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# On Obtaining Solutions to the Navier-Stokes Equations with Automatic Digital Computers

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# On Obtaining Solutions to the Navier-Stokes Equations with Automatic Digital Computers

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## 1. Object.

The purpose of this paper is to show how to obtain steady-state solutions to the Navier-Stokes equations on an automatic digital computer. First the relative merits of various finite-difference formulae are discussed. Thereafter the main part of the paper is concerned with the methods used to solve the finite-difference equations and an investigation is made of all the simpler iterative methods.

## 2. The Navier-Stokes Equations.

The Navier-Stokes equations for steady incompressible isothermal flow in two dimensions may be expressed as follows:

$$\nu \nabla_{xy}^4 \psi + \psi_x \nabla_{xy}^2 \psi_y - \psi_y \nabla_{xy}^2 \psi_x = 0 \quad (1)$$

which is subject to the boundary conditions

$$\psi_s = \psi_n = 0 \quad (2)$$

along a solid wall. Equation (1) is a non-linear fourth-order partial differential equation, which cannot be solved analytically except for a few very simple boundary shapes. (See, for example, Schlichting, Ref. 18, Chapter 5). Thus its solution must generally be carried out numerically. The first person to do this for the complete equations was Thom<sup>21</sup>, who calculated the flow past a circular cylinder at Reynolds number 10. He converted equation (1) to the form

$$\nu \nabla_{xy}^2 \zeta + \psi_x \zeta_y - \psi_y \zeta_x = 0 \quad (3a)$$

$$\nabla_{xy}^2 \psi + \zeta = 0 \quad (3b)$$

which he then solved simultaneously for  $\zeta$  and  $\psi$ . Other numerical solutions to the Navier-Stokes equations have since been obtained by Thom<sup>22</sup>, Kawaguti<sup>11,12</sup>, Allen and Southwell<sup>1</sup>, Jenson<sup>9</sup>, Apelt<sup>3</sup> and Lester<sup>13</sup>. Of these only Lester and, in his second paper, Kawaguti had the use of automatic digital computers. Now although many of the earlier techniques can be programmed on to an automatic digital computer, some of them have to be modified to make the best possible use of such an expensive piece of equipment.

Equations (3a), (3b) are two simultaneous second-order partial differential equations of the elliptic type. However their non-linearity makes them rather more difficult to solve than, for instance, Laplace's equation.

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\* Replaces A.R.C. 23,797.

### 3. Conformal Transformation of the Field.

Undoubtedly the biggest difficulties in solving an elliptic partial differential equation are those associated with the boundaries. Many of these difficulties can be overcome by first transforming the field into a rectangle. By this means the boundaries become straight lines parallel to the axes so that they can be chosen as grid lines. The problem is then solved in the transformed plane. In fact it is easiest to carry out the transformation itself in the transformed plane, as recommended by Thom and Apelt (Ref. 24, p. 43). It is not proposed to elaborate here on the pros and cons of conformal transformation in solving elliptic partial differential equations, since this is not the object of the paper. Suffice to say here that its advantages become even more apparent on an automatic digital computer.

Putting  $w = \xi + i\eta = w(z) = w(x + iy)$ , equations (3a), (3b) become

$$\nu \nabla_{\xi\eta}^2 \zeta + \psi_{\xi} \zeta_{\eta} - \psi_{\eta} \zeta_{\xi} = 0 \quad (4a)$$

$$\nabla_{\xi\eta}^2 \psi + M^2 \zeta = 0 \quad (4b)$$

which are subject to the boundary conditions

$$\psi_{\sigma} = \psi_{\nu} = 0 \quad (5)$$

along a solid wall.  $M$  is the modulus of transformation defined by

$$M = \left| \frac{dz}{dw} \right|.$$

Our transformation is so chosen that the boundaries are parallel to the axes of the transformed plane. Before we can solve equation (4), it is first necessary to find  $M$  as a function of  $\xi$ ,  $\eta$ . However it is not intended to describe here methods of obtaining  $M$ . Instead we proceed directly to the solution of the Navier-Stokes equations in the transformed plane, where  $M$  is known.

### 4. Finite-Difference Approximations to the Navier-Stokes Equations.

The choice of finite-difference equations is always a compromise between accuracy on the one hand and ease of solution on the other. In forming approximations to equations (4a), (4b), there are two types of differential operator to consider:  $\partial/\partial\xi$  and  $\nabla_{\xi\eta}^2$ .

#### 4.1. Approximations to $\partial F/\partial\xi$ .

The simplest and most generally useful approximation to  $\partial F/\partial\xi$  at a grid point  $(j, k)$  is  $\partial f_{j, k}/\partial\xi$  defined by

$$h \frac{\partial}{\partial\xi} f_{j, k} = \delta_{\xi} f_{j, k} \quad (6)$$

which neglects  $O(h^3)$ . A more accurate formula can be derived from the following expression obtained by Bickley<sup>4</sup>.

$$6h \frac{\partial}{\partial\xi} f_{j, k} = \delta_{\xi}(4f_{j, k} + f_{j, k+1} + f_{j, k-1}) - h^3 \nabla_{\xi\eta}^2 \frac{\partial}{\partial\xi} f_{j, k} - \dots$$

Making use of equation (6), we obtain

$$6h \frac{\partial}{\partial\xi} f_{j, k} = \delta_{\xi}(4f_{j, k} + f_{j, k+1} + f_{j, k-1} - h^2 \nabla_{\xi\eta}^2 f_{j, k}) \quad (7)$$

which neglects  $O(h^5)$ . However, the author finds that the time required for the application of equation (7) is about six times as much as that for equation (6). When it is included in an iterative procedure to solve equations (4a), (4b), it more than doubles the total iteration time. Moreover the author

finds it very difficult to obtain a good convergence rate using equation (7). Altogether it is not a convenient formula to use in obtaining solutions to the Navier-Stokes equations. Its only possible use might be as a check on the accuracy, after a solution has been obtained using equation (6).

Formulae involving values at points further than  $h\sqrt{2}$  from  $(j, k)$  are not considered here because they are not applicable next to the boundary, where, with the biggest neglected higher-order derivatives present, the more accurate formulae are most needed. Thus we choose to use equation (6) when solving the Navier-Stokes equations, because it is usually possible to obtain good convergence rates with it. Its accuracy can usually be improved by a reduction in grid size.

#### 4.2. Approximations to $\nabla_{\xi\eta}^2 F$ .

The simplest formula is the five-point approximation (also known as the Liebmann formula and the diamond formula), which replaces  $\nabla_{\xi\eta}^2 F$  at a grid point  $(j, k)$  by  $\nabla_{\xi\eta}^2 f_{j, k}$  defined by

$$h^2 \nabla_{\xi\eta}^2 f_{j, k} = (S_1 - 4) f_{j, k} \quad (8)$$

which neglects

$$-\frac{h^4}{12} (\nabla_{\xi\eta}^4 - 2\mathcal{D}_{\xi\eta}^4) f_{j, k} + 0(h^6).$$

Another well-known formula is the nine-point approximation (also known as the Bickley formula and the 20 formula).

$$6h^2 \nabla_{\xi\eta}^2 f_{j, k} = (4S_1 + S_2 - 20) f_{j, k} \quad (9)$$

which neglects

$$-\left[ \frac{1}{2} h^4 \nabla_{\xi\eta}^4 + \frac{h^6}{60} (\nabla_{\xi\eta}^4 + 2\mathcal{D}_{\xi\eta}^4) \nabla_{\xi\eta}^2 \right] f_{j, k} + 0(h^8).$$

As pointed out by Kantorovich and Krylov<sup>10</sup>, page 211 in Benster's translation, for Laplace's equation the first two neglected terms in equation (9) are zero, which makes it much more accurate than equation (8). Moreover for a more general elliptic differential equation it is not difficult to estimate the first neglected term. Thus

$$h^2 (4 + \frac{1}{2} S_1) \nabla_{\xi\eta}^2 f_{j, k} = (4S_1 + S_2 - 20) f_{j, k} \quad (10)$$

which neglects

$$+\frac{h^6}{2400} (3\nabla_{\xi\eta}^4 - 14\mathcal{D}_{\xi\eta}^4) \nabla_{\xi\eta}^2 f_{j, k} + 0(h^8).$$

For Poisson's equation this is well worth using because the correction terms need be calculated only once at the beginning. However for equations (4a), (4b) it is not worth using except possibly in conjunction with equation (7) as a check on accuracy of a completed solution.

Combining equations (6), (8) we obtain the following Navier-Stokes difference equations:

$$\zeta_{j, k} = 0.25 \left[ S_1 \zeta_{j, k} + \frac{1}{\nu} \{ (\delta_{\xi} \psi) (\delta_{\eta} \zeta) - (\delta_{\eta} \psi) (\delta_{\xi} \zeta) \}_{j, k} \right] \quad (11a)$$

$$\psi_{j, k} = 0.25 [ S_1 \psi_{j, k} + h^2 M_{j, k}^2 \zeta_{j, k} ]. \quad (11b)$$

Equation (5) gives us  $\psi$  along a solid boundary but not  $\zeta$ . Accordingly it becomes necessary to derive a boundary formula for  $\zeta$  using equations (4a), (4b), (5). The simplest is that due to Thom<sup>22</sup>

$$\zeta_{j, 0} = 2(\psi_{j, 0} - \psi_{j, 1}) / (h^2 M_{j, 0}^2) \quad (12)$$

which neglects

$$-\frac{h}{3M_{j, 0}^2} \frac{\partial}{\partial \nu} (M^2 \zeta)_{j, 0} + 0(h^2).$$

More accurate is the formula due to Woods<sup>25</sup>

$$\zeta_{j,0} = 3(\psi_{j,0} - \psi_{j,1})/(h^2 M_{j,0}^2) - 0.5(M^2 \zeta)_{j,1}/M_{j,0}^2 \quad (13)$$

which neglects

$$\frac{1}{8} h^2 \nabla_{\xi\eta}^2 (M^2 \zeta)_{j,0} + O(h^3).$$

This is not much more complicated than equation (12). Its first neglected term may be expressed as

$$\frac{1}{8} h^2 M^2 \{ [L_\sigma \zeta_\sigma - k M \zeta_\nu] + \zeta [(L_\sigma)^2 + k^2 M^2] \}_{j,0}$$

which is easy to estimate if a check on the accuracy is required without reducing the grid size.

The above formulae are all derived by means of Taylor's series, the validity of which depends on the function and all its derivatives being continuous and finite in the region. Thus if a singularity exists, the formulae must be modified near it. It is not intended to elaborate on this here.

### 5. Solving Linear Elliptic Difference Equations.

5.1. The non-linearity of the Navier-Stokes difference equations makes their solution possible only by indirect methods. It also makes the study of iterative treatment rather difficult. In order to simplify the problem, we first consider the equation

$$\nabla_{\xi\eta}^2 F + AF_\xi + BF_\eta = 0 \quad (14)$$

where  $A$ ,  $B$  are known constants. The simplest finite-difference approximation to this uses equations (6) and (8).

$$f_{j,k} = \frac{1}{4} [(1-a)f_{j-1,k} + (1+a)f_{j+1,k} + (1-b)f_{j,k-1} + (1+b)f_{j,k+1}] \quad (15)$$

where

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{2}. \quad (16)$$

We will consider the convergence rates of various iterative treatments for the solution of equation (15), assuming that  $f$  is known along the lines  $\xi = 0$ ,  $\xi = ph$ ,  $\eta = 0$ ,  $\eta = qh$ . The problem is thus to find  $f$  at every internal point of a grid of  $p \times q$  squares, i.e. to find  $f_{j,k}$  for

$$\begin{cases} j = 1, 2, \dots, p-1 \\ k = 1, 2, \dots, q-1. \end{cases}$$

Iterative treatments can be divided into two types.

(1) Explicit methods, for which we determine  $f_{j,k}^{(n)}$  by an explicit linear formula, e.g. equation (17). Thus we revise the values of  $f$  point by point.

(2) Implicit methods, for which we determine  $f_{j,k}^{(n)}$  by the solution of a small set of linear simultaneous equations implicit in  $f_{j,k}^{(n)}$  usually for a whole row of points. Thus we revise the value of  $f$  line by line. Each of these types can be further divided into two sub-types:

- (a) Successive methods, which use the most recently calculated value at every point.
- (b) Simultaneous methods, which use only those values calculated during the previous iteration.

## 5.2. Successive explicit methods.

These methods entail visiting every point in the grid in regular succession, calculating a new value of  $f_{j,k}$  at every point using the most recently calculated values at the neighbouring points. Thus on the  $n$ th iteration we put

$$f_{j,k}^{(n)} = (1-\omega)f_{j,k}^{(n-1)} + \frac{\omega}{4} [(1-a)f_{j-1,k}^{(n)} + (1+a)f_{j+1,k}^{(n-1)} + (1-b)f_{j,k-1}^{(n)} + (1+b)f_{j,k+1}^{(n-1)}] \quad (17)$$

where  $\omega$  is the movement or displacement factor. Subtracting  $\omega$  times equation (15) from equation (17) we get the error equation

$$e_{j,k}^{(n)} = (1-\omega)e_{j,k}^{(n-1)} + \frac{\omega}{4} [(1-a)e_{j-1,k}^{(n)} + (1+a)e_{j+1,k}^{(n-1)} + (1-b)e_{j,k-1}^{(n)} + (1+b)e_{j,k+1}^{(n-1)}] \quad (18)$$

where  $e_{j,k}^{(n)}$  is the error in  $f_{j,k}^{(n)}$  defined by

$$e_{j,k}^{(n)} = f_{j,k}^{(n)} - f_{j,k}. \quad (19)$$

Equation (18) applies for

$$\begin{cases} j = 1, 2, \dots, p-1 \\ k = 1, 2, \dots, q-1. \end{cases}$$

Of course

$$e_{0,k} = e_{p,k} = e_{j,0} = e_{j,q} = 0 \quad (20)$$

since we have assumed Dirichlet boundary conditions. We write equation (18) in the form

$$e^{(n)} = Me^{(n-1)}$$

where  $M$  is a matrix of order  $(p-1) \times (q-1)$  and  $e^{(n)}, e^{(n-1)}$  are vectors. We put

$$e^{(n)} = \lambda e^{(n-1)} \quad (21)$$

where  $\lambda$  is an eigenvalue of the matrix  $M$ . We seek solutions of the form

$$e_{j,k} = \sum_{r=1}^{p-1} \sum_{s=1}^{q-1} E_{j,k} \sin \frac{\pi r j}{p} \sin \frac{\pi s k}{q} \quad (22)$$

for

$$\begin{cases} r = 1, 2, \dots, p-1 \\ s = 1, 2, \dots, q-1. \end{cases}$$

Substituting equations (21), (22) in equation (18) we obtain

$$\begin{aligned} & (\lambda - 1 + \omega) E_{j,k} \sin \frac{\pi r j}{p} \sin \frac{\pi s k}{q} \\ &= \frac{\omega}{4} \left\{ \left[ \lambda(1-a) E_{j-1,k} \sin \frac{\pi r(j-1)}{p} + (1+a) E_{j+1,k} \sin \frac{\pi r(j+1)}{p} \right] \sin \frac{\pi s k}{q} + \right. \\ & \quad \left. + \left[ \lambda(1-b) E_{j,k-1} \sin \frac{\pi s(k-1)}{q} + (1+b) E_{j,k+1} \sin \frac{\pi s(k+1)}{q} \right] \sin \frac{\pi r j}{p} \right\}. \quad (23) \end{aligned}$$

For  $a^2 \neq 1$ ,  $b^2 \neq 1$  we may put

$$\begin{aligned} \sqrt{\left[\lambda \left(\frac{1-a}{1+a}\right)\right]} E_{j-1, k} &= \sqrt{\left[\frac{1}{\lambda} \left(\frac{1+a}{1-a}\right)\right]} E_{j+1, k} \\ &= \sqrt{\left[\lambda \left(\frac{1-b}{1+b}\right)\right]} E_{j, k-1} = \sqrt{\left[\frac{1}{\lambda} \left(\frac{1+b}{1-b}\right)\right]} E_{j, k+1} = E_{j, k} \end{aligned} \quad (24)$$

so that equation (23) becomes

$$\lambda - \omega\mu\sqrt{\lambda} - (1-\omega) = 0 \quad (25)$$

which we shall name the eigenvalue equation. It is independent of  $j$ ,  $k$  and so it is the required solution. Here

$$\mu = \frac{1}{2} \left[ \sqrt{(1-a^2) \cos \frac{\pi r}{p}} + \sqrt{(1-b^2) \cos \frac{\pi s}{q}} \right] \quad (26)$$

$$\begin{cases} r = 1, 2, \dots, p-1 \\ s = 1, 2, \dots, q-1. \end{cases}$$

For the trivial case when  $a^2 = 1$  and/or  $b^2 = 1$ , equation (24) must be modified. For example, for  $a = 1$ , we put  $E_{j+1, k} = 0$ , but equations (25), (26) are still valid. Clearly  $\mu$  may be real, imaginary or complex depending on the magnitudes of  $a$ ,  $b$ . There are thus four cases to consider

- (1)  $a^2 \leq 1$ ,  $b^2 \leq 1$ ,  $\mu$  is real.
- (2)  $a^2 < 1$ ,  $b^2 > 1$ ,  $\mu$  is complex.
- (3)  $a^2 > 1$ ,  $b^2 < 1$ ,  $\mu$  is complex.
- (4)  $a^2 \geq 1$ ,  $b^2 \geq 1$ ,  $\mu$  is imaginary

These are illustrated in Fig. 1.

### 5.3. Successive Unit Displacements by Points (*Succ. u.d.p.*).

This method is sometimes named after Gauss<sup>7</sup>, Seidel<sup>1</sup> or Liebmann<sup>16</sup>. Thom<sup>21</sup> refers to it as squaring. It consists of putting  $\omega = 1$  in equation (17). The eigenvalue equation then has the solution

$$\lambda = \mu^2. \quad (27)$$

From equation (21) it is evident that the criterion for convergence is

$$\rho < 1 \quad (28)$$

where  $\rho$  is the spectral radius of  $\lambda$  defined by

$$\rho = |\lambda|_{\max} = \rho(\omega). \quad (29)$$

It represents the factor by which the errors are being reduced every iteration. A convenient measure of the time required for any iterative treatment is  $N$ , the number of iterations required to reduce the errors through a decade.  $N$  is defined by the equation

$$N = \frac{\log(0.1)}{\log(\rho)}. \quad (30)$$

It is related to  $r$ , the rate of convergence, as introduced by Young<sup>26</sup> by

$$N = \frac{2 \cdot 30259}{r}.$$

$N$  is used here in preference to  $r$  because its practical meaning is more immediately apparent.

*Case 1*

$$a^2 \leq 1, \quad b^2 \leq 1 \quad (31)$$

$\mu$  is real and hence  $\lambda$  is real positive.  $\mu$  has a range of values spread either side of zero. Clearly

$$\rho = \frac{1}{4} \left[ \sqrt{(1-a^2)} \cos \frac{\pi}{p} + \sqrt{(1-b^2)} \cos \frac{\pi}{q} \right]^2 \quad (32)$$

which always satisfies equation (28), so that convergence always occurs. For  $a^2 \leq 1, b^2 \leq 1$ .

$$N \approx \frac{2 \cdot 3}{\phi} \propto h^{-2}$$

where

$$\phi = \frac{1}{2} [a^2 + b^2 + \pi^2(p^{-2} + q^{-2})].$$

If we put  $A = B = 0$ , equation (14) becomes Laplace's equation and equation (32) becomes

$$\rho = \frac{1}{4} \left[ \cos \frac{\pi}{p} + \cos \frac{\pi}{q} \right]^2$$

which is the result obtained by Frankel<sup>6</sup>.

*Case 2*

$$a^2 < 1, \quad b^2 > 1. \quad (33)$$

Both  $\mu$  and  $\lambda$  are complex. Equation (27) becomes

$$\lambda = \frac{1}{4} \left[ \sqrt{(1-a^2)} \cos \frac{\pi r}{p} + i\sqrt{(b^2-1)} \cos \frac{\pi s}{q} \right]^2$$

so that  $\rho$  is given by

$$\rho = \frac{1}{4} \left[ (1-a^2) \cos^2 \frac{\pi}{p} + (b^2-1) \cos^2 \frac{\pi}{q} \right]$$

which satisfies equation (28) provided that

$$(1-a^2) \cos^2 \frac{\pi}{p} + (b^2-1) \cos^2 \frac{\pi}{q} < 4. \quad (34)$$

*Case 3*

$$a^2 > 1, \quad b^2 < 1. \quad (35)$$

By comparison with Case 2, we get the convergence criterion

$$(a^2-1) \cos^2 \frac{\pi}{p} + (1-b^2) \cos^2 \frac{\pi}{q} < 4. \quad (36)$$



Case 4

$$a^2 \geq 1, b^2 \geq 1. \quad (37)$$

In this case  $\mu$  is imaginary and  $\lambda$  is real negative. Equation (27) becomes

$$\lambda = -\frac{1}{4} \left[ \sqrt{(a^2-1) \cos \frac{\pi p}{p} + \sqrt{(b^2-1) \cos \frac{\pi q}{q}} \right]^2$$

so that  $\rho$  is given by

$$\rho = \frac{1}{4} \left[ \sqrt{(a^2-1) \cos \frac{\pi}{p} + \sqrt{(b^2-1) \cos \frac{\pi}{q}} \right]^2$$

so that for convergence we require that

$$\sqrt{(a^2-1) \cos \frac{\pi}{p} + \sqrt{(b^2-1) \cos \frac{\pi}{q}} < 2. \quad (38)$$

The convergence criteria, equations (34), (36), (38), are rather complicated and so they are best shown graphically. Fig. 2 shows a graph of  $|a|$  against  $|b|$  for  $\rho = 1$ , for three sizes of square field. Convergence for successive unit displacements by points occurs within the line  $\rho = 1$  and divergence outside. An interesting feature is that for given  $a, b$  the size of the field has little effect on the convergence criteria except for very small fields. This suggests that the convergence criteria might also be valid for non-Dirichlet boundary conditions, since the boundaries have so little influence. Of course for given  $A, B$  halving the grid size halves the value of  $a, b$ . Thus by reducing the grid size, successive unit displacements by points can be made to converge where previously it would have diverged. When we are just inside the line  $\rho = 1$ , the rate of convergence is impractically slow and so it is of interest to see how far inside we must be before a satisfactory rate of convergence is obtained.

Fig. 3 shows for  $p = q = \infty$  graphs of constant  $\rho$ . The values of  $\rho$  are so chosen as to make  $N$  a convenient number. An interesting feature is that for  $a^2 < 1, b^2 < 1$  increasing  $a^2$  and/or  $b^2$  improves the rate of convergence, whereas for  $a^2 > 1, b^2 > 1$  the reverse is true. The graphs are drawn for an infinite field. For finite values of  $p, q$  there will be only a slight improvement in the convergence rate except for  $a^2 \ll 1, b^2 \ll 1$ , when the improvement will be much more marked.

Since for given  $A, B, a, b$  depend on  $h$ , it is always possible to obtain a satisfactory convergence rate by suitable choice of grid size. However this can result in very long iteration times which is undesirable at least in the early stages of the solution. It is thus desirable to seek other ways of improving the convergence rate. Until now we have used unit movement, i.e. we have put  $\omega = 1$ , but this is not necessarily the best value to use.

#### 5.4. Successive Optimum Displacements by Points (*Succ.o.p.d.*).

This method uses the optimum value of  $\omega$ , which we shall designate  $\omega_0$ , in equation (17). Since  $\omega_0$  is frequently less than unity, we shall call the method successive optimum displacements in preference to the much used name due to Young<sup>26</sup>, successive over-relaxation. The term relaxation is rejected in this context because it is already used to describe a process due to Southwell<sup>20</sup> which is very popular amongst workers performing pencil and paper calculations. The terms, extrapolated Gauss-Seidel method, extrapolated Liebmann method, are also discarded as being inept.  $\omega_0$  is defined as being that value of  $\omega$ , which makes  $\rho$  a minimum. The eigenvalue equation has the roots

$$\lambda = \left[ \frac{1}{2} \omega \mu \pm \sqrt{\left\{ \left( \frac{1}{2} \omega \mu \right)^2 + 1 - \omega \right\}} \right]^2 \quad (39)$$

where  $\mu$  is given by equation (26).

### Case 1

For  $\omega = 1$ ,  $1 > \lambda > 0$  so that we expect an increase of  $\omega$  to improve the rate of convergence. For  $\omega > 1$ , we must consider two ranges. The first is

$$\left(\frac{1}{2}\omega\mu\right)^2 - \omega + 1 < 0$$

for which  $\lambda$  consists of two complex conjugates of modulus

$$|\lambda| = \omega - 1 \quad (40)$$

which increases with increase of  $\omega$ . The second range is

$$\left(\frac{1}{2}\omega\mu\right)^2 - \omega + 1 \geq 0$$

for which the two roots of equation (39) are real, the larger of the two having magnitudes greater than that given by equation (40). Thus

$$\rho = \left[\frac{1}{2}\omega\mu' + \sqrt{\left\{\left(\frac{1}{2}\omega\mu'\right)^2 + 1 - \omega\right\}}\right]^2 \quad (41)$$

where

$$\mu' = \frac{1}{2} \left[ \sqrt{(1-a^2)} \cos \frac{\pi}{p} + \sqrt{(1-b^2)} \cos \frac{\pi}{q} \right]. \quad (42)$$

If we put

$$\omega = \frac{2}{1 + \sqrt{(1-\mu'^2)}} = \omega_0 \quad (43)$$

equation (41) reaches the minimum value of

$$\rho = \omega_0 - 1 = \frac{\mu'^2}{(2-\mu'^2) + 2\sqrt{(1-\mu'^2)}}. \quad (44)$$

This is the result obtained by Young<sup>26</sup> for  $a = b = 0$ . For small  $a^2, b^2$

$$N \approx \frac{1 \cdot 15}{\sqrt{\phi}} \propto h^{-1}. \quad (45)$$

This means that, for a given starting solution, the computational labour varies as  $h^{-3}$ . Thus if we halve the grid size, we increase the labour eightfold.

### Case 2

$\mu$  in equation (39) becomes

$$\mu = \frac{1}{2} \left[ \sqrt{(1-a^2)} \cos \frac{\pi r}{p} + i\sqrt{(b^2-1)} \cos \frac{\pi s}{q} \right] = \mu_1 + i\mu_2 \text{ (say)}$$

so that

$$\lambda = \left[ \left\{ \frac{1}{2}\omega(\mu_1 + i\mu_2) \right\}^2 + 1 - \omega \right]^2$$

It is possible to find an analytical equation for  $|\lambda|$ , to differentiate it with respect to  $\omega$ , to equate the result to zero and thus to solve for  $\omega_0$  and to find the corresponding value of  $\rho$ . However the equation for  $\omega$  is a polynomial of degree ten with very unwieldy coefficients and so it becomes preferable to find  $\omega_0$  and  $\rho$  numerically for particular values of  $\mu'$ , which is given by

$$\mu' = \frac{1}{2} \left[ \sqrt{(1-a^2)} \cos \frac{\pi}{p} + i\sqrt{(b^2-1)} \cos \frac{\pi}{q} \right] = \mu'_1 + i\mu'_2. \quad (46)$$

In Table 1 are given  $\omega_0, \rho, N$  for a range of values of  $\mu'_1, \mu'_2$ .

Case 3

This is the same as Case 2 except that equation (46) is replaced by

$$\mu'_1 + i\mu'_2 = \frac{1}{2} \left[ i\sqrt{(a^2-1)} \cos \frac{\pi}{p} + \sqrt{(1-b^2)} \cos \frac{\pi}{q} \right].$$

TABLE 1a

Values of  $10^4\omega_0$

$\mu'_2$	$\mu'_1$	0.0	0.1	0.2	0.3	0.4	0.5
0		10000	10026	10103	10237	10436	10719
0.5		9442	9312	9285	9282	9276	9233
1		8284	7969	7772	7573	7339	7043
2		6180	5762	5476	5193	4892	4556
4		3904	3555	3318	3093	2864	2622
8		2207	1983	1833	1694	1555	1411
16		1174	1048	964	887	811	733

TABLE 1b

Values of  $10^4\rho$

$\mu'_2$	$\mu'_1$	0.0	0.1	0.2	0.3	0.4	0.5
0		0	26	103	237	436	719
0.5		558	1316	2004	2771	3642	4630
1		1716	2902	3808	4701	5602	6509
2		3820	5109	5953	6699	7381	8007
4		6096	7087	7673	8157	8574	8939
8		7793	8409	8753	9027	9257	9453
16		8826	9168	9355	9501	9621	9723

TABLE 1c

Values of  $N$

$\mu'_1$	$\mu'_2$	0.0	0.1	0.2	0.3	0.4	0.5
	0	0.0	0.4	0.5	0.6	0.7	0.9
	0.5	0.8	1.1	1.4	1.8	2.3	3.0
	1	1.3	1.9	2.4	3.1	4.0	5.4
	2	2.4	3.4	4.4	5.7	7.6	10.4
	4	4.7	6.7	8.7	11.3	15.0	20.5
	8	9.2	13.3	17.3	22.5	29.8	40.9
	16	18.4	26.5	34.5	44.9	59.6	81.8

Case 4

$$\mu = \frac{i}{2} \left[ \sqrt{(a^2-1)} \cos \frac{\pi r}{p} + \sqrt{(b^2-1)} \cos \frac{\pi s}{q} \right] = i\mu_2 \text{ (say)}$$

so that equation (39) becomes

$$\lambda = [i\frac{1}{2}\omega\mu_2 \pm \sqrt{1 - \omega - (\frac{1}{2}\omega\mu_2)^2}]^2.$$

We must consider two ranges. The first is

$$(\frac{1}{2}\omega\mu_2)^2 + \omega - 1 > 0$$

for which  $\lambda$  consists of two complex conjugates of modulus

$$|\lambda| = 1 - \omega \tag{47}$$

which increases with decrease of  $\omega$ . The second range is

$$\frac{1}{2}(\omega\mu_2)^2 + \omega - 1 \leq 0$$

for which the two roots of equation (39) are real negative, the larger having modulus greater than that given by equation (47). Thus

$$\rho = [\frac{1}{2}\omega\mu'_2 + \sqrt{\{(\frac{1}{2}\omega\mu'_2)^2 + \omega - 1\}}] \tag{48}$$

where

$$\mu'_2 = \frac{1}{2} \left[ \sqrt{(a^2-1)} \cos \frac{\pi}{p} + \sqrt{(b^2-1)} \cos \frac{\pi}{q} \right].$$

If we put

$$\omega = \frac{2}{1 + \sqrt{(1 + \mu'^2_2)}} = \omega_0 \tag{49}$$

equation (48) reaches the minimum value of

$$\rho = 1 - \omega_0 = \frac{\mu'^2_2}{(2 + \mu'^2_2) + 2\sqrt{(1 + \mu'^2_2)}}. \tag{50}$$

Equation (49) may also be written

$$\omega_0 = \frac{2}{1 + \sqrt{(1 - \mu'^2)}}.$$

which is identical with equation (43).  $\mu'$  is given by equation (42). Unfortunately this relation is not valid for Cases 2 and 3. For large  $a^2, b^2$

$$N \approx 0.58 (|a| + |b|) \propto h \tag{51}$$

which means that the computational labour varies as  $h^{-1}$ . Thus halving the grid size only doubles the computational labour.

Successive optimum displacements by points always converges, which is a distinct improvement over successive unit displacements by points. However, equations (45), (51) and Table 1c do not show readily the effective range of the method and so in Fig. 4 we show plots of constant  $N$ . It will be seen that a good convergence rate is now possible even for values of  $a, b$  ten times and more those for which successive unit displacements by points diverges.

For Cases 1 and 4 we have one equation for  $\omega_0$ , whereas for Cases 2 and 3 we have had to find it numerically. The question thus arises, How important is it to be able to calculate  $\omega_0$  accurately? To answer this question we have plotted  $\rho$  against  $\omega$  for various values of  $a, b$ . These appear in

Figs. 5a, b, c for  $p = q = 20$ . For Case 1 the graphs are, of course, very similar in shape to that plotted by Forsythe and Wasow (Ref. 5, p. 257), for the five-point Laplacian difference equation. They pointed out that it was better to err on the high side when estimating  $\omega_0$  because

$$\left(\frac{d\rho}{d\omega}\right)_{\omega=\omega_0+0} = 1$$

whereas

$$\left(\frac{d\rho}{d\omega}\right)_{\omega=\omega_0-0} = -\infty.$$

This is of course true for  $a = b = 0$ , but as  $a, b$  are increased to unity

$$\left(\frac{d\rho}{d\omega}\right)_{\omega=\omega_0-0} \rightarrow -1$$

so that it matters less and less on which side of  $\omega_0$  we are. But even for very small  $a, b$  the effect of underestimating  $\omega_0$  is only to slow up the convergence of the solution. It does not prevent it.

For Cases 2 and 3 it will be seen that  $(d^2\rho/d\omega^2)_{\omega=\omega_0}$  is finite, so that an error of 5% to 10% in estimating  $\omega_0$  is not going to matter. However it is vital not to overestimate  $\omega_0$  too much because to do so makes  $\rho > 1$  with consequential divergence.

For Case 4 the situation is even more drastic. For instance, for  $a = b = 4, p = q = 20, \omega_0 = 0.4037$  which makes  $\rho = 0.5963$ . But for  $\omega = 1.03\omega_0, \rho = 1.0151 > 1$ . Thus in Case 4 it is fatal to overestimate  $\omega_0$ . In fact it is better to underestimate  $\omega_0$  by 50% or more, than to overestimate it by only 3%.

As mentioned before, the determination of  $\omega_0$  is not simple as there are four cases to consider, and for two of these no analytical relation has been found. Consequently the author sought an empirical formula for  $\omega_0$  to cover all four cases. The result is given below

$$\omega_0 = \frac{2}{1 + \sqrt{\phi}} \quad (52)$$

where  $\phi = \frac{1}{2}[a^2 + b^2 + \pi^2(p^{-2} + q^{-2})]$ . Clearly equation (52) is a much more convenient form and it reduces the effectiveness of successive optimum displacements by points only slightly as shown in Fig. 6.

### 5.5. Simultaneous Explicit Methods.

So far we have considered successive explicit methods using unit and optimum movements. There are of course several other possible treatments. The simplest of these have been considered in this and ensuing sections. For brevity we have included little more than the actual results of the analyses.

Simultaneous explicit methods entail visiting every point in the grid, calculating a new value of  $f_{j,k}$  using at the neighbouring points only those values calculated during the previous iteration. Thus the order of visiting is immaterial. In fact it would be possible to visit the points simultaneously. Thus on the  $n$ th iteration we put

$$\begin{aligned} f_{j,k}^{(n)} = & (1 - \omega)f_{j,k}^{(n-1)} + \frac{\omega}{4} [(1 - a)f_{j-1,k}^{(n-1)} + (1 + a)f_{j+1,k}^{(n-1)} + \\ & + (1 - b)f_{j,k-1}^{(n-1)} + (1 + b)f_{j,k+1}^{(n-1)}]. \end{aligned} \quad (53)$$

The eigenvalue equation is

$$\lambda = \omega\mu + 1 - \omega. \quad (54)$$

### 5.6. Simultaneous Unit Displacements by Points (Sim.u.d.p.).

This method is sometimes named after Jacobi<sup>8</sup>. It consists of putting  $\omega = 1$  in equation (53). The eigenvalue equation then becomes

$$\lambda = \mu. \quad (55)$$

Comparison with equation (27) shows that

$$\lambda_{\text{Sim. u. d. p.}} = \sqrt{\lambda_{\text{Succ. u. d. p.}}}$$

from which we may immediately deduce that the convergence criteria for the two methods are identical and that Sim.u.d.p. is only half as fast as Succ.u.d.p. when convergence occurs. (See Figs. 2 and 3.)

### 5.7. Simultaneous Optimum Displacements by Points (Sim.o.d.p.).

This method uses  $\omega = \omega_0$  in equation (53).

Case 1

$$\omega_0 = 1 \quad (56)$$

which makes

$$\rho = \mu'. \quad (57)$$

For small  $a^2, b^2$

$$N \approx \frac{4 \cdot 6}{\phi} \propto h^{-2}. \quad (58)$$

This means that for a given starting solution, the computational labour varies as  $h^{-4}$ . Thus if we halve the grid size, we increase the labour sixteenfold.

Cases 2, 3 and 4

$$\omega_0 = \frac{(1 - \mu'_1)}{\mu'^2_2 + (1 - \mu'_1)^2} \text{ or } 1 \quad (59)$$

(whichever be least).

$$\rho = \frac{\mu'_2}{\sqrt{[\mu'^2_2 + (1 - \mu'_1)^2]}} \quad (60)$$

or

$$\sqrt{(\mu'^2_1 + \mu'^2_2)} \text{ if } \omega_0 = 1.$$

In fact equations (59), (60) also hold for Case 1. For large  $\mu'_2$

$$N \approx 4 \cdot 6 \mu'^2_2 \propto h^2. \quad (61)$$

This means that the computational labour is independent of the grid size. This of course assumes that the chosen value of  $h$  makes  $a^2$  and/or  $b^2$  large. For comparison with successive optimum displacements by points, we have plotted in Fig. 7 lines of constant  $N$  for simultaneous optimum displacements by points. Comparison with Fig. 6 will show that the successive method is faster over the complete range of  $a, b$ .

### 5.8. Successive Methods Implicit by Lines.

Now we will consider methods which involve solving simultaneously for all the values of  $f$  along the line  $\eta = kh$ . (This necessitates the inversion of a tridiagonal matrix of order  $p - 1$ ). We will visit

each  $\eta$  line in the grid in regular succession, calculating new values of  $f$  along the line using the most recently calculated values along neighbouring lines. Thus on the  $n$ th iteration we put

$$\begin{aligned} & [(1-a)f_{j-1,k}^{(n)} - 4f_{j,k}^{(n)} + (1+a)f_{j+1,k}^{(n)}] \\ &= (1-\omega)[(1-a)f_{j-1,k}^{(n-1)} - 4f_{j,k}^{(n-1)} + (1+a)f_{j+1,k}^{(n-1)}] - \\ & \quad - \omega[(1-b)f_{j,k-1}^{(n)} + (1+b)f_{j,k+1}^{(n-1)}]. \end{aligned} \quad (62)$$

The eigenvalue equation is

$$\lambda - \omega\nu\sqrt{\lambda} - (1-\omega) = 0 \quad (63)$$

where

$$\nu = \frac{\sqrt{(1-b^2)} \cos \frac{\pi s}{q}}{2 - \sqrt{(1-a^2)} \cos \frac{\pi r}{p}} \quad (64)$$

for

$$\begin{cases} r = 1, 2, \dots, p-1 \\ s = 1, 2, \dots, q-1. \end{cases}$$

It is interesting to compare equation (63) with equation (25). There are four cases to consider:

- Case 1. {See equation (31)}  $\nu$  is real.
- Case 2. {See equation (33)}  $\nu$  is imaginary.
- Case 3. {See equation (35)}  $\nu$  is complex.
- Case 4. {See equation (37)}  $\nu$  is complex.

#### 5.9. Successive Unit Displacements Implicit by Lines.

This method consists of putting  $\omega = 1$  in equation (62). The eigenvalue equation then has the solution

$$\lambda = \nu^2 \quad (65)$$

Case 1

$\nu$  is real and so  $\lambda$  is real positive. Clearly

$$\rho = \nu'^2 \quad (66)$$

which always satisfies equation (28), so that convergence always occurs. Here

$$\nu' = \frac{\sqrt{(1-b^2)} \cos \frac{\pi}{q}}{2 - \sqrt{(1-a^2)} \cos \frac{\pi}{p}} \quad (67)$$

For  $a^2 \ll 1$ ,  $b^2 \ll 1$

$$N \approx \frac{1.15}{\phi} \propto h^{-2}. \quad (68)$$

*Case 2*

$\nu$  is imaginary so that  $\lambda$  is real negative.

$$\rho = \frac{\mu'_2{}^2}{(1-\mu'_1)^2}$$

so that convergence will occur if

$$\sqrt{(1-a^2)} \cos \frac{\pi}{p} + \sqrt{(b^2-1)} \cos \frac{\pi}{q} < 2. \quad (69)$$

*Case 3*

$\nu$  is complex so that  $\lambda$  is complex.

$$\rho = \mu'_1{}^2 = \frac{1}{4} (1-b^2) \cos^2 \frac{\pi}{q} \quad (70)$$

which always satisfies equation (28), so that convergence always occurs. Moreover  $\rho$  never exceeds one quarter, so that  $N$  never exceeds two.

*Case 4*

$\nu$  is complex so that  $\lambda$  is complex.

$$\rho = \frac{1}{4} (b^2-1) \cos^2 \frac{\pi}{q}$$

which satisfies equation (28) if

$$(b^2-1) \cos^2 \frac{\pi}{q} < 4. \quad (71)$$

In Fig. 8 is shown the convergence range of successive unit displacements implicit by lines. Comparing this with the range of successive unit displacements by points (Fig. 2) we see that the two criteria are very different. For Case 2 the point method has a larger convergence range. For Case 3 the line method has no restrictions and so is far superior. (Of course the criteria for Cases 2 and 3 may be interchanged by using a perpendicular direction of sweeping the grid. That is, we could solve simultaneously for all the values of  $f$  along the line  $\xi = jh$  instead of along the line  $\eta = kh$ .) For Case 4 the line method has a larger convergence range with no restrictions on  $a^2$ .

5.10. *Successive Optimum Displacements Implicit by Lines.*

This method uses  $\omega = \omega_0$  in equation (62). The eigenvalue equation then has the roots

$$\lambda = [\frac{1}{2}\omega\nu \pm \sqrt{\{(\frac{1}{2}\omega\nu)^2 + 1 - \omega\}^2}] \quad (72)$$

cf. equation (39).

*Case 1*

By analogy with Case 1 for successive optimum displacements by points, we can see that

$$\omega_0 = \frac{2}{1 + \sqrt{(1-\nu^2)}} \quad (73)$$

$$\rho = \omega_0 - 1 \quad (74)$$

which is the result obtained by Forsythe and Wasow (Ref. 5, p. 271) for  $a = b = 0$ . For small  $a^2, b^2$

$$N \approx \frac{1.15}{\sqrt{\phi}} \propto h^{-1}. \quad (75)$$



Case 2

By analogy with Case 4 for successive optimum displacements by points, we can see that

$$\omega_0 = \frac{2}{1 + \sqrt{(1 + \nu'^2_2)}} = \frac{2}{1 + \sqrt{(1 - \nu'^2)}} \quad (76)$$

$$\rho = 1 - \omega_0. \quad (77)$$

For large  $b^2$

$$N \approx 1.15 |b| \propto h. \quad (78)$$

Case 3

For  $\omega = 1$ ,  $\rho < \frac{1}{4}$  so that if  $\omega = 1$  is not optimum, then it is certainly very good.

Case 4

This, like Cases 2 and 3, for successive optimum displacements by points, is very difficult to analyse. As yet the author has found no empirical formula for  $\omega_0$  comparable with equation (52).

$$\omega_0 = \frac{2}{1 + \sqrt{(2\phi)}} \quad (79)$$

is good for Cases 1 and 2 but not for Cases 3 and 4.

5.11. *Simultaneous Methods Implicit by Lines.*

These methods entail visiting every  $\eta$  line in the grid, calculating simultaneously new values of  $f$  all along the line, using along neighbouring lines only those values calculated during the previous iteration. Thus the order of visiting is immaterial. In fact we may visit every line simultaneously. Thus on the  $n$ th iteration we put

$$\begin{aligned} f_{j,k}^{(n)} = (1 - \omega)f_{j,k}^{(n-1)} + \frac{\omega}{4} [(1 - a)f_{j-1,k}^{(n)} + (1 + a)f_{j+1,k}^{(n)} + \\ + (1 - b)f_{j,k-1}^{(n-1)} + (1 + b)f_{j,k+1}^{(n+1)}] \end{aligned} \quad (80)$$

(but see also Appendix). The eigenvalue equation is

$$\lambda = \frac{\omega\beta + 1 - \omega}{1 - \omega\alpha} \quad (81)$$

where

$$\left. \begin{aligned} 2\alpha &= \sqrt{(1 - a^2)} \cos \frac{\pi r}{p} \\ 2\beta &= \sqrt{(1 - b^2)} \cos \frac{\pi s}{q} \end{aligned} \right\} \quad (82)$$

5.12. *Simultaneous Unit Displacements Implicit by Lines.*

This method consists of putting  $\omega = 1$  in equation (80). The eigenvalue equation then becomes

$$\lambda = \nu. \quad (82a)$$

Comparison with equation (65) shows that the convergence criteria for this method is identical with that for successive unit displacements implicit by lines and that it is only half as fast when convergence occurs. (See Fig. 8.)

### 5.13. Simultaneous Optimum Displacements Implicit by Lines.

This method consists of putting  $\omega = \omega_0$  in equation (90).

Case 1

$$\omega_0 = 1 \quad (83)$$

which makes

$$\rho = \nu'. \quad (84)$$

For small  $a^2, b^2$

$$N \sim \frac{2 \cdot 3}{\phi} \propto h^{-2}. \quad (85)$$

Cases 2 and 4

$$\omega_0 = \frac{(1 - \mu'_1)}{\mu'^2_2 + (1 - \mu'_1)^2} \quad (86)$$

$$\rho = \frac{\mu'_2}{\sqrt{[\mu'^2_2 + (1 - \mu'_1)^2]}} \quad (87)$$

Case 3

$$\omega_0 = 1 \quad (88)$$

$$\rho = \mu'_1. \quad (89)$$

Fig. 9 shows lines of constant  $N$  for simultaneous optimum displacements implicit by lines. It will be seen that this method is very rapid for Case 3 but very slow for Case 2, so that the direction of sweeping is all important. For Case 1 this method is definitely inferior to successive optimum displacements by points. For Case 4 the relative merits are not so obvious and will depend on the computer available.

### 5.14. Conclusions.

Eight iterative treatments for solving equation (15) have been considered and the question naturally arises as to which is the best to use on an automatic digital computer. The requirements of a method are that:

(1) It should converge for the complete range of  $a, b$ .

(2) It should provide a good rate of convergence particularly for Case 1, since as the grid size is reduced, the point  $(a, b)$  moves towards the origin.

(3) The iteration time should not be too great.

The first requirement immediately rules out all those methods using unit movement, since they all suffer from divergence for some values of  $a, b$ . As yet it also rules out successive optimum displacements implicit by lines, because we do not know what movement to use for Case 4. Of the three remaining methods, the one that fulfils the second requirement best is successive optimum displacements by points. This method also fulfils the third requirement. All considered, the best method does seem to be successive optimum displacements by points, although a more successful investigation into its line counterpart might well promote that method, particularly for use on a computer with a small high-speed store and a long access time to the auxiliary store.

6. *Solving the Navier-Stokes Difference Equations.*

In the previous section we considered eight ways of solving equation (15) for constant  $a, b$ . We found that generally the best method to use was successive optimum displacements by points. Comparing equation (11a) with equation (15) we see that

$$f \equiv \zeta, \quad a \equiv -\frac{1}{2\nu} \delta_\eta \psi, \quad b \equiv \frac{1}{2\nu} \delta_\xi \psi \quad (90)$$

but there are three ways in which equation (11a) differs from equation (15):

- (1)  $a, b$  (as given above) are not constant but vary with  $\xi, \eta$ .
- (2)  $a, b$  (as given above) are unknown initially.
- (3) the boundary values of  $\zeta$  are unknown initially.

We will take care of these differences one by one. First let us extend our consideration to the case where  $a, b$  are known functions of  $\xi, \eta$ . Equation (15) then becomes

$$f_{j,k} = \frac{1}{4} [(1-a_{j,k})f_{j-1,k} + (1+a_{j,k})f_{j+1,k} + (1-b_{j,k})f_{j,k-1} + (1+b_{j,k})f_{j,k+1}]. \quad (91)$$

It will be remembered that for certain values of  $a, b$  the implicit methods were particularly good. However, now that  $a, b$  vary all over the field this is no longer so useful. Accordingly we try successive optimum displacements by points for solving<sup>β</sup> equation (91). Thus on the  $n$ th iteration we put

$$f_{j,k}^{(n)} = (1-\omega_{j,k})f_{j,k}^{(n-1)} + \frac{\omega_{j,k}}{4} [(1-a_{j,k})f_{j-1,k}^{(n)} + (1+a_{j,k})f_{j+1,k}^{(n-1)} + (1-b_{j,k})f_{j,k-1}^{(n)} + (1+b_{j,k})f_{j,k+1}^{(n-1)}]. \quad (92)$$

The problem is to decide on the best value(s) for  $\omega_{j,k}$ . To find this out a series of numerical tests were carried out on a small square field ( $p = q = 10$ ) with known boundary values. First of all we found out the effect of using a constant value of  $\omega$  all over the field. Now it will be noticed that, except for very small  $a, b$ ,  $\omega_0$  in equation (52) is almost independent of  $p, q$ . This means that  $\omega_0$  depends essentially on  $a, b$ , which prompts us to investigate the effect of using  $\omega_{j,k} = (\omega_0)_{j,k}$  given by

$$(\omega_0)_{j,k} = \frac{2}{1 + \sqrt{\phi_{j,k}}} \quad (93)$$

where

$$\phi_{j,k} = \frac{1}{2} [a_{j,k}^2 + b_{j,k}^2 + \pi^2(p^{-2} + q^{-2})]$$

cf. equation (52). The results of the tests are given below in Table 2. The value of  $N$  is calculated as the mean over either six decades or sixty iterations, whichever is completed first.

TABLE 2a

Case 1

$$a = 0.1j, \quad b = 0, \quad p = q = 10$$

$\omega_{j,k}$	1.32	1.36	1.38	1.40	1.44	$(\omega_0)_{j,k}$
$N$	5.2	4.4	3.7	4.0	4.7	3.2

TABLE 2a—continued

$$a = 0.1j, b = 0.1j, p = q = 10$$

$\omega_{j, k}$	1.24	1.32	1.36	1.40	1.48	$(\omega_0)_{j, k}$
$N$	6.0	4.5	3.9	4.1	5.1	3.3

TABLE 2b

Case 2 or 3

$$a = 0.1j - 0.5, b = 0.5k + 1, p = q = 10$$

$\omega_{j, k}$	0.30	0.40	0.47	0.49	0.51	0.53	0.60	0.70	$(\omega_0)_{j, k}$
$N$	16.9	9.5	8.1	7.7	7.8	8.2	18.1	$\infty$	6.6

TABLE 2c

Case 4

$$a = 0.5j + 1, b = 0.5k + 1, p = q = 10$$

$\omega_{j, k}$	0.20	0.30	0.35	0.37	0.38	0.39	0.40	0.50	$(\omega_0)_{j, k}$
$N$	11.9	7.6	6.5	6.1	6.0	5.8	16.7	$\infty$	5.6

TABLE 2d

Mixed Case

$$a = 0.9j - 4.5, b = 0.9k - 4.5, p = q = 10$$

$\omega_{j, k}$	0.40	0.58	0.60	0.62	0.65	0.70	$(\omega_0)_{j, k}$
$N$	6.2	4.7	3.5	3.3	6.7	$\infty$	3.3

From the numerical tests we may conclude that it is better to use  $\omega_{j, k} = (\omega_0)_{j, k}$  than to use any constant value, since this reduces the value of  $N$ . This is very convenient because it saves us from having to find an empirical formula for the best constant value of  $\omega$ . It is not claimed that  $\omega_{j, k} = (\omega_0)_{j, k}$  is the absolute best [for instance, slight improvement can sometimes be obtained by using  $\omega_{j, k} = \sqrt{\{\text{constant} \times (\omega_0)_{j, k}\}}$ ], but it is claimed that it is the best simple procedure to use.

Similar tests were carried out for simultaneous optimum displacements implicit by lines, but this method was eventually abandoned because it was found to be much more difficult to adapt for solving equation (91) particularly when  $a, b$  varied much along the line and because it was much

slower for Case 1. In any case, one of the advantages of the methods implicit by lines is that it is very easy to invert the tridiagonal matrix involved. The inverse can be stored as the product of two bidiagonal matrices, an upper triangular and a lower triangular matrix. For constant  $a, b$  the same matrix is used for every line in the grid. The multiplication of the matrices necessary for sweeping the field takes very little longer than an explicit method. For instance, Young and Ehrlich<sup>27</sup>, using Fortran on an IBM 704 computer, found that for  $a = b = 0$ , a single implicit sweep of the field took only about  $1\frac{1}{2}$  times as long as one explicit iteration. In fact for  $a \neq 0, b \neq 0$  the extra complication affects the explicit method more than it does the implicit method so that the factor of  $1\frac{1}{2}$  becomes more like  $1\frac{1}{5}$ . However if  $a, b$  are initially unknown, as in the Navier-Stokes difference equations, then the tridiagonal matrix must be set up and inverted separately for every grid line for every iteration. This can put up the iteration time for the  $\zeta$  field to as much as  $2\frac{1}{2}$  times that for an explicit method. Hence the implicit methods for solving the Navier-Stokes difference equations are not as attractive as they might at first appear.

We will now investigate what happens when  $a, b$  are initially unknown. We will try using successive optimum displacements by points to solve equations (11a), (11b), revising first  $\zeta$  and then  $\psi$  as we visit each point. Thus on the  $n$ th iteration we put

$$\begin{aligned}\zeta_{j,k}^{(n)} = & (1 - \omega_{j,k}^{(n)})\zeta_{j,k}^{(n-1)} + \\ & + \frac{\omega_{j,k}^{(n)}}{4} [(1 - a_{j,k}^{(n)})\zeta_{j-1,k}^{(n)} + (1 + a_{j,k}^{(n)})\zeta_{j+1,k}^{(n-1)} + \\ & + (1 - b_{j,k}^{(n)})\zeta_{j,k-1}^{(n)} + (1 + b_{j,k}^{(n)})\zeta_{j,k+1}^{(n-1)}] \end{aligned} \quad (94a)$$

where

$$\begin{aligned}a_{j,k}^{(n)} = & -\frac{1}{4\nu} (\psi_{j,k+1}^{(n-1)} - \psi_{j,k-1}^{(n)}) \\ b_{j,k}^{(n)} = & +\frac{1}{4\nu} (\psi_{j+1,k}^{(n-1)} - \psi_{j-1,k}^{(n)}) \\ \psi_{j,k}^{(n)} = & (1 - \Omega)\psi_{j,k}^{(n-1)} + \frac{\Omega}{4} [\psi_{j-1,k}^{(n)} + \psi_{j+1,k}^{(n-1)} + \psi_{j,k-1}^{(n)} + \\ & + \psi_{j,k+1}^{(n-1)} + h^2 M_{j,k}^2 \zeta_{j,k}^{(n)}]. \end{aligned} \quad (94b)$$

We put

$$e^{(n)} = \zeta^{(n)} - \zeta \quad (95a)$$

$$d^{(n)} = \psi^{(n)} - \psi \quad (95b)$$

$$\delta a^{(n)} = a^{(n)} - a \quad (95c)$$

$$\delta b^{(n)} = b^{(n)} - b. \quad (95d)$$

We will first assume that

$$|\delta a| \ll |a| \quad (96a)$$

$$|\delta b| \ll |b| \quad (96b)$$

which makes equation (94a) similar to equation (92). Thus the results of the foregoing sections are directly applicable and we may put in equation (94a).

$$\omega_{j,k}^{(n)} = \frac{2}{1 + \sqrt{\phi_{j,k}^{(n)}}} \quad (97)$$

where

$$\phi_{j,k}^{(n)} = \frac{1}{2} [(a^2 + b^2)_{j,k}^{(n)} + \pi^2(p^{-2} + q^{-2})].$$

To try and ensure that equations (96a), (96b) are valid, we must have a reasonable starting solution for  $\psi$ , (e.g. the solution for potential flow or a lower Reynolds number), but if either is invalid, then it is the other equation that is dominant and we may use equation (97). If neither equation (96a) nor (96b) is valid, then  $|a| < 1$ ,  $|b| < 1$  and so we have no divergence problems. That at least is the author's experience.

Next we will assume that

$$h^2 M_{j,k}^2 \ll 1 \quad (98)$$

which makes equation (94b) a Laplacian difference equation so we may put

$$\Omega = \frac{2}{1 + \pi\sqrt{[(p^{-2} + q^{-2})/2]}} \quad (99)$$

Now if equation (98) is not satisfied, the above movement can lead to divergence, in which case it will be necessary to reduce  $\Omega$ . The author is of the opinion that an expression of the type

$$\Omega_{j,k}^{(n)} = \Omega_0 \exp \left\{ -ch^2 M_{j,k}^2 \frac{|\zeta^{(n-1)} - \zeta^{(n-2)}|_{\max}}{|\psi^{(n-1)} - \psi^{(n-2)}|_{\max}} \right\} \quad (100)$$

might give satisfactory results. Here  $c$  is a positive constant, which must be chosen by trial and error, and  $\Omega_0$  is given by equation (99). This is of course essentially empirical, but the governing idea behind it is to dampen out the effect on equation (94b) of changes in  $\zeta_{j,k}$ . In practice equation (98) fails to be satisfied in only two circumstances: if the grid is very coarse or if the boundary is transformed into a circle, as opposed to a slit, so that  $M$  is very large away from the boundary.

For the Navier-Stokes difference equations (11a), (11b), Thom and Apelt<sup>23</sup>, and Lester<sup>14</sup> have derived convergence criteria and it is of interest to compare their results with those obtained here. Since Lester's analysis follows basically the same pattern as Thom and Apelt's, the two are described together, with those parts due solely to Lester appearing in square brackets. Their analyses were equivalent to assuming that  $\zeta, \psi$  were known along the four lines defined by

$$(\xi \pm \eta) = \pm 2h.$$

This meant that there were five internal points in a small diamond-shaped field. Initially they assumed that  $e, d$ , the errors in  $\zeta, \psi$  {see equations (95a), (95b)} were zero for all five internal points. Thom and Apelt then put

$$e_{0,0}^{(1)} = \epsilon$$

$$\left[ \begin{array}{l} \text{whereas Lester put in addition} \\ d_{0,0}^{(1)} = \delta. \end{array} \right]$$

They then calculated:  $\zeta_{1,0}^{(1)}, \psi_{1,0}^{(1)}, \zeta_{0,1}^{(1)}, \psi_{0,1}^{(1)}, \zeta_{0,-1}^{(2)}, \psi_{0,-1}^{(2)}, \zeta_{-1,0}^{(2)}, \psi_{-1,0}^{(2)}, \zeta_{0,0}^{(2)}, [\psi_{0,0}^{(2)}]$  using successive unit displacements by points. This gave them  $e_{0,0}^{(2)}, [d_{0,0}^{(2)}]$ . The criteria for convergence of  $\zeta$  [and  $\psi$ ] were

$$|e_{0,0}^{(2)}| < |e_{0,0}^{(1)}|, [|d_{0,0}^{(2)}| < |d_{0,0}^{(1)}|]$$

which became

$$0 \leq h^2 M^2 (u^2 + v^2) / \nu^2 < 40 \quad (101a)$$

for  $\delta = 0$ , and

$$[-96 < h^4 M^4 \{u(\nabla_{\xi\eta}^2 u) + v(\nabla_{\xi\eta}^2 v)\} / \nu^2 < 160] \quad (101b)$$

for  $\epsilon = 0$ . An assumption made en route was that  $u, v$  did not vary much over the field. (This assumption was, of course, made in Section 5 by assuming  $a, b$  to be constant.) This in fact ensures

that equation (101b) is satisfied unless  $hM$  is large. The form of equation (101a) prompted Thom and Apelt to define a 'local mesh Reynolds number', as

$$Re_{\text{mesh}} = hM\sqrt{(u^2 + v^2)}/\nu \quad (102)$$

which to ensure convergence had to be less than  $\sqrt{40}$ . In present notation equation (101a) becomes

$$0 \leq a^2 + b^2 < 10.$$

Incidentally the same criterion is obtained even if we iterate only the  $\zeta$  field, leaving the  $\psi$  field untouched. This is equivalent to solving equation (91). This confirms what was effectively assumed in equation (96), that changing the values of  $a$ ,  $b$  every iteration does not have much effect on the convergence.

Another assumption that they made during their analyses was that  $M$  varies little over the field. It will be seen that the assumption made here in equation (98) is more stringent, but allowance for it is taken in equation (100).

It is interesting to compare their convergence criterion with the present one. This is done in Fig. 10. It will be seen that their criterion is not unlike that obtained here for  $p = q = 3$  (i.e. with four internal points), but it is obvious that their criterion is insufficient for larger fields.

To prevent divergence, Apelt<sup>2</sup> suggested reducing the movement in the  $\zeta$  field putting

$$\left. \begin{aligned} \omega_{j,k}^{(n)} &= 8/[6 + (a^2 + b^2)_{j,k}^{(n)}] \\ \Omega &= 1 \end{aligned} \right\} \quad (103)$$

in equation (94), whereas Lester suggested controlling the movement of both the  $\zeta$  field and the  $\psi$  field using a more complicated procedure. On the  $n$ th iteration he put

$$\left. \begin{aligned} \zeta^{(n)} &= \zeta^{(n-1)} + \delta\zeta \\ \psi^{(n)} &= \psi^{(n-1)} + \delta\psi \end{aligned} \right\} \quad (104)$$

where

$$\left. \begin{aligned} \delta\zeta &= \frac{4Y\Delta\psi + (3 + h^2M^2Y)\Delta\zeta}{3X + 1.25h^2M^2Y + 2.25} \\ \delta\psi &= \frac{(3 + 4X)\Delta\psi + (X - 0.5)h^2M^2\Delta\zeta}{3X + 1.25h^2M^2Y + 2.25} \end{aligned} \right\} \quad (105)$$

where

$$\left. \begin{aligned} \Delta\zeta &= \zeta^{(n)} - \zeta^{(n-1)} \\ \Delta\psi &= \psi^{(n)} - \psi^{(n-1)} \end{aligned} \right\} \quad (106)$$

{as given by putting  $\omega = \Omega = 1$  in equation (94)} and

$$\left. \begin{aligned} X &= \frac{1}{32\nu^2} \{(\delta_\xi\psi)^2 + (\delta_\eta\psi)^2\} = (a^2 + b^2)/8 \\ Y &= \frac{1}{32\nu^2} \{(\delta_\xi\psi)(\delta_\xi\zeta) + (\delta_\eta\psi)(\delta_\eta\zeta)\}. \end{aligned} \right\} \quad (107)$$

The methods of both Apelt and Lester were evolved by observing what was the best procedure at the central point, having previously used unit movement at the four surrounding points. This is a weakness since in practice we apply the same treatment at every point in the field. A less important weakness is that they evolved their treatments for a field with only five internal points, whereas in practice we have much larger fields. However this will have little effect for large  $|a|$ ,  $|b|$ .

### 7. Solving the Boundary Formula.

There is one further feature in which the Navier-Stokes difference equations differ from the preceding analysis and numerical tests. So far we have assumed knowledge of  $\zeta$ ,  $\psi$  along the grid lines  $\xi = 0$ ,  $\xi = ph$ ,  $\eta = 0$ ,  $\eta = qh$ . The assumption that the boundaries are rectangular is perfectly valid because our transformation has ensured that. However the assumption that we know  $\zeta$  and/or  $\psi$  along these boundaries is true for only two types. If we have a centre-line parallel to the undisturbed flow (assumed to be in the  $x$  direction), about which the solid boundaries are symmetrical, then there will be symmetry of  $u$  and antisymmetry of  $v$  and hence  $\zeta = 0$ ,  $\psi = \text{constant}$  (often zero for convenience).

Along a solid boundary  $\psi$  is a known constant, but  $\zeta$  must be calculated by means of equation (13). The method of solving this equation is simply to include the boundary points in our iteration route. Thus on the  $n$ th iteration we put

$$\zeta_{j,0}^{(n)} = (1 - \omega')\zeta_{j,0}^{(n-1)} + \omega'[3(\psi_{j,0} - \psi_{j,1}^{(n-1)})/h^2 - 0.5M_{j,1}^2\zeta_{j,1}^{(n-1)}]/M_{j,0}^2. \quad (108)$$

So as to minimize the effect of the incorrectness of equation (20), we make  $\omega'$  appropriately small. Thom and Apelt (Ref. 24, p. 125) recommend putting

$$\omega' = \frac{1}{2} \quad (109)$$

and, as the tests in Section 8 reveal, this movement can give very good results. However it is rather arbitrary and so we will now investigate what is the optimum value of  $\omega'$  to use.

Let us suppose that we have a field of  $p \times q$  squares and that we are trying to solve equation (11) for  $j = 1(1)[p-1]$ ,  $k = 1(1)[q-1]$  and equation (13) for  $j = 1(1)[p-1]$ ,  $k = 0$ . The conditions we apply along  $j = 0$ ,  $j = p$ ,  $k = q$  do not affect the analysis. Like Thom and Apelt<sup>23</sup> we will suppose that the  $\zeta$ ,  $\psi$  fields are completely settled initially. We then place a disturbance  $\epsilon$  on the value of  $\zeta$  at the point  $(j, 0)$  only, so that

$$\begin{aligned} \zeta_{j,0}^{(1)} &= \zeta_{j,0} + \epsilon \\ e_{j,0}^{(1)} &= \epsilon. \end{aligned} \quad (110)$$

We then apply successive optimum displacements by points, starting at the first point after  $(j, 0)$  in the succession. Having iterated the complete field we eventually arrive back at the point  $(j, 0)$  and find that

$$e_{j,0}^{(2)} = \epsilon \left[ 1 - \omega'_{j,0} \left\{ \left( \frac{M_{j,1}^2}{M_{j,0}^2} \right) \left( \frac{\omega_{j,1}}{16} \right) (3\Omega_{j,1} + 2)(1 - b_{j,1}) + 1 \right\} \right] \quad (111)$$

where  $\omega'_{j,0}$  is the movement used on the boundary {see equation (108)}. The process is considered to be convergent if  $|e_{j,0}^{(2)}| < |e_{j,0}^{(1)}|$ . For successive unit displacements by points (i.e. for  $\omega' = \omega = \Omega = 1$ ), this condition is satisfied provided that

$$1 - 3 \cdot 2 \left( \frac{M_{j,0}^2}{M_{j,1}^2} \right) < b_{j,1} < 1 + 3 \cdot 2 \left( \frac{M_{j,0}^2}{M_{j,1}^2} \right). \quad (112)$$

This is more stringent if the boundary is (physically) convex to the fluid (i.e. if  $M_{j,0} < M_{j,1}$ ). For an  $x, y$  grid  $\xi = x$ ,  $\eta = y$ ,  $M = 1$  and the criterion becomes

$$-8 \cdot 4 < \left( \frac{hv_{j,1}}{v} \right) < 4 \cdot 4 \quad (113)$$

which is the result (with appropriate changes in notation) obtained by Thom and Apelt<sup>23</sup>. Here  $v$  is the velocity measured away from the boundary. We define the optimum movement as being that which makes  $e_{j,0}^{(2)} = 0$  and this is given by

$$(\omega'_{j,0}) = 1 / \left\{ 1 + \left( \frac{M_{j,1}^2}{M_{j,0}^2} \right) \left( \frac{\omega_{j,1}}{16} \right) (3\Omega_{j,1} + 2)(1 - b_{j,1}) \right\}. \quad (114)$$



Of course our original assumption that  $\zeta$  and  $\psi$  are initially well settled all over the field is not true in practice, but nevertheless equation (114) does give good results as is shown in the tests which follow.

Although it is intended for use with successive optimum displacements by points, equation (114) can also be applied with unit movement in the field and with Apelt's method, but it cannot easily be applied for Lester's method because this is a more complicated procedure, which does not correspond directly to the use of movements  $\omega, \Omega$  in equation (94).

Other types of boundaries are fluid boundaries. A common one which is difficult to treat, is an infinite boundary where

$$\left. \begin{aligned} \psi_x &\rightarrow \text{constant} \\ \psi_y &\rightarrow \text{constant} \\ \zeta &\rightarrow 0. \end{aligned} \right\} \quad (115)$$

Another occurs in a flow which is periodic in space, so that we may write

$$\left. \begin{aligned} \psi_{\xi, \eta} &= \pm \psi_{\xi', \eta'} + \text{constant} \\ \zeta_{\xi, \eta} &= \pm \zeta_{\xi', \eta'}. \end{aligned} \right\} \quad (116)$$

These may all require special treatment.

### 8. Numerical Tests.

In the preceding analyses it was necessary to make various assumptions {i.e. equations (20), (96), (98) in Sections 5 and 6 and the assumption that 'the  $\zeta, \psi$  fields are completely settled initially' in Section 7}, because of the non-linearity of the equations and the awkwardness of treating the boundaries. To observe the effect of these assumptions being untrue or only approximately true, a series of numerical tests has been carried out for a simple viscous-flow problem. These tests also serve to compare the different methods of treating the internal points and the boundary points of the field. To permit a large number of runs, the field was kept small. In fact we made  $p = 17$ ,  $q = 8$ . For further convenience, the boundaries were made rectangular in the physical plane so that in fact  $M$  was everywhere unity. The problem was one of steady viscous flow past an infinite array of flat plates, a small part of which is shown in Fig. 11, which also shows the grid used. Since the mean velocity and the length of the plates are both unity, the Reynolds number is given by  $Re = 1/\nu$ . The appropriate formulae are:

for  $j = 1(1)16, k = 1(1)7$

equations (11a), (11b);

for  $j = 1(1)16$

equation (13), but  $\zeta_{16, 0} = 0$

$$\zeta_{j, 8} = 0$$

$$\psi_{j, 0} = 0$$

$$\psi_{j, 8} = 0.5;$$

for  $k = 0(1)8$

$$\zeta_{0, k} = -\zeta_{16, 8-k}$$

$$\zeta_{17, k} = -\zeta_{1, 8-k}$$

$$\psi_{0, k} = -\psi_{16, 8-k} + 0.5$$

$$\psi_{17, k} = -\psi_{1, 8-k} + 0.5.$$

(117)

The following four methods of solving equation (11) were tried:

- (1) Successive unit displacements by points {i.e.  $\omega = \Omega = 1$  in equations (94)}.
- (2) Apelt's method {i.e. equations (94), (103)}.
- (3) Lester's method {i.e. equations (104) to (107)}.
- (4) Successive optimum displacements by points {i.e. equations (94), (97), (99)}.

Where possible the following three movements were used in solving equation (13) with each of the above four methods:

- (a)  $\omega' = 1$
- (b)  $\omega' = 0.5$
- (c)  $\omega'_{j,0} = (\omega'_0)_{j,0}$  {see equation (114)}.

For the initial tests unit movement was used in applying equations (117) along the lines  $j = 0, j = 17$  and the values of  $\zeta, \psi$  were revised as soon as the right-hand sides were altered.

Solutions were obtained at Reynolds numbers of 1, 10, 100, which correspond to mean values of  $\sqrt{(a^2 + b^2)} = Re_{\text{mesh}}/2$  of 0.03, 0.3, 3. The latter is outside the convergence range of successive unit displacements by points according to the present theory as can be seen in Fig. 2, but is just within the convergence range according to the theory of Thom and Apelt<sup>23</sup> as can be seen in Fig. 10. For  $Re = 1$ , the starting solution was that of potential flow past the same boundaries (i.e.  $\psi = \eta, \zeta = 0$ ). For higher Reynolds numbers, the solution at the next lowest Reynolds number was used.

The results of the tests are given in Table 3, which shows the number of iterations required to make

$$|\zeta^{(n)} - \zeta^{(n-1)}|_{\text{max}} < 10^{-3}$$

$$|\psi^{(n)} - \psi^{(n-1)}|_{\text{max}} < 10^{-5}.$$

This necessitated reducing  $e, d$  through about four decades. Where the number of iterations required was greater than fifty, it was estimated by extrapolating the graph of  $\log |\zeta^{(n)} - \zeta^{(n-1)}|_{\text{max}}$  versus  $n$ , and similarly for  $\psi$ .

TABLE 3

Method		Succ. u.d.p.			Apelt			Lester			Succ. o.d.p.		
<i>Re</i>	$\omega'$	$\zeta'$	$\zeta$	$\psi$	$\zeta'$	$\zeta$	$\psi$	$\zeta'$	$\zeta$	$\psi$	$\zeta'$	$\zeta$	$\psi$
1	1	65	70	75	47	45	46	44	40	42	150	150	100
	0.5	65	65	70	48	45	46	45	42	39	41	38	39
	$\omega'_0$	65	65	70	48	45	46	—	—	—	39	36	37
10	1	54	52	56	49	45	49	42	39	40	85	80	56
	0.5	54	52	57	50	45	47	42	39	40	31	29	30
	$\omega'_0$	54	52	56	48	44	50	—	—	—	30	28	29
100	1	Diverged			95	75	90	60	65	70	48	41	43
	0.5	Diverged			95	75	90	60	65	75	48	41	42
	$\omega'_0$	Diverged			95	75	90	—	—	—	49	41	42

(The ' on the  $\zeta$  here refers to boundary values.) Attempts at solutions for  $Re = 1000$  were all very slow and would have required at least 200 iterations even using the fastest (to date) methods.

From Table 3 the following points should be noticed:

- (1) At higher Reynolds numbers, successive unit displacements by points diverges whatever the boundary treatment. Moreover the new criterion for divergence is more accurate than that of Thom and Apelt.
- (2) The boundary treatment has little overall effect except for successive optimum displacements by points.
- (3) For successive optimum displacements by points the use of unit movement on the boundary can greatly retard the convergence of the solution particularly at the lower Reynolds numbers.
- (4) On an iteration basis, successive optimum displacements by points is certainly faster than any of the other methods, provided that we use  $\omega' = 0.5$  or  $\omega' = \omega'_0$  on the boundaries. (There is little to choose between these two in the present tests.)

On a small computer much time is spent in transferring numbers to and from the auxiliary store and so it is the number of iterations which determines the time requirement. Thus successive optimum displacements by points is the fastest method for a small computer. However, on a large computer no transferring of numbers will be necessary and so we must compare the iteration times of the different methods. For a large field the number of internal points will greatly exceed the number of boundary points and so we need only consider the time it takes to revise the values of  $\zeta$ ,  $\psi$  at a typical internal point. The iteration times, of course, depend very much on the type of computer used and on the language in which the subroutines are written. Typical times are given in Table 4. The first column of figures has been assessed using the Symbolic Language of a Ferranti Mercury

TABLE 4

Method	Iteration time in millisecc/point	
	on Mercury	on similar computer with fast division
Succ. u.d.p.	6.8	6.8
Apelt	12.4	9.4
Lester	20.1	17.1
Succ. o.d.p.	19.4	16.4
Modified Succ. o.d.p.	13.7	10.4

Computer. Unfortunately Mercury is very slow at performing division, an operation necessary for all the methods except Succ.u.d.p., and so a second column of figures has been added giving the times Mercury would take if the speed of its division were increased to half that of its multiplication. Thus the second figures give a better indication of the relative times of the different methods on a computer such as an IBM 7090, which performs division almost as fast as multiplication. A fifth method has also been added. This is called Modified Succ.o.d.p. and is a version of Succ.o.d.p., modified, as explained in Section 9, to reduce the iteration time per point.

On the strength of the preceding tests, Succ. u.d.p. can immediately be eliminated as a suitable method because of its divergent nature at higher Reynolds numbers. So also can Lester's method

because its iteration time is longer than Succ.o.d.p., the fastest method on an iteration basis. The relative merits of Apelt's method and Succ.o.d.p. are not quite so obvious from Tables 3 and 4, and so in Table 5 we give the relative time requirements (excluding the times of transfers to and from the auxiliary store) for solving the test problem. It will be seen that Apelt's method is as fast as Succ.o.d.p., but it is slower than Modified Succ.o.d.p., which we shall describe in the next section.

It must be stressed that in the preceding numerical tests  $hM$  was chosen to be small. Its value was  $1/256$  all over the field, so that equation (98) was satisfied. When  $hM$  is large, divergence can easily occur unless  $\Omega$  is reduced. The exact manner in which  $\Omega$  should be controlled is not known and equation (100) is only a tentative suggestion. It might be thought that Lester's method would provide the answer, controlling, as it does, the movement in the  $\psi$  field. However Lester<sup>15</sup> quotes a problem, for which his method fails in a particular region of a field where, in fact,  $hM \approx 4$ .

Movements other than unity were tried along the lines  $j = 0, j = 17$ , sometimes with slight improvement in convergence, but generally unity was found to be the best value to use.

TABLE 5

Method	Re	Relative time requirements	
		on Mercury	on similar computer with fast division
Succ. u.d.p.	1	0.87	1.15
	10	0.93	1.23
	100	$\infty$	$\infty$
Apelt	1	1.09	1.09
	10	1.48	1.48
	100	1.70	1.70
Lester	1	1.66	1.86
	10	2.06	2.32
	100	2.15	2.40
Succ. o.d.p.	1	1.42	1.58
	10	1.42	1.58
	100	1.42	1.58
Modified Succ. o.d.p.	1	1.00	1.00
	10	1.00	1.00
	100	1.00	1.00

9. *Modified Successive Optimum Displacements by Points.*

Much of the iteration time for successive optimum displacements by points is spent in finding the square root of  $\phi$  {see equation (97)} and it is obviously desirable to reduce this. One method, convenient for some computers, is to store  $\sigma \approx \sqrt{\phi}$  for every internal point of the field and to use this to apply one iteration of Newton's method of calculating square roots. Thus on the  $n$ th iteration we put:

$$\sigma^{(n)} = \frac{1}{2}[\sigma^{(n-1)} + \phi^{(n)}/\sigma^{(n-1)}]. \tag{118}$$

In this application Newton's method is particularly good, for if

$$\sigma^{(n-1)} = (1 + \epsilon)\sqrt{\phi^{(n)}}$$

then

$$\sigma^{(n)} = \sqrt{\phi^{(n)}}[1 + \frac{1}{2}\epsilon^2\sqrt{\phi^{(n)}}/\sigma^{(n-1)}]$$

so that whether  $\sigma^{(n-1)}$  is a high or a low estimate of  $\sqrt{\phi^{(n)}}$ , i.e. whether  $\epsilon$  is positive or negative, then  $\sigma^{(n)}$  is always a high estimate. This means that our estimate for  $\omega$ ,

$$\omega^{(n)} = 2/(1 + \sigma^{(n)})$$

will always be on the low side. As has been shown in Section 5.4, this is just what is required. Moreover Newton's method is convergent, no matter the magnitude of  $\epsilon$ . For  $|\epsilon| \ll 1$

$$\sigma^{(n)} \approx (1 + \frac{1}{2}\epsilon^2)\sqrt{\phi^{(n)}}$$

so that the number of correct significant figures in  $\sigma$  approximately doubles each iteration.

However, on Mercury, Newton's method is unsuitable because it uses division. So instead we store  $\sigma \approx 1/\sqrt{\phi}$  for every internal point of the field. Then on the  $n$ th iteration we put

$$\sigma^{(n)} = \sigma^{(n-1)} + \frac{1}{2}\sigma^{(n-1)}(1 - \phi^{(n)}[\sigma^{(n-1)}]^2) \quad (119)$$

At first sight this 'quicky' might seem dangerous, for if

$$\sigma^{(n-1)} = (1 + \epsilon)/\sqrt{\phi^{(n)}}$$

then

$$\sigma^{(n)} = (1 - \frac{3}{2}\epsilon^2 - \frac{1}{2}\epsilon^3)/\sqrt{\phi^{(n)}}$$

which for  $|\epsilon| \ll 1$  is less than  $1/\sqrt{\phi^{(n)}}$ . Thus our estimate of  $\omega$  will be on the high side, which could cause divergence of our solution. Fortunately the process is highly convergent (subject to the limitations described below), so that it will soon correct itself. Like Newton's method, the number of correct significant figures approximately doubles each iteration. Moreover our division 'quicky' underestimates and so this tends to counteract the effect of the underestimation of  $1/\sqrt{\phi^{(n)}}$ . For division we store  $\omega$  and on the  $n$ th iteration we put

$$\omega^{(n)} = \omega^{(n-1)} + \omega^{(n-1)}(1 - \omega^{(n-1)}\Phi^{(n)}) \quad (120)$$

where

$$\Phi^{(n)} = \frac{1}{2}(1 + \sigma^{(n)}\phi^{(n)}).$$

Now if

$$\omega^{(n-1)} = (1 + \delta)/\Phi^{(n)}$$

then

$$\omega^{(n)} = (1 - \delta^2)/\Phi^{(n)}$$

so that whether  $\omega^{(n-1)}$  is a high or a low estimate of  $1/\Phi^{(n)}$ , then  $\omega^{(n)}$  is a low estimate, which is what is required.

Provided that  $|\delta| < 1$ , the process converges, and once again the number of correct significant figures doubles every iteration. If and only if  $|\delta| \geq 1$ , then  $\omega^{(n)} \leq 0$ . Thus having applied equation (120), we must first check that  $\omega^{(n)}$  is positive. If it is not we must revert to Autocode division for that iteration, which will automatically make  $|\delta|$  small for the next iteration. It is thus likely that equation (120) will be satisfactory for all but the first iteration.

The condition that  $\sigma^{(n)}$  in equation (119) should be a better approximation than  $\sigma^{(n-1)}$  to  $1/\sqrt{\phi}$  is

$$\epsilon < \frac{1}{2}(\sqrt{17}-3) = 0.56155\dots$$

but the condition that  $\sigma^{(n)} \rightarrow 1/\sqrt{\phi}$  as  $n \rightarrow \infty$  is a little less stringent

$$\epsilon < \sqrt{3} - 1 = 0.73205\dots$$

If the latter condition is not satisfied, then  $\sigma^{(n+1)}$  will be negative. Thus, having calculated  $\sigma^{(n)}$  from equation (119), we check that it is positive. If it is not we must revert to the Autocode square root subroutine, which will make  $|\epsilon|$  small for the next iteration. It is thus likely that equation (119) will be satisfactory for all but the first iteration.

The author has used this modified form of successive optimum displacements by points for a number of solutions and has found that the procedure, with its extra safeguards, tends, if anything, to hasten the convergence. It does, however, increase the storage requirements.

#### 10. *Concluding Remarks.*

It has been found that the most convenient finite-difference approximations to use for solving the Navier-Stokes equations are the simpler formulae {see equations (11)}, since they require little time for their application and since with them good convergence rates can be obtained.

Of the many possible iterative treatments for solving the Navier-Stokes difference equations on a square grid, modified successive optimum displacements by points {see Sections 5.4, 6, 9) has much to commend itself over other methods. It is not difficult to programme on to a computer, it has a short iteration time and it provides solutions reasonably rapidly. However the work of Lester<sup>14</sup> would be useful, if it were intended to use a rectangular grid so as to obtain greater detail in the boundary layer at higher Reynolds numbers.

Of the possible boundary treatments, there is little to choose between the use of equation (109) and the use of equation (114). The former has the merit of simplicity, but the latter, by taking account of grid size and curvature of boundary, is more likely to have universal success.

#### 11. *Acknowledgements.*

The author would like to express his gratitude to Professor Thom for introducing him to the subject and for his patience and guidance during the early stages of the work, to the staff of the Oxford University Computing Laboratory for its assistance and for the use of the Ferranti Mercury computer, and to D.S.I.R. and to Brasenose College for their financial support at different times during the work.

NOTATION (*unless otherwise stated*)

*Physical Plane*

$x, y$	Cartesian co-ordinates
$s, n$	Tangential and normal co-ordinates at boundary
$\kappa = \frac{d\Theta}{ds}$	curvature of the boundary
$z$	Complex co-ordinate, $z = x + iy$
$\nabla_{xy}^2$	Laplacian operator, $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ .

*Transformed Plane*

$\xi, \eta$	Cartesian co-ordinates
$\sigma, \nu$	Tangential and normal co-ordinates at boundary
$w$	Complex co-ordinate, $w = \xi + i\eta$
$\nabla_{\xi\eta}^2$	Laplacian operator, $(\partial^2/\partial \xi^2 + \partial^2/\partial \eta^2)$
$\mathcal{D}_{\xi\eta}^2 = \partial^2/\partial \xi \partial \eta$	
$M$	Modulus of transformation, $M = \left  \frac{dz}{dw} \right $
$L = \log(M)$	
$\Theta = \arg \left( \frac{dz}{dw} \right)$	
$\tau = L + i\Theta$	

*Fluid-Flow Equations*

$u, v$	Components of velocity in $x, y$ directions
$\psi$	Stream function, $u = \psi_y, v = -\psi_x$
$\zeta$	Vorticity, $\zeta = v_x - u_y$
$\nu$	Kinematic viscosity
$Re$	Reynolds number

*Finite-Difference Equations*

$F$	Function defined by a partial differential equation
$f$	Function defined by a difference equation, such that $f$ approximates to $F$
$h$	Grid size
$j, k$	Co-ordinates in grid, $x = jh, y = kh$ or $\xi = jh, \eta = kh$
$S_1 f_{j,k} =$	$(f_{j,k-1} + f_{j,k+1} + f_{j-1,k} + f_{j+1,k})$
$S_2 f_{j,k} =$	$(f_{j-1,k-1} + f_{j-1,k+1} + f_{j+1,k-1} + f_{j+1,k+1})$
$\delta_\xi f_{j,k} =$	$\frac{1}{2}(f_{j+1,k} - f_{j-1,k})$
$\delta_\eta f_{j,k} =$	$\frac{1}{2}(f_{j,k+1} - f_{j,k-1})$

NOTATION—*continued*

*Iterative Treatments*

$A, B$	Constants defined by equation (14)
$a, b$	Constants defined by equation (16)
$h$	Grid size
$p, q$	The number of intervals along the sides of a rectangular field
$r, s$	Positive integers less than $p, q$ respectively
$e$	Error in $f$ defined by equation (19), or error in $\zeta$ defined by equation (95a)
$d$	Error in $\psi$ defined by equation (95b)
$E$	Component of $e$ defined by equation (22)
$\lambda$	Eigenvalue
$\rho$	Spectral radius of $\lambda$ , as defined by equation (29)
$N$	Number of iterations required to reduce the maximum error through a decade. <i>See equation (30).</i>
$\omega$	Movement or displacement factor of $f$ or $\zeta$
$\omega_0$	Optimum value of $\omega$
$\omega'$	Movement of $\zeta$ on boundary
$\Omega$	Movement of $\psi$
$\mu$	$= \frac{1}{2} \left[ \sqrt{(1-a^2)} \cos \frac{\pi r}{p} + \sqrt{(1-b^2)} \cos \frac{\pi s}{q} \right]$
$\mu_1, \mu_2$	Real and imaginary parts of $\mu$
$\mu'$	Value of $\mu$ for $r = s = 1$
$\mu'_1, \mu'_2$	Real and imaginary parts of $\mu'$
$\nu$	$= \frac{\left[ \sqrt{(1-b^2)} \cos \frac{\pi s}{q} \right]}{\left[ 2 - \sqrt{(1-a^2)} \cos \frac{\pi r}{p} \right]}$
$\nu_1, \nu_2$	Real and imaginary parts of $\nu$
$\nu'$	Value of $\nu$ for $r = s = 1$
$\nu'_1, \nu'_2$	Real and imaginary parts of $\nu'$
$\phi$	$= \frac{1}{2} [a^2 + b^2 + \pi^2(p^{-2} + q^{-2})]$

*Subscripts and Superscripts*

$f_{j,k}^{(n)}$  Value assigned to  $f$  at the point  $(jh, kh)$  on the  $n$ th iteration

The following subscripts all indicate partial differentiation:

$x, y, s, n, \xi, \eta, \sigma, \nu$

For example

$$F_x = \frac{\partial F}{\partial x}, F_{\xi\xi} + F_{\eta\eta} = \nabla_{\xi\eta}^2 F, \text{ etc.}$$



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## APPENDIX

It should be noted that there are other interpretations of implicit methods by lines than those given in Sections 5.11, 5.8. For instance, it is possible to define simultaneous methods implicit by lines either as

$$\begin{aligned}
 & [(1-a)f_{j-1,k} - 4f_{j,k} + (1+a)f_{j+1,k}]^{(n)} \\
 & = (1-\omega)[(1-a)f_{j-1,k} - 4f_{j,k} + (1+a)f_{j+1,k}]^{(n-1)} - \\
 & \quad - \omega[(1-b)f_{j,k-1} + (1+b)f_{j,k+1}]^{(n-1)}
 \end{aligned} \tag{121}$$

for which

$$\lambda = \nu + 1 - \omega \tag{122}$$

or else as the following, which is the first half of an alternating-direction method due to Peaceman and Rachford<sup>17</sup>.

$$\begin{aligned}
 (1+\omega)f_{j,k}^{(n)} & = (1-\omega)f_{j,k}^{(n-1)} + \\
 & \quad + \frac{\omega}{2} [(1-a)f_{j-1,k}^{(n)} + (1+a)f_{j+1,k}^{(n)} + \\
 & \quad + (1-b)f_{j,k-1}^{(n-1)} + (1+b)f_{j,k+1}^{(n-1)}]
 \end{aligned} \tag{123}$$

for which

$$\lambda = \frac{1 - \omega(1 - 2\beta)}{1 + \omega(1 + 2\alpha)} \tag{124}$$

For  $\omega = 1$ , equations (121), (122) reduce to equations (80), (82a). However these methods have yet to be suitably adapted for solving the Navier-Stokes difference equations.

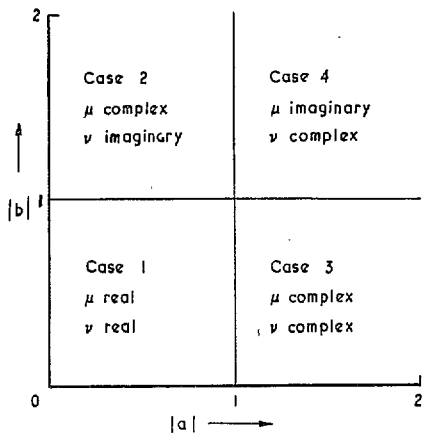


FIG. 1. The four cases.

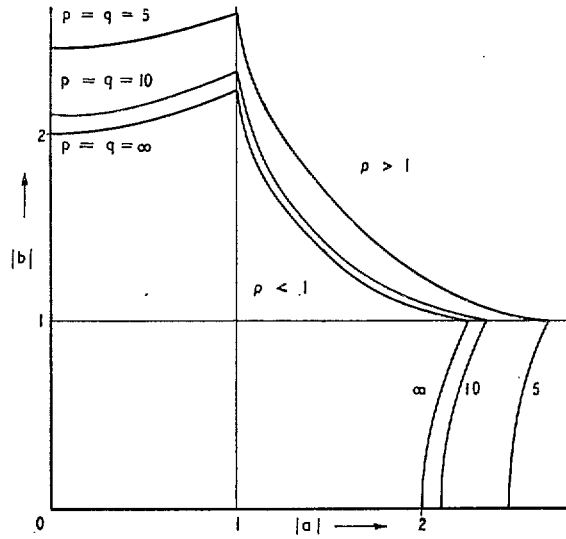


FIG. 2. The convergence range of successive and simultaneous unit displacements by points for three sizes of square field.

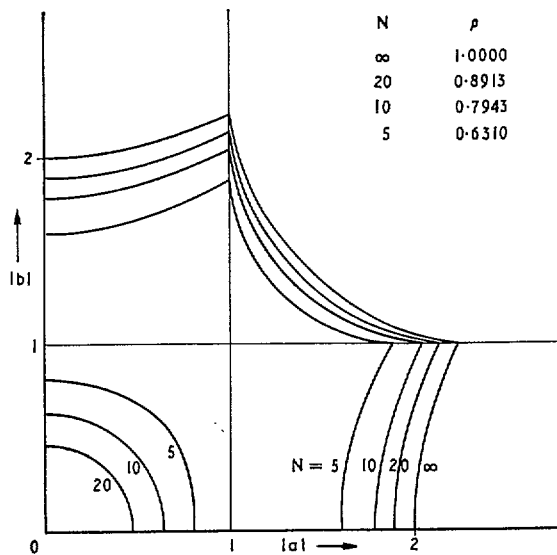


FIG. 3. The number of iterations of successive unit displacements by points required to reduce the errors through a decade for an infinite field.

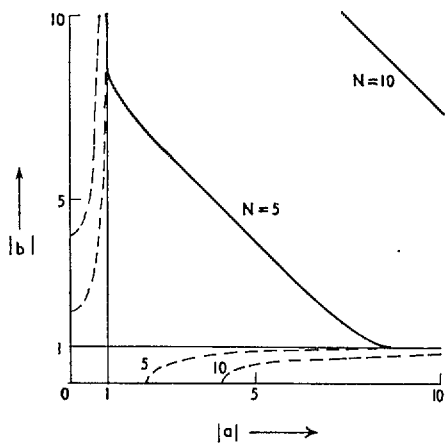


FIG. 4a. The number of iterations of successive optimum displacements by points required to reduce the errors through a decade for an infinite field. Cases 2, 3 and 4.

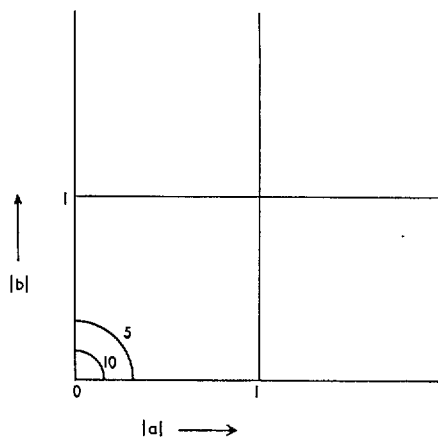


FIG. 4b. Lines of constant  $N$ . Case 1.

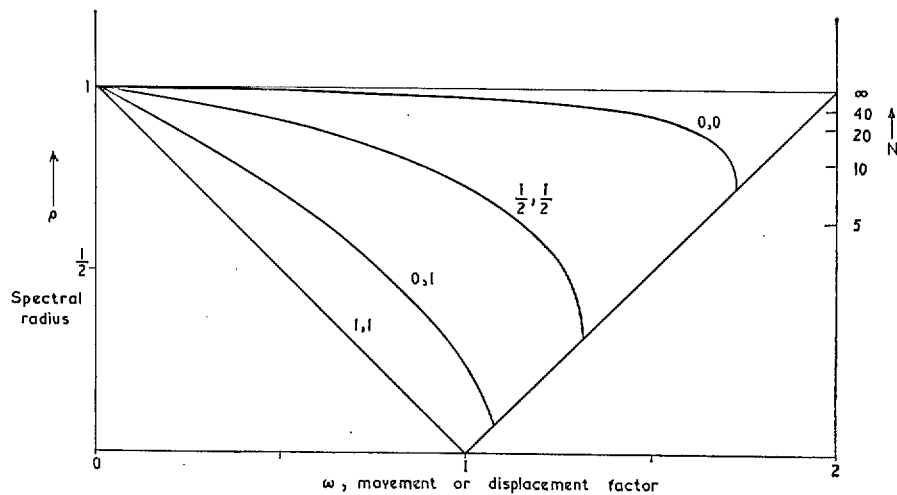


FIG. 5a. Graphs of spectral radius against movement, using successive displacements by points, for different values of  $a$ ,  $b$  or  $b$ ,  $a$ . Case 1.

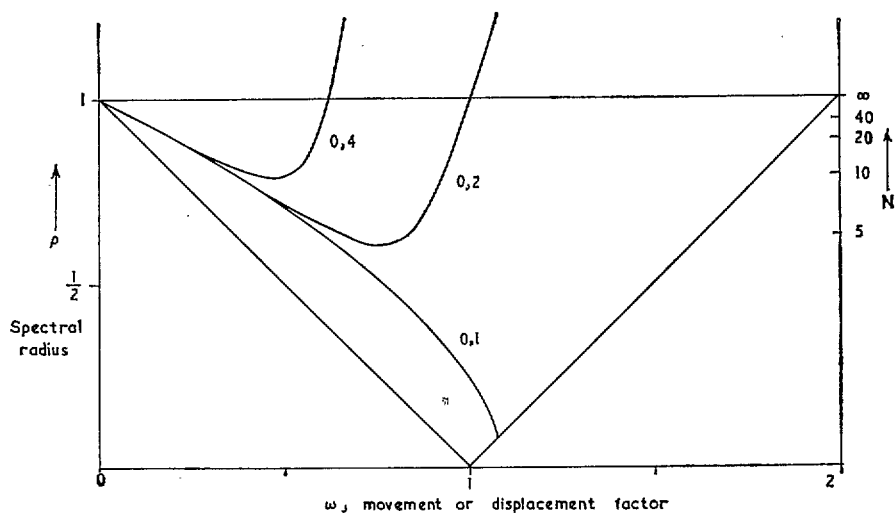


FIG. 5b. Graphs of spectral radius against movement, using successive displacements by points, for different values of  $a, b$  or  $b, a$ . Cases 2 and 3.

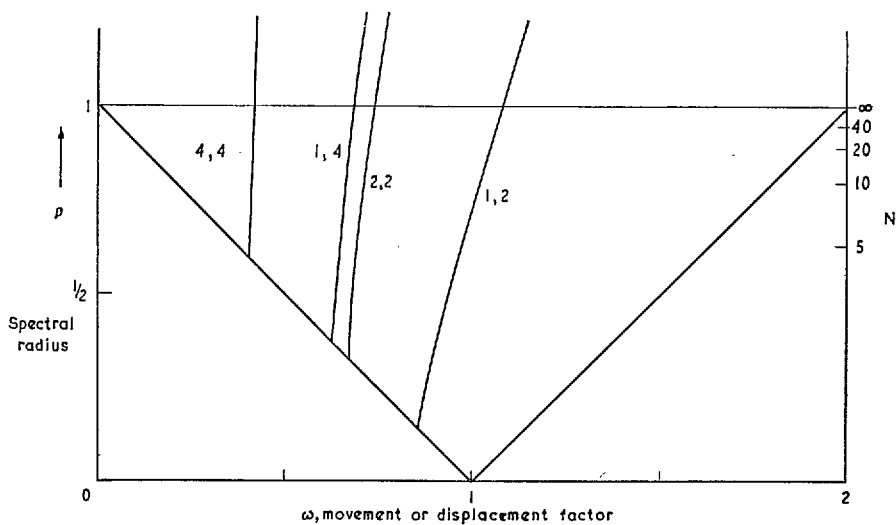


FIG. 5c. Graphs of spectral radius against movement, using successive displacements by points, for different values of  $a, b$  or  $b, a$ . Case 4.

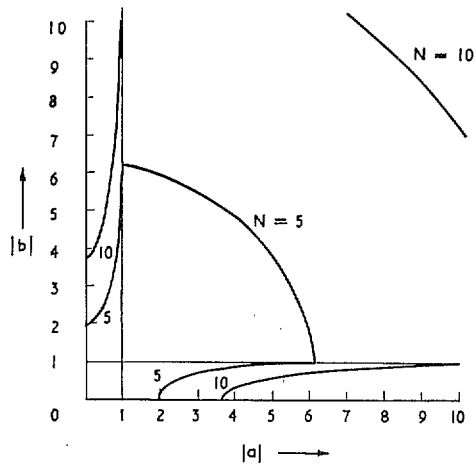


FIG. 6. Plots of  $N$  for successive optimum displacements by points but with  $\omega = 2/(1 + \sqrt{\phi})$ . Cases 2, 3 and 4.

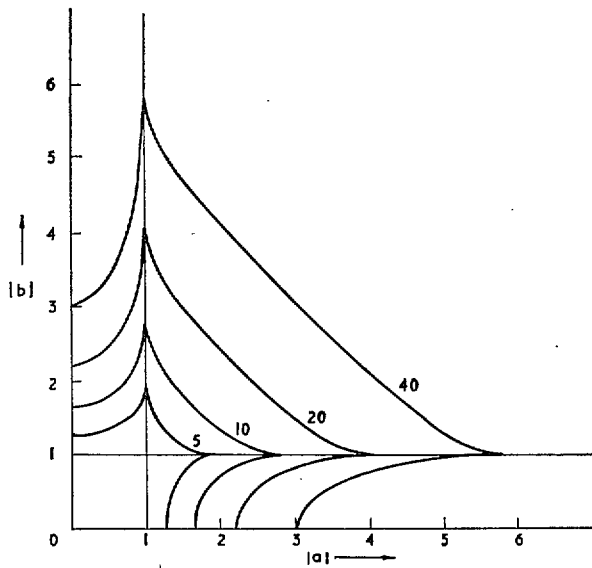


FIG. 7a. Plots of  $N$  for simultaneous optimum displacements by points. Cases 2, 3 and 4.

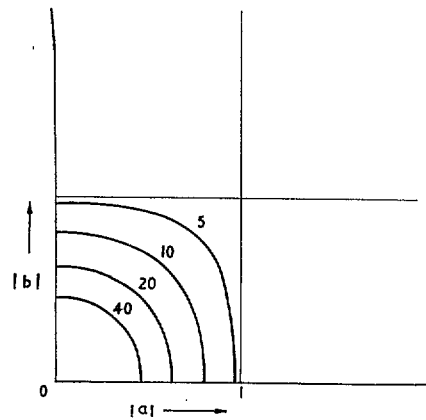


FIG. 7b. Lines of constant  $N$ . Case 1.

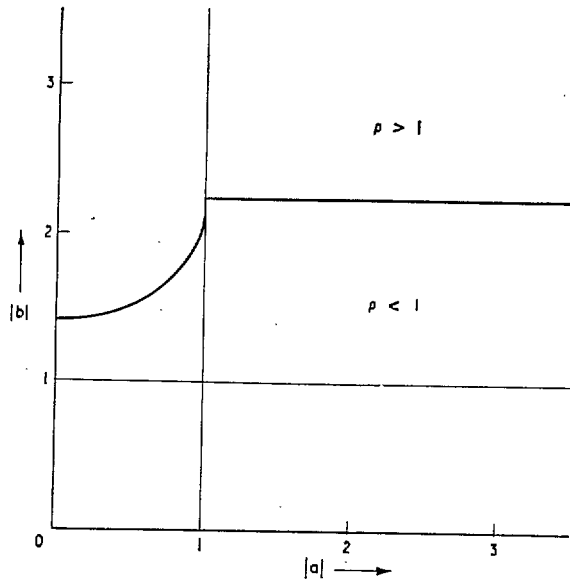


FIG. 8. The convergence range of successive and simultaneous unit displacements implicit by lines for an infinite field.

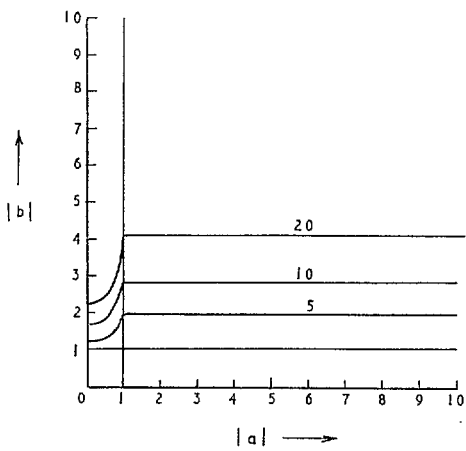


FIG. 9a. Plots of  $N$  for simultaneous optimum displacements implicit by lines. Cases 2, 3 and 4.

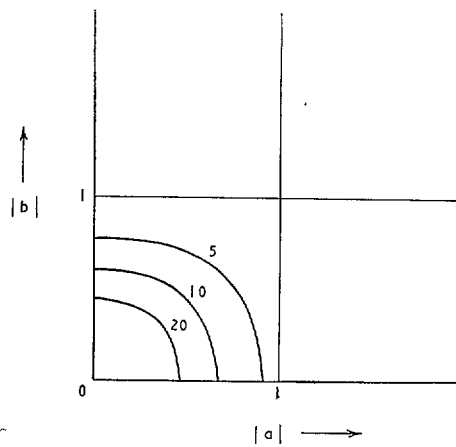


FIG. 9b. Lines of constant  $N$ . Case 1.



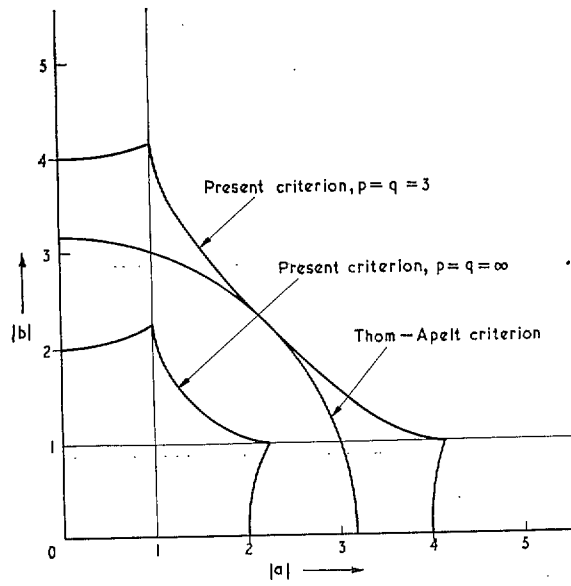


FIG. 10. Comparison of divergence criteria.

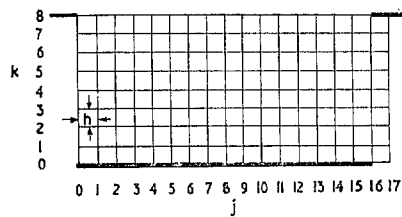
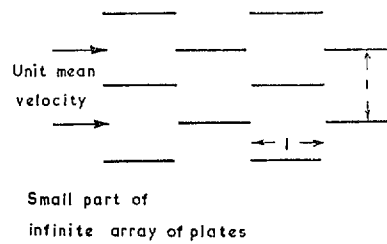


FIG. 11. Grid used.  $p = 17$ ,  
 $q = 8$ ,  $h = 1/16$ .

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