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The Airforces on the Low-Aspect-Ratio Rectangular Wing Oscillating in Sonic Flow

By D. E. Davies

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The Airforces on the Low-Aspect-Ratio Rectangular Wing Oscillating in Sonic Flow

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Summary.

Approximate expressions for the generalised airforces acting on a rectangular wing of low aspect ratio oscillating harmonically in sonic flow at low frequencies are derived in this paper. The modes of oscillation considered are rigid modes and a small selection of flexible modes. Results are presented as the first few terms of infinite expansions.

A brief description of the modes of oscillation and of the generalised airforces is given towards the end of the paper so that the results may be used without the main text of the paper having to be read.

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1. *Introduction.*

The reliable prediction of wing flutter speeds involves the proper assessment of the airforces acting on the wing when it is oscillating in various modes. The transonic speed regime is often the most critical from the flutter standpoint but it is also the regime in which theoretical assessment of the airforces is most difficult. However, with certain restrictions (*see* Ref. 1) on the thickness ratio, aspect ratio and frequency parameter it is possible to deal with wings of particular shapes in this regime. This paper is concerned with the assessment of these airforces for a low-aspect-ratio rectangular wing oscillating in the transonic regime.

Mangler[2] and Landahl[1] have considered the low-aspect-ratio delta wing; Miles[3], using a method different from that of the present paper, has treated the low-aspect-ratio rectangular wing. Since the work of this paper was completed, Landahl[4] has given a treatment of low-aspect-ratio rectangular wings, which, although similar to that of this paper, differs in detail and produces slightly different results. The results of this paper and of Landahl's contain terms of higher order in frequency than do those of Mangler[2] and Miles[3].

The procedure is first outlined to draw attention to the assumptions and approximations that underlie the solution. Approximate expressions are derived for generalised airforces acting on the low-aspect-ratio wing oscillating harmonically in sonic flow at low frequencies. The modes of oscillation considered are rigid modes and a small selection of flexible modes. Results are presented as the first few terms of infinite expansions. A final resumé permits the results to be used without the main text having to be read.

2. *Procedure.*

With certain restrictions on the thickness ratio, aspect ratio, and frequency parameter, it can be shown[1] that the partial differential equation governing the oscillatory flow about a wing may be linearised even when the main-stream speed is sonic relative to the mean position of the oscillating wing.

The procedure of this paper is to take Fourier transforms of the velocity-potential function and other appropriate functions with respect to distance in the main-stream flow direction so that the governing partial differential equation is transformed into a simpler one. This simpler differential equation together with the transforms of the boundary conditions are then replaced by an integral equation, originally obtained by Landahl[1], which relates the transform of the known upwash on the wing surface with the transform of the velocity potential on the wing surface. It does not seem likely that an exact analytical solution of this equation can be obtained so an approximate solution must be sought. If the Hankel function appearing in the kernel of the integral equation is expanded in a series and term by term integration is allowed, then a solution of the integral equation valid for small values of the transformed variable u may be obtained by the method described in Section 4.

As it has not been possible to obtain the solution of the integral equation over the whole of the u-plane certain assumptions about the behaviour of the solution at large distances from the origin have to be made in order to make any progress. These assumptions are stated where they occur in the text. When the flow is steady these assumptions can be readily justified, but for oscillatory flow it is not possible to do this and they then remain intuitive assumptions. Using these assumptions it is found in Section 4 of this paper that, with certain restrictions, an approximation to the original functions can be obtained from the approximation to the transformed function valid for small values of u only.

Fourier transforms of generalised airforces may be given as simple expressions in the transforms of the velocity potentials on the wing surface, so that once the expressions for the velocity potential on the wing surface, valid for small values of u, have been obtained then the expressions for the transforms of the generalised airforces, valid for small values of u, can be obtained immediately. The generalised airforces themselves can then be obtained without having to obtain the velocity potentials themselves on the wing surface.

The derivation of one particular generalised airforce is given in detail in Section 5, and an indication of how others can be obtained is also given. Derivations of other generalised forces are usually more complicated than the one given in Section 5, so an example of a more complicated derivation is given in Appendix II.

Results are given in Section 6 for generalised airforces associated with the modes described in Section 5, and a discussion of the validity of these results is given in Section 7. A brief description of the modes of oscillation and of the generalised airforces is repeated in Section 8 so that the results may be used without the main text of the paper having to be read.

3. *An Integral Equation for Sonic Flow.*

When the flow about a wing oscillating in an airstream is governed by linear equations, we can, by the principle of superposition separate the steady and unsteady parts of the disturbed motion created by the presence of the wing. For flutter purposes we are interested in the unsteady part of the motion only. This is independent of the thickness distribution of the wing (provided always that the thickness parameters are small enough for linearisation to be permissible), so for its analysis the wing may be considered to be a flat plate in its undisturbed state, and in this undisturbed state its surface is parallel to the main-stream flow.

A Cartesian co-ordinate system (x, y, z) as shown in Fig. 1 is chosen fixed relative to the undisturbed position of the wing, with x-axis in the direction of the undisturbed flow, y-axis to starboard in the plane of the undisturbed flow, and z-axis vertically upwards to complete a right-handed system. The origin is taken at the undisturbed position of the mid-point of the leading edge.

In linearised theory, the velocity potential $\phi(x, y, z, t)$ satisfies the second-order linear partial differential equation

$$\frac{1}{a^2}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 \phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{1}$$

and the linear boundary condition of tangential flow over the wing surface

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = W(x, y, t) = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right) Z(x, y, t) \tag{2}$$

where $Z(x, y, t)$ represents the vertical displacement of a point (x, y) on the wing at time t.

If we assume that conditions are varying harmonically with time we can take

$$\phi(x, y, z, t) = \varphi(x, y, z)e^{i\omega t} \tag{3}$$

$$W(x, y, t) = w(x, y)e^{i\omega t}. \tag{4}$$

Also when the main-stream speed is sonic, as is here the case, we have $V = a$. The governing partial differential equation then becomes

$$\varphi_{yy} + \varphi_{zz} - 2\frac{i\omega}{a}\varphi_x + \frac{\omega^2}{a^2}\varphi = 0 \tag{5}$$

and the boundary condition becomes

$$Z_x + \frac{i\omega}{a}Z = \frac{w(x, y)}{a} \tag{6}$$

on the wing planform.

Since the flow is sonic no disturbances are transmitted from the wing forward of the leading edge, hence

$$\varphi(x, y, 0) = \varphi_Z(x, y, 0) = 0, \quad x < 0 \tag{7}$$

and also there is no influence from the wake on the potential on the surface of the wing so that the velocities in the wake may be replaced by other velocities which are more convenient for our purposes without affecting results, and this will be done to simplify some of the analysis.

Outside the wing planform the pressure across the plane $z = 0$ is continuous. Since the velocity potential is continuous across this plane except over the wing planform and wake we must have

$$\varphi(x, y, 0) = 0 \qquad \text{for } |y| \geqslant b. \tag{8}$$

The solution of the partial differential equation (5) under the conditions (6), (7) and (8) will be obtained with the aid of Fourier Integral analysis.

If, for u in some domain of the complex u-plane, the integral

$$\bar{g}(u; y, z) = \int_{-\infty}^{+\infty} e^{iux} g(x, y, z) dx \tag{9}$$

is absolutely integrable, then $\bar{g}(u; y, z)$ is the Fourier Integral of $g(x, y, z)$ with respect to x. The function $\bar{g}(u; y, z)$ can usually be continued analytically to a larger domain of the u-plane than that for which the integral (9) is absolutely integrable. Similar definitions hold for functions with a different number of variables.

In the applications which follow the functions $g(x, y, z)$ will be zero for $x < 0$, and the domain in which the integral (9) is absolutely integrable will be an upper half-plane $\mathrm{I}(u) \geqslant$ a constant. In some cases it will be quite evident that $\bar{g}(u; y, z)$ may be continued analytically to the whole complex u-plane except for isolated singularities. In other cases the $\bar{g}(u; y, z)$ will be known by means of infinite series only in a restricted area, and then it will be assumed that analytic continuation to the whole complex u-plane excluding isolated singularities is permissible.

On multiplying equations (5), (6) and (7) by e^{iux}, integrating from $-\infty$ to $+\infty$, and simplifying we get respectively

$$\bar{\varphi}_{yy}(u; y, z) + \bar{\varphi}_{zz}(u; y, z) + \left(\frac{\omega^2}{a^2} - 2\frac{\omega}{a}u\right)\bar{\varphi}(u; y, z) = 0 \tag{10}$$

$$\bar{\varphi}_z(u; y, z) = \bar{w}(u; y) \quad |y| \leqslant b \tag{11}$$

$$\bar{\varphi}(u; y, 0) = 0 \qquad |y| \geqslant b. \tag{12}$$

To obtain the Fourier Integral $\bar{w}(u; y)$ the function $w(x, y)$ must be assumed in the wake. If $w(x, y)$ is a polynomial in x and y on the wing, then it is convenient to extend this polynomial to the region of the wake.

A solution of (10) in $z > 0$ which vanishes as $z \to \infty$ when u is in the upper half-plane is found in Appendix I to be

$$\bar{\varphi}(u; y, z) = -i\left(\frac{\omega^2}{a^2} - 2\frac{w}{a}u\right)^{1/2} \int_{-b}^{+b} \frac{z}{\bar{r}} H_1^{(2)}\left\{\bar{r}\left(\frac{\omega^2}{a^2} - 2\frac{\omega}{a}u\right)^{1/2}\right\} \bar{\varphi}(u; \eta, +0) d\eta \tag{13}$$

where

$$\bar{r} = \sqrt{\{(y-\eta)^2 + z^2\}}. \tag{14}$$

To make $\{(\omega^2/a^2) - 2(\omega/a)u\}^{1/2}$ single valued a cut must be introduced into the u-plane from $u = \omega/2a$ to infinity. Since the Fourier Integral $\bar{\varphi}$ is regular in an upper half-plane $\mathrm{I}(u) \geqslant$ a constant the cut must not enter the upper half-plane. It will be inserted between the point $u = \omega/2a$ and infinity along a straight line parallel to the negative imaginary axis. If the branch of $\{(\omega^2/a^2) - (2\omega/a)u\}^{1/2}$ is obtained by putting

$$u = \frac{\omega}{2a} - qe^{i\theta}, \quad -\frac{3\pi}{2} < \theta < \frac{\pi}{2} \tag{15}$$

then in the upper half u-plane the imaginary part of $\{(\omega^2/a^2) - (2\omega/a)u\}^{1/2}$ is negative.

The function $\hat{H}_1^{(2)}$ is defined by

$$\hat{H}_1^{(2)}\left\{\bar{r}\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\right\} = H_1^{(2)}\left\{\bar{r}\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\right\},\ \mathrm{I}(u) > 0 \tag{16}$$

and by analytic continuation in the whole cut u-plane, where $H_1^{(2)}$ is Hankel's function of the first order and first kind.

Differentiating equation (13) with respect to z and proceeding to the limit $z = +0$ we obtain the integral equation

$$\bar{\varphi}_z(u; y, 0) = \bar{w}(u; y) = -\frac{i}{2}\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\int_{-b}^{+b}\frac{\hat{H}_1^{(2)}\left\{|y-\eta|\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\right\}}{|y-\eta|}\bar{\varphi}(u; \eta, +0)d\eta. \tag{17}$$

An expanded form of this integral equation is obtained if the function

$$\hat{H}_1^{(2)}[|y-\eta|\{(\omega^2/a^2)-(2\omega/a)u\}^{1/2}]$$

is replaced by a series expansion and integration carried out term by term.

The Hankel function $H_1^{(2)}(\lambda)$ of the complex variable λ may be expressed as

$$H_1^{(2)}(\lambda) = \frac{1}{\lambda}\alpha(\lambda^2) + \lambda\log\left(\frac{\lambda}{2}\right)\beta(\lambda^2) \tag{18}$$

where $\alpha(\lambda^2)$ and $\beta(\lambda^2)$ are integral functions of the complex variable λ^2 the expansions of which as power series in λ^2 are well known. The logarithm is, as is usual, real on the positive real axis of the complex λ-plane and has a branch line along the negative real axis.

In the upper half u-plane we therefore have

$$\begin{aligned}&\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\hat{H}_1^{(2)}\left\{|y-\eta|\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\right\}\\ &= \frac{1}{|y-\eta|}\alpha\left\{(y-\eta)^2\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)\right\} + |y-\eta|\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)\times\\ &\times\left\{\log|y-\eta| + \tfrac{1}{2}\log\left(\frac{\omega^2}{4a^2}-\frac{\omega u}{a}\right)\right\}\beta\left\{(y-\eta)^2\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)\right\}.\end{aligned} \tag{19}$$

Let the analytic continuation of $\log\{(\omega^2/4a^2)-(\omega u/a)\}$ to the whole cut u-plane be called $\mathrm{Log}\,\{(\omega^2/4a^2)-(\omega u/a)\}$. Using the expression (15) for u, this is defined as

$$\mathrm{Log}\left(\frac{\omega^2}{4a^2}-\frac{\omega u}{a}\right) = \log\left(\frac{\omega q}{2a}\right) + i\theta,\ -\frac{3\pi}{2} < \theta < \frac{\pi}{2}. \tag{20}$$

Then for the complete cut u-plane

$$\begin{aligned}&\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\hat{H}_1^{(2)}\left\{|y-\eta|\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)^{1/2}\right\}\\ &= \frac{1}{|y-\eta|}\alpha\left\{(y-\eta)^2\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)\right\} + |y-\eta|\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)\times\\ &\times\left\{\log|y-\eta| + \tfrac{1}{2}\mathrm{Log}\left(\frac{\omega^2}{4a^2}-\frac{\omega u}{a}\right)\right\}\beta\left\{(y-\eta)^2\left(\frac{\omega^2}{a^2}-\frac{2\omega}{a}u\right)\right\}.\end{aligned} \tag{21}$$

Substituting (21) into (17) and using the power-series expansions of α and β defined in equation (18) we get

$$\overline{w}(u, y) = \frac{1}{\pi}\int_{-b}^{+b} \frac{\bar{\varphi}(u;\, \eta,\, +0)}{(y-\eta)^2}\, d\eta -$$

$$-\frac{1}{\pi}\left[\left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a}-u\right)\right]\left\{\left(\mathrm{Log}\left[\left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a}-u\right)\right] + 2\gamma + \pi i - 1\right) \times\right.$$

$$\left.\times \int_{-b}^{+b} \bar{\varphi}(u;\, \eta,\, +0)d\eta + 2\int_{-b}^{+b} \bar{\varphi}(u;\, \eta,\, +0) \log|y-\eta| d\eta\right\} +$$

$$+\frac{1}{2\pi}\left[\left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a}-u\right)\right]^2\left\{\left(\mathrm{Log}\left[\left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a}-u\right)\right] + 2\gamma + \pi i - \frac{5}{2}\right) \times\right.$$

$$\left.\times \int_{-b}^{+b} \bar{\varphi}(u;\, \eta,\, +0)(y-\eta)^2 d\eta + 2\int_{-b}^{+b} \bar{\varphi}(u;\, \eta,\, +0)(y-\eta)^2 \log|y-\eta| d\eta\right\} -$$

$$-\frac{1}{2\pi}\left[\left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a}-u\right)\right]^3\left\{\left(\mathrm{Log}\left[\left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a}-u\right)\right] + 2\gamma + \pi i - \frac{10}{3}\right) \times\right.$$

$$\left.\times \int_{-b}^{+b} \bar{\varphi}(u;\, \eta,\, +0)(y-\eta)^4 d\eta + 2\int_{-b}^{+b} \psi(u;\, \eta,\, +0)(y-\eta)^4 \log|y-\eta| d\eta\right\} + \dots \quad (22)$$

4. *Solution of the Integral Equation.*

Corresponding to the Fourier Integral formula (9) there is the Inverse Fourier Integral formula

$$g(x, y, z) = \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N+i\epsilon}^{N+i\epsilon} \bar{g}(u;\, y,\, z) e^{-iux} du \quad (23)$$

where $\epsilon > 0$ is a real constant such that $\bar{g}(u; y, z)$ is a regular function for $\mathrm{I}(u) > \epsilon$. The path of integration is along a straight line parallel to the real axis.

A more powerful form of inverse formula may be obtained by applying contour integration in the complex u-plane. The function $\bar{g}(u; y, z)$ is assumed to have a branch point at $u = \omega/2a$.

The contour of integration is shown in Fig. 3.

A, G, C and B lie on a circle of radius $R\{= \sqrt{(N^2+\epsilon^2)}\}$ and centre $u = 0$.

D, E and F lie on a circle of arbitrarily small radius δ and centre at the branch point $u = \omega/2a$.

GF and CD are two lines, indefinitely close together, but separated by a slit in the u-plane.

If we apply Cauchy's theorem of residues to the integral of $\bar{g}(u; y, +0)e^{-iux}$ around the contour, we obtain

$$\frac{1}{2\pi}\int_{-N+i\epsilon}^{N+i\epsilon} \bar{g}(u;\, y,\, z)e^{-iux}du = \frac{1}{2\pi}\left\{\int_{\mathrm{arc\,AG}} + \int_{\mathrm{GF}} + \int_{\mathrm{arc\,FED}} +\right.$$

$$\left. + \int_{\mathrm{DC}} + \int_{\mathrm{arc\,CB}} \bar{g}(u;\, y,\, z)e^{-iux}du\right\} -$$

$$- i\sum_{s} R_s(x, y, z) \quad (24)$$

where $R_s(x, y, z)$ is the residue of $\bar{g}(u, y, z)e^{-iux}$ at any pole $u = u_s$ which may occur inside the contour. Any poles on the slit could be accounted for by deforming the lines GF and DC appropriately near these poles.

The transform functions $\bar{g}(u, y, z)$ will be replaced later by the functions $\bar{Q}_{j,k}(u)$ defined in equation (60). In the case of steady flow these functions can be obtained exactly and tend to zero on the large circle as $R \to \infty$, so that

$$\lim_{R \to \infty} \left\{ \int_{AG} + \int_{CB} \bar{g}(u; y, z) e^{-iux} du \right\} = 0 \tag{25}$$

when $x > 0$.

We make the assumption that the limit formula (25) still holds when the flow is oscillatory.

Proceeding, therefore, to the limit $R = \infty$ in formula (24) and writing

$$u = \frac{\omega}{2a} - qe^{i\theta}$$

we obtain

$$g(x, y, z) = -\frac{i}{2\pi} e^{-i\omega x/2a} \int_{\delta}^{\infty} \left[\bar{g}\left(\frac{\omega}{2a} - qe^{-3i\pi/2}; y, z\right) - \bar{g}\left(\frac{\omega}{2a} - qe^{i\pi/2}; y, z\right) \right] e^{-qx} dq +$$

$$+ \frac{i\delta}{2\pi} e^{-i\omega x/2a} \int_{-3\pi/2}^{\pi/2} \bar{g}\left(\frac{\omega}{2a} - \delta e^{i\theta}; y, z\right) e^{ix\delta e^{i\theta}} e^{i\theta} d\theta - i \sum_{s} R_s(x, y, z). \tag{26}$$

The first integral on the right-hand side of (26) is very powerfully convergent for large positive x and it is assumed that it may be used when $\bar{g}$ is given by an infinite expansion to give a result as an asymptotic expansion valid for large positive x.

We solve equation (22) by using the procedure of Stewartson[6], rather than that of Adams and Sears[7] since the former procedure reveals the existence of poles, which the latter does not. Corresponding to the Fourier Integral of the symmetric normal velocity

$$\bar{w}(u; y) = \sum_{r=0}^{\infty} A_r(u) \frac{y^{2r}}{b^{2r}} \tag{27}$$

the Fourier Integral $\bar{\varphi}(u; y, +0)$ of the surface velocity potential is given by

$$\bar{\varphi}(u; y, +0) = \sum_{r=0}^{\infty} \frac{B_r(u)}{b^{2r}} (b^2 - y^2)^{r+1/2} \tag{28}$$

as substitution in (22) will show. If the series (27) and (28) are substituted into equation (22), and the coefficients of all the different powers of y compared, an infinite set of simultaneous equations for the $B_r(u)$ results. An approximate solution of this set is obtained by taking the first few equations only (N in number say) and assuming that all but the first N of the $B_r(u)$ are negligibly small. By solving this finite set of equations the $B_r(u)$ are obtained as the ratio of a numerator and denominator function, the denominator function being the same for all the $B_r(u)$. The numerator and denominator functions so obtained are only approximations to the actual numerator and denominator functions and represent only the first few terms of infinite series expansions. The effect of increasing N is to add higher-order terms to these series expansions without altering the lower-order terms.

The poles of $\bar{\varphi}(u; y, +0)$ are the zeros of the denominator function, and with only an approximation to this function available it is possible to determine the position of the pole nearest the origin only.

If the first two terms only of (27) are non-zero, and if we take $N = 4$, then we obtain the following expressions for the first four of the $B_r(u)$:

$$B_0(u) = \left\{-1 + 2\Theta - \Theta^2(\psi+\tfrac{1}{2}) + 2\Theta^3\left(\psi - \frac{17}{18}\right) + \ldots\right\}\frac{A_0(u)}{\Delta_0(\Theta)} +$$

$$+\frac{1}{2}\left\{-1 - \Theta(\psi-3) + \Theta^2\left(\psi - \frac{5}{2}\right) - \frac{1}{4}\Theta^3\left(\psi - \frac{119}{36}\right) + \ldots\right\}\frac{A_1(u)}{\Delta_0(\Theta)} \tag{29}$$

$$B_1(u) = \frac{4}{3}\left\{-\Theta + \Theta^2(\psi+1) - 3\Theta^3\left(\psi - \frac{1}{2}\right) + \ldots\right\}\frac{A_0(u)}{\Delta_0(\Theta)} +$$

$$+\frac{1}{3}\left\{1 + 2\Theta\left(\psi - \frac{5}{2}\right) - 3\Theta^2\left(\psi - \frac{13}{6}\right) + \Theta^3\left(\psi - \frac{67}{18}\right) + \ldots\right\}\frac{A_1(u)}{\Delta_0(\Theta)} \tag{30}$$

$$B_2(u) = \frac{4}{5}\left\{-\Theta^2 + \frac{4}{3}\Theta^3\left(\psi - \frac{1}{2}\right) + \ldots\right\}\frac{A_0(u)}{\Delta_0(\Theta)} +$$

$$+\frac{2}{15}\left\{\Theta + 2\Theta^2\left(\psi - \frac{5}{2}\right) - \Theta^3\left(\psi - \frac{29}{6}\right) + \ldots\right\}\frac{A_1(u)}{\Delta_0(\Theta)} \tag{31}$$

$$B_3(u) = \frac{16}{63}\left\{-\Theta^3 + \ldots\right\}\frac{A_0(u)}{\Delta_0(\Theta)} + \frac{4}{105}\left\{\Theta^2 + 2\Theta^3(\psi-2) + \ldots\right\}\frac{A_1(u)}{\Delta_0(\Theta)}, \tag{32}$$

where

$$\Delta_0(\Theta) = 1 + 2\Theta(\psi-2) - 4\Theta^2\left(\psi - \frac{5}{4}\right) + \frac{14}{3}\Theta^3(\psi-1) + \ldots \tag{33}$$

$$\Theta = \left(\frac{\omega}{2a}\right)\left(\frac{\omega}{2a} - u\right)\left(\frac{b^2}{4}\right) \tag{34}$$

$$\psi = \operatorname{Log}\Theta + 2\gamma + \pi i. \tag{35}$$

The zero of $\Delta_0(\Theta)$ nearest the origin is at

$$\Theta_0 = 0\cdot0482 + i0\cdot0956$$

and corresponding to this zero the $B_r(u)$ will have a pole at

$$u_0 = \frac{\omega}{2a} - \frac{8a}{\omega b^2}\Theta_0.$$

Corresponding to the Fourier Integral of the antisymmetric normal velocity

$$\bar{w}(u; y) = \sum_{r=0}^{\infty} C_r(u)\frac{y^{2r+1}}{b^{2r+1}} \tag{36}$$

the Fourier Integral $\bar{\varphi}(u; y, +0)$ of the surface velocity potential is given by

$$\bar{\varphi}(u; y, +0) = \sum_{r=0}^{\infty} \frac{D_r(u)}{b^{2r+1}}\, y(b^2-y^2)^{r+1/2} \tag{37}$$

where, following the procedure for the symmetric case and assuming only the first term in (36) non-zero and $N = 4$, it is found that

$$D_0(u) = \frac{1}{2}\left\{ -1 + \Theta - \frac{1}{3}\Theta^3\left(\psi - \frac{1}{12}\right) + \ldots \right\} \frac{C_0(u)}{\Delta_1(\Theta)}, \tag{38}$$

$$D_1(u) = \frac{1}{3}\left\{ -\Theta + \Theta^3\left(\psi + \frac{1}{4}\right) + \ldots \right\} \frac{C_0(u)}{\Delta_1(\Theta)}, \tag{39}$$

$$D_2(u) = \frac{2}{15}\left\{ -\Theta^2 - \frac{1}{2}\Theta^3 + \ldots \right\} \frac{C_0(u)}{\Delta_1(\Theta)}, \tag{40}$$

$$D_3(u) = \frac{2}{63}\left\{ -\Theta^3 + \ldots \right\} \frac{C_0(u)}{\Delta_1(\Theta)}, \tag{41}$$

where

$$\Delta_1(\Theta) = 1 - 2\Theta + \Theta^2\left(\psi + \frac{1}{2}\right) - 2\Theta^3\left(\psi - \frac{5}{6}\right) + \ldots . \tag{42}$$

The zero of $\Delta_1(\Theta)$ nearest the origin is at

$$\Theta_1 = 0\cdot 2901 + i0\cdot 3151 \tag{43}$$

and corresponding to this zero, the $D_r(u)$ will have a pole at

$$u_1 = \frac{\omega}{2a} - \frac{8a}{\omega b^2}\Theta_1 . \tag{44}$$

5. *The Generalised Forces Associated with Different Modes of Oscillation.*

Let the displacement of a point (x, y) on the surface of the wing at time t be given by the second-degree polynomial equation in x and y the coefficients of which are functions of time

$$Z(x, y, t) = q_1(t) + q_2(t)x + q_3(t)y + q_4(t)x^2 + q_5(t)xy + q_6(t)y^2 . \tag{45}$$

The q's may be regarded as generalised co-ordinates for the wing motion. If the displacement varies harmonically about the undisturbed state with circular frequency ω, we may write

$$q_j = q_{j0}e^{i\omega t} \qquad j = 1, 2, \ldots, 6 \tag{46}$$

where the q_{j0}'s are constants which may be complex.

Since the governing equations are linear, the potential about a wing oscillating in the form defined by (45) and (46) may be written

$$\phi = \varphi e^{i\omega t} = \sum_{j=1}^{6} \varphi_j q_{j0} e^{i\omega t} \tag{47}$$

where $\varphi_j e^{i\omega t}$ is the velocity potential associated with the displacement

$$Z_j = f_j(x, y)e^{i\omega t} \tag{48}$$

and $f_j(x, y)$ is the coefficient of $q_j(t)$ in equation (45).

Corresponding to the displacements Z_j of equation (48) there are the normal velocity distributions on the wing surface

$$w_j(x, y)e^{i\omega t} = \left(V\frac{\partial f_j}{\partial x} + i\omega f_j\right) e^{i\omega t}, \tag{49}$$

and the Fourier Integrals of the w_j are

$$\overline{w}_1(u; y) = -\frac{i\omega}{iu} \tag{50}$$

$$\overline{w}_2(u; y) = -\frac{1}{iu}\left(a - \frac{i\omega}{iu}\right) \tag{51}$$

$$\overline{w}_3(u; y) = -\frac{i\omega}{iu}y \tag{52}$$

$$\overline{w}_4(u; y) = \frac{2}{(iu)^2}\left(a - \frac{i\omega}{iu}\right) \tag{53}$$

$$\overline{w}_5(u; y) = -\frac{1}{iu}\left(a - \frac{i\omega}{iu}\right) y \tag{54}$$

$$\overline{w}_6(u; y) = -\frac{i\omega}{iu}y^2 . \tag{55}$$

The potentials far downstream of the leading edge could now be obtained by using the formulae of the last section and the Inverse Fourier Integral in the form (26), but these are not suitable for obtaining the generalised forces using Bernoulli's equation over the whole wing. Instead the Fourier Integrals of the generalised forces as functions of wing chord will be obtained in terms of the Fourier Integrals of the velocity potentials and the inverse of these taken.

From the linearized Bernoulli equation, the aerodynamic loading distribution (positive upwards) on the wing is given by $l(x, y)e^{i\omega t}$ where

$$\begin{aligned} l(x, y)e^{i\omega t} &= 2\rho\left\{a\frac{\partial\phi}{\partial x}(x, y, +0, t) + \frac{\partial\phi}{\partial t}(x, y, +0, t)\right\} \\ &= 2\rho\left\{a\frac{\partial\varphi}{\partial x}(x, y, +0) + i\omega\varphi(x, y, +0)\right\}. \end{aligned} \tag{56}$$

The generalised airforces on the wing are given by $Q_j e^{i\omega t}$, $j = 1, 2, \ldots, 6$, where

$$Q_j = \iint_{\text{area of wing}} l(x, y)f_j(x, y)dx\,dy \qquad j = 1, 2, \ldots, 6. \tag{57}$$

We may write further

$$Q_j = \sum_{k=1}^{6} Q_{j,k}q_{k0}, \tag{58}$$

where

$$Q_{j,k} = 2\rho\iint_{\text{area of wing}}\left\{a\frac{\partial\varphi_k}{\partial x}(x, y, +0) + i\omega\varphi_k(x, y, +0)\right\}f_j(x, y)dx\,dy. \tag{59}$$

The generalised forces will be required for a wing of chord length c. For the present consider the generalised forces on a wing of arbitrary chord length X, so that X may be considered to be a variable. The generalised forces are functions of the chord length and may then be written $Q_{j,k}(X)$. If $Q_{j,k}(X)$ is defined to be zero for $X < 0$ then we may write down the Fourier Integrals of $Q_{j,k}(X)$ as

$$\bar{Q}_{j,k}(u) = \int_0^\infty Q_{j,k}(X)e^{iuX}dX \qquad \begin{matrix} j = 1, 2, \ldots, 6 \\ k = 1, 2, \ldots, 6. \end{matrix} \tag{60}$$

These Fourier Integrals reduce to

$$\bar{Q}_{1,k}(u) = 2\rho\left(a - \frac{i\omega}{iu}\right)\int_{-b}^{+b} \bar{\varphi}_k(u; y, +0)dy \tag{61}$$

$$\bar{Q}_{2,k}(u) = \frac{2\rho}{iu}\frac{\partial}{i\partial u}\left\{iu\left(a - \frac{i\omega}{iu}\right)\int_{-b}^{+b} \bar{\varphi}_k(u; y, +0)dy\right\} \tag{62}$$

$$\bar{Q}_{3,k}(u) = 2\rho\left(a - \frac{i\omega}{iu}\right)\int_{-b}^{+b} y\bar{\varphi}_k(u; y, +0)dy \tag{63}$$

$$\bar{Q}_{4,k}(u) = \frac{2\rho}{iu}\frac{\partial^2}{i^2\partial u^2}\left\{iu\left(a - \frac{i\omega}{iu}\right)\int_{-b}^{+b} \bar{\varphi}_k(u; y, +0)dy\right\} \tag{64}$$

$$\bar{Q}_{5,k}(u) = \frac{2\rho}{iu}\frac{\partial}{i\partial u}\left\{iu\left(a - \frac{i\omega}{iu}\right)\int_{-b}^{+b} y\bar{\varphi}_k(u; y, +0)dy\right\} \tag{65}$$

$$\bar{Q}_{6,k}(u) = 2\rho\left(a - \frac{i\omega}{iu}\right)\int_{-b}^{+b} y^2\bar{\varphi}_k(u; y, +0)dy. \tag{66}$$

The following integrals are therefore required. For the symmetric oscillations,

$$\begin{aligned}\int_{-b}^{+b} \bar{\varphi}(u; y, +0)dy &= \pi b^2\left\{\frac{1}{2}B_0(u) + \frac{3}{8}B_1(u) + \frac{5}{16}B_2(u) + \frac{35}{128}B_3(u) + \ldots\right\} \\ &= \frac{\pi b^2}{\Delta_0(\Theta)}A_0(u)\left\{-\frac{1}{2} + \frac{1}{2}\Theta - \Theta^3\left(\frac{1}{6}\psi + \frac{7}{72}\right) + \ldots\right\} + \\ &\quad + \frac{\pi b^2}{\Delta_0(\Theta)}A_1(u)\left\{-\frac{1}{8} + \frac{1}{6}\Theta - \Theta^2\left(\frac{1}{24}\psi + \frac{1}{96}\right) + \ldots\right\}\end{aligned} \tag{67}$$

$$\begin{aligned}\int_{-b}^{+b} y^2\bar{\varphi}(u; y, +0)dy &= \pi b^4\left\{\frac{1}{8}B_0(u) + \frac{1}{16}B_1(u) + \frac{5}{128}B_2(u) + \frac{7}{256}B_3(u) + \ldots\right\} \\ &= \frac{\pi b^4}{\Delta_0(\Theta)}A_0(u)\left\{-\frac{1}{8} + \frac{1}{6}\Theta - \Theta^2\left(\frac{1}{24}\psi + \frac{1}{96}\right) + \right. \\ &\quad \left. + \Theta^3\left(\frac{1}{24}\psi - \frac{7}{72}\right) + \ldots\right\} \\ &\quad + \frac{\pi b^4}{\Delta_0(\Theta)}A_1(u)\left\{-\frac{1}{24} - \Theta\left(\frac{1}{48}\psi - \frac{17}{192}\right) + \right. \\ &\quad \left. + \Theta^2\left(\frac{1}{96}\psi - \frac{11}{240}\right) + \ldots\right\}.\end{aligned} \tag{68}$$

For the antisymmetric oscillations,

$$\int_{-b}^{+b} y\bar{\varphi}(u;\, y,\, +0)dy = \pi b^3 \left\{\frac{1}{8} D_0(u) + \frac{1}{16} D_1(u) + \frac{5}{128} D_2(u) + \frac{7}{256} D_3(u) + \ldots\right\}$$

$$= \frac{\pi b^3 C_0(u)}{\Delta_1(\Theta)} \left\{-\frac{1}{16} + \frac{1}{24}\Theta - \frac{1}{192}\Theta^2 + \frac{1}{288}\Theta^3 + \ldots\right\}. \qquad (69)$$

The residues of the functions

$$\frac{-\frac{1}{2} + \frac{1}{2}\Theta - \Theta^3\left(\frac{1}{6}\psi + \frac{7}{72}\right) + \ldots}{\Delta_0(\Theta)}, \qquad (70)$$

$$\frac{-\frac{1}{8} + \frac{1}{6}\Theta - \Theta^2\left(\frac{1}{24}\psi + \frac{1}{96}\right) + \ldots}{\Delta_0(\Theta)}, \qquad (71)$$

$$\frac{-\frac{1}{8} + \frac{1}{6}\Theta - \Theta^2\left(\frac{1}{24}\psi + \frac{1}{96}\right) + \Theta^3\left(\frac{1}{24}\psi - \frac{7}{72}\right) + \ldots}{\Delta_0(\Theta)}, \qquad (72)$$

$$\frac{-\frac{1}{24} - \Theta\left(\frac{1}{48}\psi - \frac{17}{192}\right) + \Theta^2\left(\frac{1}{96}\psi - \frac{11}{240}\right) + \ldots}{\Delta_0(\Theta)}, \qquad (73)$$

at the pole u_0 are respectively

$$\frac{8a}{\omega b^2} a_{0,0} = \frac{8a}{\omega b^2}(0\cdot0111 + i0\cdot0616) \qquad (74)$$

$$\frac{8a}{\omega b^2} a_{1,0} = \frac{8a}{\omega b^2}(0\cdot0034 + i0\cdot0149) \qquad (75)$$

$$\frac{8a}{\omega b^2} a_{0,1} = \frac{8a}{\omega b^2}(0\cdot0034 + i0\cdot0149) \qquad (76)$$

$$\frac{8a}{\omega b^2} a_{1,1} = \frac{8a}{\omega b^2}(-0\cdot0011 - i0\cdot0049). \qquad (77)$$

The residue of the function

$$\frac{-\frac{1}{16} + \frac{1}{24}\Theta - \frac{1}{192}\Theta^2 + \frac{1}{288}\Theta^3 + \ldots}{\Delta_1(\Theta)} \qquad (78)$$

at the pole u_1 is

$$\frac{8a}{\omega b^2} c_{0,0} = \frac{8a}{\omega b^2}(0\cdot0011 + i0\cdot0139). \qquad (79)$$

The functions (67), (68) and (69) when expanded as infinite series in the variable Θ become, in the symmetric case,

$$\int_{-b}^{+b} \bar{\varphi}(u; y, +0)dy = \left\{ -\frac{1}{2} + \Theta\left(\psi - \frac{3}{2}\right) - \Theta^2\left(2\psi^2 - 5\psi + \frac{7}{2}\right) + \ldots \right\} \pi b^2 A_0(u) +$$
$$+ \left\{ -\frac{1}{8} + \Theta\left(\frac{1}{4}\psi - \frac{1}{3}\right) - \Theta^2\left(\frac{1}{2}\psi^2 - \frac{9}{8}\psi + \frac{73}{96}\right) + \ldots \right\} \pi b^2 A_1(u) \qquad (80)$$

$$\int_{-b}^{+b} y^2\bar{\varphi}(u; y, +0)dy = \left\{ -\frac{1}{8} + \Theta\left(\frac{1}{4}\psi - \frac{1}{3}\right) - \Theta^2\left(\frac{1}{2}\psi^2 - \frac{9}{8}\psi + \frac{73}{96}\right) + \ldots \right\} \pi b^2 A_0(u) +$$
$$+ \left\{ -\frac{1}{24} + \Theta\left(\frac{1}{16}\psi - \frac{5}{64}\right) - \Theta^2\left(\frac{1}{8}\psi^2 - \frac{1}{4}\psi + \frac{3}{20}\right) + \ldots \right\} \pi b^2 A_1(u), \qquad (81)$$

and in the antisymmetric case

$$\int_{-b}^{+b} y\bar{\varphi}(u; y, +0)dy = \left\{ -\frac{1}{16} - \frac{1}{12}\Theta + \Theta^2\left(\frac{1}{16}\psi - \frac{27}{32}\right) + \ldots \right\} \pi b^2 C_0(u). \qquad (82)$$

The expressions (80), (81) and (82) are now inserted into the formulae (61) to (66) with the appropriate expressions for $A_0(u)$, $A_1(u)$ and $C_0(u)$ to obtain the first few terms of infinite expansions suitable for insertion in the integrals of the inversion formula (26). These first few terms of the infinite expansions are adequate approximations to the functions they represent only for small values of Θ. However, for sufficiently small values of ω and large values of X the exponential term in the infinite integrals will reduce the values of the integrands to negligibly small quantities while Θ is still small {*see* equation (34)} so that the final results should be a good approximation. When the contributions of the residues at the poles are also included with the contributions from the integrals in formula (26), approximate expressions for the generalised forces are obtained. The values of the generalised forces are required only at $X = c$, where c is the chord length of the actual wing under consideration.

Generalised force coefficients $P_{j,k}$ may be defined by

$$P_{j,k} = \frac{Q_{j,k}}{\rho a^2 b f_j(c, b) f_k(c, b)} \qquad (83)$$

and these expressions may be written in terms of the frequency parameter

$$\nu = \frac{\omega c}{a} \qquad (84)$$

and the aspect ratio

$$A = \frac{2b}{c} \qquad (85)$$

only.

The expression for $P_{1,1}$ will now be derived. The derivations of expressions for the other $P_{j,k}$'s are similar to, though more complicated than that of $P_{1,1}$. The derivation of $P_{4,4}$ is given in Appendix II as an illustration of one of the more complicated derivations.

From (61) and (67) with $k = 1$ the following expression for $\bar{Q}_{1,1}(u)$ is obtained:

$$\bar{Q}_{1,1}(u) = -2\rho\pi b^2 \left(\frac{i\omega}{iu}\right)\left(a - \frac{i\omega}{iu}\right) \frac{\left\{ -\frac{1}{2} + \frac{1}{2}\Theta - \Theta^3\left(\frac{1}{6}\psi + \frac{7}{72}\right) + \ldots \right\}}{\Delta_0(\Theta)}. \qquad (86)$$

The residue of $\bar{Q}_{1,1}(u)e^{-iuc}$ at the pole

$$u_0 = -\frac{8a}{\omega b^2}\left(\Theta_0 - \frac{\omega^2 b^2}{16a^2}\right) \tag{87}$$

is

$$R_{1,1} = \rho\pi b a^2 \frac{\nu A a_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\left\{1 - \frac{(i\nu)\left(\frac{i\nu A^2}{32}\right)}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right\} \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\}. \tag{88}$$

The function $\bar{Q}_{1,1}(u)e^{-iuc}$ has also a pole of order 2 at $u = 0$. Its residue there is therefore

$$\hat{R}_{1,1} = -\left[\frac{\partial}{\partial u}\left\{u^2 \bar{Q}_{1,1}(u)e^{-iuc}\right\}\right]_{u=0} \tag{89}$$

$$= -\left[\frac{\partial}{\partial u}\left\{u^2 \bar{Q}_{1,1}(u)\right\}\right]_{u=0} + ic\left[u^2 \bar{Q}_{1,1}(u)\right]_{u=0} \tag{90}$$

and this may be written

$$\hat{R}_{1,1} = i\pi\rho a^2 b\,[h(\nu A) + i\nu k(\nu A)] \tag{91}$$

where h and k are functions of a single variable. Approximate expressions for h and k are obtained by using the first few terms of the infinite expansion

$$\bar{Q}_{1,1}(u) = -2\rho\pi b^2\left(\frac{i\omega}{iu}\right)\left(a - \frac{i\omega}{iu}\right)\left\{-\frac{1}{2} + \Theta\left(\psi - \frac{3}{2}\right) - \Theta^2\left(2\psi^2 - 5\psi + \frac{7}{2}\right) + \ldots\right\} \tag{92}$$

and are

$$h(\nu A) = \left(\frac{i\nu A}{2}\right) + \frac{3}{4}\left(\frac{i\nu A}{2}\right)^3\left\{\log\frac{\nu A}{8} + \gamma + \frac{\pi i}{2} - \frac{5}{12}\right\} +$$
$$+ \frac{5}{16}\left(\frac{i\nu A}{2}\right)^5\left\{\left(\log\frac{\nu A}{2} + \frac{\pi i}{2}\right)^2 + \left(2\gamma - \frac{17}{20}\right)\left(\log\frac{\nu A}{8} + \frac{\pi i}{2}\right) + \left(\gamma^2 - \frac{17}{20}\gamma + \frac{3}{16}\right)\right\} \tag{93}$$

$$k(\nu A) = \left(\frac{i\nu A}{2}\right) + \frac{1}{4}\left(\frac{i\nu A}{2}\right)^3\left\{\log\frac{\nu A}{8} + \gamma + \frac{\pi i}{2} - \frac{3}{4}\right\} +$$
$$+ \frac{1}{8}\left(\frac{i\nu A}{2}\right)^5\left\{\left(\log\frac{\nu A}{2} + \frac{\pi i}{2}\right)^2 + \left(2\gamma - \frac{5}{4}\right)\left(\log\frac{\nu A}{8} + \frac{\pi i}{2}\right) + \left(\gamma^2 - \frac{5}{4}\gamma + \frac{7}{16}\right)\right\} \tag{94}$$

correct to within $O\{(\nu A)^7 \log(\nu A)\}$.

The contributions of the terms containing integrals in the inversion formula (24) must now be determined.

Since $\bar{Q}_{1,1}(u)$ is finite at $u = (\omega/2a)$ we may take $\delta = 0$, and then the contribution from the second term is zero.

Let us write

$$\bar{Q}_{1,1}(u) = \bar{Q}_{1,1}^{(1)}(u) + \bar{Q}_{1,1}^{(2)}(u) + \bar{Q}_{1,1}^{(3)}(u) + \ldots \tag{95}$$

where

$$\bar{Q}_{1,1}^{(1)}(u) = \pi\rho b^2\left(\frac{i\omega}{iu}\right)\left(a - \frac{i\omega}{iu}\right) \tag{96}$$

$$\bar{Q}_{1,1}^{(2)}(u) = -2\pi\rho b^2\left(\frac{i\omega}{iu}\right)\left(a - \frac{i\omega}{iu}\right)\left(\psi - \frac{3}{2}\right)\Theta \tag{97}$$

$$\bar{Q}_{1,1}^{(3)}(u) = 4\pi\rho b^2\left(\frac{i\omega}{iu}\right)\left(a - \frac{i\omega}{iu}\right)\left(\psi^2 - \frac{5}{2}\psi + \frac{7}{2}\right)\Theta^2. \tag{98}$$

The contribution of $\bar{Q}_{1,1}^{(1)}$ to the first term in (24) is

$$I_{1,1}^{(1)} = 0. \tag{99}$$

The contribution of $\bar{Q}_{1,1}^{(2)}$ to the first term in (24) is

$$I_{1,1}^{(2)} = -\frac{i}{2\pi} e^{-i\nu/2} \int_0^\infty -2\pi\rho b^2 \frac{i\omega a\left(q-\dfrac{i\omega}{2a}\right)}{\left(q+\dfrac{i\omega}{2a}\right)^2} (-2i\pi) \frac{\omega b^2}{8a} iqe^{-qc}dq$$

$$= -\frac{\pi}{4}\rho b^4\omega^2 e^{-i\nu/2} \int_0^\infty \frac{q\left(q-\dfrac{i\nu}{2a}\right)}{\left(q+\dfrac{i\nu}{2a}\right)^2} e^{-qc}dq$$

$$= -\frac{\pi\rho b^4\omega^3}{8a} e^{-i\nu/2} \int_0^\infty \frac{q(q-i)}{(q+i)^2} e^{-(\nu/2)q}dq$$

$$= \frac{1}{8}\pi\rho b a^2 \left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)\left(\frac{\nu}{2}\right) e^{-i\nu/2} \int_0^\infty \left\{1 - \frac{3i}{(q+i)} - \frac{2}{(q+i)^2}\right\} e^{-(\nu/2)q}dq$$

$$= \frac{1}{8}\pi\rho b a^2 \left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)\left(\frac{\nu}{2}\right)\left[e^{-i\nu/2}\int_0^\infty e^{-(\nu/2)q}\,dq - 3i\int_{\nu/2}^\infty \frac{e^{-iq}}{q}\,dq + \right.$$

$$\left. + 2\left(\frac{i\nu}{2}\right)\int_{\nu/2}^\infty \frac{e^{-iq}}{q^2}\,dq\right]$$

$$= \frac{1}{8}\pi\rho b a^2 \left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)\left[\left\{1+2\left(\frac{i\nu}{2}\right)\right\}e^{-i\nu/2} - \left(\frac{i\nu}{2}\right)\left\{3+2\left(\frac{i\nu}{2}\right)\right\}\int_{\nu/2}^\infty \frac{e^{-iq}}{q}\,dq\right]. \tag{100}$$

The contribution of $\bar{Q}_{1,1}^{(3)}(u)$ to the first term in (24) is

$$I_{1,1}^{(3)} = +\frac{i}{2\pi} e^{-i\nu/2} \int_0^\infty \frac{\pi\rho b^6\omega^3}{64ia}\left\{q - \frac{3}{2}\frac{i\omega}{a} + \frac{5}{4}\left(\frac{i\omega}{a}\right)^2 \frac{1}{\left(q+\dfrac{i\omega}{2a}\right)} - \frac{1}{4}\left(\frac{i\omega}{a}\right)^3 \frac{1}{\left(q+\dfrac{i\omega}{2a}\right)^2}\right\} \times$$

$$\times 16i\pi\left\{\log\left(\frac{\omega b^2 q}{16a^2}\right) + 2\gamma + \frac{\pi i}{2} - \frac{5}{4}\right\} e^{-qc}dq$$

$$= \frac{i\pi\rho b^6\omega^5}{32a^3} e^{-i\nu/2} \int_0^\infty \left\{q - 3i - \frac{5}{q+i} + \frac{2i}{(q+i)^2}\right\}\left\{\log\left(\frac{\omega^2 b^2 q}{16a^2}\right) + 2\gamma + \frac{\pi i}{2} - \frac{5}{4}\right\} e^{-(\nu/2)q}dq$$

$$= -\frac{1}{32}\pi\rho b a^2 \left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)^2\left(\frac{\nu}{2}\right)^2\left[e^{-i\nu/2}\int_0^\infty (q-3i)\left\{\log\left(\frac{\omega^2 b^2 q}{16a^2}\right) + \right.\right.$$

$$+ 2\gamma + \frac{\pi i}{2} - \frac{5}{4}\right\} e^{-(\nu/2)q}dq - 5\int_{\nu/2}^\infty\left\{\log\left(\frac{\nu A^2}{32}\right) + \log\left(q-\frac{\nu}{2}\right) + 2\gamma + \pi i - \frac{5}{4}\right\}\frac{e^{-iq}}{q}\,dq -$$

$$\left. - 2i\left(\frac{i\nu}{2}\right)\int_{\nu/2}^\infty\left\{\log\left(\frac{\nu A^2}{32}\right) + \log\left(q-\frac{\nu}{2}\right) + 2\gamma + \pi i - \frac{5}{4}\right\}\frac{e^{-iq}}{q^2}\,dq\right]$$

[*Continued*

$$= -\frac{1}{32}\pi\rho ba^2\left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)^2\left(\frac{\nu}{2}\right)^2\Bigg[\left(\frac{2}{\nu}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{1}{4}\right\}e^{-i\nu/2} -$$

$$- 3i\left(\frac{2}{\nu}\right)\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{5}{4}\right\}e^{-i\nu/2} -$$

$$- 5\left\{\log\left(\frac{\nu A^2}{32}\right)+2\gamma+\pi i-\frac{5}{4}\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq - 5\int_{\nu/2}^{\infty}\log\left(q-\frac{\nu}{2}\right)\frac{e^{-iq}}{q}\,dq -$$

$$- 2i\left(\frac{i\nu}{2}\right)\left\{\log\left(\frac{\nu A^2}{32}\right)+2\gamma+\pi i-\frac{5}{4}\right\}\left\{\frac{e^{-i\nu/2}}{\nu/2}-i\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq\right\} -$$

$$- 2i\left(\frac{i\nu}{2}\right)\left\{-\left(\gamma+\frac{i\pi}{2}\right)\frac{e^{-i\nu/2}}{\nu/2}-\frac{2}{\nu}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq - i\int_{\nu/2}^{\infty}\frac{\log\left(q-\frac{\nu}{2}\right)}{q}e^{-iq}dq\right\}\Bigg]$$

$$= -\frac{1}{32}\pi\rho ba^2\left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)^2\Bigg[\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{1}{4}\right\}e^{-i\nu/2} -$$

$$- 3\left(\frac{i\nu}{2}\right)\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{5}{4}\right\}e^{-i\nu/2} -$$

$$- 2\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\gamma i}{2}-\frac{5}{4}\right\}e^{-i\nu/2} +$$

$$+ \left(\frac{i\nu}{2}\right)^2\left\{5+2\left(\frac{i\nu}{2}\right)\right\}\left\{\log\left(\frac{\nu A^2}{32}\right)+2\gamma+\pi i-\frac{5}{4}\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq +$$

$$+ 2\left(\frac{i\nu}{2}\right)^2\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq + \left(\frac{i\nu}{2}\right)^2\left\{5+2\left(\frac{i\nu}{2}\right)\right\}\int_{\nu/2}^{\infty}\frac{\log\left(q-\frac{\nu}{2}\right)}{q}e^{-iq}dq\Bigg]. \qquad (101)$$

Then adding all the contributions to equation (24) we get

$$P_{1\,1} = \frac{Q_{1\,1}}{\rho ba^2} = -\pi\frac{(i\nu A)a_{0,\,0}}{\left(\Theta_0-\frac{\nu^2A^2}{64}\right)}\left[1+\frac{\nu^2A^2}{32\left(\Theta_0-\frac{\nu^2A^2}{64}\right)}\right]\exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0-\frac{\nu^2A^2}{64}\right)\right\} -$$

$$- \pi\Bigg[\left(\frac{i\nu A}{2}\right)+\frac{3}{4}\left(\frac{i\nu A}{2}\right)^3\left\{\log\left(\frac{\nu A}{8}\right)+\gamma+\frac{\pi i}{2}-\frac{5}{12}\right\} +$$

$$+ \frac{5}{16}\left(\frac{i\nu A}{2}\right)^5\left\{\left(\log\left(\frac{\nu A}{8}\right)+\frac{\pi i}{2}\right)^2+\left(2\gamma-\frac{17}{20}\right)\left(\log\left(\frac{\nu A}{8}\right)+\frac{\pi i}{2}\right) +$$

$$+ \left(\gamma^2-\frac{17}{20}\gamma+\frac{3}{16}\right)\right\}\Bigg] -$$

$$- \pi(i\nu)\Bigg[\left(\frac{i\nu A}{2}\right)+\frac{1}{4}\left(\frac{i\nu A}{2}\right)^3\left\{\log\left(\frac{\nu A}{8}\right)+\gamma+\frac{\pi i}{2}-\frac{3}{4}\right\} +$$

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$$+\frac{1}{16}\left(\frac{i\nu A}{2}\right)^5\left\{\left(\log\left(\frac{\nu A}{8}\right)+\frac{\pi i}{2}\right)^2+\left(2\gamma-\frac{5}{4}\right)\left(\log\left(\frac{\nu A}{8}\right)+\frac{\pi i}{2}\right)+\right.$$

$$\left.+\left(\gamma^2-\frac{5}{4}\gamma+\frac{7}{16}\right)\right\}\Bigg]+$$

$$+\frac{1}{8}\pi\left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)\left[\left\{1+2\left(\frac{i\nu}{2}\right)\right\}e^{-i\nu/2}-\left(\frac{i\nu}{2}\right)\left\{3+2\left(\frac{i\nu}{2}\right)\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq\right]+$$

$$+\frac{1}{32}\pi\left(\frac{i\nu A}{2}\right)\left(\frac{i\nu A^2}{2}\right)^2\Bigg[-\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{1}{4}\right\}e^{-i\nu/2}+$$

$$+3\left(\frac{i\nu}{2}\right)\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{5}{4}\right\}e^{-i\nu/2}+$$

$$+2\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}-\frac{5}{4}\right\}e^{-i\nu/2}-$$

$$-\left(\frac{i\nu}{2}\right)^2\left\{5+\left(\frac{i\nu}{2}\right)\right\}\left\{\log\left(\frac{\nu A^2}{32}\right)+2\gamma+\pi i-\frac{5}{4}\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq-$$

$$-2\left(\frac{i\nu}{2}\right)^2\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq-\left(\frac{i\nu}{2}\right)^2\left\{5+2\left(\frac{i\nu}{2}\right)\right\}\int_{\nu/2}^{\infty}\frac{\log\left(q-\frac{\nu}{2}\right)}{q}\,e^{-iq}dq\Bigg] \qquad (102)$$

provided ν and A are sufficiently small.

To evaluate the integrals which occur the following rapidly convergent series may be used:

$$\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq=-\left(\gamma+\frac{i\pi}{2}\right)-\log\left(\frac{\nu}{2}\right)-\sum_{n=1}^{\infty}\frac{\left(-\frac{i\nu}{2}\right)^n}{n\times n!} \qquad (103)$$

$$\int_{\nu/2}^{\infty}\frac{\log\left(q-\frac{\nu}{2}\right)}{q}\,e^{-iq}dq=\frac{1}{2}\gamma^2-\frac{5\pi^2}{24}+\frac{i\pi\gamma}{2}-\frac{1}{2}\log^2\left(\frac{\nu}{2}\right)+\left(\gamma+\frac{i\pi}{2}\right)\sum_{n=1}^{\infty}\frac{\left(-\frac{i\nu}{2}\right)^n}{n\times m!}+$$

$$+\sum_{n=1}^{\infty}\frac{\left(-\frac{i\nu}{2}\right)^n}{n^2\times n!}+\left\{\sum_{n=1}^{\infty}\frac{\left(-\frac{i\nu}{2}\right)^n}{n\times n!}\right\}^2-$$

$$-e^{-i\nu/2}\sum_{r=0}^{\infty}\left(-\frac{i\nu}{2}\right)^r\sum_{n=r+1}^{\infty}\frac{(n-r-1)!}{n\times n!} \qquad (104)$$

where $\gamma = 0\cdot577216$ is Euler's constant.

When ν is very small these series may be truncated after a few terms. Also $e^{-i\nu/2}$ may be replaced by the first few terms of its power-series expansion. In the results given in the following section such expansion and truncation has been carried out.

6. *Results.*

When ν and A are small approximate values of the generalised airforces are:

$$P_{1,1} = \frac{Q_{1,1}}{\rho b a^2} = -\pi \frac{(i\nu A)a_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A \left[\left(\frac{i\nu}{2}\right) + 2\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3 \left[-\frac{1}{8}\left(\frac{i\nu}{2}\right)^2 + \frac{3}{8}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{7}{6}\right\} +\right.$$

$$\left. + \frac{1}{4}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{3}{4}\right\} + \frac{5}{96}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right] -$$

$$-\pi A^5 \left[\frac{1}{32}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{1}{4}\right\} -\right.$$

$$-\frac{1}{8}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - 1\right\} +$$

$$+\left(\frac{i\nu}{2}\right)^5 \left\{\frac{5}{64}\log^2\left(\frac{\nu A^2}{32}\right) + \left(\frac{5i\pi}{64} + \frac{5\gamma}{32} - \frac{11}{128}\right)\log\left(\frac{\nu A^2}{32}\right) -\right.$$

$$\left. -\frac{15\pi^2}{256} + \frac{5}{64}(i\pi\gamma) + \frac{5}{64}\gamma^2 - \frac{11}{256}i\pi - \frac{11}{128}\gamma + \frac{1}{64}\right\} +$$

$$\left. + O\left\{\left(\frac{\nu}{2}\right)^6 \log\left(\frac{\nu A^2}{32}\right)\right\}\right]. \qquad (105)$$

$$P_{1,2} = \frac{Q_{1,2}}{\rho b c a^2} = -\pi \frac{A a_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right]^2 \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A \left[\frac{1}{2} + 2\left(\frac{i\nu}{2}\right) + \left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3 \left[-\frac{1}{16}\left(\frac{i\nu}{2}\right) + \frac{5}{16}\left(\frac{i\nu}{2}\right)^2 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{13}{10}\right\} +\right.$$

$$+ \frac{1}{2}\left(\frac{i\nu}{2}\right)^3 \left[\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{1}{16}\right] +$$

$$\left. + \frac{1}{8}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{35}{24}\right\} + \frac{17}{1152}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right] -$$

$$-\pi A^5 \left[\frac{1}{64}\left(\frac{i\nu}{2}\right)^2 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{1}{4}\right\} -\right.$$

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$$- \frac{3}{32}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{13}{12}\right\} +$$

$$+ \left(\frac{i\nu}{2}\right)^4 \left\{\frac{13}{128}\log^2\left(\frac{\nu A^2}{32}\right) + \left(\frac{13 i\pi}{128} + \frac{13}{64}\gamma - \frac{43}{256}\right)\log\left(\frac{\nu A^2}{32}\right) +\right.$$

$$\left. + \frac{39}{512}(i\pi)^2 + \frac{13}{128} i\pi\gamma + \frac{13}{128}\gamma^2 - \frac{43}{512} i\pi - \frac{43}{256}\gamma + \frac{5}{64}\right\} +$$

$$+ \left(\frac{i\nu}{2}\right)^5 \left\{\frac{3}{32}\log^2\left(\frac{\nu A^2}{32}\right) + \left(\frac{3 i\pi}{32} + \frac{3\gamma}{16} + \frac{11}{96}\right)\log\left(\frac{\nu A^2}{32}\right) +\right.$$

$$\left. + \frac{9}{128}(i\pi)^2 + \frac{3}{32} i\pi\gamma + \frac{3}{32}\gamma^2 + \frac{11}{192} i\pi + \frac{11}{96}\gamma + \frac{89}{768}\right\} +$$

$$+ O\left\{\left(\frac{\nu}{2}\right)^6 \log\left(\frac{\nu A^2}{32}\right)\right\}\Bigg]. \tag{106}$$

$$P_{1,3} = \frac{Q_{1,3}}{\rho b^2 a^2} = 0. \tag{107}$$

$$P_{1,4} = \frac{Q_{1,4}}{\rho b c^2 a^2} = +\frac{\pi}{16}\frac{i\nu A^3 a_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^2}\left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right]^2 \exp\left\{\frac{32 i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$- \pi A\left[1 + 2\left(\frac{i\nu}{2}\right) + \frac{2}{3}\left(\frac{i\nu}{2}\right)^2\right] - \pi A^3\left[\frac{1}{8}\left(\frac{i\nu}{2}\right)\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{i\pi}{2} - \frac{3}{16}\right\} +\right.$$

$$+ \frac{5}{8}\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{i\pi}{2} - \frac{3}{10}\right\} + \frac{1}{2}\left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{i\pi}{2} + \frac{9}{16}\right\} +$$

$$\left. + \frac{1}{12}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{i\pi}{2} + \frac{43}{24}\right\} + \frac{17}{2304}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right] + \ldots \tag{108}$$

$$P_{1,5} = \frac{Q_{1,5}}{\rho b^2 c a^2} = 0. \tag{109}$$

$$P_{1,6} = \frac{Q_{1,6}}{\rho b^3 a^2} = -\pi\frac{i\nu A a_{1,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32 i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$- \pi A\left[\frac{1}{4}\left(\frac{i\nu}{2}\right) + \frac{1}{2}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$- \pi A^3\left[-\frac{1}{32}\left(\frac{i\nu}{2}\right)^2 + \frac{3}{32}\left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - 1\right\} +\right.$$

$$\left. + \frac{1}{16}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{15}{12}\right\} + \frac{5}{384}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right] + \ldots \tag{110}$$

$$P_{2,1} = \frac{Q_{2,1}}{\rho bca^2} = -\frac{\pi}{1024}\frac{\nu^4 A^5 a_{0,0}}{\left(\Theta_0 - \frac{\nu^2A^2}{64}\right)^3}\exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2A^2}{64}\right)\right\} - \pi A\left[\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3\left[-\frac{1}{8}\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{1}{2}\right\} - \frac{3}{8}\left(\frac{i\nu}{2}\right)^3 +\right.$$

$$\left. + \frac{1}{8}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{1}{4}\right\} + \frac{5}{144}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right] -$$

$$-\pi A^5\left[\frac{1}{16}\left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{3}{4}\right\} +\right.$$

$$+\left(\frac{i\nu}{2}\right)^4\left\{-\frac{1}{16}\log^2\left(\frac{\nu A^2}{32}\right) + \left(-\frac{1}{16}i\pi - \frac{1}{8}\gamma\right)\log\left(\frac{\nu A^2}{32}\right) -\right.$$

$$\left. -\frac{3}{64}(i\pi)^2 - \frac{1}{16}(\pi i\gamma) - \frac{1}{16}\gamma^2 + \frac{7}{128}\right\} +$$

$$\left. +\left(\frac{i\nu}{2}\right)^5\left\{-\frac{5}{32}\log\left(\frac{\nu A^2}{32}\right) - \frac{1}{64}\pi i - \frac{1}{32}\gamma - \frac{9}{128}\right\} + O\left(\frac{\nu}{2}\right)^6\log^2\left(\frac{\nu A^2}{32}\right)\right]. \tag{111}$$

$$P_{2,2} = \frac{Q_{2,2}}{\rho bc^2a^2} = \frac{\pi}{512}\frac{i\nu^3A^5a_{0,0}}{\left(\Theta_0 - \frac{\nu^2A^2}{64}\right)^3}\left[1 + \frac{\nu^2A^2}{32\left(\Theta_0 - \frac{\nu^2A^2}{64}\right)}\right]\exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2A^2}{64}\right)\right\} -$$

$$-\pi A\left[\left(\frac{i\nu}{2}\right) + \frac{2}{3}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3\left[-\frac{1}{16}\left(\frac{i\nu}{2}\right)\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{1}{2}\right\} - \frac{5}{16}\left(\frac{i\nu}{2}\right)^2 +\right.$$

$$+\frac{1}{4}\left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{7}{16}\right\} +$$

$$\left. +\frac{1}{12}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{31}{24}\right\} + \frac{17}{1536}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right] -$$

$$-\pi A^5\left[\frac{1}{32}\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{3}{4}\right\} +\right.$$

$$+\left(\frac{i\nu}{2}\right)^3\left\{-\frac{3}{64}\log^2\left(\frac{\nu A^2}{32}\right) + \left(-\frac{3}{64}\pi i - \frac{3\gamma}{32} + \frac{1}{128}\right)\log\left(\frac{\nu A^2}{32}\right) -\right.$$

$$\left. -\frac{9}{256}(i\pi)^2 - \frac{3}{64}(\pi i\gamma) - \frac{3}{64}\gamma^2 + \frac{1}{256}\pi i + \frac{1}{128}\gamma + \frac{5}{96}\right\} -$$

$$-\frac{13}{64}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{3}{4}\right\} +$$

$$+\left(\frac{i\nu}{2}\right)^5\left\{\frac{3}{64}\log^2\left(\frac{\nu A^2}{32}\right) + \left(\frac{3}{64}\pi i + \frac{3}{32}\gamma + \frac{1}{96}\right)\log\left(\frac{\nu A^2}{32}\right) +\right.$$

$$\left. +\frac{9}{256}(i\pi)^2 + \frac{3}{64}(\pi i\gamma) + \frac{3}{64}\gamma^2 + \frac{1}{192}i\pi + \frac{1}{96}\gamma + \frac{7}{1536}\right\} +$$

$$\left. + O\left\{\left(\frac{\nu}{2}\right)^6\log^2\left(\frac{\nu A^2}{32}\right)\right\}\right]. \tag{112}$$

$$P_{2,3} = \frac{Q_{2,3}}{\rho b^2 c a^2} = 0$$

$$P_{2,4} = \frac{Q_{2,4}}{\rho b c^3 a^2} = + \frac{\pi}{572} \frac{\nu^2 A^5 a_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^3} \times$$

$$\times \left[1 + \frac{\nu^2 A^2}{8\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} + \frac{3}{1024} \frac{\nu^4 A^4}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^2}\right] \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$- \pi A \left[\frac{1}{2} + \frac{4}{3}\left(\frac{i\nu}{2}\right) + \frac{1}{2}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$- \pi A^3 \left[\frac{1}{48} - \frac{1}{8}\left(\frac{i\nu}{2}\right) + \frac{5}{16}\left(\frac{i\nu}{2}\right)^2 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{4}{5}\right\} + \right.$$

$$+ \frac{1}{3}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{19}{48}\right\} +$$

$$\left. + \frac{1}{16}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{41}{24}\right\} + \frac{17}{2880}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{113}$$

$$P_{2,5} = \frac{Q_{2,5}}{\rho b^2 c^2 a^2} = 0. \tag{114}$$

$$P_{2,6} = \frac{Q_{2,6}}{\rho b^3 c a^2} = - \frac{\pi}{512} \frac{\nu^4 A^5 a_{1,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^3} \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} - \pi A \left[\frac{1}{4}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$- \pi A^3 \left[- \frac{1}{32}\left(\frac{i\nu}{2}\right)^2 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{1}{3}\right\} - \frac{3}{32}\left(\frac{i\nu}{2}\right)^3 + \right.$$

$$\left. + \frac{1}{32}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{5}{12}\right\} + \frac{5}{576}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{115}$$

$$P_{3,1} = \frac{Q_{3,1}}{\rho b^2 a^2} = 0. \tag{116}$$

$$P_{3,2} = \frac{Q_{3,2}}{\rho b^2 c a^2} = 0. \tag{117}$$

$$P_{3,3} = \frac{Q_{3,3}}{\rho b^3 a^2} = - \pi \frac{i\nu A c_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$- \pi A \left[\frac{1}{8}\left(\frac{i\nu}{2}\right) + \frac{1}{4}\left(\frac{i\nu}{2}\right)^2\right] - \pi A^3 \left[- \frac{1}{32}\left(\frac{i\nu}{2}\right) - \frac{1}{48}\left(\frac{i\nu}{2}\right)^2\right]. \tag{118}$$

$$P_{3,4} = \frac{Q_{3,4}}{\rho b^3 c a^2} = 0. \tag{119}$$

$$P_{3,5} = \frac{Q_{3,5}}{\rho b^3 c a^2} = -\pi \frac{A c_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right]^2 \exp\left\{\frac{32 i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A\left[-\frac{1}{16} + \frac{1}{4}\left(\frac{i\nu}{2}\right) + \frac{1}{8}\left(\frac{i\nu}{2}\right)^2\right] - \pi A^3\left[\frac{5}{192}\left(\frac{i\nu}{2}\right)^2 - \frac{1}{24}\left(\frac{i\nu}{2}\right)^3 - \frac{1}{96}\left(\frac{i\nu}{2}\right)^4\right]. \tag{120}$$

$$P_{3,6} = \frac{Q_{3,6}}{\rho b^4 a^2} = 0. \tag{121}$$

$$P_{4,1} = \frac{Q_{4,1}}{\rho b c^2 a^2} = \frac{\pi}{16{,}384} \frac{i \nu^5 A^7}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^4} \exp\left\{\frac{32 i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} - \pi A\left[\frac{2}{3}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3\left[\frac{1}{8}\left(\frac{i\nu}{2}\right)^2 - \frac{3}{16}\left(\frac{i\nu}{2}\right)^3 + \frac{1}{12}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{1}{12}\right\} +\right.$$

$$\left. + \frac{5}{192}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{122}$$

$$P_{4,2} = \frac{Q_{4,2}}{\rho b c^3 a^2} = \frac{\pi}{8192} \frac{\nu^4 A^7}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^4} \left[1 + \frac{3\nu^2 A^2}{64\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32 i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A\left[\frac{2}{3}\left(\frac{i\nu}{2}\right) + \frac{1}{2}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3\left[\frac{1}{16}\left(\frac{i\nu}{2}\right) - \frac{5}{32}\left(\frac{i\nu}{2}\right)^2 + \frac{1}{6}\left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{29}{48}\right\} +\right.$$

$$\left. + \frac{1}{16}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{29}{24}\right\} + \frac{17}{1920}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{123}$$

$$P_{4,3} = \frac{Q_{4,3}}{\rho b^2 c^3 a^2} = 0. \tag{124}$$

$$P_{4,4} = \frac{Q_{4,4}}{\rho b c^4 a^2} = -\frac{\pi}{8192} \frac{i \nu^3 A^7}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^4} \times$$

$$\times \left[1 + \frac{3\nu^2 A^2}{16\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} + \frac{3}{512} \frac{\nu^4 A^4}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32 i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A\left[\frac{1}{3} + \left(\frac{i\nu}{2}\right) + \frac{2}{5}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3\left[-\frac{1}{16}\left(\frac{i\nu}{2}\right) + \frac{5}{24}\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{29}{30}\right\} +\right.$$

$$+ \frac{1}{4}\left(\frac{i\nu}{2}\right)^2\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{5}{16}\right\} +$$

$$\left. + \frac{1}{20}\left(\frac{i\nu}{2}\right)^4\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{199}{120}\right\} + \frac{17}{3456}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{125}$$

$$P_{4,5} = \frac{Q_{4,5}}{\rho b^2 c^3 a^2} = 0. \tag{126}$$

$$P_{4,6} = \frac{Q_{4,6}}{\rho b^3 c^2 a^2} = \frac{\pi}{16{,}384} \frac{i\nu^5 A^7 a_{1,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^4} \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} - \pi A \left[\frac{1}{6}\left(\frac{i\nu}{2}\right)\right] -$$

$$- \pi A^3 \left[\frac{1}{32}\left(\frac{i\nu}{2}\right)^2 - \frac{3}{64}\left(\frac{i\nu}{2}\right)^3 + \frac{1}{48}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{1}{4}\right\} + \right.$$

$$\left. + \frac{5}{768}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{127}$$

$$P_{5,1} = \frac{Q_{5,1}}{\rho b^2 c a^2} = 0. \tag{128}$$

$$P_{5,2} = \frac{Q_{5,2}}{\rho b^2 c^2 a^2} = 0. \tag{129}$$

$$P_{5,3} = \frac{Q_{5,3}}{\rho b^3 c a^2} = -\frac{\pi}{1024} \frac{\nu^4 A^5 c_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^3} \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} - \pi A \left[\frac{1}{8}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$- \pi A^3 \left[-\frac{1}{96}\left(\frac{i\nu}{2}\right)^2 + \frac{1}{96}\left(\frac{i\nu}{2}\right)^4\right]. \tag{130}$$

$$P_{5,4} = \frac{Q_{5,4}}{\rho b^2 c^3 a^2} = 0. \tag{131}$$

$$P_{5,5} = \frac{Q_{5,5}}{\rho b^3 c^2 a^2} = \frac{\pi}{512} \frac{i\nu^3 A^5 c_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^3} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$- \pi A \left[\frac{1}{8}\left(\frac{i\nu}{2}\right) - \frac{1}{12}\left(\frac{i\nu}{2}\right)^2\right] - \pi A^3 \left[\frac{1}{192}\left(\frac{i\nu}{2}\right) - \frac{1}{48}\left(\frac{i\nu}{2}\right)^3 - \frac{1}{144}\left(\frac{i\nu}{2}\right)^4\right]. \tag{132}$$

$$P_{5,6} = \frac{Q_{5,6}}{\rho b^4 c a^2} = 0. \tag{133}$$

$$P_{6,1} = \frac{Q_{6,1}}{\rho b^3 a^2} = -\pi \frac{i\nu A a_{0,1}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$- \pi A \left[\frac{1}{4}\left(\frac{i\nu}{2}\right) + \frac{1}{2}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$- \pi A^3 \left[-\frac{1}{32}\left(\frac{i\nu}{2}\right)^2 + \frac{3}{32}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - 1\right\} + \right.$$

$$\left. + \frac{1}{16}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{11}{12}\right\} + \frac{5}{384}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \tag{134}$$

$$P_{6,2} = \frac{Q_{6,2}}{\rho b^3 c a^2} = -\pi \frac{A a_{0,1}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right]^2 \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A \left[\frac{1}{8} + \frac{1}{2}\left(\frac{i\nu}{2}\right) + \frac{1}{4}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3 \left[-\frac{1}{128}\left(\frac{i\nu}{2}\right) + \frac{5}{128}\left(\frac{i\nu}{2}\right)^2 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{17}{15}\right\} + \right.$$

$$+ \frac{1}{16}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{11}{48}\right\} +$$

$$\left. + \frac{1}{64}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{13}{8}\right\} + \frac{17}{9216}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \quad (135)$$

$$P_{6,3} = \frac{Q_{6,3}}{\rho b^4 a^2} = 0. \quad (136)$$

$$P_{6,4} = \frac{Q_{6,4}}{\rho b^3 c^2 a} = \frac{\pi}{16} \frac{i\nu A^3 a_{0,1}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^2} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right]^2 \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A \left[\frac{1}{4} + \frac{1}{2}\left(\frac{i\nu}{2}\right) + \frac{1}{6}\left(\frac{i\nu}{2}\right)^2\right] - \pi A^3 \left[\frac{1}{64}\left(\frac{i\nu}{2}\right)\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{4}{3}\right\} + \right.$$

$$+ \frac{5}{64}\left(\frac{i\nu}{2}\right)^2 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{2}{15}\right\} + \frac{1}{16}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{35}{48}\right\} +$$

$$\left. + \frac{1}{96}\left(\frac{i\nu}{2}\right)^4 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} + \frac{47}{24}\right\} + \frac{17}{18432}\left(\frac{i\nu}{2}\right)^5 + O\left(\frac{\nu}{2}\right)^6\right]. \quad (137)$$

$$P_{6,5} = \frac{Q_{6,5}}{\rho b^4 c a^2} = 0. \quad (138)$$

$$P_{6,6} = \frac{Q_{6,6}}{\rho b^5 a^2} = -\pi \frac{i\nu A a_{1,1}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} \left[1 + \frac{\nu^2 A^2}{32\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)}\right] \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\} -$$

$$-\pi A \left[\frac{1}{12}\left(\frac{i\nu}{2}\right) + \frac{1}{6}\left(\frac{i\nu}{2}\right)^2\right] -$$

$$-\pi A^3 \left[-\frac{1}{128}\left(\frac{i\nu}{2}\right)^2 + \frac{3}{128}\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2} - \frac{11}{12}\right\}\right]. \quad (139)$$

7. *Discussion of Results.*

The results given in Section 5 are approximately correct only when νA and νA^2 are both small and ν is not large. An idea of the ranges of ν and A over which these results are applicable may be obtained by drawing graphs of the different approximations to the generalised forces corresponding to the different number of terms retained in equations (28) and (37). These approximations are easily identified since the highest power of A retained is increased by two when the order of approximation is increased by one. The contribution of the residue at the pole nearest the origin is taken into account in each approximation and is also drawn separately and labelled R. The value of this residue is also obtained only approximately, but only the approximations based on all the terms quoted in equations (29) to (33) and (38) to (42) have been used.

The graphs plotted are a selection of different order approximations to the generalised forces against frequency parameter ν for different values of aspect ratio, and they display characteristics of asymptotic expansions.

An example of an asymptotic expansion is

$$F(\nu) = \int_0^\nu \sin\left(\frac{\nu - s}{\nu s}\right) ds \sim \nu - 2!\nu^3 + 4!\nu^5 - 6!\nu^7 + \ldots .$$

This asymptotic expansion is divergent but nevertheless the n'th approximation given by the sum of n terms of this series is an arbitrarily good approximation to the actual value of $F(\nu)$ in a limited range of values of ν near $\nu = 0$. This agreement deteriorates as ν increases and becomes very bad for large ν. However, for the range of values of ν over which different approximations are close to each other they are also close to the actual value of the function.

Accordingly we surmise that when curves corresponding to two consecutive approximations are close together then it may be assumed that a good approximation to the generalised force has been obtained.

It is seen from these graphs that the results are applicable over a fairly wide range of frequency parameter when the aspect ratio is small, but that this range of frequency parameter rapidly diminishes when the aspect ratio is increased. This range of frequency parameter also depends on the particular generalised force coefficient under examination.

When the frequency parameter is so large that the results of the different approximations are all in poor agreement with each other, then it is not to be expected that taking still higher-order approximations will improve the position.

In Fig. 5, for example, the second and third approximations to the real part of $P_{1,1}$ are quite near to each other for ν between 0 and 1, but for ν greater than 1 they diverge. The values of the real part of $P_{1,1}$ obtained from either the second or third approximations may then be expected to be fair approximations to its actual values for ν between 0 and 1 but not otherwise. For $\nu > 1$ good approximations to the values of the real part of $P_{1,1}$ cannot be obtained by the method of this paper.

The second and third approximations to the imaginary part of $P_{1,1}$ are, on the other hand, quite near to each other over the whole range of ν from 0 to 2 shown in Fig. 5, so these should be fair approximations to the actual values of this imaginary part for ν between 0 and 2.

8. *Resumé.*

A thin flat rectangular wing of aspect ratio A is placed in a uniform sonic airstream with its surface parallel to and its leading edge at right angles to the stream. A Cartesian co-ordinate system is chosen with x-axis in the direction of the flow, y-axis to starboard in the plane of the wing and z-axis vertically upwards. The origin is taken at the mid-point of the leading edge.

In a small disturbance the vertical displacement of any point on the surface is assumed to be given by the second-degree equation

$$Z(x, y, t) = q_1(t) + xq_2(t) + yq_3(t) + x^2q_4(t) + xyq_5(t) + y^2q_6(t) \tag{140}$$

$$= \sum_{j=1}^{6} f_j(x, y)q_j(t). \tag{141}$$

The q's may be regarded as generalised co-ordinates for the wing displacement. If the wing experiences an incremental virtual displacement at time t given by

$$\delta Z = \sum_{j=1}^{6} f_j(x, y)\, \delta q_j \tag{142}$$

where the δq_j are incrementally small and arbitrary, then the virtual work done by the airforces on the wing in this displacement is

$$\begin{aligned} \delta W &= \iint\limits_{\substack{\text{area of}\\ \text{wing}}} L(x, y, t)\delta Z\, dx\, dy \\ &= \sum_{j=1}^{6} \delta q_j \iint\limits_{\substack{\text{area of}\\ \text{wing}}} L(x, y, t) f_j(x, y) dx\, dy \end{aligned} \tag{143}$$

where $L(x, y, t)$ is the loading distribution on the wing at time t.

The virtual work may however be written as

$$\delta W = \sum_{j=1}^{6} Q_j\, \delta q_j \tag{144}$$

where Q_j is the generalised force in mode j. Hence by comparing (143) and (144) and noting that the δq_j are arbitrary we get

$$Q_j = \iint\limits_{\substack{\text{area of}\\ \text{wing}}} L(x, y, t) f_j(x, y) dx\, dy\,. \tag{145}$$

If the disturbed motion is simple harmonic with circular frequency ω, we may write, as is usual with linear theory,

$$q_j = q_{j0} e^{i\omega t} \qquad j = 1, 2, \ldots, 6\,. \tag{146}$$

The corresponding loading distribution may then be written

$$L(x, y, t) = \sum_{k=1}^{6} l_k(x, y) q_{k0} e^{i\omega t} \tag{147}$$

where $l_k(x, y)e^{i\omega t}$ is the loading distribution corresponding to the wing oscillation

$$Z_k = f_k(x, y) e^{i\omega t}\,. \tag{148}$$

Substituting expression (147) into (145) leads to

$$Q_j = \sum_{k=1}^{6} Q_{j,k} q_{k0} e^{i\omega t} \tag{149}$$

where

$$Q_{j,k} = \iint\limits_{\substack{\text{area of}\\ \text{wing}}} f_j(x, y) l_k(x, y) dx\, dy \tag{150}$$

is a generalised aerodynamic force coefficient. We can define a non-dimensional aerodynamic force coefficient $P_{j,k}$ by

$$P_{j,k} = \frac{Q_{j,k}}{\rho a^2 b f_j(c, b) f_k(c, b)} \tag{151}$$

where b is the wing semi-span, c is the wing chord, a is the speed of sound and ρ is the density of the air. The coefficients $P_{j,k}$ are functions of aspect ratio

$$A = \frac{2b}{c} \tag{152}$$

and frequency parameter

$$\nu = \frac{\omega c}{a} \tag{153}$$

only. Expressions for the $P_{j,k}$'s are given in Section 5.

We may compare some of the quantities $P_{j,k}$ with ordinary aerodynamic derivatives (stability derivatives) when the wing is oscillating rigidly. In this case we retain only the first two terms in equation (140).

The total lift on the wing is $Le^{i\omega t}$ where

$$\begin{aligned} L &= \iint_{\substack{\text{area of}\\ \text{wing}}} l(x, y)dx\,dy \\ &= Q_1 \{\text{since } f_1(x, y) = 1\} \\ &= Q_{1,1}q_{10} + Q_{1,2}q_{20} \\ &= \rho a^2 b(P_{1,1}q_{10} + cP_{1,2}q_{20}). \end{aligned} \tag{154}$$

In the derivative notation we have

$$L = 2\rho a^2 b\{z(0)l_z + c\alpha(0)l_\alpha\} \tag{155}$$

where

$$z = z(0)e^{i\omega t} \tag{156}$$

is the vertical displacement of the wing leading edge, and

$$\alpha = \alpha(0)e^{i\omega t} \tag{157}$$

is the angle of inclination of the wing to the horizontal. If $\alpha(0)$ is small then the displacement of a point on the wing is

$$Z = \{z(0) + x\alpha(0)\}e^{i\omega t}. \tag{158}$$

Hence

$$z(0) = q_{10},\ \alpha(0) = q_{20} \tag{159}$$

and

$$L = 2\rho a^2 b(q_{10}l_z + q_{20}cl_\alpha) \tag{160}$$

from which we identify

$$P_{1,1} = 2l_z,\ P_{1,2} = 2l_\alpha. \tag{161}$$

The pitching moment about the leading edge, and positive in the sense which would make the leading edge move upwards relative to the trailing edge, is $Me^{i\omega t}$, where

$$\begin{aligned} M &= -\iint_{\substack{\text{area of}\\ \text{wing}}} xl(x, y)dx\,dy \\ &= -Q_2 \\ &= -Q_{2,1}q_{10} - Q_{2,2}q_{20} \\ &= -\rho a^2 Cb(P_{2,1}q_{1,0} + P_{2,2}cq_{2,0}). \end{aligned} \tag{162}$$

In the derivative notation (derivatives about the leading edge) we have

$$M = 2\rho a^2 cb\{z(0)m_z + (0)cm_\alpha\} \tag{163}$$

and so

$$P_{2,1} = -2m_z,\ P_{2,2} = -2m_\alpha . \tag{164}$$

9. *Conclusion.*

Results have been obtained for the generalised forces on a low-aspect-ratio rectangular wing oscillating in sonic flow at low frequencies. The modes of oscillation considered were rigid modes and a small selection of flexible modes. These results are valid only for low frequency unless the aspect ratio is extremely small, in which case the frequency may be quite large.

The results of the present work reduce to those of slender-body first-order theory for very small aspect ratio and frequency parameter. The results given by Miles[3] are more restricted than the present ones but are in agreement with them as far as they go.

10. *Acknowledgement.*

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LIST OF SYMBOLS

x, y, z		Cartesian co-ordinates fixed relative to undisturbed wing
t		Time
$Z(x, y, t)$		Vertical upward displacement of a point on the wing
$w(x, y, t)$		Upwash velocity at a point on the wing
V		Free-stream velocity
a		Speed of sound
M	=	V/a, Mach number of free stream
ω		Circular frequency
b		Semi-span of wing
X		Chord length of an arbitrary rectangular wing
c		Chord length of the actual rectangular wing
ν	=	$\dfrac{\omega c}{a}$, frequency parameter in sonic stream
ρ		Free-stream density
A	=	$\dfrac{2b}{c}$, aspect ratio
$\phi(x, y, z, t)$		Velocity potential
$\varphi(x, y, z)$	=	$\phi(x, y, z, t)e^{-i\omega t}$
u		Fourier Transform variable
ξ, η		Variables cognate with x, y
r	=	$\sqrt{[(x-\xi)^2 - (M^2-1)\{(y-\eta)^2 + z^2\}]}$
σ	=	$\sqrt{\{(y-\eta)^2 + z^2\}}$
λ	=	$\sqrt{\left\{(M^2-1)u^2 - 2Mu\dfrac{\omega}{a} + \dfrac{\omega^2}{a^2}\right\}}$
$q_k^{(t)}$		generalised co-ordinates
Z_i	=	$f_i(x, y)e^{i\omega t}$, mode of oscillation
Q_j	=	$\Sigma\, Q_{j,k}q_k$, generalised airforces
$P_{j,k}$	=	$\dfrac{Q_{j,k}}{\rho a^2 b f_j(c, b) f_k(c, b)}$
R		Contribution to $P_{j,k}$ from residue at the pole nearest the origin
Θ	=	$\left(\dfrac{\omega}{2a}\right)\left(\dfrac{\omega}{2a} - u\right)\left(\dfrac{b^2}{4}\right)$
ψ	=	$\operatorname{Log}\Theta + 2\gamma + \pi i$, where Log is defined immediately after equation (19)

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APPENDIX I

Solution of Partial Differential Equation

When u is in the upper half-plane $I(u) > 0$ an elementary solution of the partial differential equation

$$\bar{\varphi}_{yy} + \bar{\varphi}_{zz} + \left(\frac{\omega^2}{a^2} - 2\frac{\omega}{a}u\right)\bar{\varphi} = 0 \tag{165}$$

which vanishes as $z \to \infty$, and is singular at the point (η, ζ), is

$$\bar{\varphi}_0 = H_0^{(2)}\left\{\sigma\left(\frac{\omega^2}{a^2} - 2\frac{\omega}{a}u\right)^{1/2}\right\} \tag{166}$$

where

$$\sigma = \sqrt{\{(y-\eta)^2 + (z-\zeta)^2\}}. \tag{167}$$

If $\bar{\varphi}$ is a solution of (165) which is non-singular in an area A of the y, z-plane, then provided the area A does not include (η, ζ), Green's theorem gives

$$\int_s \left(\bar{\varphi}\frac{\partial\bar{\varphi}_0}{\partial n} - \bar{\varphi}_0\frac{\partial\bar{\varphi}}{\partial n}\right)ds = \int_A (\bar{\varphi}\nabla^2\bar{\varphi}_0 - \bar{\varphi}_0\nabla^2\bar{\varphi})dA$$
$$= 0 \tag{168}$$

where s is the contour bounding A and $\partial/\partial n$ is the derivative in the direction normal to this contour and pointing out of the area A.

Let us take $\zeta > 0$. Let $\sum_1$ be a circle of small radius ϵ and centre (η, ζ). Let $\sum_2$ and $\sum_3$ be respectively the parts of the circle of large radius R and centre (η, ζ) in $z > 0$ and $z < 0$. Let $\sum_4$ be the part of the line $\zeta = 0$ lying between the points of intersection of this line with the circle of radius R.

Let the area A_1 be the area enclosed between $\sum_1$, $\sum_2$ and $\sum_4$, and let area A_2 be the area enclosed between $\sum_2$ and $\sum_3$ as shown in Fig. 2.

We have, with suffixes 1 and 2 referring to areas A_1 and A_2,

$$\lim_{\epsilon\to 0}\int_{\Sigma_1}\left(\bar{\varphi}\frac{\partial\bar{\varphi}_0}{\partial n_1} - \bar{\varphi}_0\frac{\partial\bar{\varphi}}{\partial n_1}\right)ds_1 = 4i\bar{\varphi}(u;\,\eta,\,\zeta) \tag{169}$$

$$\lim_{R\to\infty}\int_{\Sigma_4}\left(\bar{\varphi}\frac{\partial\bar{\varphi}_0}{\partial n_1} - \bar{\varphi}_0\frac{\partial\bar{\varphi}}{\partial n_1}\right)ds_1 = -\zeta\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}u\right)^{1/2}\int_{-\infty}^{+\infty}\frac{H_1^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}u\right)^{1/2}\right\}}{\bar{\sigma}}\times$$
$$\times\,\bar{\varphi}(u;\,y,\,+0)dy + \int_{-\infty}^{+\infty}H_0^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}u\right)^{1/2}\right\}\bar{w}(u;\,y)dy \tag{170}$$

$$\lim_{R\to\infty}\int_{\Sigma_4}\left(\bar{\varphi}\frac{\partial\bar{\varphi}_0}{\partial n_2} - \bar{\varphi}_0\frac{\partial\bar{\varphi}}{\partial n_2}\right)ds_2 = \zeta\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}u\right)^{1/2}\int_{-\infty}^{+\infty}\frac{H_1^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}u\right)\right\}}{\bar{\sigma}}\times$$
$$\times\,\bar{\varphi}(u;\,y,\,-0)dy - \int_{-\infty}^{+\infty}H_0^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}u\right)^{1/2}\right\}\bar{w}(u;\,y)dy \tag{171}$$

where

$$\bar{\sigma} = \sqrt{\{(y-\eta)^2 + \zeta^2\}}. \tag{172}$$

Also if $\bar{\varphi} = O(\log R)$ and $\partial\bar{\varphi}/\partial R = O(1/R)$ for large R

$$\lim_{R\to\infty} \int_{\Sigma_2} \left(\bar{\varphi}\,\frac{\partial\bar{\varphi}_0}{\partial n_1} - \bar{\varphi}_0\,\frac{\partial\bar{\varphi}}{\partial n_1}\right) ds_1 = 0 \tag{173}$$

$$\lim_{R\to\infty} \int_{\Sigma_3} \left(\bar{\varphi}\,\frac{\partial\bar{\varphi}_0}{\partial n_2} - \bar{\varphi}_0\,\frac{\partial\bar{\varphi}}{\partial n_2}\right) ds_1 = 0\,. \tag{174}$$

Hence applying Green's theorem and taking the limits $\epsilon \to 0$ and $R \to \infty$ for the two areas A_1 and A_2 we get respectively

$$4i\bar{\varphi}(u;\,\eta,\,\zeta) = +\,\zeta\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2} \int_{-\infty}^{+\infty} \frac{H_1^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2}\right\}}{\bar{\sigma}}\,\bar{\varphi}(u;\,y,\,+0)dy\, -$$
$$-\int_{-\infty}^{\infty} H_0^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2}\right\}\bar{w}(u;\,y)dy \tag{175}$$

$$0 = -\,\zeta\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2} \int_{-\infty}^{+\infty} \frac{H_1^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2}\right\}}{\bar{\sigma}}\,\bar{\varphi}(u;\,y,\,-0)dy\, +$$
$$+\int_{-\infty}^{\infty} H_0^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2}\right\}\bar{w}(u;\,y)dy\,. \tag{176}$$

Then from (175) and (176) on using the fact that

$$\bar{\varphi}(u;\,y,\,-0) = -\,\bar{\varphi}(u;\,y,\,+0) \tag{177}$$

we get

$$\bar{\varphi}(u;\,\eta,\,\zeta) = -\,\frac{i\zeta}{2}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2} \int_{-\infty}^{+\infty} \frac{H_1^{(2)}\left\{\bar{\sigma}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2}\right\}}{\bar{\sigma}}\,\bar{\varphi}(u;\,y,\,+0)dy\,. \tag{178}$$

This result may be extended analytically to the whole cut u-plane, described in the main text, and then since

$$\bar{\varphi}(u;\,y,\,+0) = 0 \qquad |y| \geqslant b \tag{179}$$

it may be written

$$\bar{\varphi}(u;\,y,\,z) = -\,\frac{iz}{2}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2} \int_{-b}^{+b} \frac{\hat{H}_1^{(2)}\left\{\bar{r}\left(\frac{\omega^2}{a^2} - \frac{2\omega}{a}\,u\right)^{1/2}\right\}}{\bar{r}}\,\bar{\varphi}(u;\,\eta,\,+0)d\eta \tag{180}$$

where

$$\bar{r} = \sqrt{\{(y-\eta)^2 + z^2\}} \tag{181}$$

and $\hat{H}_1^{(2)}\,[\bar{r}\{(\omega^2/a^2) - (2\omega/a)u\}^{1/2}]$ is the analytic continuation of $H_1^{(2)}[\bar{r}\{(\omega^2/a^2) - (2\omega/a)u\}^{1/2}]$ to the whole of the cut u-plane.

APPENDIX II

Derivation of the Expression for $P_{4,4}$

From (64) and (67) with $k = 4$ the following expression for $\bar{Q}_{4,4}(u)$ is obtained:

$$\bar{Q}_{4,4}(u) = 4\rho\pi b^2 \frac{1}{iu}\frac{\partial^2}{i^2\partial u^2}\left[\frac{1}{(iu)}\left(a - \frac{i\omega}{iu}\right)^2 \frac{\left\{-\frac{1}{2}+\frac{1}{2}\Theta - \Theta^3\left(\frac{1}{6}\psi + \frac{7}{72}\right)\right\}}{\Delta_0(\Theta)}\right].$$

The residue of $\bar{Q}_{4,4}(u)e^{-iuc}$ at the pole

$$u_0 = -\frac{8a}{\omega b^2}\left(\Theta_0 - \frac{\omega^2 b^2}{16a^2}\right)$$

is

$$R_{4,4} = 4\rho\pi b^2 \frac{8a}{\omega b^2} a_{0,0}\left[\frac{1}{iu}\frac{\partial^2}{i^2\partial u^2}\left\{\frac{1}{iu}\left(a - \frac{i\omega}{iu}\right)^2\right\}\right]_{u=u_0} \exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\}$$

$$= \frac{1}{8192}\rho\pi b c^4 a^2 \frac{\nu^3 A^7 a_{0,0}}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^4} \times$$

$$\times\left\{1 + \frac{3}{16}\frac{\nu^2 A^2}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)} + \frac{3}{512}\frac{\nu^4 A^4}{\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)^2}\right\}\exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0 - \frac{\nu^2 A^2}{64}\right)\right\}.$$

Using the first few terms of the infinite expansion of

$$\frac{-\frac{1}{2}+\frac{1}{2}\Theta - \Theta^3\left(\frac{1}{6}\psi + \frac{7}{72}\right) + \ldots}{\Delta_0(\Theta)}$$

and carrying out the differentiations we obtain

$$\bar{Q}_{4,4}(u) = \bar{Q}_{4,4}^{(1)}(u) + \bar{Q}_{4,4}^{(2)}(u) + \bar{Q}_{4,4}^{(3)}(u) + \ldots$$

where

$$\bar{Q}_{4,4}^{(1)}(u) = \pi\rho b^2 a^2\left[-\frac{4}{u^4} + 24\frac{\omega}{a}\frac{1}{u^5} - 24\frac{\omega^2}{a^2}\frac{1}{u^6}\right]$$

$$\bar{Q}_{4,4}^{(2)}(u) = \frac{1}{2}\pi\rho b^4\omega a\left[\left(\psi - \frac{3}{2}\right)\left\{5\frac{\omega}{a}\frac{1}{u^4} - 12\frac{\omega^2}{a^2}\frac{1}{u^5} + 6\frac{\omega^3}{a^3}\frac{1}{u^6}\right\} - \right.$$

$$\left. - \frac{\omega b^2}{8a}\frac{1}{\Theta}\left\{-\frac{5}{2}\frac{\omega}{a}\frac{1}{u^3} + 4\frac{\omega^2}{a^2}\frac{1}{u^4} - \frac{3}{2}\frac{\omega^3}{a^3}\frac{1}{u^5}\right\} + \left\{\frac{1}{u^3} - 4\frac{\omega}{a}\frac{1}{u^4} + 3\frac{\omega^2}{a^2}\frac{1}{u^5}\right\}\right].$$

$$\bar{Q}_{4,4}^{(3)}(u) = \frac{1}{32}\pi\rho b^6\omega^2\left[\psi^2\left\{26\frac{\omega^2}{a^2}\frac{1}{u^4} - 36\frac{\omega^3}{a^3}\frac{1}{u^5} + 12\frac{\omega^4}{a^4}\frac{1}{u^6}\right\} + \right.$$

$$+ \psi\left\{8\frac{1}{u^2} + 24\frac{\omega}{a}\frac{1}{u^3} - 121\frac{\omega^2}{a^2}\frac{1}{u^4} + 114\frac{\omega^3}{a^3}\frac{1}{u^5} - 30\frac{\omega^4}{a^4}\frac{1}{u^6}\right\} +$$

$$\left. + \left\{-2\frac{1}{u^2} - 46\frac{\omega}{a}\frac{1}{u^3} + \frac{247}{2}\frac{\omega^2}{a^2}\frac{1}{u^4} - 93\frac{\omega^3}{a^3}\frac{1}{u^5} + 21\frac{\omega^4}{a^4}\frac{1}{u^6}\right\}\right].$$

The value of the residue of $\bar{Q}_{4,4}(u)e^{-iuc}$ at $u = 0$ is approximately equal to the value of the residue of $\{\bar{Q}_{4,4}^{(1)}(u) + \bar{Q}_{4,4}^{(2)}(u) + \bar{Q}_{4,4}^{(3)}(u)\}e^{-iuc}$ at $u = 0$. The residue may be written

$$\hat{R}_{4,4} = i\pi\rho b^5 a^2 \left[h_1(\nu A) + \left(\frac{i\nu}{2}\right)^3 h_3(\nu A) + \left(\frac{i\nu}{2}\right)^4 h_4(\nu A) + \left(\frac{i\nu}{2}\right)^5 h_5(\nu A)\right]$$

where

$$h_1(\nu A) = \frac{2}{15}\frac{1}{\left(\frac{i\nu A}{2}\right)} - \frac{4}{15}\left(\frac{i\nu A}{2}\right)\left\{\log\left(\frac{\nu A}{8}\right) + \gamma + \frac{\pi i}{2} + \frac{5}{8}\right\}$$

$$h_3(\nu A) = -\frac{16}{3}\frac{1}{\left(\frac{i\nu A}{2}\right)^3} - \frac{20}{3}\frac{1}{\left(\frac{i\nu A}{2}\right)}\left\{\log\left(\frac{\nu A}{8}\right) + \gamma + \frac{\pi i}{2} - \frac{3}{20}\right\} -$$

$$-\frac{13}{3}\left(\frac{i\nu A}{2}\right)\left[\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\}^2 + \left(2\gamma - \frac{25}{52}\right)\left\{\log\left(\frac{\nu A}{2}\right) + \frac{\pi i}{2}\right\} + \right.$$

$$\left. + \left\{\gamma^2 - \frac{25}{52}\gamma + \frac{7}{208}\right\}\right]$$

$$h_4(\nu A) = -16\frac{1}{\left(\frac{i\nu A}{2}\right)^3} - 8\frac{1}{\left(\frac{i\nu A}{2}\right)}\left\{\log\left(\frac{\nu A}{8}\right) + \gamma + \frac{\pi i}{2} - \frac{1}{2}\right\} -$$

$$-3\left(\frac{i\nu A}{2}\right)\left[\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\}^2 + \left(2\gamma - \frac{11}{12}\right)\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\} + \right.$$

$$\left. + \left(\gamma^2 - \frac{11}{12}\gamma + \frac{11}{48}\right)\right]$$

$$h_5(\nu A) = -\frac{32}{5}\frac{1}{\left(\frac{i\nu A}{2}\right)^3} - \frac{8}{5}\frac{1}{\left(\frac{i\nu A}{2}\right)}\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2} - \frac{3}{4}\right\} -$$

$$-\frac{2}{5}\left(\frac{i\nu A}{2}\right)\left[\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\}^2 + \left(2\gamma - \frac{5}{4}\right)\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\} + \right.$$

$$\left. + \left(\gamma^2 - \frac{5}{4}\gamma + \frac{7}{16}\right)\right].$$

The contributions of $\bar{Q}_{4,4}^{(1)}$ to the first term in (26) is

$$I_{4,4}^{(1)} = 0.$$

The contribution of $\bar{Q}_{4,4}^{(2)}$ to the first term in (26) is

$$I_{4,4}^{(2)} = -\frac{ie^{-i\nu/2}}{2\pi}\int_0^\infty \frac{1}{2}\pi\rho b^4\omega a\left\{5\frac{\omega}{a}\frac{1}{\left(q + \frac{i\omega}{2a}\right)^4} - 12\frac{\omega^2}{a^2}\frac{i}{\left(q + \frac{i\omega}{2a}\right)^5} - \right.$$

$$\left. - 6\frac{\omega^3}{a^3}\frac{1}{\left(q + \frac{i\omega}{2a}\right)^6}\right\}(-2i\pi)e^{-qc}dq$$

where, since the integrand is integrable at $q = 0$, the lower limit of integration δ has been replaced by zero in anticipation of the passage to the limit $\delta \to 0$.

Then

$$I_{4,4}^{(2)} = -\frac{4\pi\rho b^4 a^3}{\omega} e^{-i\nu/2} \int_0^\infty \left\{5\frac{1}{(q+i)^4} - 24\frac{i}{(q+i)^5} - 24\frac{1}{(q+i)^6}\right\} e^{-(\nu/2)q} dq$$

$$= 4\pi\rho b^5 a^2 \frac{1}{\left(\frac{i\nu A}{2}\right)} \int_{\nu/2}^\infty \left\{5\frac{1}{q^4} + 24i\left(\frac{i\nu}{2}\right)\frac{1}{q^5} - 24\left(\frac{i\nu}{2}\right)^2\frac{1}{q^6}\right\} e^{-iq} dq$$

$$= 4\pi\rho b^5 a^2 \frac{1}{\left(\frac{i\nu A}{2}\right)} \left[\left\{\frac{7}{15} - \frac{1}{30}\left(\frac{i\nu}{2}\right) + \frac{7}{30}\left(\frac{i\nu}{2}\right)^2 + \frac{4}{5}\left(\frac{i\nu}{2}\right)^3 + \frac{1}{5}\left(\frac{i\nu}{2}\right)^4\right\} e^{-i\nu/2} - \right.$$

$$\left. - \left(\frac{i\nu}{2}\right)^3 \left\{\frac{5}{6} + \left(\frac{i\nu}{2}\right) + \frac{1}{5}\left(\frac{i\nu}{2}\right)^2\right\} \int_{\nu/2}^\infty \frac{e^{-iq}}{q} dq\right].$$

The contribution of $\bar{Q}_{4,4}^{(2)}$ to the second term in (26) is

$$J_{4,4}^{(2)} = \lim_{\delta\to 0} + \frac{i\delta}{2\pi} e^{-i\nu/2} \int_{-3\pi/2}^{\pi/2} \left\{\frac{\pi\rho b^4 \omega a}{2} - \frac{1}{\delta} - \frac{5}{2}\frac{\omega}{a}\frac{1}{u^3} + 4\frac{\omega^2}{a^2}\frac{1}{u^4} - \frac{3}{2}\frac{\omega^3}{a^3}\frac{1}{u^4}\right\} e^{ic\delta e^{i\theta}} d\theta$$

$$\left(u = \frac{\omega}{2a} - \delta e^{i\theta}\right)$$

$$= -2\pi\rho b^5 a^2 e^{-i\nu/2} \frac{1}{\left(\frac{i\nu A}{2}\right)}.$$

The contribution of $\bar{Q}_{4,4}^{(3)}$ to the first term in (26) is

$$I_{4,4}^{(3)} = \frac{1}{4}\pi\rho b^6 \omega a e^{-i\nu/2} \int_0^\infty \left[\left\{\frac{52}{(q+i)^4} - \frac{144i}{(q+i)^5} - \frac{96}{(q+i)^6}\right\}\left\{\log\left(\frac{\omega^2 b^2}{16a^2} q\right) + 2\gamma + \frac{\pi i}{2}\right\} + \right.$$

$$\left. + \left\{-\frac{2}{(q+i)^2} - \frac{12i}{(q+i)^3} - \frac{121}{(q+i)^4} - \frac{228i}{(q+i)^5} + \frac{120}{(q+i)^6}\right\}\right] e^{-\nu/2q} dq$$

$$= i\pi\rho b^5 a^2 \frac{1}{\left(\frac{i\nu A}{2}\right)} \left(\frac{i\nu A^2}{2}\right)\left(\frac{i\nu}{2}\right)\left[\left(\frac{i\nu}{2}\right)^3 \left\{\log\left(\frac{\nu A^2}{32}\right) + 2\gamma + \pi i\right\} \times \right.$$

$$\times \int_{\nu/2}^\infty \left\{\frac{52}{q^4} + 144\left(\frac{i\nu}{2}\right)\frac{i}{q^5} - 96\left(\frac{i\nu}{2}\right)^2\frac{1}{q^6}\right\} e^{-iq} dq +$$

$$+ \left(\frac{i\nu}{2}\right) \int_{\nu/2}^\infty \left\{-\frac{2}{q^2} + 12\left(\frac{i\nu}{2}\right)\frac{i}{q^3} - 12\left(\frac{i\nu}{2}\right)^2\frac{1}{q^4} - \right.$$

$$\left. - 228\left(\frac{i\nu}{2}\right)^3\frac{i}{q^5} + 120\left(\frac{i\nu}{2}\right)^4\frac{1}{q^6}\right\} e^{-iq} dq +$$

$$\left. + \left(\frac{i\nu}{2}\right)^3 \int_{\nu/2}^\infty \left\{\frac{52}{q^4} + 144\left(\frac{i\nu}{2}\right)\frac{i}{q^5} - 96\left(\frac{i\nu}{2}\right)^2\frac{1}{q^6}\right\} \log\left(q - \frac{\nu}{2}\right) e^{-iq} dq\right]$$

$$= \pi\rho b^5 a^2 \frac{1}{\left(\frac{i\nu A}{2}\right)} \left(\frac{i\nu A^2}{2}\right)\left(\frac{i\nu}{2}\right)\left[\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}\right\}\times\right.$$

$$\times\left\{\frac{8}{15}-\frac{22}{15}\left(\frac{i\nu}{2}\right)+\frac{64}{15}\left(\frac{i\nu}{2}\right)^2+\frac{26}{5}\left(\frac{i\nu}{2}\right)^3+\frac{4}{5}\left(\frac{i\nu}{2}\right)^4\right\}e^{-i\nu/2}-$$

$$-\left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right)+\gamma+\frac{\pi i}{2}\right\}\left\{\frac{26}{3}+6\left(\frac{i\nu}{2}\right)+\frac{4}{15}\left(\frac{i\nu}{2}\right)^2\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq+$$

$$+\left\{\frac{2}{3}-\frac{3}{10}\left(\frac{i\nu}{2}\right)+\frac{157}{30}\left(\frac{i\nu}{2}\right)^2-\frac{53}{10}\left(\frac{i\nu}{2}\right)^3-\left(\frac{i\nu}{2}\right)^4\right\}e^{-i\nu/2}-$$

$$-\left\{\frac{8}{15}-\frac{25}{6}\left(\frac{i\nu}{2}\right)^3-\frac{q}{2}\left(\frac{i\nu}{2}\right)^4+\left(\frac{i\nu}{2}\right)^5\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq-$$

$$\left.-\left(\frac{i\nu}{2}\right)^3\left\{\frac{13}{9}-\frac{1}{4}\left(\frac{i\nu}{2}\right)+\frac{4}{5}\left(\frac{i\nu}{2}\right)^2\right\}\int_{\nu/2}^{\infty}\frac{\log\left(q-\frac{\nu}{2}\right)}{q}\,e^{-iq}dq\right]$$

The contribution of $\bar{Q}_{4,4}^{(3)}$ to the second term in (24) is

$$J_{4,4}^{(3)} = 0\,.$$

Then adding all the contributions to equation (24) we get

$$P_{4,4} = \frac{Q_{4,4}}{\rho b c^4 a^2} = -\frac{\pi}{64}\,\frac{i\nu^3 A^7 a_{0,0}}{\left(\Theta_0-\frac{\nu^2A^2}{64}\right)^4}\times$$

$$\times\left\{1+\frac{6\nu^2A^2}{\left(\Theta_0-\frac{\nu^2A^2}{64}\right)}+\frac{6\nu^4A^4}{\left(\Theta_0-\frac{\nu^2A^2}{64}\right)^2}\right\}\exp\left\{\frac{32i}{\nu A^2}\left(\Theta_0-\frac{\nu^2A^2}{64}\right)\right\}-$$

$$-\pi A^4\left[\left(\frac{1}{8}e^{-i\nu/2}-\frac{1}{120}\right)\frac{1}{\left(\frac{i\nu A}{2}\right)}-\frac{1}{60}\left(\frac{i\nu A}{2}\right)\left\{\log\left(\frac{\nu A}{8}\right)+\gamma+\frac{\pi i}{2}+\frac{5}{8}\right\}\right]-$$

$$-\left(\frac{i\nu}{2}\right)^3\pi A^4\left[\frac{1}{3}\frac{1}{\left(\frac{i\nu A}{2}\right)^3}+\frac{5}{12}\frac{1}{\left(\frac{i\nu A}{2}\right)}\left\{\log\left(\frac{\nu A}{8}\right)+\gamma+\frac{\pi i}{2}-\frac{3}{20}\right\}+\right.$$

$$+\frac{13}{48}\left(\frac{i\nu A}{2}\right)\left[\left\{\log\left(\frac{\nu A}{8}\right)+\frac{\pi i}{2}\right\}^2+\right.$$

$$\left.\left.+\left(2\gamma-\frac{25}{52}\right)\left\{\log\left(\frac{\nu A}{8}\right)+\frac{\pi i}{2}\right\}+\left(\gamma^2-\frac{25}{52}\gamma+\frac{7}{208}\right)\right]\right]-$$

$$-\left(\frac{i\nu}{2}\right)^4\pi A^4\left[\frac{1}{\left(\frac{i\nu A}{2}\right)^3}+\frac{1}{2}\frac{1}{\left(\frac{i\nu A}{2}\right)}\left\{\log\left(\frac{\nu A}{8}\right)+\gamma+\frac{\pi i}{2}-\frac{1}{2}\right\}+\right.$$

$$+ \frac{3}{16}\left(\frac{i\nu A}{2}\right)\left[\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\}^2 + \right.$$

$$\left.\left. + \left(2\gamma - \frac{11}{12}\right)\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\} + \left(\gamma^2 - \frac{11}{12}\gamma + \frac{11}{48}\right)\right]\right] -$$

$$- \left(\frac{i\nu}{2}\right)^5 \pi A^4 \left[\frac{2}{5}\frac{1}{\left(\frac{i\nu A}{2}\right)^3} + \frac{1}{10}\frac{1}{\left(\frac{i\nu A}{2}\right)}\left\{\log\left(\frac{\nu A}{8}\right) + \gamma + \frac{\pi i}{2} - \frac{3}{4}\right\} + \right.$$

$$+ \frac{1}{40}\left(\frac{i\nu A}{2}\right)\left[\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\}^2 + \right.$$

$$\left.\left. + \left(2\gamma - \frac{5}{4}\right)\left\{\log\left(\frac{\nu A}{8}\right) + \frac{\pi i}{2}\right\} + \left(\gamma^2 - \frac{5}{4}\gamma + \frac{7}{16}\right)\right]\right] +$$

$$+ \frac{\pi}{4}\frac{\left(\frac{i\nu A^2}{2}\right)^4}{\left(\frac{i\nu A}{2}\right)^5}\left[\left\{\frac{7}{15} - \frac{1}{30}\left(\frac{i\nu}{2}\right) + \frac{7}{30}\left(\frac{i\nu}{2}\right)^2 + \frac{4}{15}\left(\frac{i\nu}{2}\right)^3 + \frac{1}{5}\left(\frac{i\nu}{2}\right)^4\right\} e^{-i\nu/2} - \right.$$

$$\left. - \left(\frac{i\nu}{2}\right)^3\left\{\frac{5}{6} + \left(\frac{i\nu}{2}\right) + \frac{1}{5}\left(\frac{i\nu}{2}\right)^2\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq\right] +$$

$$+ \frac{\pi}{16}\frac{\left(\frac{i\nu A^2}{2}\right)^4}{\left(\frac{i\nu A}{2}\right)^3}\left[\left[\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2}\right\}\left\{\frac{8}{15} - \frac{22}{15}\left(\frac{i\nu}{2}\right) + \frac{64}{15}\left(\frac{i\nu}{2}\right)^2 + \frac{26}{5}\left(\frac{i\nu}{2}\right)^3 + \right.\right.\right.$$

$$\left. + \frac{4}{5}\left(\frac{i\nu}{2}\right)^4\right\} e^{-i\nu/2} - \left(\frac{i\nu}{2}\right)^3\left\{\log\left(\frac{\nu A^2}{32}\right) + \gamma + \frac{\pi i}{2}\right\} \times$$

$$\times \left\{\frac{26}{3} + 6\left(\frac{i\nu}{2}\right) + \frac{4}{5}\left(\frac{i\nu}{2}\right)^2\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq +$$

$$+ \left\{\frac{2}{3} - \frac{3}{10}\left(\frac{i\nu}{2}\right) + \frac{157}{30}\left(\frac{i\nu}{2}\right)^2 - \frac{53}{10}\left(\frac{i\nu}{2}\right)^3 - \left(\frac{i\nu}{2}\right)^4\right\} e^{-i\nu/2} -$$

$$- \left\{\frac{8}{15} - \frac{25}{6}\left(\frac{i\nu}{2}\right)^3 + \frac{9}{2}\left(\frac{i\nu}{2}\right)^4 + \left(\frac{i\nu}{2}\right)^5\right\}\int_{\nu/2}^{\infty}\frac{e^{-iq}}{q}\,dq -$$

$$\left. - \left(\frac{i\nu}{2}\right)^3\left\{\frac{13}{9} - \frac{1}{4}\left(\frac{i\nu}{2}\right) + \frac{4}{5}\left(\frac{i\nu}{2}\right)^2\right\}\int_{\nu/2}^{\infty}\frac{\log\left(q - \frac{\nu}{2}\right)}{q}\,e^{-iq}dq\right].$$

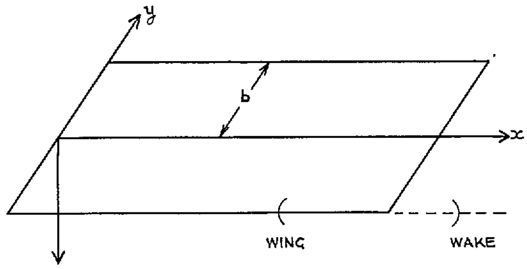

FIG. 1. The wing and co-ordinate system.

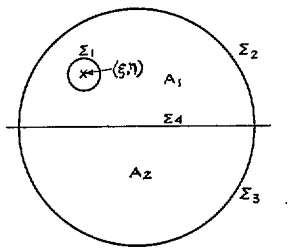

FIG. 2. The areas A_1 and A_2 described in Appendix I.

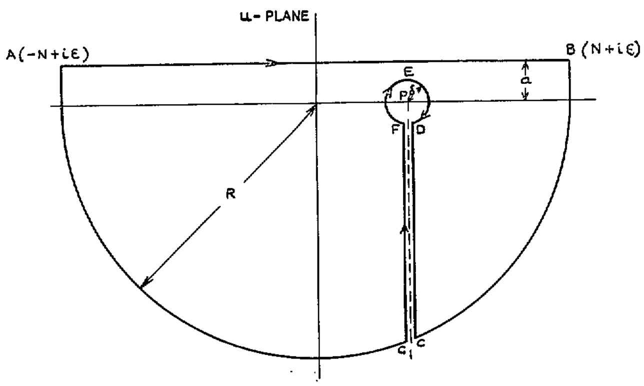

FIG. 3. The contour in the u-plane.

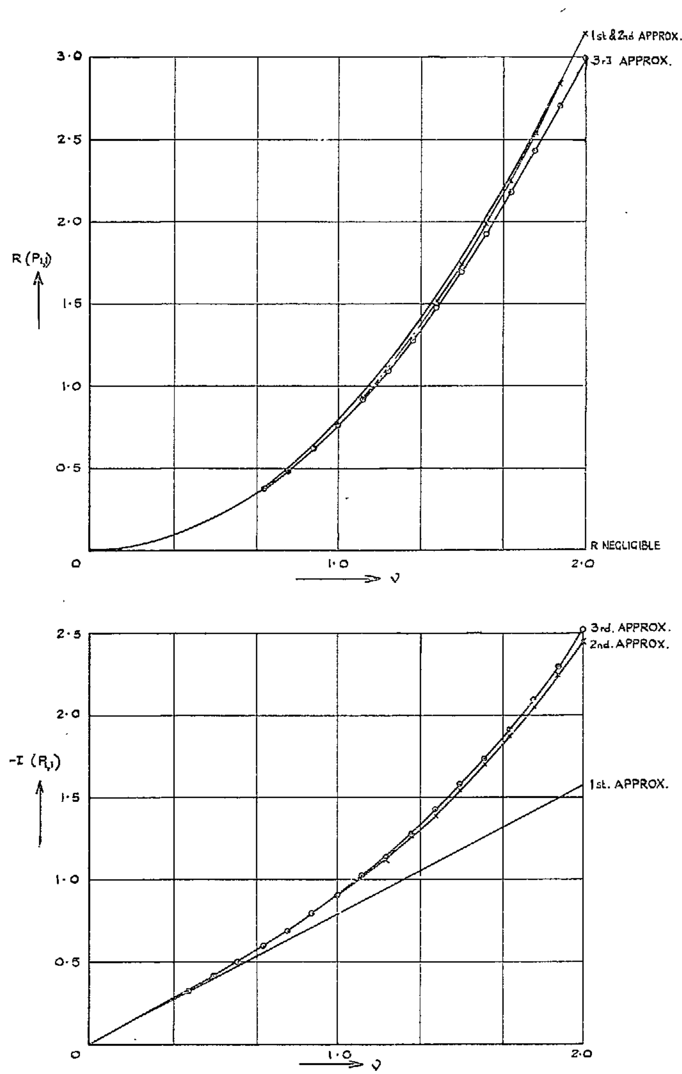

FIG. 4. Graphs of $P_{1,1}$ for $A = 0 \cdot 5$.

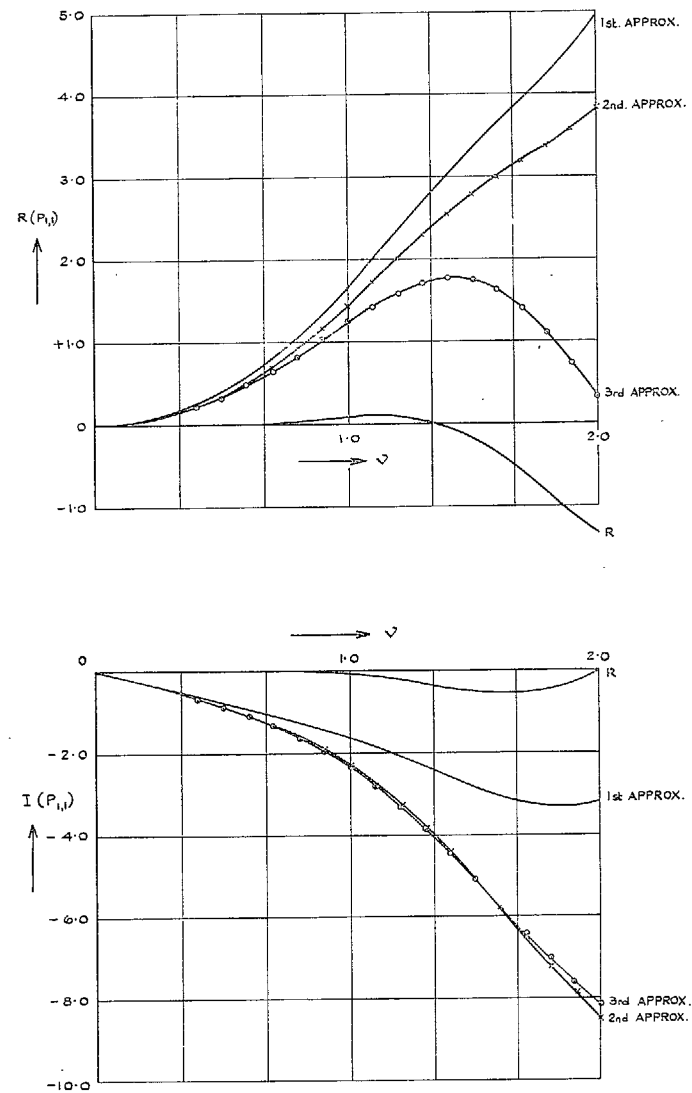

FIG. 5. Graphs of $P_{1,1}$ for $A = 1 \cdot 0$.

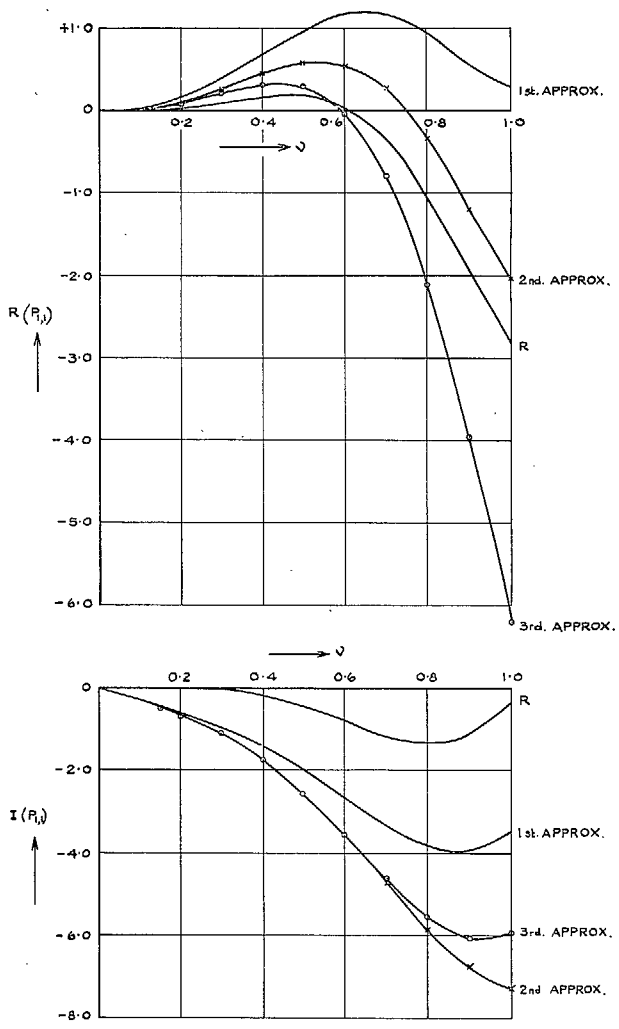

FIG. 6. Graphs of $P_{1,1}$ for $A = 2{\cdot}0$.

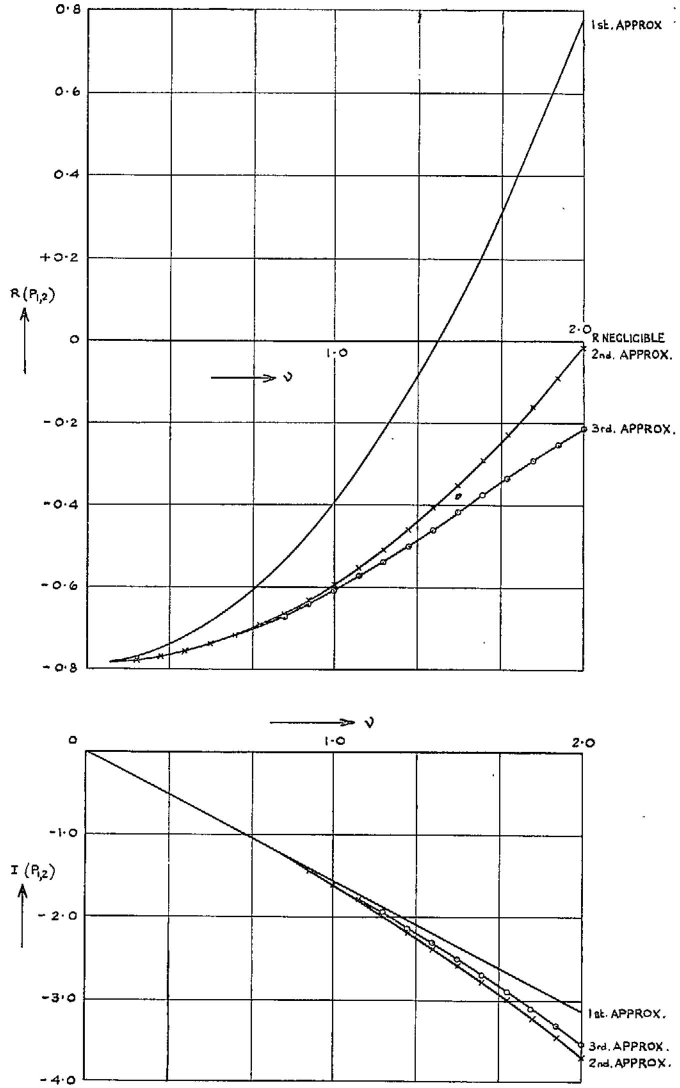

FIG. 7. Graphs of $P_{1,2}$ for $A = 0{\cdot}5$.

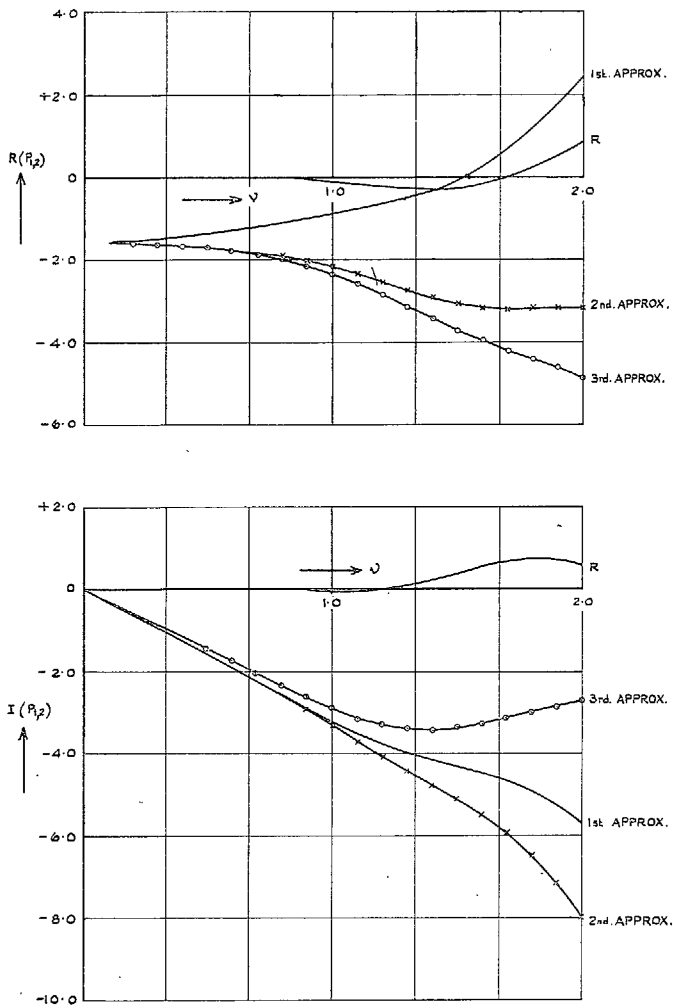

FIG. 8. Graphs of $P_{1,2}$ for $A = 1{\cdot}0$.

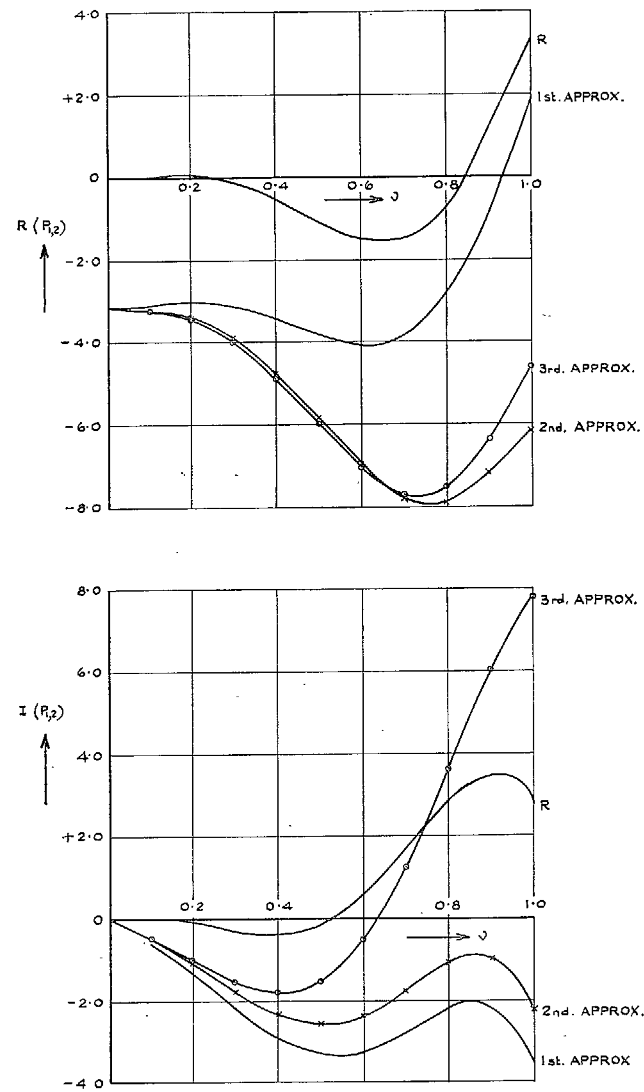

FIG. 9. Graphs of $P_{1,2}$ for $A = 2{\cdot}0$.

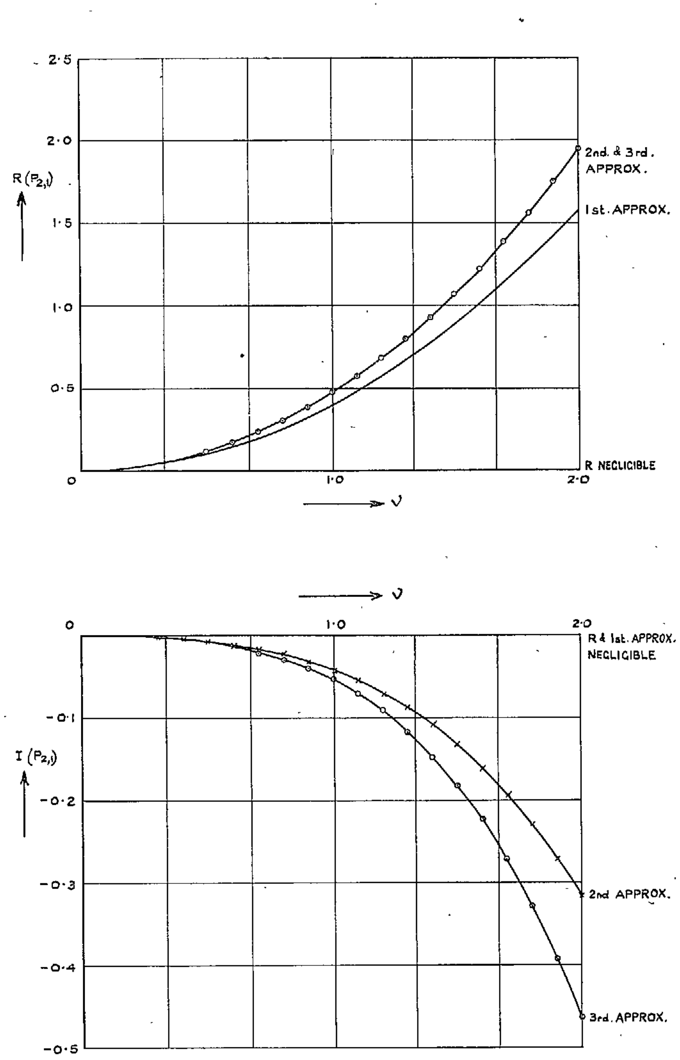

FIG. 10. Graphs of $P_{2,1}$ for $A = 0 \cdot 5$.

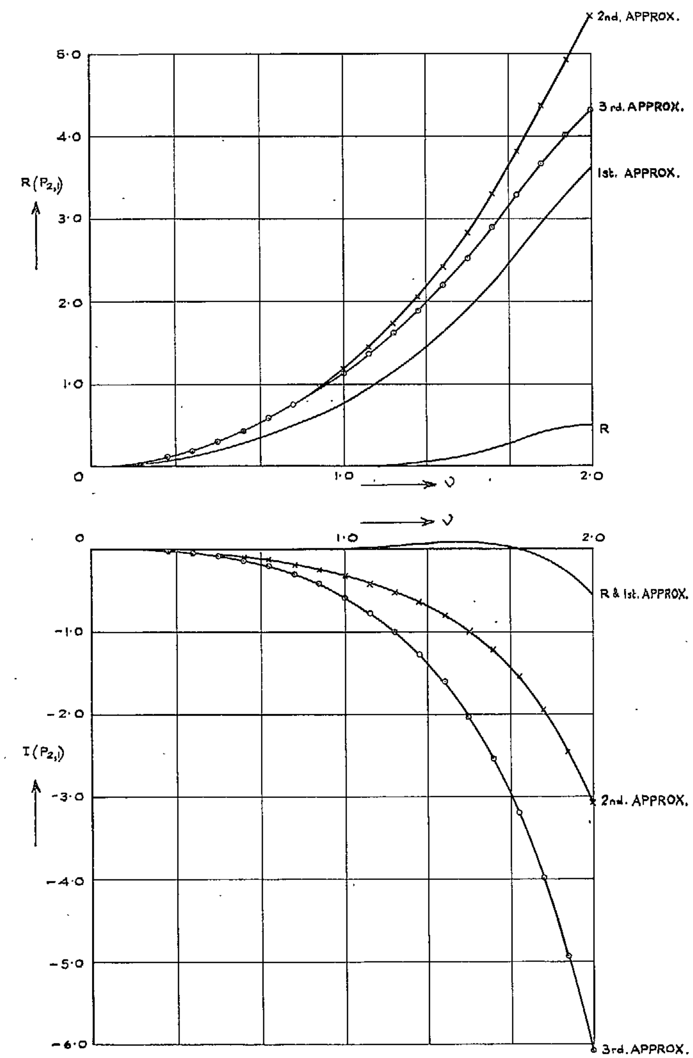

FIG. 11. Graphs of $P_{2,1}$ for $A = 1 \cdot 0$.

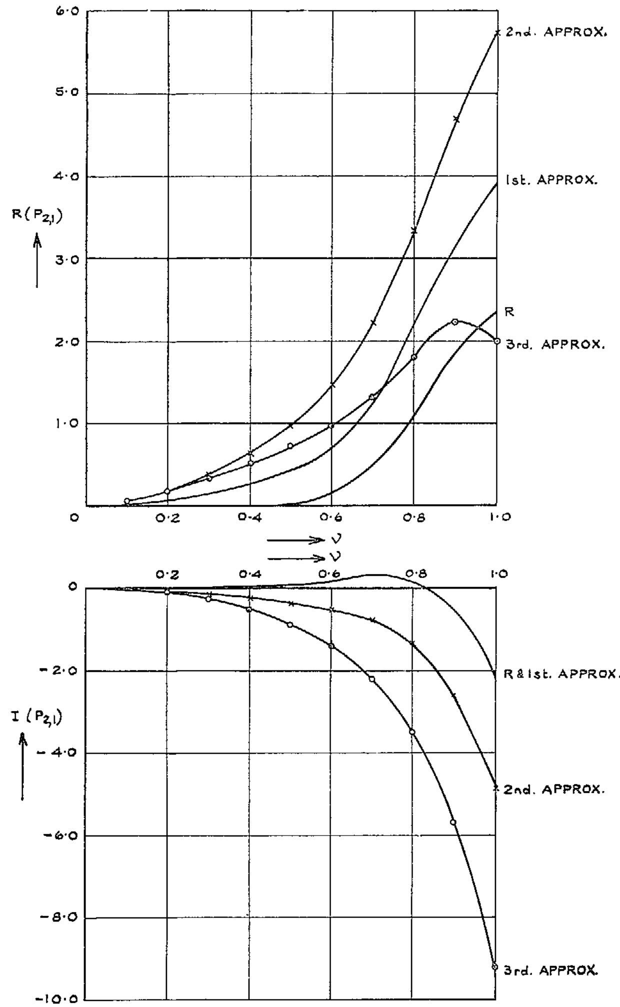

FIG. 12. Graphs of $P_{2,1}$ for $A = 2 \cdot 0$.

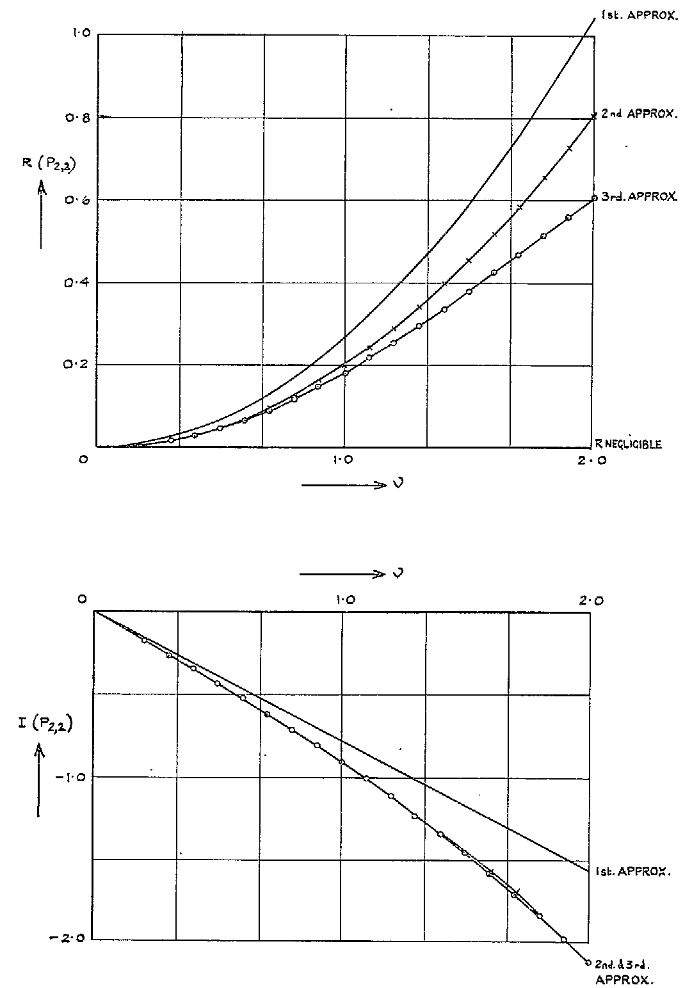

FIG. 13. Graphs of $P_{2,2}$ for $A = 0 \cdot 5$.

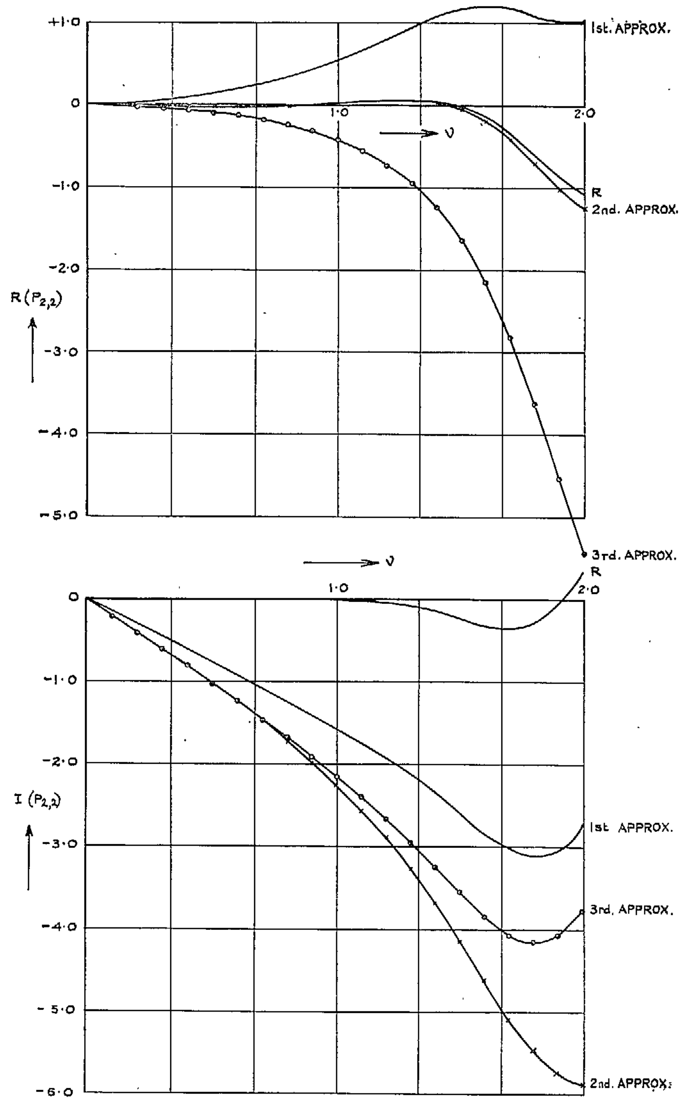

Fig. 14. Graphs of $P_{2,\,2}$ for $A = 1 \cdot 0$.

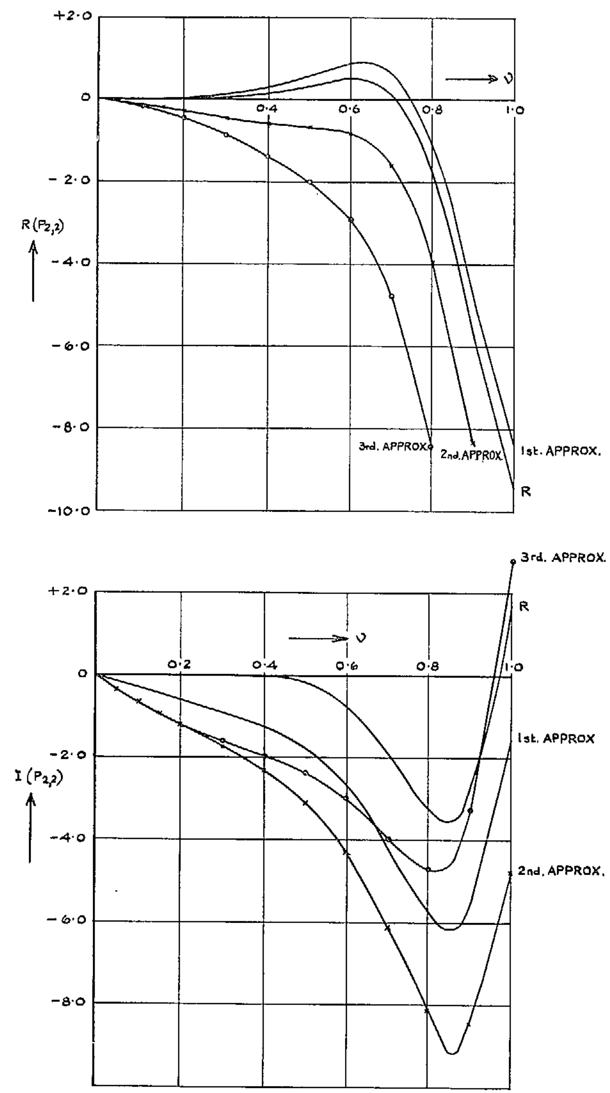

Fig. 15. Graphs of $P_{2\,2}$ for $A = 2 \cdot 0$.

(87657) Wt. 65/1418 K.5 10/63 Hw.

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