexibility is included is given in Section 5.

Let $EI(x)$ be the bending stiffness of the wing when treated as a slender beam of variable cross-section; then the Influence Function for the beam considered as built-in at the trailing edge is most conveniently expressed by the Unit Load Equation (Principle of Virtual Complementary Work) in the form

$$\begin{aligned} G(x, \xi) &= \int_0^x \frac{(x-x')(\xi-x')}{EI(x')}\,dx', \quad x < \xi \\ &= \int_0^\xi \frac{(x-x')(\xi-x')}{EI(x')}\,dx', \quad \xi < x. \end{aligned} \tag{2.3, 1}$$

In the general case these integrals will be evaluated numerically to yield a set of influence coefficients for a chosen set of collocation points (the root station contributes a null row and column, *see* Part I, Section 2.4.4).

The Influence Function for the unrestrained wing is given by {equation (2.4, 11), Part I}

$$G'(x, \xi) = G(x, \xi) - \int_0^l G(x, \xi')\,[a(\xi) + b(\xi)\xi']\,d\xi' + A(\xi) + B(\xi)x \tag{2.3, 2}$$

where

$$a(\xi) = \frac{2}{l^2}(2l-3\xi), \; b(\xi) = \frac{6}{l^3}(2\xi-l).$$

The second integral may be evaluated numerically using the influence coefficients G_{ij}.

The unrestrained Influence Function referred to Attached Axes at the trailing edge is given from (2.3, 2) by taking $A = B \equiv 0$. The unrestrained Influence Function referred to Mean Axes at the trailing edge is given by taking A, B as in equations (2.4, 12), Part I: these equations may be evaluated numerically.

It will be seen in Section 3 that in setting up the equations of motion the quantities

$$\frac{\partial G'(x, \xi)}{\partial \xi} \text{ and } \frac{\partial^2 G'(x, \xi)}{\partial \xi^2}$$

are required for the unrestrained Influence Function referred to Attached Axes. These are given from equation (2.3, 1) as

$$\begin{aligned} \frac{\partial G'}{\partial \xi} &= \frac{\partial G}{\partial \xi} - \int_0^l G(x, \xi')\left[\frac{da}{d\xi} + \frac{db}{d\xi}\xi'\right] d\xi'; \\ \frac{\partial^2 G'}{\partial \xi^2} &= \frac{\partial^2 G}{\partial \xi^2}, \end{aligned} \tag{2.3, 3}$$

where

$$\begin{aligned} \frac{\partial G}{\partial \xi} &= \int_0^x \frac{(x-x')}{EI(x')}\,dx', \quad x < \xi \\ &= \int_0^\xi \frac{(x-x')}{EI(x')}\,dx', \quad \xi < x \end{aligned} \tag{2.3, 4}$$

and

$$\begin{aligned} \frac{\partial^2 G}{\partial \xi^2} &= 0 \quad, \quad x < \xi \\ &= \frac{x-\xi}{EI(\xi)}, \quad \xi < x. \end{aligned} \tag{2.3, 5}$$

The foregoing relations (2.3, 1), (2.3, 4) and (2.3, 5) constitute all the information required on the elastic properties of the wing.

In writing the equations of motion and the trim equations the Attached-Axes unrestrained Influence Function is used. The reasons for adopting Attached Axes are given in Section 3.1.

2.4. *The Aerodynamic Loading.*

For the calculation of the deviant, unsteady aerodynamic pressure loading {item (4) of Section 2.3.3, Part I} the Slender-Wing Theory[18, 19] is employed; spanwise sections being rigid then the local loading per unit root chord is dependent only on the downwash at that section.

Although the same theory need not necessarily be used in calculating the trim state it is convenient to do so. However, significant non-linear effects may be present in this type of wing due to leading-edge separation so that in calculating the trim state a non-linear theory might be preferable. But there seems no alternative at present to the use of linearised unsteady aerofoil theory for zero mean incidence in calculating the deviant forces. The use of non-linear aerodynamic theory in calculating the trim state is dealt with in Ref. 15.

In application it is assumed, with resulting considerable simplification, that the frequency parameter of the motion is not too high so that, in the cross-flow plane, the velocity potential satisfies Laplace's Equation (*see* Ref. 20 for these conditions in detail). Then contrary to almost all other unsteady theories it is feasible to dispense with the restriction of simple harmonic motion and since a general motion may be dealt with the deviant equations of motion may be solved completely in the sense that the frequency and damping of each constituent mode of the total motion may be determined.

The deviant aerodynamic forces are derived as for a flat wing but no difficulty ensues if the cross-section is taken as a wing-body combination. The inclusion of spanwise flexibility is discussed in Section 5.

The deviant aerodynamic loading per unit length $l(x)$, taken positive in the negative z-direction (Fig. 2), is given by

$$l^*(x^*, t^*) = \pi \left(\frac{\partial}{\partial t^*} - \frac{\partial}{\partial x^*}\right) [s^{*2} w_f^*] \tag{2.4, 1}$$

where the non-dimensional scheme of Section 2.2.5 of Part I is employed and $w_f^*(x^*, t^*)$ is the fluid velocity normal to the wing surface (downwash velocity). This velocity is given in terms of w^*, q^* and ζ^* by (Ref. 8)

$$w_f^* = w^* - q^* x^* - \frac{\partial \zeta^*}{\partial x^*} + \frac{\partial \zeta^*}{\partial t^*} \tag{2.4, 2}$$

and finally,

$$l^*(x^*, t^*) = \pi \left(\frac{\partial}{\partial t^*} - \frac{\partial}{\partial x^*}\right) \left[s^{*2} \left(w^* - q^* x^* - \frac{\partial \zeta^*}{\partial x^*} + \frac{\partial \zeta^*}{\partial t^*}\right)\right]. \tag{2.4, 3}$$

The derivatives with respect to x^* do not lend themselves to accurate numerical evaluation; however, it will be seen that in setting up the equations of motion these derivatives may be eliminated by repeated integration by parts.

3. *The Symmetric, Deviant Equations of Motion for the Slender Configuration referred to Attached Axes at the Wing Trailing Edge.*

3.1. *The Deviant Equations of Motion.*

For the deviant equations of motion the attached axes are most conveniently taken to be 'wind' (or stability) axes: further, in conformity with the adoption of a linear aerodynamic theory, it is assumed that the trim-state, local incidence of any section of the wing is small. Then in the general deviant equations of motion (2.2, 6) of Part I we have, for symmetric motion,

$$\mathbf{v}^* = (u^*, 0, w^*)\,;\; \mathbf{\Omega}^* = (0, q^*, 0)\,;\; \phi_1 = \phi_3 = 0$$
$$\phi_2 = \theta$$

$$\mathbf{F}^* = (X^*, 0, Z^*)\,;\; \mathbf{L}^* = (0, Q^*, 0)$$

and for simple longitudinal bending of the wing as a beam,

$$\mathbf{r}' = \left(-z\frac{\partial\zeta(x)}{\partial x},\, 0,\, \zeta(x)\right)$$

to first order.

Neglecting rotatory inertia terms and setting

$$\delta\zeta(\xi) = G'(x, \xi)\delta c$$

in the variational equation (2.2, 6c) Part I, then the deviant equations of motion are:

$$m_r^*\mu\frac{du^*}{dt^*} + w^*\theta = X^* \qquad (3.1,\ 1)$$

$$m_r^*\left\{\mu\left[\frac{dw^*}{dt^*} - q^* - \frac{dq^*}{dt^*}x_g^*\right] + \int_0^1 f_m(x^*)\frac{\partial^2\zeta^*}{\partial t^{*2}}\,dx^*\right\} = Z^* \qquad (3.1,\ 2)$$

$$m_r^*\left\{\mu\left[-x_g^*\frac{dw^*}{dt^*} + x_g^* q^* + k^{*2}\frac{dq^*}{dt^*}\right] - \int_0^1 f_m(x^*)x^*\frac{\partial^2\zeta^*}{\partial t^{*2}}\,dx^*\right\} = Q^* \qquad (3.1,\ 3)$$

$$\zeta^*(x^*) + \int_0^1 c_r^* f_G(x^*, \xi^*)\left\{l^*(\xi^*) + m_r^* f_m(\xi^*)\left(\frac{dw^*}{dt^*} - q^* - \frac{dq^*}{dt^*}\xi^* + \frac{\partial\zeta^*}{\partial t^{*2}}\right)\right\}d\xi^* = 0 \qquad (3.1,\ 4)$$

where the reference length l is taken as the root chord (Fig. 1) and k^* is the (non-dimensional) radius of gyration of the wing about the y-axis. The mass and flexibility parameters m_r^*, c_r^* are defined at the reference section as

$$m_r^* = \frac{m_r}{\rho l^2},\quad c_r^* = \frac{\rho V^2 l^4}{EI_r}. \qquad (3.1,\ 5\text{ and }6)$$

The mass distribution and influence function are expressed in terms of these parameters by writing

$$m^*(x^*) = m_r^* f_m(x^*),\; G'^*(x^*, \xi^*) = c_r^* f_G(x^*, \xi^*)$$

where the 'f' functions are purely numerical functions of x^*, ξ^*. The mass of the aeroplane is given by

$$M^* = \int_0^1 m^*(x^*)dx^* = m_r^*\int_0^1 f_m(x^*)dx^* = \mu m_r^* \qquad (3.1,\ 7)$$

where μ is constant for a given mass distribution.

The aerodynamic forces are the sum of the four contributions outlined in Section 2.3.3 of Part I where, in this case, the unsteady contribution is given by equation (2.4, 3) based on Slender-Wing Theory.

The trimmed-state aerodynamic forces are

$$\frac{Z_1}{\rho V_1^2 l^2} = C_{L1}' + C_{T1}' w_1^* = \frac{W}{\rho V_1^2 l^2} = W_1^* \tag{3.1, 8a}$$

$$\frac{X_1}{\rho V_1^2 l^2} = 0 = C_{T1}' - C_{D1}' \tag{3.1, 8b}$$

$$\frac{Q_1}{\rho V_1^2 l^3} = C_{M1}' = \frac{W x_g}{\rho V_1^2 l^3} = W_1^* x_g^* \tag{3.1, 8c}$$

$$\frac{l_1}{\rho V_1^2} = l_1^*(x^*) + P_1^* \delta(x^*) \tag{3.1, 8d}$$

where $\delta(x^*)$ is the Dirac Function representing the (idealised) control force and it has been assumed that the thrust line is along the tangent to the wing at the trailing edge. The dash on the lift, drag and moment coefficients is to denote that they are based on l^2 and not on wing area: the more usual coefficients are given by

$$C_L = C_L' \frac{A.R.}{2} \left(\frac{l}{s_0}\right)^2, \text{ etc.}$$

The control coefficient P_1^* and steady loading $l_1^*(x^*)$ can only be determined by solving the trim problem for the flexible configuration; this is done in Section 4.

It is assumed that thrust remains constant throughout the deviant motion and any change in the trim-state drag coefficient due to the deviant deformation is ignored.

Finally

$$\left\{\mu m_r^* \frac{d}{dt^*} + \left(\frac{\partial C_D'}{\partial M}\right)_1 M_1 + 2C_{D1}'\right\} u^* - \left\{C_{L1}' - \left(\frac{\partial C_D'}{\partial \alpha}\right)_1\right\} w^* + C_{L1}'\theta = 0$$

$$\left\{\left(\frac{\partial C_L'}{\partial M}\right)_1 M_1 + 2C_{L1}'\right\} u^* + \left\{\mu m_r^* \frac{d}{dt^*} + C_{D1}' + \int_0^1 l_{w^*}^* dx^*\right\} w^* -$$
$$- \left\{\mu m_r^* \left(1 + x_g^* \frac{d}{dt^*}\right) - \int_0^1 l_{q^*}^* dx^*\right\} q^* + m_r^* \int_0^1 f_m \frac{\partial^2 \zeta^*}{\partial t^{*2}} dx^* + \int_0^1 l_{\zeta^*}^* \zeta^* dx^* = 0.$$

$$\left\{- \left(\frac{\partial C_M'}{\partial M}\right)_1 M_1 - 2C_{L1}' x_g^*\right\} u^* - \left\{\mu m_r^* x_g^* \frac{d}{dt^*} + \int_0^1 l_{w^*}^* x^* dx^*\right\} w^* +$$
$$+ \left\{\mu m_r^* \left(x_g^* + k^{*2} \frac{d}{dt^*}\right) - \int_0^1 l_{q^*}^* x^* dx^*\right\} q^* - m_r^* \int_0^1 f_m x^* \frac{\partial^2 \zeta^*}{\partial t^{*2}} dx^* - \int_0^1 l_{\zeta^*}^* x^* \zeta^* dx^* = 0$$

$$\frac{\zeta^*(x^*)}{c_r^*} + \int_0^1 f_G(x^*, \xi^*) \left\{\left[\left(\frac{\partial}{\partial M}[l_1^*(\xi^*) + P_1^*\delta(\xi^*)]\right)_1 M_1 + 2[l_1^*(\xi^*) + P_1^*\delta(\xi^*)]\right] u^* +\right.$$
$$+ l_{w^*}^*(\xi^*) w^* + l_{q^*}^*(\xi^*) q^* + l_{\zeta^*}^*(\xi^*) \zeta^*(\xi^*) +$$
$$\left. + m_r^* f_m(\xi^*) \left[\frac{dw^*}{dt^*} - q^* - \frac{dq^*}{dt^*} \xi^* + \frac{\partial^2 \zeta^*(\xi^*)}{\partial t^{*2}}\right]\right\} d\xi^* = 0. \tag{3.1, 9}$$

are the complete deviant equations of motion for the slender configuration where $l_{w*}{}^*$, $l_{q*}{}^*$, $l_{\zeta*}{}^*$ are the linear operators,

$$l_{w*}{}^* \equiv \pi\left(\frac{\partial}{\partial t^*} - \frac{\partial}{\partial x^*}\right) s^{*2} \tag{3.1, 10a}$$

$$l_{q*}{}^* \equiv -\pi\left(\frac{\partial}{\partial t^*} - \frac{\partial}{\partial x^*}\right) s^{*2}x^* \tag{3.1, 10b}$$

$$l_{\zeta*}{}^* \equiv \pi\left(\frac{\partial}{\partial t^*} - \frac{\partial}{\partial x^*}\right)\left[s^{*2}\left(\frac{\partial}{\partial t^*} - \frac{\partial}{\partial x^*}\right)\right]. \tag{3.1, 10c}$$

Since the integrals involving the aerodynamic operators $l_{w*}{}^*$, $l_{q*}{}^*$ and $l_{\zeta*}{}^*$ are to be evaluated finally by numerical integration it is clearly important that only $\zeta^*(x^*)$ and not its derivatives should appear in these integrals since the presence of a derivative of the unknown ζ^* would require the use of numerical differentiation which is notoriously inaccurate. It is possible to achieve this by repeated integration by parts at the expense of introducing derivatives of the influence function G' and the semi-span s^*: however, equations (2.3, 4), (2.3, 5) show that the derivatives of G are available as integral expressions and it is assumed that the wing planform will be known closely enough to allow calculation of the derivatives of s. But the complete elimination of the derivatives of ζ^* from these integrals depends on the use of Attached Axes situated at the trailing edge of the wing and the fact that the wing (or wing-body combination) is pointed. These conditions are explicitly

$$s^* = 0 \text{ at } x^* = 1,$$

$$\frac{\partial \zeta^*}{\partial x^*} = \zeta^* \equiv 0 \text{ at } x^* = 0.$$

Details of the above reduction are not given here {but *see* equation (3.1, 12)}.

The integro-differential equations of motion are now replaced by a finite set of ordinary differential equations by replacing the function $\zeta^*(x^*)$ by the vector $\{\zeta^*\}$ the elements of which are the values of $\zeta^*(x^*)$ at the $(n+1)$ collocation points, x_j, $j = 0$ to n. The integrals are evaluated by numerical integration using a weighting matrix $[\int]_D$ as outlined in Section 2.2. Equations

$$\begin{bmatrix} (Da_{11}+b_{11}) & b_{12} & c_{13} & \\ b_{21} & (Da_{22}+b_{22}) & D(Da_{23}+b_{23}) & (D^2\{a\}'_2+D\{b\}'_2+\{c\}'_2) \\ b_{31} & (Da_{32}+b_{32}) & D(Da_{33}+b_{33}) & (D^2\{a\}'_3+D\{b\}'_3+\{c\}'_3) \\ \{b\}_1 & (D\{a\}_2+\{b\}_2) & D(D\{a\}_3+\{b\}_3) & \left(D^2[a]+D[b]+[c]+\dfrac{I}{c_r{}^*}\right) \end{bmatrix} \begin{bmatrix} u^* \\ w^* \\ \theta \\ \{\zeta^*\} \end{bmatrix} = 0 \tag{3.1, 11}$$

are now the (dynamical) equations of motion for the aeroplane where, for convenience, the symbol D replaces d/dt^*. The origin of axes at the trailing edge will normally be a collocation point but since, by definition of attached axes $\zeta^* = 0$ there, the vector $\{\zeta^*\}$ need not contain a value of ζ^* for this point. Hence the vector $\{\zeta^*\}$ contains only n elements for $n+1$ collocation points one of which is at the trailing edge. However, the numerical integration and hence the weighting matrix $[\int]_D$ are carried over $n+1$ stations.

The scalars and row, column and square matrices appearing in the dynamical equations (3.1, 11) are as follows: unless otherwise stated row, column and square matrices are of order 1, $(n+1)$; $(n+1)$, 1; $(n+1)$, $(n+1)$ respectively; a dagger † indicates that the element, row or column appropriate to the trailing-edge station has been omitted.

$$a_{11} = \mu m_r^*; \quad b_{11} = \left(\frac{\partial C_D'}{\partial M}\right)_1 M_1 + 2C_{D1}'$$

$$b_{12} = -C_{L1}' - \left(\frac{\partial C_D'}{\partial \alpha}\right)_1; \quad c_{13} = C_{L1}'$$

$$b_{21} = \left(\frac{\partial C_L'}{\partial M}\right)_1 M_1 + 2C_{L1}'$$

$$a_{22} = \mu m_r^* + \pi \{1\}' \left[\int\right]_D \{s^{*2}\}; \quad b_{22} = C_{D1}' + \pi s^{*2}\Big|_{x^*=0}$$

$$a_{23} = -\mu m_r^* x_g^* - \pi \{1\}' \left[\int\right]_D \{s^{*2}x^*\}; \quad b_{23} = -\mu m_r^*$$

$$\dagger\{a\}'_2 = m_r^* \{f_m\}' \left[\int\right]_D + \pi \{s^{*2}\}' \left[\int\right]_D$$

$$\dagger\{b\}'_2 = \pi \left\{\frac{ds^{*2}}{dx^*}\right\}' \left[\int\right]_D; \quad \{c\}'_2 = 0$$

$$b_{31} = -\left(\frac{\partial C_M'}{\partial M}\right)_1 M_1 - 2C_{L1}' x_g^*$$

$$a_{32} = -\mu m_r^* x_g^* - \pi \{x^*\}' \left[\int\right]_D \{s^{*2}\}; \quad b_{32} = -\pi \{1\}' \left[\int\right]_D \{s^{*2}\}$$

$$a_{33} = \mu m_r^* k^{*2} + \pi \{x^{*2}\}' \left[\int\right]_D \{s^{*2}\}; \quad b_{33} = \mu m_r^* x_g^* + \pi \{x^*\}' \left[\int\right]_D \{s^{*2}\}$$

$$\dagger\{a\}'_3 = -m_r^* \{f_m x^*\}' \left[\int\right]_D - \pi \{x^* s^{*2}\}' \left[\int\right]_D$$

$$\dagger\{b\}'_3 = -\pi \left\{2s^{*2} + x^* \frac{ds^{*2}}{dx^*}\right\}' \left[\int\right]_D$$

$$\dagger\{c\}'_3 = -\pi \left\{\frac{ds^{*2}}{dx^*}\right\}' \left[\int\right]_D$$

$$\dagger\{b\}_1 = [f_G] \left[\int\right]_D \left(M \frac{\partial}{\partial M} + 2\right)_1 l_1^* + \left(M \frac{\partial P^*}{\partial M} + 2P^*\right)_1 \{f_G\}_{j=0}$$

$$\dagger\{a\}_2 = m_r^* [f_G] \left[\iint\right]_D \{f_m\} + \pi [f_G] \left[\iint\right]_D \{s^{*2}\}$$

$$\dagger\{b\}_2 = -\pi [f_G] \left[\iint\right]_D \left\{\frac{ds^{*2}}{d\xi^*}\right\}$$

$$\dagger\{a\}_3 = -m_r^* [f_G] \left[\iint\right]_D \{\xi^* f_m\} - \pi [f_G] \left[\iint\right]_D \{s^{*2}\xi^*\}$$

$$\dagger\{b\}_3 = -m_r^* [f_G] \left[\iint\right]_D \{f_m\} + \pi [f_G] \left[\iint\right]_D \left\{\frac{d}{d\xi^*}(s^{*2}\xi^*)\right\}$$

$$\dagger[a] = m_r^* [f_G] \left[\iint\right]_D [f_m]_D + \pi [f_G] \left[\iint\right]_D [s^{*2}]_D$$

$$\dagger[b] = \pi \left(2\left[\frac{\partial f_G}{\partial \xi^*}\right] \left[\iint\right]_D [s^{*2}]_D + [f_G] \left[\iint\right]_D \left[\frac{ds^{*2}}{d\xi^*}\right]_D\right)$$

$$\dagger[c] = \pi \left(\left[\frac{\partial^2 f_G}{\partial \xi^{*2}}\right] \left[\iint\right]_D [s^{*2}]_D + \left[\frac{\partial f_G}{\partial \xi^*}\right] \left[\iint\right]_D \left[\frac{ds^{*2}}{d\xi^*}\right]\right). \qquad (3.1,\ 12)$$

3.1.1. *Steady-state Mach number.*—All the aerodynamic forces derived from unsteady aerofoil theory are fundamentally functions of the steady-state Mach number. However, in this instance for the particular form of Slender-Wing Theory used the dependence on Mach number is absent: this fact will be used in the ensuing development but the restriction is not necessary to the analysis.

In addition since the configuration is slender the variation of the steady-state aerodynamic forces with Mach number is likely to be small; that is, we may take b_{11}, b_{21}, b_{31}, and $\{b\}_1$ to be independent of Mach number.

3.2. *Solution of the Dynamical Equations of Motion.*

The solution of the set of homogeneous equations (3.1, 11) is of the form

$$u^*,\ w^*,\ \theta,\ \zeta_i^* \propto e^{\nu t^*}$$

where ν is in general complex. It is convenient to use the symbols u^*, w^*, θ, ζ_i^* also as the complex amplitudes of the motion $e^{\nu t^*}$ and then the algebraic equations for the determination of the modal columns

$$\begin{bmatrix} u^* \\ w^* \\ \theta \\ \zeta_i^* \end{bmatrix}_k$$

and the characteristic roots ν_k are simply equations (3.1, 11) with D replaced by ν. The presence of a positive real root ν_k indicates a divergence while the presence of a complex root with positive real part indicates an oscillatory instability.

The equations are most conveniently dealt with by reducing the second-order equations to an equivalent set of first-order equations when the ν_k and the modal columns appear as the eigenvalues and eigenvectors of a single matrix with real elements. The form of the equations of motion (3.1, 11) is, in terms of partitioned matrices,

$$\begin{bmatrix} A_{11}D + B_{11}, & A_{12}D^2 + B_{12}D + C_{12} \\ A_{21}D + B_{21}, & A_{22}D^2 + B_{22}D + C_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \tag{3.2, 1}$$

where

$$q_1 \equiv \begin{bmatrix} u^* \\ w^* \end{bmatrix} \text{ and } q_2 \equiv \begin{bmatrix} \theta \\ \{\zeta^*\} \end{bmatrix}.$$

The vector q_1 contains the ignorable co-ordinates u^* and w^*.

Introducing the velocities corresponding to the non-ignorable co-ordinates $\dot{q}_2 \equiv Dq_2$ as subsidiary variables then equation (3.2, 1) may be rewritten

$$(D\Phi + \Psi)y = 0 \tag{3.2, 2}$$

where

$$\Phi = \begin{bmatrix} A_{11} & 0 & A_{21} \\ 0 & I & 0 \\ A_{21} & 0 & A_{22} \end{bmatrix}_1, \quad \Psi = \begin{bmatrix} B_{11} & C_{12} & B_{12} \\ 0 & 0 & -I \\ B_{21} & C_{22} & B_{22} \end{bmatrix}$$

and

$$y = \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}.$$

The order of the matrix equation (3.2, 2) is $(4+2n)$ where $(n+1)$ is the number of collocation points covering the wing root chord.

The standard eigenvalue problem is usually stated as

$$(\nu I + U)x = 0 \tag{3.2, 3}$$

and a variety of methods are available, suitably embodied in digital computer programmes, to deal with this equation.

Programmes do not seem to be available to deal directly with equation (3.2, 2). The point seems trivial since a premultiplication by Φ^{-1} or Ψ^{-1} will yield (3.2, 3). However, in this case the use of the unrestrained influence coefficients renders Φ singular; in fact Φ is of rank $2(1+n)$ (*see* Part I, Section 2.4.4). This is seen immediately if it is noted that parts of the matrices A_{21} and A_{22} are derived from the influence matrix $[f_G]$ by multiplication by non-singular matrices. The matrix Ψ is not generally singular due to the presence of the unit matrix in C_{22}: however, it will certainly be singular whenever the static stability is limiting, that is, when $\nu = 0$ is a root of equation (3.2, 2).

In view of these considerations the following course is adopted. Let β be any arbitrary constant, real or complex; then equation (3.2, 2) may be written

$$((\nu - \beta)\Phi + (\Psi + \beta\Phi))y = 0. \tag{3.2, 4}$$

Let $\mu = 1/(\nu - \beta)$ then the equivalent eigenvalue problem is

$$((\Psi + \beta\Phi)^{-1}\Phi + \mu I)y = 0 \tag{3.2, 5}$$

wherein the root $\nu = 0$ appears simply as $-1/\beta$. The matrix $\Psi + \beta\Phi$ will only be singular if it happens that β is an eigenvalue of equation (3.2, 2). The choice of β is best dictated by a knowledge of the probable value of the roots of smallest modulus of equation (3.2, 2). It is clearly convenient and indeed essential for many digital-computer programmes to take β real.

For a given aeroplane layout the non-dimensional scale parameters to be specified for a solution of the equations are:

$$m_r^* = \frac{m_r}{\rho l^2}, \quad c_r^* = \frac{\rho V_1^2 l^4}{EI_r} \text{ and } W_1^* = \frac{W}{\rho V_1^2 l^2}.$$

Also, before the equations may be solved the trim state must be known: we thus study the stability of the trim state appropriate to a forward speed V_1 and weight parameter W_1^*. The parameter W_1^* is thus necessarily a variable parameter for a complete study of stability. Should the altitude be fixed (at least for one series of calculations) then the parameter m_r^* is fixed: the parameter c_r^* is most conveniently replaced by the quotient

$$c_r^* = \frac{e_r^*}{W_1^*}$$

where

$$e_r^* = \frac{Wl^2}{EI_r} \tag{3.2, 6}$$

is a fixed parameter for the aeroplane. It may be noted that since $W = Mg$ then

$$W_1^* = \mu m_r^* g_1^* \tag{3.2, 7}$$

where g_1^* is the Froude number gl/V_1^2.

Thus in the equations of motion (3.1, 11) the factor $1/c_r^*$ multiplying the unit matrix is replaced by W_1^*/e_r^*. Then the coefficients b_{11}, b_{12}, c_{13}, b_{21}, b_{31}, the column $\{b\}_1$ and the factor W_1^*/e_r^* vary according to the trim state considered: all other coefficients are fixed except for b_{22} but if we neglect C_{D1}' in comparison to $\pi(s^{*2})_{x^*=0}$ (the lift-curve slope of the rigid aeroplane) then this coefficient may also be considered constant.

It may be noted that when the change in forward speed u^* is suppressed the first row and first column of equations (3.1, 11) are eliminated. These contain all the coefficients dependent on the trim state with the result that reference to the trim state is now completely absent from the deviant equations of motion; the most significant term dependent on the trim state is $\{b\}_1$. The significant parameter for this reduced set of equations is c_r^*: this is a purely aeroelastic parameter in the sense that change in stiffness EI_r is indistinguishable from change in V^2 (for constant altitude).

The free vibrations of the aeroplane are given from equations (3.1, 11) by taking all forces except inertia forces to be zero. The two parameters m_r^* and c_r^* now combine to yield the single parameter

$$v_r^* = \nu^2 m_r^* c_r^* \tag{3.2, 8}$$

provided the contributions to the inertia coefficients due to aerodynamic inertia are ignored (*in-vacuo* vibrations). It is much more convenient for the calculation of vibration modes and frequencies to replace the attached-axes influence coefficients by influence coefficients referred to mean axes (at the trailing edge); the equations for the *in-vacuo* vibration modes are then simply

$$\left[\frac{1}{m_r^*}[a_M] + \frac{1}{v_r^*} I\right]\{\zeta^*\} = 0 \tag{3.2, 9}$$

where $[a_M]$ is given by the appropriate equation of the set (3.1, 12) with the mean-axes influence coefficients substituted for those derived for attached axes. It may be recalled that, as defined, the inertia coefficients $[\dot{a}_M]$ are proportional to m_r^*.

4. *The Trimmed-Flight State for the Slender Configuration.*

4.1. *Trimmed Level Flight.*

The calculation of the trim state is based on the application of Slender-Wing Theory for rigid spanwise sections as for the deviant equations of motion. Accordingly it is assumed that the aerodynamic forces are not dependent on Mach number.

The aeroplane structure is again represented by the influence function for attached axes at the trailing edge, the control force is represented by a concentrated load at the trailing edge and the thrust is assumed to be adjusted to give level flight at a given airspeed.

As defined in Part I, Section 2.1.1 suffix 0 is used to designate a reference configuration which is not necessarily a real equilibrium configuration for the aeroplane. Here it is taken to mean the aeroplane configuration when completely unloaded. Thus the specification that the aeroplane has a certain 'built-in' camber refers to this idealised state: the uncambered aeroplane is defined to have a plane mean surface in the reference configuration. It may be imagined that the reference configuration will result from the aeroplane being supported at a great many points so that the weight is locally equilibrated. When reference is made to the 'rigid' aeroplane it is to be understood that the corresponding invariable configuration is the reference configuration with or without built-in camber as the case may be.

For the calculation of the trim state the attached axes are most conveniently taken so that the x-axis is tangential to the wing mean surface at the trailing edge; the (x, y) plane then defines the mean surface of the uncambered reference configuration. The overall incidence of the aeroplane is the incidence of the trailing-edge section for this choice of axis orientation.

The aeroplane is taken to have a built-in longitudinal camber $\zeta_0^*(x^*)$ which leads to the aerodynamic loading $l_0^*(x^*)$: the total aerodynamic load is thus

$$l_1^*(x^*) = l_0^*(x^*) - \pi \frac{d}{dx^*}\left[s^{*2}\left(w_1^* - \frac{d\zeta_1^*}{dx^*}\right)\right] \tag{4.1, 1}$$

and if l_0^* is also calculated on the basis of Slender-Wing Theory then

$$l_1^*(x^*) = \pi \frac{d}{dx^*}\left(s^{*2}\frac{d\zeta_0^*}{dx^*}\right) - \pi \frac{d}{dx^*}\left[s^{*2}\left(w_1^* - \frac{d\zeta_1^*}{dx^*}\right)\right]. \tag{4.1, 2}$$

Assuming the local incidence to be everywhere small, the equations of equilibrium are {Part I, equations (2.2, 1)}

$$M^*g_1^* + Z_1^* = 0 \tag{4.1, 3a}$$

$$M^*g_1^*x_g^* - Q_1^* = 0 \tag{4.1, 3b}$$

$$\zeta_1^*(x^*) - \int_0^1 c_r^* f_G(x^*, \xi^*)\,\{z_1^*(\xi) + m^*(\xi^*)g_1^*\}\,d\xi^* = 0 \tag{4.1, 3c}$$

where

$$z_1^*(\xi^*) = -\,l_1^*(\xi^*) - P_1^*\delta(\xi^*) \tag{4.1, 4}$$

is the total aerodynamic loading.

The expanded forms of these equations are:

$$W_1^* - \pi s^{*2}\Big|_{x^*=0} w_1^* - C_{D1}'w_1^* - \int_0^1 l_0^* dx^* - P_1^* = 0 \tag{4.1, 5a}$$

$$W_1^* x_g^* - \pi \int_0^1 s^{*2} dx^* \, w_1^* - \pi \int_0^1 \frac{ds^{*2}}{dx^*} \zeta_1^* dx^* - \int_0^1 l_0^* x^* dx^* = 0 \tag{4.1, 5b}$$

$$\frac{\zeta_1^*(x^*)}{c_r^*} + \int_0^1 f_G(x^*, \xi^*) \left\{ -\pi \frac{ds^{*2}}{d\xi^*} w_1^* + \pi \frac{d}{d\xi^*}\left(s^{*2} \frac{d\zeta_1^*}{d\xi^*}\right) + \right.$$

$$\left. + \, l_0^*(\xi^*) + P_1^* \delta(\xi^*) - w_r^* f_m(\xi^*) \right\} d\xi^* = 0 \tag{4.1, 5c}$$

where

$$W_1^* = M^* g_1^* = \mu m_r^* g_1^* = \mu w_{r1}^*.$$

As with the deviant equations of motion these equations of equilibrium are replaced by a finite set of algebraic equations in the unknowns P_1^*, w_1^* and the vector $\{\zeta^*\}_1$ using the same set of collocation points: as a consequence some of the resulting matrices are identical to those already derived and where this is the case the same symbol is employed. The resulting inhomogeneous, algebraic equations of equilibrium are:

$$\begin{bmatrix} b_{22} & 1 & \{0\}' \\ b_{32} & 0 & \{c\}'_3 \\ \{b\}_2 & \{f_{G0}\} & \left([c] + \frac{W_1^*}{e_r^*} I\right) \end{bmatrix} \begin{bmatrix} w_1^* \\ P_1^* \\ \{\zeta^*\}_1 \end{bmatrix} = W_1^* \begin{bmatrix} 1 \\ -x_g^* \\ -\{k\} \end{bmatrix} + \begin{bmatrix} b_{20} \\ b_{30} \\ \{b\}_0 \end{bmatrix} \tag{4.1, 6}$$

where $\{f_{G0}\}$ is the first column of $[f_G]$ and

$$\{k\} = -\frac{1}{\mu}[f_G]\left[\int\right]_D \{f_m\} \tag{4.1, 7}$$

$$b_{20} = 0$$

$$b_{30} = -\{c\}'_3 \{\zeta^*\}_0 \tag{4.1, 8}$$

$$\{b\}_0 = -[c]\{\zeta^*\}_0$$

when l_0^* is calculated from Slender-Wing Theory {equation (4.1, 2)}. It should be noted that the expressions (4.1, 8) apply only for attached axes which are tangential to the wing at the trailing edge: the fact that b_{20} is zero for these axes for example is merely reflected in the particular meaning given to overall incidence w^*. Attached axes could equally well be chosen so that the x-axis joined the trailing edge to the wing apex for the fixed camber shape ζ_0^*.

4.2. *Solution of the Trim Equations for Level Flight.*

Since interest is fixed in high-speed-flight conditions it is convenient, with close approximation, to replace W_1^* by the lift coefficient C_{L1}'.

Equation 4.1, 6 shows that the trim-state solution is the sum of two parts:

(1) the trim state of an uncambered aeroplane,

(2) the equilibrium state of a weightless, cambered aeroplane.

It is convenient to retain this division of the complete solution and for this purpose the suffices u and c are used to refer to solutions 1 and 2 above respectively.

The distance of the aerodynamic centre of the rigid aeroplane from the trailing edge appears naturally as a parameter in the trim solution. When C_{D1}' is neglected the coefficient b_{22} is the lift of the rigid aeroplane per unit angle of attack {equations (3.1, 12)} while b_{32} is the (aerodynamic) moment about the trailing edge per unit angle of attack: thus if we define $\bar{x}^*$ as the distance of the aerodynamic centre of the rigid aeroplane from the trailing edge then

$$b_{32} = -\bar{x}^* b_{22}.$$

The quantity $x_g^* - \bar{x}^*$ will be recognised as the c.g. margin of the aeroplane.

Rearranging the force and moment equations of the set (4.1, 6) {and using the first of equations (4.1, 8)} we have

$$P_1^* \bar{x}^* = C_{L1}' (\bar{x}^* - x_g^*) + b_{30} - \{c\}'_3 \{\zeta^*\}_1 \qquad (4.2, 1a)$$

$$w_1^* = \frac{C_{L1}' x_g^*}{b_{22}\bar{x}^*} - \frac{b_{30}}{b_{22}\bar{x}^*} + \frac{\{c\}'_3}{b_{22}\bar{x}^*} \{\zeta^*\}_1 \qquad (4.2, 1b)$$

and substituting in the last n equations of (4.1, 6),

$$\left[\frac{C_{L1}'}{e_r^*} I + [c] - \frac{\{f_{G0}\} - \frac{1}{b_{22}}\{b\}_2}{\bar{x}^*} \{c\}'_3\right] \{\zeta^*\}_1$$

$$= C_{L1}' \left[-\left(\{k\} + \frac{\{b\}_2 x_g^*}{b_{22}\bar{x}^*}\right) + \{f_{G0}\}\left(\frac{x_g^*}{\bar{x}^*} - 1\right)\right] +$$

$$+ \left[\left(\{b\}_0 + \{b\}_2 \frac{b_{30}}{b_{22}\bar{x}^*}\right) - \{f_{G0}\} \frac{b_{30}}{\bar{x}^*}\right]. \qquad (4.2, 2)$$

The calculation of $\{\zeta^*\}_1$ from (4.2, 2) involves simply the solution of n simultaneous equations: the left-hand-side matrix depends only on the stiffness and aerodynamic characteristics of the aeroplane. Substitution of the solution $\{\zeta^*\}_1$ in equation (4.2, 1b) then gives w_1^* and thence P_1^* is obtained from equation (4.2, 1a).

At low speeds the left-hand side is effectively $(C_{L1}'/e_r^*)I$ so that the part solutions $\{\zeta^*\}_u$ and $\{\zeta^*\}_c$ tend to

$$\{\zeta^*\}_{u\ \text{low speed}} = e_r^* \left[-\left(\{k\} + \frac{\{b\}_2 x_g^*}{b_{22}\bar{x}^*}\right) + \{f_{G0}\}\left(\frac{x_g^*}{\bar{x}^*} - 1\right)\right] \qquad (4.2, 3)$$

and

$$\{\zeta^*\}_{c\ \text{low speed}} = \frac{e_r^*}{C_{L1}'} \left[\left(\{b\}_0 + \{b\}_2 \frac{b_{30}}{b_{22}\bar{x}^*}\right) - \{f_{G0}\} \frac{b_{30}}{\bar{x}^*}\right]. \qquad (4.2, 4)$$

Thus at low speeds $(P_1^*)_u \bar{x}^*$ is effectively that for a rigid aeroplane with the constant camber shape $\{\zeta^*\}_{u\ \text{low speed}}$ while $(P_1^*)_c \bar{x}^*$ is effectively that for a rigid aeroplane with the total camber shape $\{\zeta^*\}_0 + \{\zeta^*\}_{c\ \text{low speed}}$ ($\{\zeta_1^*\}_c$ is inversely proportional to C_{L1}'). Equation (4.2, 2) can now be written in the alternative form

$$\left[\frac{C_{L1}'}{e_r^*} I + [K]\right] \{\zeta^*\}_1 = \frac{C_{L1}'}{e_r^*} \{\zeta^*\}_{u\ \text{low speed}} + \frac{1}{e_r^*} (C_{L1}' \{\zeta^*\}_{c\ \text{low speed}}) \qquad (4.2, 5)$$

where

$$[K] = \left[[c] - \frac{\{f_{G0}\} - \frac{1}{b_{22}}\{b\}_2}{\bar{x}^*} \{c\}'_3\right]. \qquad (4.2, 6)$$

The expression in the bracket on the right-hand side of equation (4.2, 5) is constant.

Finally, substituting the expressions (4.1, 8) in equation (4.2, 4) it is seen that

$$-\frac{1}{e_r^*}(C_{L1}'\{\zeta^*\}_{c\text{ low speed}}) = [K]\{\zeta^*\}_0. \tag{4.2, 7}$$

4.3. *Behaviour of the Trim Solution.*

The following is a brief discussion of the typical behaviour of the trim solution for the slender configuration with variation in speed. This behaviour is most clearly illustrated by a consideration of the trim curve of the aeroplane, that is, the curve of control coefficient, P_1^*, against lift coefficient, C_{L1}.

A quite general picture of the probable behaviour of the trim state for the slender configuration can be deduced by consideration of equation (4.2, 5). It may be shown for example that the shape of the trim curve at high speed is determined largely by the low-speed camber shapes $\{\zeta^*\}_{u\text{ low speed}}$ and $\{\zeta^*\}_{c\text{ low speed}}$. This conclusion appears in Ref. 21 wherein a full discussion of the trim state is undertaken together with the connection between the shape of the trim curve and the static stability of the aeroplane. Ref. 21 also deals with the application of the usual ideas of manoeuvre theory[16] to the slender configuration, normally embodied in the concept of 'elevator angle per g'.

The control coefficient P_1^* (proportional to elevator angle), incidence w_1^* and displacement $\{\zeta^*\}_1$ become indefinitely large for zeros of the determinant

$$\left|\frac{C_{L1}'}{e_r^*}I + [K]\right|.$$

Thus the speed, $V_{1\max}$ which gives the first zero {i.e. $(C_{L1}')_{\min}$} of this determinant represents a theoretical maximum for a possible trimmed state of the aeroplane: from a practical point of view P_1^*, w_1^* and $\{\zeta^*\}_1$ will become large as this speed is approached: Hancock[15] has termed this the Maximum Trim Speed. Since linear aerodynamic theory is being used coupled with the assumption of small relative deformation such effects need to be suitably interpreted: the numerical example of Section 6 shows that deformations remain quite small up to near the Maximum Trim Speed although the effect on control coefficient is considerable.

Clearly

$$\frac{(C_{L1}')_{\min}}{e_r^*} = \frac{EI_r}{\rho(V_{1\max})^2 l^4}$$

is the largest (dominant) eigenvalue of the matrix $[K]$. The Maximum Trim Speed depends only on the aerodynamic and stiffness properties of the aeroplane and not on the weight distribution. A variation in stiffness EI_r is indistinguishable in this context from a variation in ρV^2. The eigenvalue itself depends only on the relative distributions of stiffness and local aerodynamic loading.

Let V_s be the (lowest) speed for limiting static stability of the aeroplane, then (Appendix I) at this speed the slope of the trim curve, dP^*/dC_L', is zero. If $V_s < V_{\max}$ then (Appendix I) the slope of the trim curve suffers a change in sign between some low speed and the Maximum Trim Speed while if $V_s > V_{\max}$ there is no such change in sign.

Some typical trim curves for an aeroplane without built-in camber are shown in Fig. 3. Curve 1 consists of the C_L-axis and the two branches of the Maximum-Trim-Speed line: it will occur in the very particular case when the local weight is exactly balanced by the local aerodynamic loading of the rigid aeroplane, that is, $-\{k\} = \{b\}_2/b_{22}$ and $x_g{}^* = \bar{x}^*$. If $x_g{}^* = \bar{x}^*$ but the weight and aerodynamic loading (due to incidence) do not coincide the trim curve will be as curves 2 depending on the sign of $\{c\}'_3\,\{\zeta_1{}^*\}$. In the usual case $\{k\}$ and $\{b\}_2/b_{22}$ will differ and for an aeroplane which is statically stable at low speed $x_g{}^* > \bar{x}^*$. The basic trim curve of the rigid aeroplane is a straight line of slope $-(x_g{}^*-\bar{x}^*)$ passing through the origin $C_L' = 0$, $P^* = 0$: such a line is shown in Fig. 3. Generally speaking the trim curve for the flexible aeroplane will be like curve 3 or 4, the shape of these curves being determined primarily by the 'natural', low-speed camber shape, $\{\zeta^*\}_{u \text{ low speed}}$. While these curves are probably typical it cannot be asserted that a trim curve cannot cross or re-cross the corresponding rigid-aeroplane trim curve (Ref. 21). Trim curve 3 indicates that static instability occurs before the Maximum Trim Speed.

5. *Allowance for Spanwise Flexibility.*

When spanwise flexibility is to be allowed for, the matrix of influence coefficients refers to an ordered grid of collocation points distributed over the wing surface. Similarly, the mass and aerodynamic loadings are functions of position in a plane. A consistent integrating scheme for integrals applying over the wing planform is required to replace that applying only along the root chord of the wing.

When Slender-Wing Theory is used then the pressure loading over any spanwise section is dependent only on the (spanwise) variation of downwash over that section. When the 'low-frequency' form of Slender-Wing Theory is adopted the determination of the pressure at a collocation point in terms of the downwash is straightforward. Thus as in Refs. 18 and 19 the velocity potential is expressed, on the wing or wing-body combination, as a Fourier sine series whose coefficients are determined by definite spanwise integrals of the downwash. These coefficients may thus be expressed in the form $\{k\}'\,\{w_f\}$ where w_{fi} is the downwash at station i and i carries only the values pertaining to stations on that section. The velocity potential and hence the pressure at any point in the cross-section is given by

$$\{p\} = [a]\,\{w_f\}$$

for a single spanwise section.

Finally the matrix of aerodynamic influence coefficients consists essentially of a partitioned matrix whose matrix elements (of different order) lie only along the diagonal: each 'element matrix' refers to one spanwise section. The downwash w_{fi} at any point is then expressed in terms of w^*, q^* and $\{\zeta^*\}$ as in equation (2.4, 2).

In carrying out the spanwise integrations it is more important that the scheme of numerical integration used be dictated by aerodynamic rather than structural considerations.

6. *A Numerical Example.*

6.1. *Introduction.*

The following numerical example illustrates the application of the general analysis for the slender configuration to a specific case and the numerical results obtained serve to illustrate some of the conclusions already drawn concerning the behaviour of this type of aeroplane. No attempt has been made to choose stiffness and mass distributions which are likely to be met in practice but the overall stiffness, mass and weight parameters have been given values which are probably typical for a possible supersonic transport.

The configuration is chosen to be a delta wing; the stiffness distribution varies directly with the local span and is therefore linear. Two mass distributions having the same total mass and c.g. position but which give very different low-speed camber shapes are assumed.

More realistic configurations will differ from this in having higher stiffness over the central parts of the wing and less stiffness at the trailing edge. At the apex there will, in a practical case, be a nose extension having a not inconsiderable mass and some stiffness. The general mass distribution is likely to resemble the stiffness distribution being somewhat concentrated in the central part of the wing. All these points are however incidental to the presentation and illustration of the general analysis for the slender configuration and belong properly to an extended design study of this type of aeroplane.

The calculations are carried out for a fixed altitude of 40,000 ft. at which height the cruising Mach number would be expected to be close to 2.

6.2. *General Specification.*

The reference cross-section is taken to be at the wing trailing edge. The stiffness distribution $EI(x^*)$ is taken as

$$EI(x^*) = EI_r(1-x^*) \qquad (6.2,\ 1)$$

The two mass distributions, referred to as (A) and (B) respectively, are taken as

$$m^*(x^*) = \frac{m_r^*}{2}(2-x^*-x^{*2}), \qquad \text{(A)} \qquad (6.2,\ 2a)$$

and

$$m^*(x^*) = \frac{m_r^*}{2}(1+9x^*-25x^{*2}+20x^{*3}-5x^{*4}) \qquad \text{(B)} \qquad (6.2,\ 2b)$$

giving a total mass $M^* = (7/12)m_r^*$ or {equation (3.1, 7)} $\mu = 7/12$.

The centre of mass of both distributions (6.2, 2) is at $x_g^* = 5/14$ while the aerodynamic centre due to incidence for a delta wing is at $\bar{x}^* = 1/3$ so that c.g. margin $= (x_g^*-\bar{x}^*) \doteqdot 0\cdot0238\ldots$

The weight/stiffness parameter $e_r^* = Wl^2/EI_r$ is taken to be unity: this value gives static deflections of the wing due to loads of the order of the weight of the wing of order 1/20. The stiffness parameter c_r^* for $e_r^* = 1$ is then

$$c_r^* = \frac{\rho V^2 l^4}{EI_r} = \frac{1}{C_L'}.$$

The relative mass parameter $M^* = M/\rho l^3$ is chosen to be $3\cdot5$ at 40,000 ft (about $0\cdot9$ at sea level). The wing loading is also, in effect, fixed by the choice of M^* since

$$\text{wing loading} = \frac{W}{s_r l} = \frac{4Mg}{l^2} = 4gM^*\rho l$$

so that, at 40,000 ft, wing loading $\approx 0\cdot27l$. While an actual specification of l is not necessary for a solution of the non-dimensional equations of motion and equilibrium it will be convenient to choose a typical value for l; this is done by fixing on a wing loading of about 55 lb/ft^2 giving $l = 200$ ft. The cruising lift coefficient (based on wing area) at 40,000 ft and Mach 2 is then $0\cdot05$.

The foregoing specification is summarised in Table 1 together with a diagram showing the two mass distributions.

6.3. *Numerical Integration.*

The number of collocation points used is seven, distributed evenly along the wing root chord with the end points at trailing edge and wing apex. The numerical integration formula used is Weddle's Rule giving the weighting matrix

$$\frac{1}{20}\begin{bmatrix} 1 & & & & & & \\ & 5 & & & & & \\ & & 1 & & & & \\ & & & 6 & & & \\ & & & & 1 & & \\ & & & & & 5 & \\ & & & & & & 1 \end{bmatrix}$$

for integration with respect to x^*.

6.4. *The Influence Coefficients.*

For the simple stiffness variation of equation (6.2, 1) the cantilever influence function for the wing 'built-in' at the trailing edge is simply

$$\begin{aligned} f_G(x^*, \xi^*) &= (1-\xi^*)\{(x^*-1)\ln(1-x^*) - x^*\} + \frac{x^{*2}}{2}, \; x^* \leqslant \xi^* \\ &= (1-x^*)\{\xi^*-1)\ln(1-\xi^*) - \xi^*\} + \frac{\xi^{*2}}{2}, \; \xi^* \leqslant x^*. \qquad (6.4,\ 1) \end{aligned}$$

An evaluation of these expressions gives the matrix of influence coefficients referred to the collocation points; these are given in Table 2.

The unrestrained influence function for attached axes at the trailing edge is given, through an application of equation (2.3, 2), as

$$\begin{aligned} f_G(x^*, \xi^*)\big|_{\text{attached axes}} &= (1-\xi^*)\{(x^*-1)\ln(1-x^*) - x^*\} + \frac{x^{*2}}{2} - \\ &\quad - \frac{x^{*2}}{12}\{(x^{*2}-2x^*) + \xi^*(6+2x^*-2x^{*2})\}, \; x^* \leqslant \xi^* \\ &= (1-x^*)\{(\xi^*-1)\ln(1-\xi^*) - \xi^*\} + \frac{\xi^{*2}}{2} - \\ &\quad - \frac{x^{*2}}{12}\{6\xi^* + 2(\xi^*-1)x^* + (1-2\xi^*)x^{*2}\}, \; x^* \geqslant \xi^*. \qquad (6.4,\ 2) \end{aligned}$$

The matrix of unrestrained influence coefficients referred to attached axes at the trailing edge is given in Table 3.

The unrestrained influence function for mean axes at the trailing edge is given through an application of equation (2.4, 12) of Part I; there is no advantage in giving the analytical expressions explicitly. The matrix of unrestrained influence coefficients referred to mean axes at the trailing edge is given in Table 4.

The simple stiffness variation chosen has allowed the analytical determination of the influence functions but in the general case the matrices of influence coefficients will be the outcome of numerical integrations. However, the general appearance of the matrices of influence coefficients will always be similar to those matrices given as Tables 2, 3 and 4.

The derivatives of $f_G(x^*, \xi^*)$ {equations (2.3, 3), (2.3, 4) and (2.3, 5)} are also simply determined; these matrices are not given here.

6.5. *The Deviant Equations of Motion.*

Little comment is required on the particular form taken by equations (3.1, 11) and (3.1, 12) for this numerical example.

However, three points require brief mention. First, the drag polar is taken as

$$C_D = 0{\cdot}020 + \frac{1}{\pi} C_L^2 \tag{6.5, 1}$$

giving

$$\frac{\partial C_D}{\partial \alpha} = C_L .$$

In evaluating b_{22} {equations (3.1, 12)} the contribution from C_D is neglected; this gives a maximum error in b_{22} at $C_L = 0{\cdot}4$ of 5%.

Secondly, since for the delta wing

$$\frac{ds^{*2}}{d\xi^*} = -\,2s_r^*(1-\xi^*)$$

then {equations (3.1, 12)}

$$\{b\}_2 \propto [f_G] \left[\int\right]_D \{2(1-\xi^*)\} .$$

But by the definition of the unrestrained influence function used (Section 2.4.4 of Part I)

$$\int_0^1 f_G^* \,(x^*, \xi^*)(1-\xi^*)d\xi^* = 0$$

so that, for these particular circumstances, $\{b\}_2 \equiv 0$.

Thirdly, at the wing apex the mass, weight and aerodynamic loading are always zero. The result is that the apex point does not constitute an independent collocation point although it is of importance when carrying out the numerical integrations. Thus the displacement of the apex point ζ_6^* may be completely determined in terms of the remaining $(n-1)$ displacements and the variables u^*, w^* and θ. In effect the last column of the deviant equations of motion (3.1, 11) consists of zeros except for the diagonal term which is simply C_{L1}'/e_r^*. The last row and column may be omitted and the remaining set of equations solved; the last row then gives ζ_6^* in terms of the remaining variables. This circumstance is a result of the unreal conditions existing at the wing apex in this idealised example.

6.6. *The Equations of Equilibrium.*

The remarks made in the previous paragraph concerning $\{b\}_2$ and the role of the apex station also apply to the equations of equilibrium (4.1, 6). The solution of five simultaneous equations and substitution in the sixth gives $\{\zeta^*\}_1$; w_1^* and P_1^* are then found from equations (4.2, 1). These calculations are easily performed on a desk calculator.

6.7. *Solution of the Trim State.*

6.7.1. *Maximum trim speed.*

—The Maximum Trim Speed was found by determining the dominant eigenvalue and eigenvector of the matrix $[K]$ {equation (4.2, 6)} by simple matrix iteration performed on a desk calculator. This gave

$$\frac{EI_r}{\rho(V_{\max})^2 l^4} = \frac{1}{c_r^*} = \frac{1}{164} .$$

When the connection between C_L' and c_r* is made by the choice of e_r* (Table 1) then

$$(C_L')_{\min} = \frac{1}{164}$$

or

$$(C_L)_{\min} = 0 \cdot 049.$$

Thus the Maximum Trim Speed is fractionally above the nominal cruising speed of this particular aeroplane.

The corresponding eigenvector (relative to attached axes which are tangential to the trailing edge) is plotted in Fig 4 normalised to unit amplitude at the apex station. The x-axis is rotated through the appropriate angle w_1* to give the shape of the aeroplane relative to a horizontal velocity vector.

6.7.2. *Trim curves in level flight.*—For each mass distribution the trim curve $P^* \sim C_L$ for level flight is plotted in Fig. 5. It is seen that for mass distribution A the aeroplane is statically unstable at speeds greater than that corresponding to $C_L \approx 0 \cdot 1$.

The deformed shapes of the aeroplane at a series of speeds are shown in Figs. 6a and 6b wherein the deformation is plotted relative to the 'tangential' attached axes while in Fig. 7 curves are plotted of incidence w_1* against C_L. The order of magnitude of the relative deformation even at speeds approaching the maximum trim speed fully justifies the use of small-deflection theory.

6.8. *Solution of the Deviant Equations of Motion.*

6.8.1. *The rigid aeroplane.*—As a basis of comparison the deviant motion of the idealised rigid aeroplane appropriate to mass distribution A was computed. The roots of the resultant quartic characteristic equation typically representing the short-period and phugoid motions are represented by dotted curves in Figs. 9 and 10. The idealised rigid aeroplane appropriate to mass distribution B differs only in the value of the radius of gyration and since this difference is small (Table 1) the roots are little different from those for mass distribution A: the frequency of the short-period motion may be expected to be increased by the factor k_{gA}/k_{gB}, that is, by about 8%.

6.8.2. *The normal modes of free vibration.*—The normal, free (*in-vacuo*) vibration modes were computed from equation (3.2, 9) which uses the matrix of influence coefficients referred to mean axes. Only the first two modes and first three frequencies are considered to be of acceptable accuracy. The normal modes are plotted in Figs. 8a and 8b for each mass distribution together with the corresponding values of the non-dimensional mass-stiffness-frequency parameter v_r* {equation (3.2, 8)}: the true frequencies are also given assuming a reference length of 200 ft, other parameters being as given in Table 1. At 40,000 ft the still-air natural frequencies are of the order of 98% of these frequencies.

6.8.3. *The flexible aeroplane.*—The complete deviant equations of motion when reduced to an equivalent first-order system as detailed in Section 3.2, yield a matrix equation of order 14×14. The characteristic roots were obtained by the application of a matrix iterative programme to the matrix $(\Psi + \beta\Phi)^{-1}\Phi$ of equation (3.2, 5). The complete programme received the matrices Φ and Ψ as data.

These calculations were performed for a series of values of C_L, the first eight roots only being found: higher roots would be of doubtful value with the small number of collocation points employed. The roots computed thus included those roots corresponding to the second normal mode. The results are presented in Figs. 9 to 12 as curves of the inverse of (real) time to half (or double) amplitude and curves of frequency in cycles per (real) second plotted against C_L. The roots are easily identifiable as stemming from either the rigid-aeroplane roots or the free-vibration motion and these figures are titled in this sense. This type of root label is used for convenience in presentation only and should not be taken to imply that the mode of motion associated with any particular root or root-pair remains similar in character at all speeds.

Since the model is not particularly representative there is little point in refining the calculation by employing more collocation points.

6.8.4. *Trim curves in shallow pull-out.*—The relation of the manoeuvre theory of Gates and Lyon to the possible dynamic behaviour of the short-period motion is discussed in Ref. 21. The curve of control coefficient per g (equivalent to 'elevator angle per g') against C_L is easily deduced from the equations of equilibrium for a shallow pull-out with constant centripetal acceleration when the variation in the direction of the gravity vector relative to the body axes is ignored. These equations are identical to equations (4.1, 6) except that the column

$$nC_{L1}' \begin{bmatrix} 1 \\ -\dfrac{1}{M^*} b_{33} \\ -\dfrac{1}{M^*} \{b\}_3 \end{bmatrix}$$

is added to the right-hand side. Thus for this 'trim' state we may write

$$P_1^* = (P_1^*)_u + (P_1^*)_c + n(P_1^*)_n$$

where n is the centripetal acceleration. Since, at constant forward speed,

$$\frac{dP_1^*}{dn} = (P_1^*)_n$$

this last is effectively the 'elevator angle per g' of Manoeuvre Theory. Vanishing of the 'elevator angle per g' indicates limiting static stability of the deviant equations of motion when the change in forward speed is suppressed but it is precisely when this occurs that the exclusion of this variable (and with it the attitude angle θ) is inadmissible. The connection between 'elevator angle per g' and dynamic stability needs to be established for the slender configuration by the investigation of many numerical examples.

The curve of 'control coefficient per g' against C_L for the particular aeroplane considered here is given in Fig. 13 (for 40,000 ft).

7. *Discussion and Conclusions.*

This discussion is concerned more with the application of the general method presented in Part II for assessing the dynamic behaviour of the slender configuration than with the particular numerical results found in Section 6.

Although the scalars and matrices appearing in the equations of motion were obtained on a desk calculator for the example of Section 6 it will be clear that this stage of the calculation could readily be programmed for a digital computer, the only operations involved being scalar and matrix multiplications. The basic data would then consist of:

(1) a set of weighting numbers relating to the collocation points;

(2) the values of bending stiffness, EI, at the collocation points;

(3) the values of semi-span and its derivatives at the collocation points;

(4) the values of mass per unit length at the collocation points;

(5) the relevant non-dimensional scale parameters;

(6) distribution of built-in camber, if any.

The trim problem being readily amenable to programming then the solution of this and the deviant motion become available from one simple set of basic data.

The drawbacks of the method are, first, the neglect of spanwise flexibility and, secondly, the use of Slender-Wing Theory. On the first count the main defence is one of ease of application and simplicity. It was indicated in Section 5 how an allowance could be made for spanwise flexibility and although this is straightforward the directness of equations (2.3, 1) etc. is lost. Similarly the use of Slender-Wing Theory is justified by its simplicity compared to other low-aspect-ratio theories for unsteady flow. Also, in the region of interest the main flow is supersonic so that the main drawback of this wing theory that it does not satisfy the Kutta condition in subsonic flow is not serious. Furthermore the use of any other unsteady-wing theory leads to the usual restriction to simple harmonic motion.

A third criticism may be directed at the large number of degrees of freedom required to obtain reasonable accuracy up to say the third or fourth pair of roots of the deviant equations of motion compared with the use of normal modes as co-ordinate functions. This is, of course, true but it must be remembered that the calculation of the normal modes will have involved in general the use of three to four times the number of degrees of freedom as the number of modes obtained.

Fourthly there is the representation of a flap control by an unspecified concentrated force applied in the immediate region of the trailing edge. This defect is easily overcome by replacing this force by that derived from a flap control using Slender-Wing Theory and the introduction of a finite stiffness connection to the wing proper.

The author considers the method as presented to be suitable to the assessment of the effects of flexibility on the overall motion of a slender configuration in the vital early design stage when the structure is largely unknown in detail. At this stage the application of an involved analysis is impossible both for lack of information and for lack of time. The ability to more or less arbitrarily change the basic data outlined at the beginning of this section quickly and easily is the paramount consideration.

For the very simple example considered interest in the numerical results lies in the change in behaviour of the perturbed motion with change in mass distribution. In particular, how far does the low-speed camber shape reflect the probable dynamic behaviour of the aeroplane?

Mass distribution A gives a trim camber shape which is always convex upward (Fig. 6a) and the trim curve (Fig. 5) indicates a static instability at $C_L \approx 0 \cdot 1$: the transition from a pair of complex roots to a real pair is extremely rapid and in the region $0 \cdot 10 < C_L < 0 \cdot 125$ the iteration of these roots is very slowly convergent. Normally a static instability is accepted with some equanimity but in this case the time to double amplitude is down to 5 sec by $C_L \approx 0 \cdot 06$ (Fig. 9a) and is rapidly decreasing with further decrease in C_L. The modes associated with the 'phugoid' roots alter in character throughout the speed range and at high speed both the stable and unstable modes show increased contributions from change in incidence and elastic deformation; the mode associated with the 'short-period' roots does not change in character.

Mass distribution B gives trim camber shapes which are (largely) concave upward (Fig. 6b) and the trim curve (Fig. 5) indicates no static instability. The curve of control coefficient per g (Fig. 13) however shows first a reduction and then a change in sign of P_n^* at $C_L \approx 0 \cdot 06$: this would indicate a static instability of the equations of motion with change in forward speed suppressed. The full equations of motion cannot show an initial static instability with decrease of C_L and any instability must be of a dynamic nature. For this mass distribution the modes associated with both the 'phugoid' and 'short-period' roots change in character as speed is increased, the former having increased contributions from change in incidence and elastic deformation, the latter having increased contributions from elastic deformation only. The 'short-period' frequency drops to zero at almost exactly that value of C_L for which the control coefficient per g is zero (Figs. 10a and 13) and subsequently this motion becomes a pair of subsidences. The 'phugoid' frequency (Fig. 9b) increases rapidly for speeds greater than that for $C_L \approx 0 \cdot 08$ and a dynamic instability appears in this mode at $C_L \approx 0 \cdot 065$ giving roughly the same order of time to double amplitude as for mass distribution A.

Two general points should be mentioned. First, any change in the low-speed camber shape (or indeed the trim camber shape at any speed) depends on the difference between the mass distribution and the steady aerodynamic loading and hence any uncertainty in the steady aerodynamic loading will be reflected in the dynamic behaviour of the aeroplane: this leaves aside the question of the accuracy of the unsteady aerodynamic loading. An accurate assessment of the trim state and dynamic behaviour for this type of aeroplane places a heavy demand on the aerodynamic theory and in this respect the position, at present, is far from satisfactory. An additional point of importance for the slender aeroplane is the fact that since weight and aerodynamic loading are reacted locally then strength considerations cannot be expected to yield the same order of stiffness margins as may be expected from a conventional aeroplane configuration.

The second point concerns the representation of the aeroplane by its normal-modes either for static or dynamic calculations. The change in first normal-mode frequency for the two mass distributions is not large nor is the first normal mode shape very different (Figs. 8a and 8b). The result is that the equations of motion for these two cases in terms of the overall degrees of freedom plus one normal mode may not be adequate to reflect the large differences in the dynamic behaviour of the aeroplane. It is clear that a representation in terms of small translation, small rotation and first normal mode will certainly be inadequate to describe the dynamic behaviour. While the use of the first normal mode in estimating the trim curve might be adequate for mass distribution A the presence of the reflexed region in the trim camber shape for mass distribution B (Fig. 6b) means that this representation would be inadequate in this case.

The question of the response of the aeroplane to controls or gusts has not been discussed. When the equations of motion are linearised for these cases the deviant equations of motion simply become an inhomogeneous set and the usual methods of solution are available. While the vanishing of the control coefficient per g is not necessarily a serious stability consideration it probably indicates undesirable response characteristics: this example, since it does not include response calculations, cannot shed light on this point.

NOTATION

PART I

ρ	Air density
V	Airspeed
l	Reference length
σ	Mass per unit volume
M	Total mass
ϕ_1, ϕ_2, ϕ_3	Euler angles
$\mathbf{r}$	Position vector
$\mathbf{r}'$	Displacement vector
$\mathbf{v}$	Velocity of origin of body axes
$\mathbf{\Omega}$	Angular velocity of body axes
$\mathbf{g}$	Gravitational acceleration vector
$\mathbf{\Phi}$	Inertia tensor
$\mathbf{\Phi}'$	Change in inertia tensor due to displacement
$\mathbf{F}$	Overall force
$\mathbf{L}$	Overall moment
$\boldsymbol{\psi}$	Surface traction
$\mathbf{\Sigma}$	Stress tensor
$\mathbf{\Psi}$	Strain tensor
w	Transverse displacement of slender beam
p	Loading on slender beam
$G(x, \xi)$	Influence function for slender beam
$\dfrac{\partial}{\partial t}$	Denotes rate of change of vector relative to moving axes
$\dfrac{d}{dt}$	Denotes rate of change of vector relative to inertial axes

Subscripts

$_0$	Reference state
$_1$	Equilibrium state
F	Fixed axis system
M	Moving axis system
$\{\ \}', \{\ \}, [\ ], [\ ]_D$	indicate row, column, square and diagonal matrices respectively

PART II (in addition to the Notation of Part I)

l	Reference length—length of slender configuration
g	Gravitational acceleration
$l(x)$	Aerodynamic loading per unit root-chord length
w_f	Fluid downwash velocity
ν	Complex frequency parameter
$s(x)$	Local semi-span
$m(x)$	Mass per unit length
M	Total mass of aeroplane
W	Total weight of aeroplane
x_g	Position of centre of mass
$\bar{x}$	Position of aerodynamic centre due to incidence
$\zeta(x)$	Bending deflection
$C_L{}'$, $C_D{}'$	Lift, drag, etc. coefficients based on l^2
C_L, C_D	Lift, drag, etc. coefficients based on wing area
*	Asterisk denotes corresponding non-dimensional quantity

Subscripts

$_r$	Reference section for definition of overall parameters
$_u$	Trim solution for uncambered aeroplane
$_c$	Trim solution for weightless, cambered aeroplane
$_n$	Trim solution for shallow pull-out
$\left[\int\right]_D$	Weighting (or integrating) matrix

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TABLE 1

General Specification

Parameter	Reference Value	Remarks
m_r^*	6 at 40,000 ft	$m^*(x^*) = \frac{m_r^*}{2}(2-x^*-x^{*2})$ (A)
		$m^*(x^*) = \frac{m_r^*}{2}(1+9x^*-25x^{*2}+20x^{*3}-5x^{*4})$ (B)
M^*	3·5 at 40,000 ft	$\mu = \frac{7}{12}$
k_{gA}^*	0·239	Radii of gyration about c.g.
k_{gB}^*	0·218	
Aspect Ratio	1·0	$s_r^* = \frac{1}{4}$, $s^*(x^*) = \frac{1}{4}(1-x^*)$
c.g. Margin	0·023 . . .	$= (x_g^*-\bar{x}^*) = \frac{5}{14} - \frac{1}{3}$
e_r^*	1·0	Weight-Stiffness Parameter
C_L cruising	0·05	$M = 2$ at 40,000 ft
Landing Speed	125 m.p.h.	$C_{L\,\max} \approx 1\cdot0$
Wing loading	55 lb/ft²	Assuming $l = 200$ ft
Wing Area	10,000 ft²	
A.U.W.	550,000 lb	
$C_L = 8C_L'$ etc. $1/c_r^* = C_L'$		Relation between coefficients based on $\frac{1}{2}\rho V^2 S$ and $\rho V^2 l^2$ respectively. $\frac{EI}{\rho V^2 l^4} = C_L'(1-x^*)$

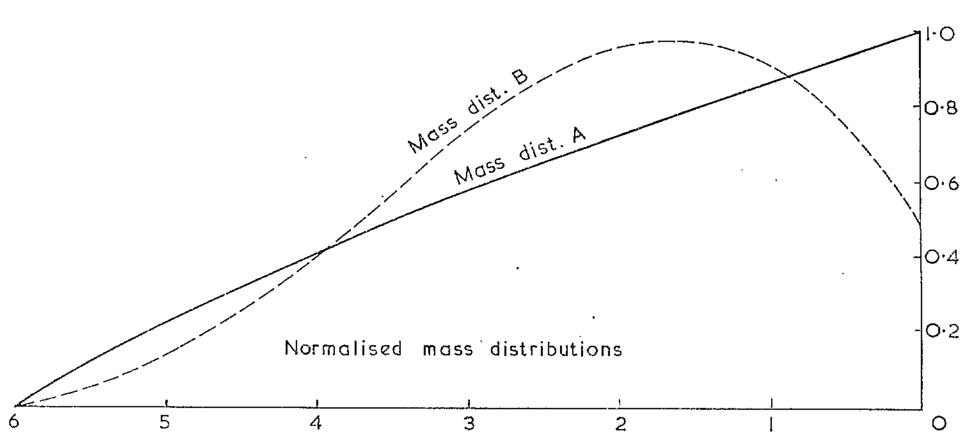


TABLE 2

Symmetrical Matrix of Influence Coefficients for Delta Wing Cantilevered at Trailing Edge

0	0	0	0	0	0	0
	0·00161219	0·00406753	0·00652287	0·00897921	0·01143355	0·01388889
		0·01354005	0·02404392	0·03454780	0·04505168	0·05555556
			0·04828679	0·07385786	0·09942893	0·12500000
				0·12206803	0·17214513	0·22222222
					0·25810443	0·34722222
						0·50000000

TABLE 3

Matrix of Influence Coefficients for Attached Axes at Trailing Edge of Delta Wing

0	0	0	0	0	0	0	
0·0070730	−0·0010248	−0·0006913	−0·0003577	−0·0000241	0·0003094	0·0006430	
0·0514403	−0·0073356	−0·0120618	−0·0064743	−0·0008869	0·0047006	0·0102881	
0·1562500	−0·0042157	−0·0546996	−0·0379654	−0·0079491	0·0220671	0·0520833	$\times 10^{-1}$
0·3292181	+0·0211950	−0·1209143	−0·1256189	−0·0413224	0·0616433	0·1646091	
0·5626286	+0·0714685	−0·1978457	−0·2595687	−0·1379023	0·1161952	0·4018776	
0·8333333	+0·1388889	−0·2777778	−0·4166667	−0·2777778	0·1388889	0·8333333	

TABLE 4

Matrix of Influence Coefficients for Mean Axes at Trailing Edge of Delta Wing
(Mass Distribution A)

+1·046366	+0·166566	−0·396318	−0·525533	−0·269922	+0·226458	+0·808271	
+0·041491	+0·052128	−0·017616	−0·077264	−0·060916	+0·023401	+0·135921	
−0·590442	−0·115168	+0·254293	+0·313415	+0·139705	−0·138839	−0·446409	
−0·617951	−0·188158	+0·213530	+0·450351	+0·278330	−0·171325	−0·707236	$\times 10^{-2}$
+0·036124	−0·038240	−0·063002	+0·025661	+0·153846	+0·001829	−0·260760	
+1·294623	+0·360306	−0·446701	−0·861991	−0·602705	+0·357653	+1·433145	
+2·926065	+0·930321	−0·860407	−1·981125	−1·792212	+0·378439	+5·068922	

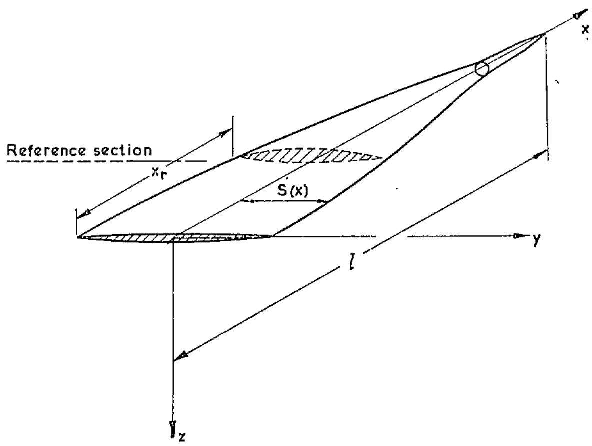


FIG. 1. Typical slender configuration.

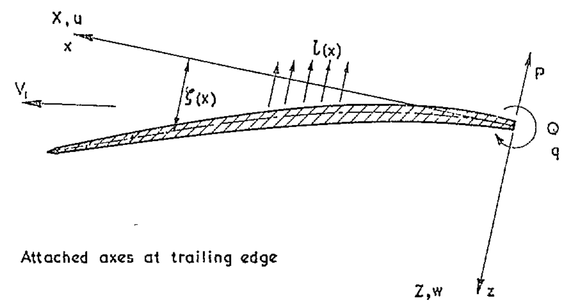


FIG. 2. Typical slender configuration.

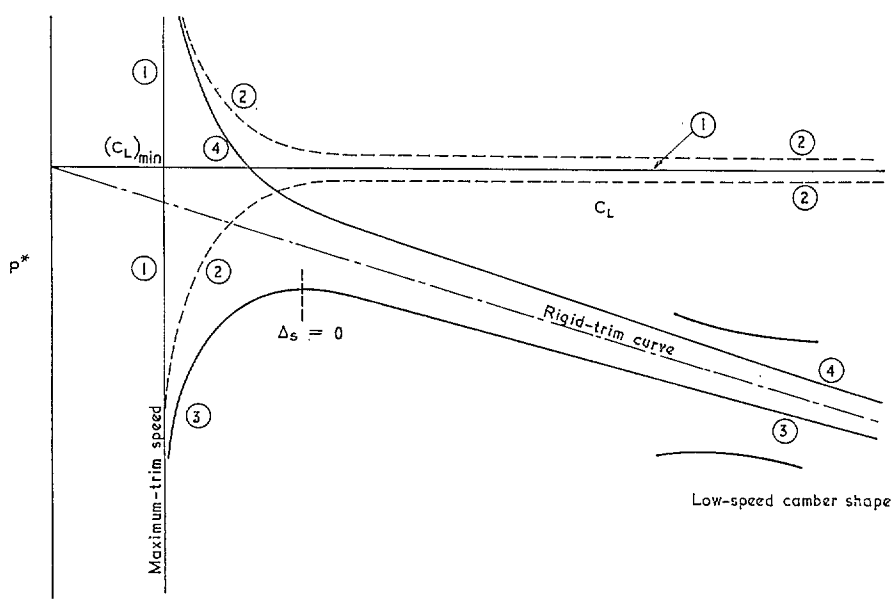


FIG. 3. Trim curves for uncambered aeroplane.

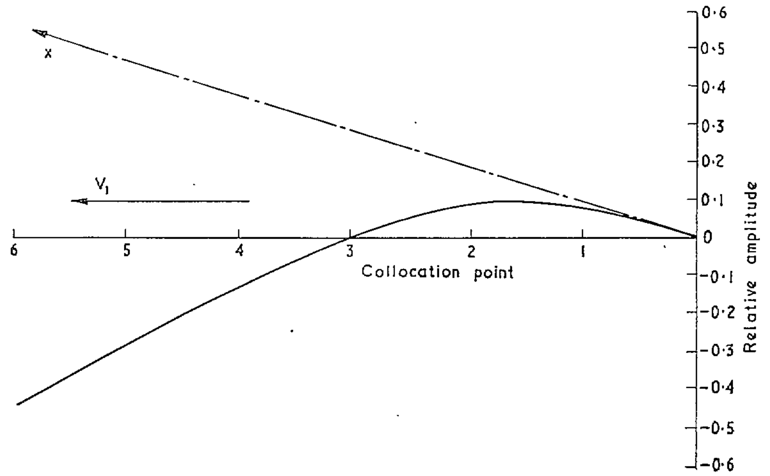


FIG. 4. Configuration at maximum trim speed.

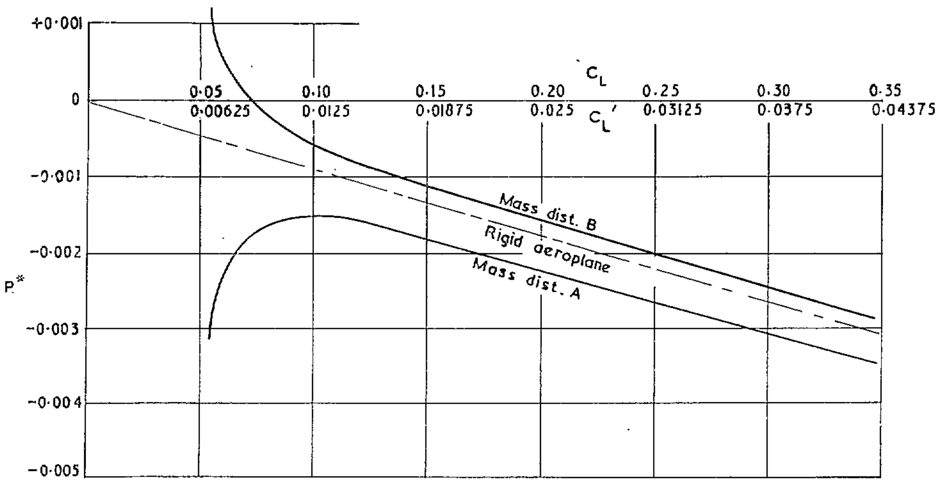


FIG. 5. Level-flight-trim curve.

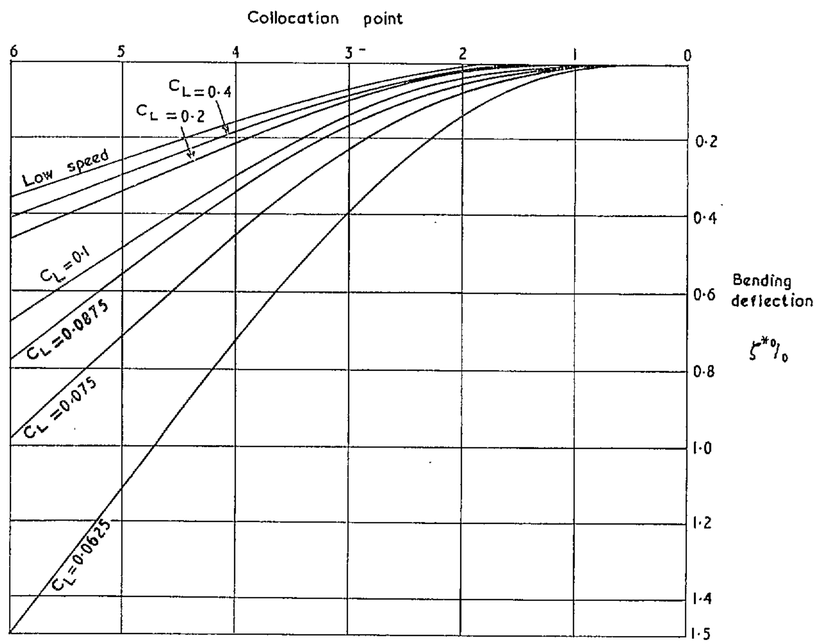


FIG. 6a. Configuration in level flight—mass distribution A.

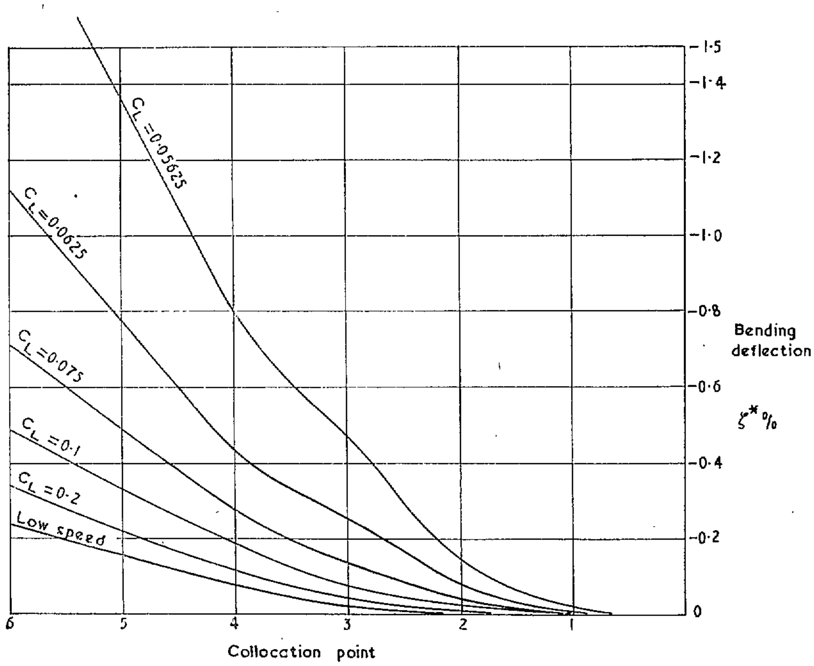


FIG. 6b. Configuration in level flight—mass distribution B.

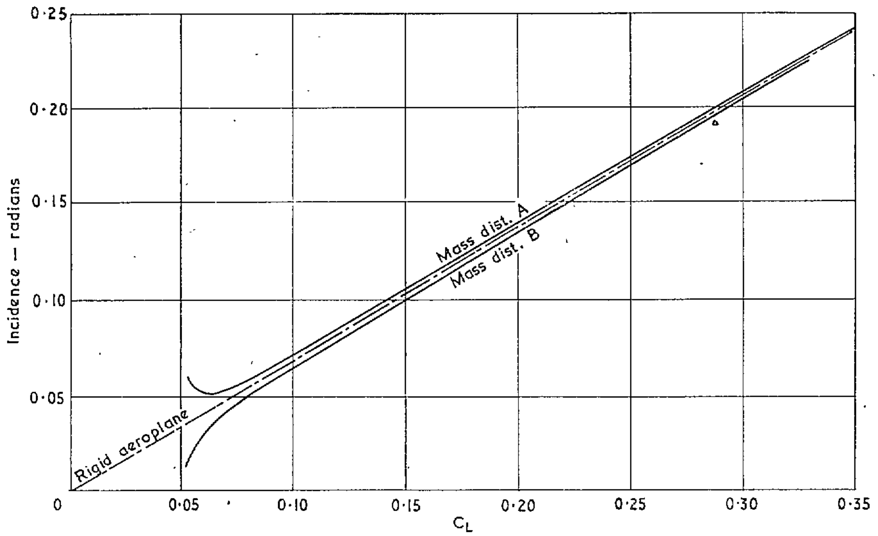
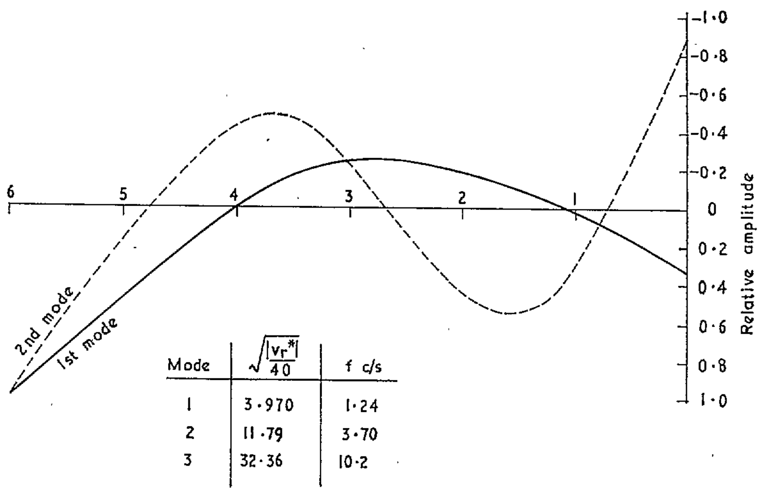
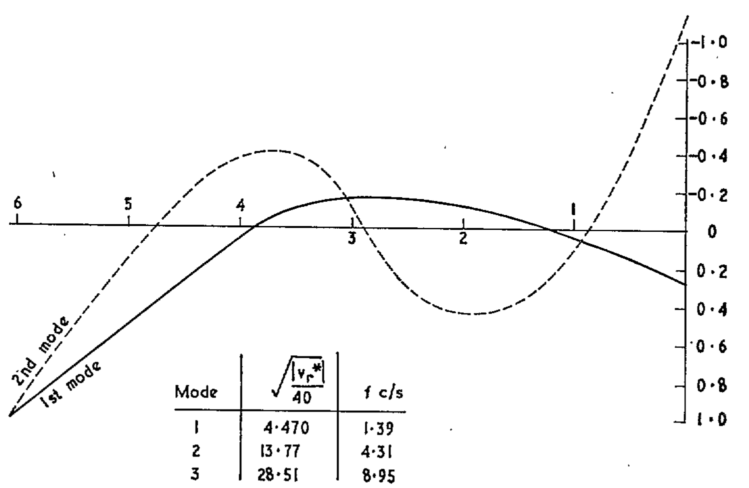


FIG. 7. Incidence in level flight.



Mode	$\sqrt{\frac{\lvert v_r^* \rvert}{40}}$	f c/s
1	3·970	1·24
2	11·79	3·70
3	32·36	10·2

FIG. 8a. Normal modes—mass distribution A.



Mode	$\sqrt{\frac{\lvert v_r^* \rvert}{40}}$	f c/s
1	4·470	1·39
2	13·77	4·31
3	28·51	8·95

FIG. 8b. Normal modes—mass distribution B.

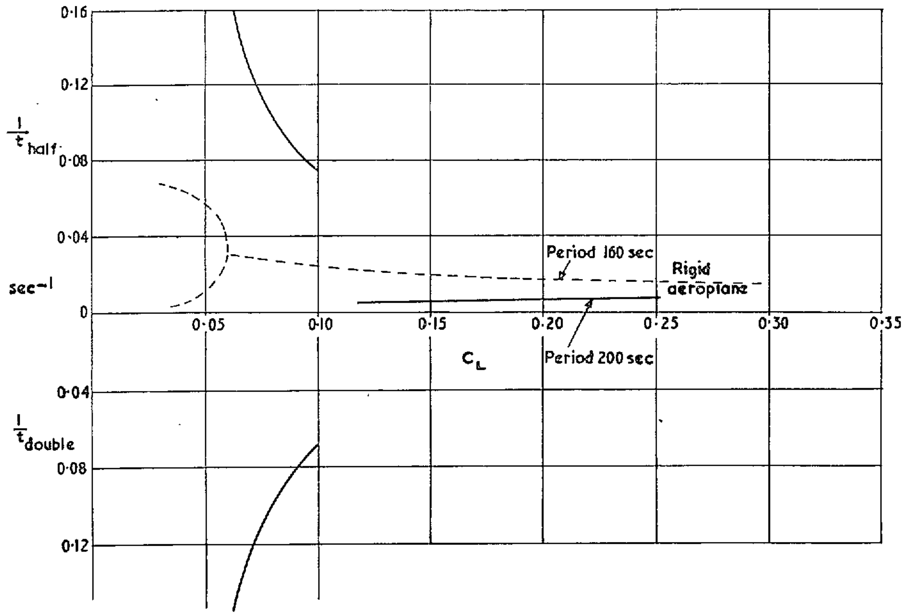


FIG. 9a. 'Phugoid' mass distribution A.

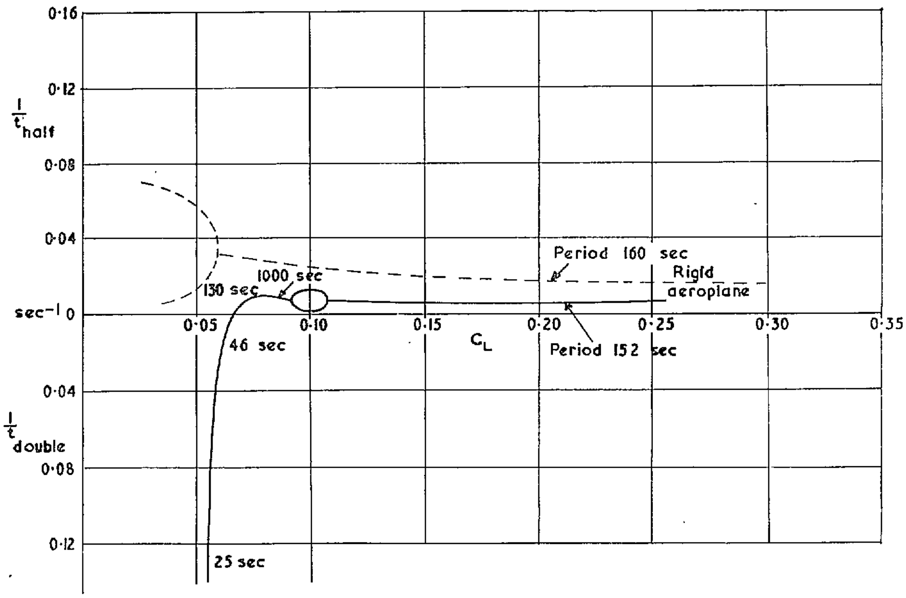


FIG. 9b. 'Phugoid' mass distribution B.

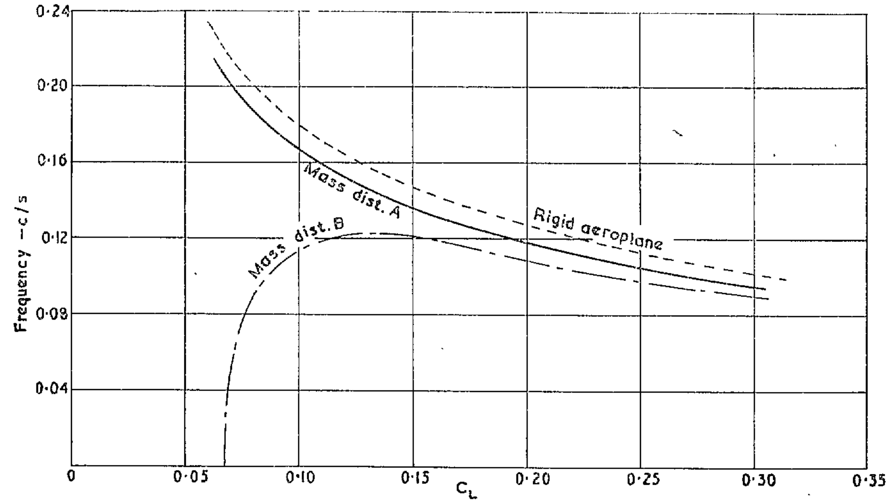


FIG. 10a. 'Short period' frequency.

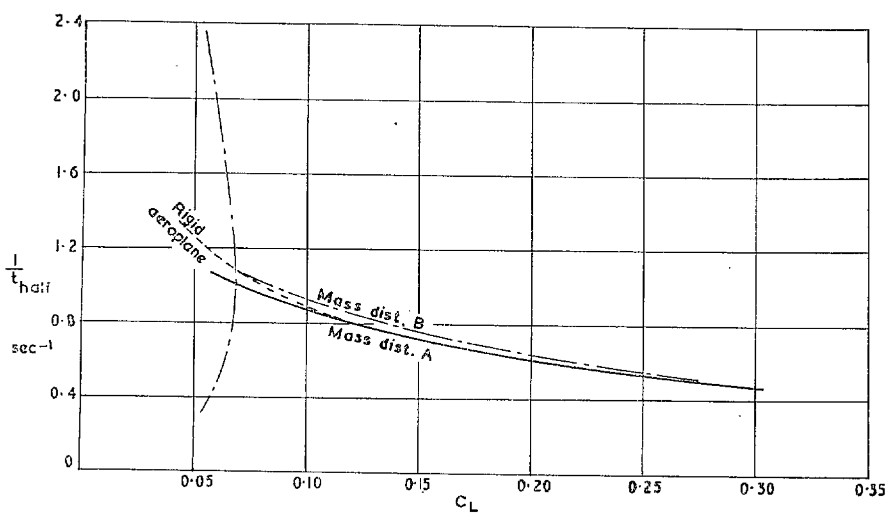


FIG. 10b. 'Short period' damping.

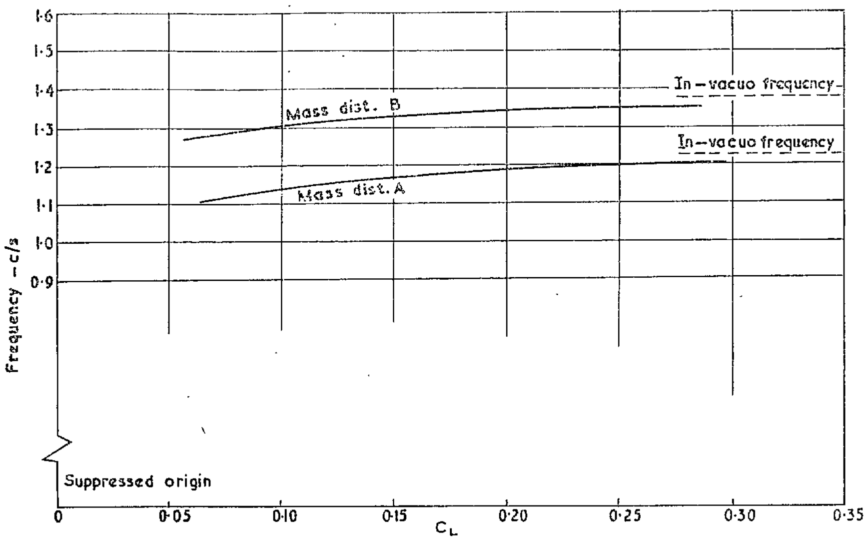


FIG. 11a. 'First normal mode' frequency.

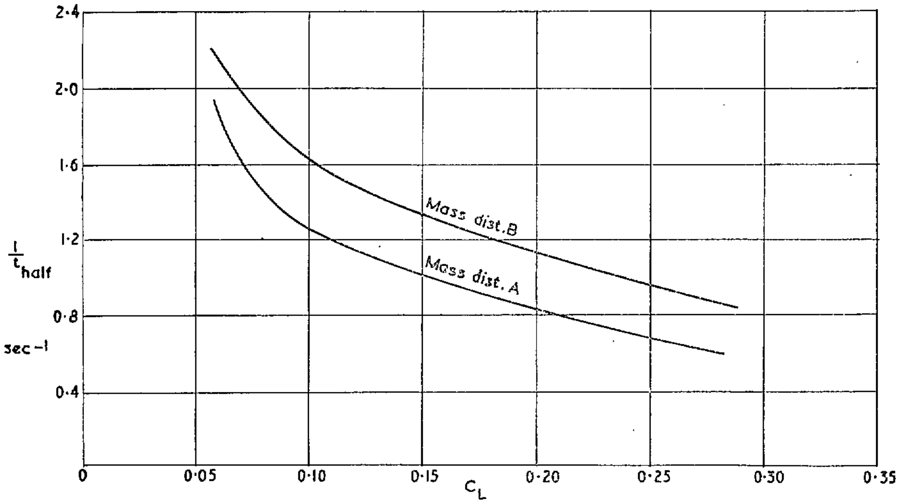


FIG. 11b. 'First normal mode' damping.

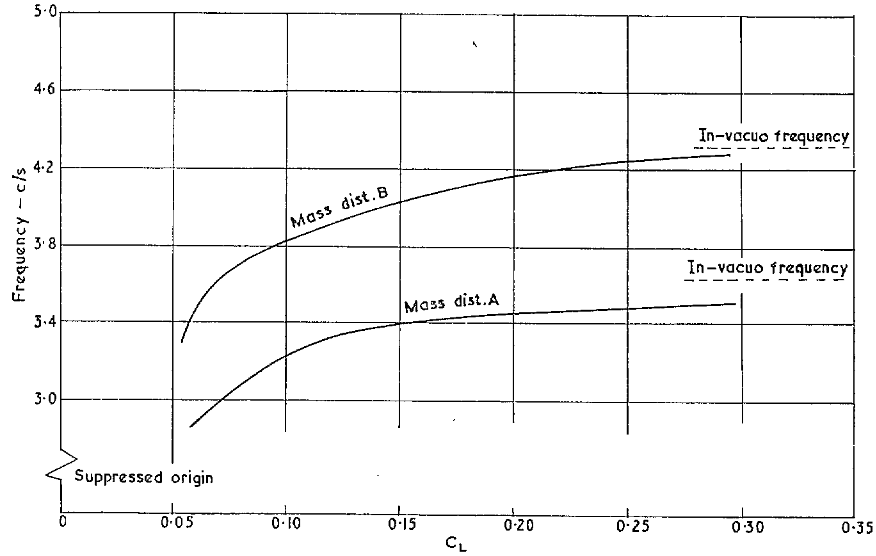


FIG. 12a. 'Second normal mode' frequency.

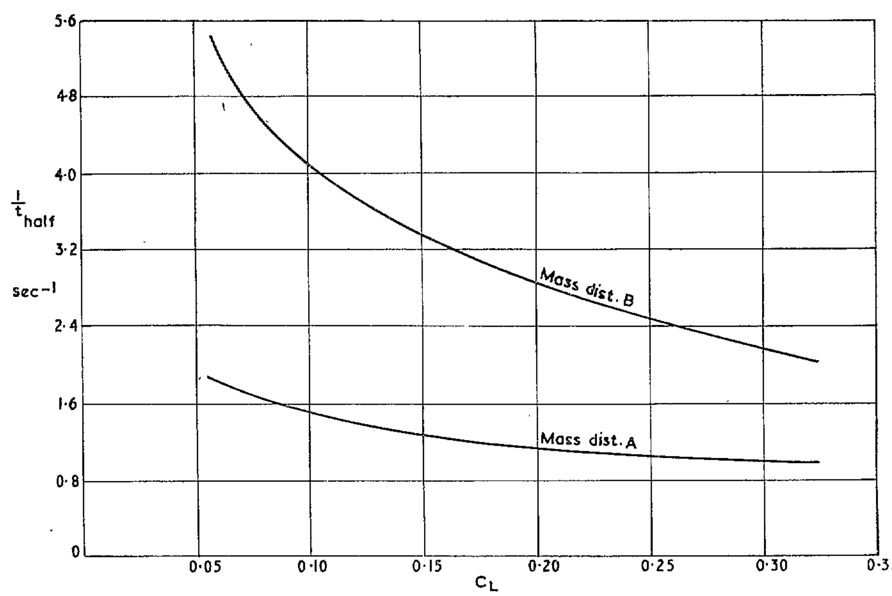


FIG. 12b. 'Second normal mode' damping.

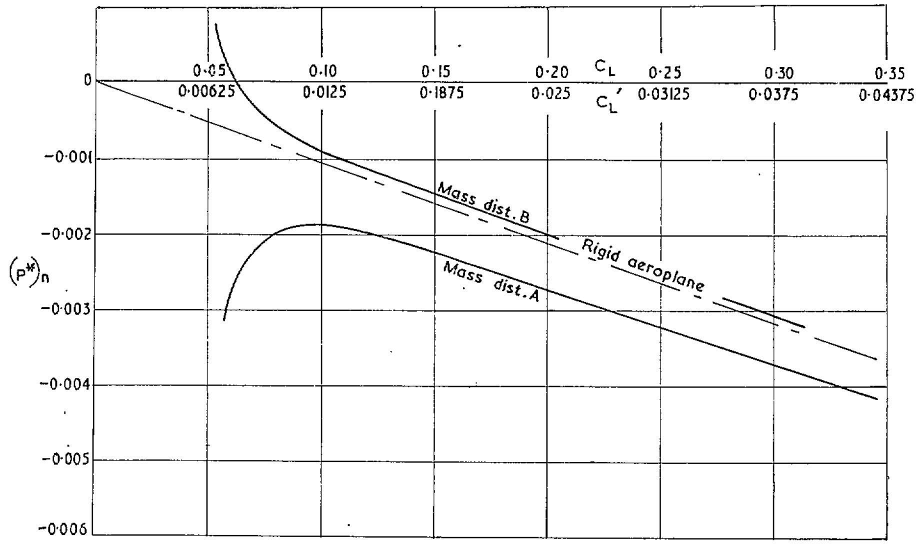


FIG. 13. Control coefficient per *g*.

(88240) Wt. 65/1418 K.5 1/64 Hw.

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