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Principles of Attitude Control of Artificial Satellites

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Principles of Attitude Control of Artificial Satellites

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Summary. The equations of attitude motion are derived for small angular displacements from the equilibrium position of an earth-pointing satellite employing reaction-flywheel damping. This is followed by a discussion on the attitude control of a space-stabilised satellite, with particular reference to attitude control against the gravitational torque due to the earth and the use of reaction-jets for the control of a spherical satellite configuration; finally a single-plane analytical account is given, of a method of eliminating any undesirable angular momentum which the satellite may possess immediately it leaves the carrier missile for its future orbit.

1. *Introduction.* It is expected that perturbation torques tending to produce turning moments about the centre of mass of an orbiting satellite will arise from many different causes; among the more important may be listed:

The torque due to the earth's gravitational field.

Torques due to movement of equipment inside the satellite.

The effect of the sun's radiation pressure.

Residual air drag.

Bombardment by meteoroids.

Electric and magnetic fields around the earth.

The effect of these torque disturbances on the satellite is greatly dependent on its configuration. Relatively long thin satellites will generally be more influenced by perturbation torques than satellites which possess some degree of spherical symmetry.

Perturbation torques may be divided into three classes:

(1) Short-lived torques.

(2) Torques which vary in an oscillatory manner as a result of the orbital motion of the satellite round the earth.

(3) Torques which tend to produce a persistent turning moment about the centre of mass of the satellite.

The effect of perturbation torques on a satellite's attitude depends upon the reference frame chosen; the space-stabilised vehicle will be subject to a turning moment due to the variation of the

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earth's gravitational field over the satellite configuration, unless the satellite is a homogeneous sphere for example, when the perturbation torque vanishes. On the other hand an earth-pointing satellite can utilise this gravitational torque as a stabilising effect, so as to provide a natural position of equilibrium.

Some satellite missions will be expected to demand much greater precision in attitude control than others. A mission characterised primarily for making very accurate astronomical observations will in general require much more stringent control of attitude motion than a satellite whose primary concern is to make weather observations.

For a satellite orbiting close to the earth the three major disturbing torques will be those due to the earth's gravitational field, air drag and movement of any kind within the satellite. For more distant satellites both the gravitational torque and that due to air drag will become very small, and the chief perturbation torques will generally originate from internal moving parts and the radiation pressure of the sun's radiation. However, in both cases, a collision with a meteoroid will need to be catered for, and the attitude control should be capable of coping with the disturbance initiated by the largest meteoroid that is likely to be encountered.

The main object of this Report is to derive the equations of motion for small angular displacements from the equilibrium position of an earth-pointing satellite employing reaction-flywheel damping. This is followed by a discussion on the space-stabilised satellite and finally a single-plane analytical account is given of a method for eliminating any undesirable angular momentum which the satellite may possess at orbit injection.

2. *Reaction-Flywheel Control.* In the theory that follows, attitude control is investigated on the concept that reaction-flywheels are to be used in helping to provide any desirable control of the satellite's orientation in space. The spin axes of the flywheels are along the directions of the principal axes of inertia of the satellite. It will be assumed that the principal axes maintain a fixed orientation in the satellite, any fluctuations of the axes due to distribution changes of matter inside the satellite being omitted. Such flywheels, mounted in fixed bearings, provide an inertial reaction torque when accelerated relative to the satellite body. Control of roll, pitch and yaw motions from the desired reference orientation may be provided by the use of three flywheels, one along each of the satellite's principal axes. For example, the control-flywheel along the roll axis of the vehicle when accelerated in the same direction as the rolling motion of the satellite, will result in an equal and opposite reaction torque being exerted on the satellite, thus attempting to counteract the roll motion.

2.1. *Derivation of the Equations of Attitude Motion.* Let XYZ be an orthogonal system of right-handed axes with origin at the centre of mass of the satellite, and let these define the principal axes of inertia of the satellite (subsequently referred to simply as 'satellite axes'). Define the unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ along the XYZ axes respectively. The angular velocity of the satellite relative to space axes will be denoted by the vector $\boldsymbol{\omega}$, with components $(\omega_X, \omega_Y, \omega_Z)$ along $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. The quantities (I_X, I_Y, I_Z) will represent the principal moments of inertia of the whole satellite configuration, *i.e.*, satellite body plus flywheels. The flywheels may be positioned anywhere along the satellite axes.

If the flywheels possess angular velocities $(\omega_1, \omega_2, \omega_3)$ relative to the vehicle frame the total angular-momentum vector of the satellite is given by:

$$\mathbf{H}_T = (I_X\omega_X + I\omega_1)\mathbf{i} + (I_Y\omega_Y + I\omega_2)\mathbf{j} + (I_Z\omega_Z + I\omega_3)\mathbf{k} \tag{1}$$

assuming all flywheels to have equal polar moments of inertia I.

The equation of rotational motion of the satellite is therefore:

$$\mathbf{L} = \frac{d}{dt}\mathbf{H}_T \tag{2}$$

$$= \frac{\partial}{\partial t}\mathbf{H}_T + \boldsymbol{\omega} \times \mathbf{H}_T, \tag{3}$$

where $\mathbf{L}$ is the resultant external torque acting on the satellite, and $\frac{\partial}{\partial t}\mathbf{A}$ denotes differentiation of $\mathbf{A}$ with respect to satellite axes.

If (L_X, L_Y, L_Z) are the components of $\mathbf{L}$ along the satellite axes, the component equations of motion are:

$$I_X\dot{\omega}_X + I\dot{\omega}_1 + (I_Z - I_Y)\omega_Y\omega_Z + I(\omega_Y\omega_3 - \omega_Z\omega_2) = L_X, \tag{4}$$

$$I_Y\dot{\omega}_Y + I\dot{\omega}_2 + (I_X - I_Z)\omega_Z\omega_X + I(\omega_Z\omega_1 - \omega_X\omega_3) = L_Y, \tag{5}$$

$$I_Z\dot{\omega}_Z + I\dot{\omega}_3 + (I_Y - I_X)\omega_X\omega_Y + I(\omega_X\omega_2 - \omega_Y\omega_1) = L_Z. \tag{6}$$

Now $(\omega_X, \omega_Y, \omega_Z)$ represent the component angular velocities of the satellite axes in space. Attitude deviations will, however, be measured from a set of reference axes which may have an angular velocity $\boldsymbol{\Omega} = (\Omega_x\mathbf{i}'' + \Omega_y\mathbf{j}'' + \Omega_z\mathbf{k}'')$, where the orthogonal unit vectors $(\mathbf{i}'', \mathbf{j}'', \mathbf{k}'')$ are along reference axes (x, y, z), origin at the centre of mass of the satellite. For a space-fixed reference frame $\boldsymbol{\Omega} = 0$, but the general case of reference-axes motion will be treated here.

The orientation of the satellite axes (X, Y, Z) relative to the reference axes (x, y, z) may be defined by the angles θ_1, θ_2 and θ_3, and the rotations necessary to transfer from reference axes to satellite axes are illustrated with the help of Fig. 1:

Rotation about Ox through angle θ_1 brings y to Y_1 and z to Z_1.

Rotation about OY_1 through angle θ_2 brings Z_1 to Z and x to X_1.

Rotation about OZ through angle θ_3 brings Y_1 to Y and X_1 to X.

The angular-velocity components $(\omega_X'\mathbf{i}, \omega_Y'\mathbf{j}, \omega_Z'\mathbf{k})$ of satellite axes relative to reference axes are given by:

$$\omega_X' = \dot{\theta}_1 \cos\theta_2 \cos\theta_3 + \dot{\theta}_2 \sin\theta_3,$$

$$\omega_Y' = \dot{\theta}_2 \cos\theta_3 - \dot{\theta}_1 \cos\theta_2 \sin\theta_3,$$

$$\omega_Z' = \dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2.$$

But the reference axes have an angular velocity $\boldsymbol{\Omega}$ relative to space-fixed axes and in order to determine the true angular velocity of the satellite, $\boldsymbol{\Omega} = (\Omega_x\mathbf{i}'' + \Omega_y\mathbf{j}'' + \Omega_z\mathbf{k}'')$ must be expressed in terms of components along satellite axes.

The transformation is governed by:

$$\Omega_X = H\Omega_x + J\Omega_y + G\Omega_z,$$

$$\Omega_Y = F\Omega_x + E\Omega_y + D\Omega_z,$$

$$\Omega_Z = A\Omega_x + B\Omega_y + C\Omega_z,$$

where $(\Omega_X, \Omega_Y, \Omega_Z)$ are the components of $\mathbf{\Omega}$ along satellite axes and:

$$
\begin{aligned}
A &= \sin\theta_2, \\
B &= -\cos\theta_2 \sin\theta_1, \\
C &= \cos\theta_2 \cos\theta_1, \\
D &= \cos\theta_3 \sin\theta_1 + \sin\theta_3 \sin\theta_2 \cos\theta_1, \\
E &= \cos\theta_3 \cos\theta_1 - \sin\theta_3 \sin\theta_2 \sin\theta_1, \\
F &= -\sin\theta_3 \cos\theta_2, \\
G &= \sin\theta_3 \sin\theta_1 - \cos\theta_3 \sin\theta_2 \cos\theta_1, \\
H &= \cos\theta_3 \cos\theta_2, \\
J &= \sin\theta_3 \cos\theta_1 + \cos\theta_3 \sin\theta_2 \sin\theta_1.
\end{aligned}
$$

For small attitude deviations of satellite axes from the reference axes $\cos\theta \to 1$ and $\sin\theta \to \theta$; using this approximation for small angles we arrive at the following:

$$\omega_X{}' = \dot{\theta}_1 + \dot{\theta}_2\theta_3, \qquad \omega_Y{}' = \dot{\theta}_2 - \dot{\theta}_1\theta_3, \qquad \omega_Z{}' = \dot{\theta}_3 + \dot{\theta}_1\theta_2.$$

A convenient choice for the reference axes is such that $\mathbf{k}''$ is directed towards the centre of the earth and $\mathbf{j}''$ normal to the orbital plane. Due to the earth's oblateness the orbital plane in general precesses about the earth's axis, but the angular velocity of the reference frame due to this precession is small compared with that due to the motion of the satellite in its orbit. The more important dynamical characteristics of the system may, therefore, be expected to be illustrated by considering the angular velocity of the reference frame to be due entirely to the orbital motion of the satellite, *i.e.*, $\mathbf{\Omega} = \Omega_y \mathbf{j}''$, where $\mathbf{j}''$ is normal to the orbital plane and Ω_y the angular velocity of the satellite in its orbit. The problem will therefore be reduced to a development of the equations of motion for a satellite moving in an orbit which remains fixed in space *i.e.*, a non-precessing orbit.

Hence we have (for small angles)

$$
\begin{aligned}
\Omega_X &= \Omega_y\theta_3, \\
\Omega_Y &= \Omega_y, \\
\Omega_Z &= -\Omega_y\theta_1,
\end{aligned}
$$

neglecting terms involving cross-products of the attitude angles.

Consequently we may write

$$
\begin{aligned}
\omega_X &= \dot{\theta}_1 + \dot{\theta}_2\theta_3 + \Omega_y\theta_3, \\
\omega_Y &= \dot{\theta}_2 - \dot{\theta}_1\theta_3 + \Omega_y, \\
\omega_Z &= \dot{\theta}_3 + \dot{\theta}_1\theta_2 - \Omega_y\theta_1
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\omega}_X &= \ddot{\theta}_1 + \ddot{\theta}_2\theta_3 + \dot{\theta}_2\dot{\theta}_3 + \Omega_y\dot{\theta}_3 + \dot{\Omega}_y\theta_3, \\
\dot{\omega}_Y &= \ddot{\theta}_2 - \ddot{\theta}_1\theta_3 - \dot{\theta}_1\dot{\theta}_3 + \dot{\Omega}_y, \\
\dot{\omega}_Z &= \ddot{\theta}_3 + \ddot{\theta}_1\theta_2 + \dot{\theta}_1\dot{\theta}_2 - \Omega_y\dot{\theta}_1 - \dot{\Omega}_y\theta_1.
\end{aligned}
$$

If terms involving products of the attitude angles are neglected then Equation (4) becomes

$$\begin{aligned} I\dot{\omega}_1 &+ I_X(\ddot{\theta}_1 + \ddot{\theta}_2\theta_3 + \dot{\theta}_2\dot{\theta}_3 + \Omega_y\dot{\theta}_3 + \dot{\Omega}_y\theta_3) + \\ &+ (I_Z - I_Y)[(\Omega_y + \dot{\theta}_2)(\dot{\theta}_3 + \dot{\theta}_1\theta_2 - \Omega_y\theta_1) - \dot{\theta}_1\dot{\theta}_3\theta_3] + \\ &+ I[(\dot{\theta}_2 - \dot{\theta}_1\theta_3 + \Omega_y)\omega_3 - (\dot{\theta}_3 + \dot{\theta}_1\theta_2 - \Omega_y\theta_1)\omega_2] = L_X \end{aligned}$$

with similar expressions corresponding to Equations (5) and (6).

For a circular orbit $\dot{\Omega}_y = 0$ since Ω_y remains constant; a circular orbit will therefore be assumed in order to keep the analysis as simple as possible. The further assumption that all attitude rates are small compared with Ω_y greatly simplifies the attitude equations and the resulting equations are

$$I\dot{\omega}_1 + I_X\ddot{\theta}_1 + (I_Y - I_Z)\Omega_y^2\theta_1 + (I_X + I_Z - I_Y)\Omega_y\dot{\theta}_3 = L_X - I[\Omega_y\omega_3 - (\dot{\theta}_3 - \Omega_y\theta_1)\omega_2], \tag{7}$$

$$I\dot{\omega}_2 + I_Y\ddot{\theta}_2 = L_Y - I[(\dot{\theta}_3 - \Omega_y\theta_1)\omega_1 - (\dot{\theta}_1 + \Omega_y\theta_3)\omega_3], \tag{8}$$

$$I\dot{\omega}_3 + I_Z\ddot{\theta}_3 + (I_Y - I_X)\Omega_y^2\theta_3 - (I_X + I_Z - I_Y)\Omega_y\dot{\theta}_1 = L_Z - I[(\dot{\theta}_1 + \Omega_y\theta_3)\omega_2 - \Omega_y\omega_1]. \tag{9}$$

At this stage it is convenient to introduce the effect of the variation of the earth's gravitational field over the satellite configuration.

2.2. *General Attitude Departure from the Radius Vector.* To the order of approximation given in Appendix I, the gravitational torque acting about the centre of mass of the satellite is

$$\mathbf{\Gamma} = -3\frac{GM_E}{|r_0|^5}[(I_Y - I_Z)r_{Y0}r_{Z0}\mathbf{i} + (I_Z - I_X)r_{X0}r_{Z0}\mathbf{j} + (I_X - I_Y)r_{X0}r_{Y0}\mathbf{k}], \tag{10}$$

where G is the gravitational constant and M_E the mass of the earth.

Since $\mathbf{r}_0 = r_0\mathbf{k}''$, the reference—satellite axis transformation (section 2.1) gives

$$r_{X0} = r_0(\sin\theta_3\sin\theta_1 - \cos\theta_3\sin\theta_2\cos\theta_1), \tag{11}$$

$$r_{Y0} = r_0(\cos\theta_3\sin\theta_1 + \sin\theta_3\sin\theta_2\cos\theta_1), \tag{12}$$

$$r_{Z0} = r_0(\cos\theta_2\cos\theta_1), \tag{13}$$

which for small angles approximates to

$$r_{X0} = -r_0\theta_2,$$

$$r_{Y0} = r_0\theta_1,$$

$$r_{Z0} = 1.$$

Since the orbit is circular we also have $GM_E/|r_0|^3 \doteq \Omega_y^2$ and the gravitational torque is

$$\mathbf{\Gamma} = -3\Omega_y^2[(I_Y - I_Z)\theta_1\mathbf{i} + (I_X - I_Z)\theta_2\mathbf{j}],$$

again neglecting the cross-product of the attitude angles.

Including the gravitational terms in the equations, we have

$$I\dot{\omega}_1 + I_X\ddot{\theta}_1 + 4(I_Y - I_Z)\Omega_y^2\theta_1 + (I_X + I_Z - I_Y)\Omega_y\dot{\theta}_3 = L_X' - I[\Omega_y\omega_3 - (\dot{\theta}_3 - \Omega_y\theta_1)\omega_2], \tag{14}$$

$$I\dot{\omega}_2 + I_Y\ddot{\theta}_2 + 3(I_X - I_Z)\Omega_y^2\theta_2 = L_Y' - I[(\dot{\theta}_3 - \Omega_y\theta_1)\omega_1 - (\dot{\theta}_1 + \Omega_y\theta_3)\omega_3], \tag{15}$$

$$I\dot{\omega}_3 + I_Z\ddot{\theta}_3 + (I_Y - I_X)\Omega_y^2\theta_3 - (I_X + I_Z - I_Y)\Omega_y\dot{\theta}_1 = L_Z' - I[(\dot{\theta}_1 + \Omega_y\theta_3)\omega_2 - \Omega_y\omega_1], \tag{16}$$

where (L_X, L_Y, L_Z) have been replaced by (L_X', L_Y', L_Z') to indicate that the gravitational torque has been included in the equations and is no longer regarded as a part of $\mathbf{L}$.

In the absence of the flywheels and any external torque and with $I_Y = I_X + I_Z$ (*e.g.*, a lamina with its mass in the XZ-plane) Equations (14) to (16) represent undamped oscillatory modes provided that $I_Y > I_X > I_Z$.

Added stability for the θ_1 and θ_3 modes can be achieved by the use of a constant-speed wheel (regarded as constant relative to satellite axes) whose spin axis lies along the Y-axis of the satellite. Such a wheel will possess angular momentum $H_Y\mathbf{j}$ relative to the vehicle. Owing to the rotational motion of the satellite, such an angular momentum will result in the development of the additional terms $(\dot{\theta}_1 + \Omega_y\theta_3)H_Y\mathbf{k}$ and $-(\dot{\theta}_3 - \Omega_y\theta_1)H_Y\mathbf{i}$ to the L.H.S's of the attitude equations, since

$$\frac{d}{dt}\mathbf{H}_Y = \frac{\partial}{\partial t}\mathbf{H}_Y + \boldsymbol{\omega} \times \mathbf{H}_Y \text{ and } \frac{\partial}{\partial t}\mathbf{H}_Y = 0, \quad (\mathbf{H}_Y = H_Y\mathbf{j}).$$

The L.H.S's of Equations (14) to (16) are now

$$I\dot{\omega}_1 + I_X\ddot{\theta}_1 + \left[4(I_Y - I_Z) + \frac{H_Y}{\Omega_y}\right]\Omega_y{}^2\theta_1 + \left[I_X + I_Z - I_Y - \frac{H_Y}{\Omega_y}\right]\Omega_y\dot{\theta}_3$$

$$I\dot{\omega}_2 + I_Y\ddot{\theta}_2 + 3(I_X - I_Z)\Omega_y{}^2\theta_2$$

$$I\dot{\omega}_3 + I_Z\ddot{\theta}_3 + \left[(I_Y - I_X) + \frac{H_Y}{\Omega_y}\right]\Omega_y{}^2\theta_3 - \left[I_X + I_Z - I_Y - \frac{H_Y}{\Omega_y}\right]\Omega_y\dot{\theta}_1$$

Furthermore, if $(I_X + I_Z) > I_Y$ and the angular momentum of the wheel is determined by the relation $H_Y = (I_X + I_Z - I_Y)\Omega_y$, the attitude equations become

$$I\dot{\omega}_1 + I_X\ddot{\theta}_1 + [3(I_Y - I_Z) + I_X]\Omega_y{}^2\theta_1 = L_X' - I[\Omega_y\omega_3 - (\dot{\theta}_3 - \Omega_y\theta_1)\omega_2], \tag{17}$$

$$I\dot{\omega}_2 + I_Y\ddot{\theta}_2 + 3(I_X - I_Z)\Omega_y{}^2\theta_2 = L_Y' - I[(\dot{\theta}_3 - \Omega_y\theta_1)\omega_1 - (\dot{\theta}_1 + \Omega_y\theta_3)\omega_3], \tag{18}$$

$$I\dot{\omega}_3 + I_Z\ddot{\theta}_3 + I_Z\Omega_y{}^2\theta_3 = L_Z' - I[(\dot{\theta}_1 + \Omega_y\theta_3)\omega_2 - \Omega_y\omega_1]. \tag{19}$$

In the absence of the flywheels Equations (17) to (19) define undamped motions so the control flywheels may serve a very useful purpose by providing damping of any oscillations by making flywheel angular acceleration proportional to angular deviation rate.

$$\text{i.e., } I\dot{\omega}_1 = k_1\dot{\theta}_1, \qquad I\dot{\omega}_2 = k_2\dot{\theta}_2, \qquad I\dot{\omega}_3 = k_3\dot{\theta}_3,$$

where k_1, k_2 and k_3 are control parameters. If the system is designed for critical damping in the absence of the terms arising as a result of flywheel cross-couplings then

$$k_1 = 2\Omega_y\sqrt{\{I_X[3(I_Y - I_Z) + I_X]\}},$$

$$k_2 = 2\Omega_y\sqrt{\{3I_Y(I_X - I_Z)\}},$$

$$k_3 = 2\Omega_y I_Z.$$

The general effect of cross-coupling terms, and the effect of Ω_y variation for an elliptic orbit, on the dynamic response of a particular system is best studied by solution of the attitude equations using an electronic computer.

The response of the system to the various perturbation torques that are likely to arise should be investigated. Short-lived disturbances are equivalent to imparting an initial angular velocity to the satellite, and the control-flywheels should be capable of providing sufficient damping for the satellite to return to rest again at its equilibrium position. In the case of a cylindrical type of satellite (Fig. 2), if the projection of the centre of pressure onto the 'long axis' does not coincide with the centre of mass, it is clear that any significant torque which arises as a result of upper atmospheric molecules will tend to produce a persistent deviation of the satellite X- and Z-axes from the desired reference. However, it may very well be that the deviations produced by this or any other persistent torque disturbance, are small enough to be insignificant in regard to the specific mission to which the satellite has been assigned.

The angular velocity ω_H of the constant-speed flywheel necessary to produce the constant angular momentum H_Y may be determined numerically for typical satellite mass and dimensions.

Suppose that $I_Y = I_X$ (*e.g.*, homogeneous cylindrical satellite), the mass of the satellite is 500 lb and the time taken to describe the orbit is 5 hours, *i.e.*, $\Omega_y = 2\pi/(5 \times 3600) \simeq 0 \cdot 00035$ rad/sec. For a homogeneous cylindrical satellite $I_Z = \frac{1}{2} M_s r_s^2$ where M_s is the mass and r_s the radius of the cylinder. If $r_s = 2$ ft, then $I_Z = 1000$ lb ft^2. These give $H_Y = \Omega_y I_Z = 0 \cdot 35$ lb ft^2 sec^{-1}. If the flywheel has a mass of $0 \cdot 5$ lb which is supposed concentrated in the rim of radius 2 in., then the polar moment of inertia of the flywheel is $0 \cdot 5(1/6)^2 = 1/72$ lb ft^2. These figures give $\omega_H \simeq 25$ rad/sec.

The theory and equations developed in this section have assumed the presence of an additional flywheel to enhance the stability of the θ_1, θ_3 modes; in general, however, it is not necessary to include a separate flywheel for this purpose since the control-flywheel situated along the Y-axis of the satellite may be given this constant angular velocity as a bias, about which angular acceleration for control purposes may be developed.

3. *Reference Axes Fixed in Space.* By making sightings on fixed stars it is possible to define an attitude reference system which maintains a definite orientation in space. The use of such a reference may be ideally suited for missions where it is desired to make certain optical observations, for example by the use of an astronomical telescope on board the satellite. For such observations, greater precision in the attitude control of the satellite will generally be demanded than in many satellite missions.

One problem, associated with the attitude control of a satellite tied to such a space-stabilised reference, arises from the fact that the orientation of the satellite will not be such that the gravitational perturbation torque due to the earth acts as a directional reference; in fact the general effect of the earth's gravitation will be to rotate the satellite away from its desired reference, in order to align the principal axis having the smallest moment of inertia into a direction perpendicular to the local gravitational equipotential surface.

3.1. *Space-Stabilised Satellite Subjected to the Earth's Gravitational Torque.* Suppose the satellite to move in a fixed plane along an elliptical orbit of semi-major axis a and eccentricity e. It will be assumed that the only perturbation torque acting on the satellite is due to the earth's gravitational field, and that the attitude control maintains complete space-stabilisation of the satellite. This means that the satellite axes X, Y, Z are coincident with the reference axes x, y, z. Equation (68) of Appendix I gives the gravitational torque acting on the satellite. As the satellite orbits the earth the gravitational torque will vary and in order to perform integrations round the orbit it is convenient to introduce a set of axes which allow the gravitational torque to be expressed in terms of the angle θ, where θ is measured in the orbital plane. Let X′, Y′, Z′ define a system of space-fixed axes, origin at the centre of mass of the satellite, and having Y′ perpendicular to the orbital plane and Z′ and X′ in the orbital plane, and suppose Z′ to be inclined at some arbitrary angle β with the

direction parallel to the major axis of the ellipse (Fig. 3). Let $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ be unit vectors along (X', Y', Z') and have direction cosines $[(l_1, m_1, n_1);\ (l_2, m_2, n_2);\ (l_3, m_3, n_3)]$ relative to $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ respectively,
i.e.,

$$\mathbf{i}' = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k}\,,$$
$$\mathbf{j}' = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k}\,,$$
$$\mathbf{k}' = l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k}\,.$$

If $\mathbf{r}_0$ is the radius vector from the centre of mass of the satellite to the centre of the earth then,

$$\mathbf{r}_0 = r_0(\cos\theta\mathbf{k}' + \sin\theta\mathbf{i}')\,, \tag{20}$$

where θ is the angle between the Z'-axis and the radius vector. Thus,

$$\begin{aligned}\mathbf{r}_0 &= r_0[(l_3\mathbf{i}+m_3\mathbf{j}+n_3\mathbf{k})\cos\theta + (l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k})\sin\theta] \\ &= r_0[(l_3\cos\theta+l_1\sin\theta)\mathbf{i} + (m_3\cos\theta+m_1\sin\theta)\mathbf{j} + (n_3\cos\theta+n_1\sin\theta)\mathbf{k}]\,.\end{aligned} \tag{21}$$

The components of the radius vector along satellite axes are therefore

$$r_{X0} = r_0(l_3\cos\theta+l_1\sin\theta)\,,$$
$$r_{Y0} = r_0(m_3\cos\theta+m_1\sin\theta)\,,$$
$$r_{Z0} = r_0(n_3\cos\theta+n_1\sin\theta)\,.$$

and the equation for the gravitational torque is

$$\begin{aligned}\mathbf{\Gamma} = -3\frac{GM_E}{r_0^{\,3}}\{&(I_Y-I_Z)(n_3\cos\theta+n_1\sin\theta)(m_3\cos\theta+m_1\sin\theta)\mathbf{i} + \\ &+(I_Z-I_X)(l_3\cos\theta+l_1\sin\theta)(n_3\cos\theta+n_1\sin\theta)\mathbf{j} + \\ &+(I_X-I_Y)(m_3\cos\theta+m_1\sin\theta)(l_3\cos\theta+l_1\sin\theta)\mathbf{k}\}\,.\end{aligned} \tag{22}$$

If the attitude control is to keep the satellite completely space-stabilised against the gravitational torque the mean angular impulse imparted by the control during one orbit is:

$$\mathbf{J}_R = \int_{\theta=0}^{2\pi} \Gamma(\theta)\left(\frac{dt}{d\theta}\right)d\theta\,,\quad \text{where}\quad \left(\frac{dt}{d\theta}\right) = \frac{r_0^{\,2}}{h}\,,\quad r_0 = \frac{l}{1+e\cos(\theta+\beta)}$$

and h is the angular momentum/unit mass of satellite in orbit.
Using the expression for $\mathbf{\Gamma}(\theta)$ given by Equation (22) this becomes:

$$\begin{aligned}\mathbf{J}_R = -3\frac{GM_E}{lh}\int_{\theta=0}^{2\pi}\Big[&(I_Y-I_Z)(n_3m_3\cos^2\theta+\overline{n_1m_3+n_3m_1}\sin\theta\cos\theta+n_1m_1\sin^2\theta)\mathbf{i} + \\ &+(I_Z-I_X)(l_3n_3\cos^2\theta+\overline{l_1n_3+l_3n_1}\sin\theta\cos\theta+l_1n_1\sin^2\theta)\mathbf{j} + \\ &+(I_X-I_Y)(m_3l_3\cos^2\theta+\overline{m_1l_3+m_3l_1}\sin\theta\cos\theta+m_1l_1\sin^2\theta)\mathbf{k}\Big]\times \\ &\times[1+e\cos(\theta+\beta)]\,d\theta\,.\end{aligned} \tag{23}$$

The integrals $\int_0^{2\pi}\cos^3\theta\,d\theta$, $\int_0^{2\pi}\sin^3\theta\,d\theta$, $\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta$, $\int_0^{2\pi}\sin^2\theta\cos\theta\,d\theta$, $\int_0^{2\pi}\sin\theta\cos\theta\,d\theta$, are all zero and the integrals $\int_0^{2\pi}\cos^2\theta\,d\theta$, $\int_0^{2\pi}\sin^2\theta\,d\theta$, are both equal to π.

Hence:

$$\mathbf{J}_R = -3\frac{GM_E\pi}{lh}[(I_Y-I_Z)(n_3m_3+n_1m_1)\mathbf{i} + (I_Z-I_X)(l_3n_3+l_1n_1)\mathbf{j} + + (I_X-I_Y)(m_3l_3+m_1l_1)\mathbf{k}]. \tag{24}$$

If $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are parallel to $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ then $l_1 = m_2 = n_3 = 1$ (all the other direction cosines being zero) and the mean angular impulse per orbit is zero. This result is independent of the angle β so that if the X- and Z-axes of the satellite lie in the orbital plane the mean angular impulse per orbit will be zero, and in general if any two of the satellite axes lie in the plane of the orbit there will be no unidirectional build up of angular momentum during one complete orbit; during each orbit equal amounts of positive and negative angular impulse will be imparted to the satellite. If the satellite axes are not so orientated the mean angular impulse per orbit is not zero and the attitude control must be capable of providing a compensating resultant impulse. Control by reaction-flywheels would eventually cease to be of any use owing to the continual demand for angular acceleration to balance the ever increasing resultant impulse.

3.2. *Maximum Value of* $\mathbf{\Gamma}$ *per Orbit.* When $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ are coincident the magnitude of the gravitational torque is:

$$|\Gamma| = \left|-3\frac{GM_E}{l^3}(I_Z-I_X)\sin\theta\cos\theta(1+e\cos\theta)^3\right|, \quad \text{assuming } \beta = 0,$$

and this is zero when $\theta = n\pi/2$, $n = 0, 1, 2, \ldots$.

To find the maximum value we have

$$\frac{d\Gamma}{d\theta} = -3\frac{GM_E}{l^3}(I_Z-I_X)[3(1+e\cos\theta)^2(-e\sin\theta)\sin\theta\cos\theta + + (1+e\cos\theta)^3(\cos^2\theta-\sin^2\theta)],$$

and this is zero when $(5e\cos^3\theta+2\cos^2\theta-4e\cos\theta-1) = 0$.

Taking the particular case of a circular orbit ($e = 0$) the maxima occur when $\cos\theta = \pm 1/\sqrt{2}$ *i.e.*, for $\theta = (2n+1)\pi/4$, $n = 0, 1, 2 \ldots$, and the magnitude of this maximum torque is:

$$|\Gamma|_{\max} = \left|\frac{3}{2}\frac{GM_E}{l^3}(I_Z-I_X)\right|. \tag{25}$$

3.3. *Total Angular Impulse per Orbit* [$(\mathbf{i}, \mathbf{j}, \mathbf{k})$ *and* $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ *Coincident*]. For a circular orbit, the total amount of angular momentum which must be supplied by the attitude-control system in order that the satellite may remain completely space-stabilised is:

$$J = \left|4\int_{\theta=0}^{\pi/2}\mathbf{\Gamma}(\theta)\left(\frac{dt}{d\theta}\right)d\theta\right|, \tag{26}$$

i.e.,

$$J = \left|4\frac{3GM_E}{lh}(I_Z-I_X)\int_0^{\pi/2}\tfrac{1}{2}\sin 2\theta\, d\theta\right| \tag{27}$$

giving

$$J = \left|6\frac{GM_E}{lh}(I_Z-I_X)\right|. \tag{28}$$

Equation (28) gives the total change of angular momentum that would have to be provided by the attitude control in order to balance the gravitational torque for each complete orbit. For a non-precessing orbit the angular-momentum storage capacity for any number of complete orbits can be evaluated. However, the situation will not in general be quite so simple, for it is known that the effect of the oblateness of the earth is to cause precession of the orbital plane about the earth's axis. A further analysis on the effect of such a precessing orbit with regard to this particular attitude control problem is outside the scope of the present Report, but the principles involved are the same.

3.4. *The Attitude Equations of Motion for a Space-Stabilised Satellite.* Neglecting second-order products for the case of small angles and angular rates as in Section 2, Equations (4) to (6) in the absence of any form of control reduce to the simple form:

$$I_X \ddot{\theta}_1 = L_X \,, \tag{29}$$

$$I_Y \ddot{\theta}_2 = L_Y \,, \tag{30}$$

$$I_Z \ddot{\theta}_3 = L_Z \,. \tag{31}$$

In order to provide a usable system, some means of damping and position stabilisation must be supplied by a suitable control mechanism.

3.5. *The Use of Reaction-Flywheels.* To the order of approximation quoted in Section 2, Equations (7) to (9) become:

$$I\dot{\omega}_1 + I_X\ddot{\theta}_1 + I(\dot{\theta}_2\omega_3 - \dot{\theta}_3\omega_2) = L_X \,, \tag{32}$$

$$I\dot{\omega}_2 + I_Y\ddot{\theta}_2 + I(\dot{\theta}_3\omega_1 - \dot{\theta}_1\omega_3) = L_Y \,, \tag{33}$$

$$I\dot{\omega}_3 + I_Z\ddot{\theta}_3 + I(\dot{\theta}_1\omega_2 - \dot{\theta}_2\omega_1) = L_Z \,. \tag{34}$$

As there is now no position stabilisation in the equations of motion the angular accelerations of the flywheels must be made some function of $\dot{\theta}_j$ and θ_j, $(j = 1, 2, 3)$; any unidirectional torque will eventually cause saturation of the flywheel control owing to the continual demand for increase of angular acceleration to oppose the external torque.

3.6. *Control by Ejection of Gaseous Matter from the Satellite.* This is an obvious method of imparting control torques to a satellite vehicle, for the escape of gas at pressure will result in a reaction on the satellite body. A method of control about the three mutually perpendicular satellite axes can be envisaged by supplying the satellite with three pairs of diametrically opposite orifices for the ejection of gaseous matter. The members of a pair should be diametrically opposite with respect to the centre of mass so as to produce no nett force on the satellite but only a couple. The reaction torques available by such means will provide for control about each of the three body axes. Since the magnitude of such a torque is the product of force and length of torque arm, a greater torque magnitude can be attained for a given velocity of ejection if the orifices are positioned some distance away from the main body of the satellite.

The use of a single jet orifice to provide a control torque about a given axis is undesirable from the point of view of the translational thrust that is experienced at the centre of mass. The existence of such a force during the long time duration of attitude control may introduce undesirable perturbations in the satellite's orbit.

3.7. *The Jet-Reaction Equations.* Ejection of matter at a constant velocity, using the amount of matter ejected per second as the controlling element, will provide a change of angular momentum about the centre of mass of the satellite. The change of angular momentum may be made to take place about each of the three satellite axes by employing diametrically opposite jets, the orifices being positioned at the surface of the satellite so as to coincide with the points where the axes would emerge from the satellite body, each orifice being arranged so that the ejected gas is tangential to the satellite surface, *see* Fig. 4 (spherical outer shell assumed).

Consider first control of motion about the Z-axis, with only the control jets for θ_3 operative. If V_E is the ejection velocity relative to the satellite and r the radius of the sphere, then the angular velocity of the ejected matter relative to satellite axes is $(V_E/r)\mathbf{k}$ for either of the jets parallel to the direction of the X-axis. But the axes themselves have angular velocity $\boldsymbol{\omega} = (\dot{\theta}_1\mathbf{i}+\dot{\theta}_2\mathbf{j}+\dot{\theta}_3\mathbf{k})$ for small angular displacements. Hence the resultant angular velocity of the ejected matter at the instant of ejection will be $\{\dot{\theta}_1\mathbf{i} + \dot{\theta}_2\mathbf{j} + [(V_E/r) + \dot{\theta}_3]\mathbf{k}\}$ about the centre of mass.

For the orifice situated along the positive direction of the Y-axis, the actual velocity of the jet material is therefore:

$$\left[\dot{\theta}_1\mathbf{i} + \dot{\theta}_2\mathbf{j} + \left(\frac{V_E}{r} + \dot{\theta}_3\right)\mathbf{k}\right] \times r\mathbf{j} = \left[r\dot{\theta}_1\mathbf{k} - \left(\frac{V_E}{r} + \dot{\theta}_3\right)r\mathbf{i}\right]. \tag{35}$$

Hence, the change of angular momentum per second about the centre of mass due to material having this velocity being ejected at a rate $\dot{m}_X$ from each of the two orifices is:

$$2\dot{m}_X\left[r\mathbf{j} \times \left\{r\dot{\theta}_1\mathbf{k} - \left(\frac{V_E}{r} + \dot{\theta}_3\right)r\mathbf{i}\right\}\right] = 2\dot{m}_X V_E r\mathbf{k} + 2\dot{m}_X r^2(\dot{\theta}_1\mathbf{i}+\dot{\theta}_3\mathbf{k}), \tag{36}$$

where $\dot{m}_X$ is the controllable quantity.

Similarly, control of motion about the X- and Y-axes may be achieved by ejection parallel to the Y- and Z-axes respectively. For θ_1 control, the change of angular momentum per second about the centre of mass is:

$$2\dot{m}_Y V_E r\mathbf{i} + 2\dot{m}_Y r^2(\dot{\theta}_2\mathbf{j}+\dot{\theta}_1\mathbf{i}) \tag{37}$$

and for θ_2 control the corresponding expression is:

$$2\dot{m}_Z V_E r\mathbf{j} + 2\dot{m}_Z r^2(\dot{\theta}_3\mathbf{k}+\dot{\theta}_2\mathbf{j}). \tag{38}$$

Equations (29) to (31) with jet-reaction control may therefore be written:

$$I_X\ddot{\theta}_1 + 2\dot{m}_Y V_E r + 2r^2(\dot{m}_Y+\dot{m}_X)\dot{\theta}_1 = L_X, \tag{39}$$

$$I_Y\ddot{\theta}_2 + 2\dot{m}_Z V_E r + 2r^2(\dot{m}_Z+\dot{m}_Y)\dot{\theta}_2 = L_Y, \tag{40}$$

$$I_Z\ddot{\theta}_3 + 2\dot{m}_X V_E r + 2r^2(\dot{m}_X+\dot{m}_Z)\dot{\theta}_3 = L_Z. \tag{41}$$

The satellite's moments of inertia (I_X, I_Y, I_Z) must now be regarded as instantaneous values since they will be decreasing as a result of loss of mass by ejection. The centre-of-mass position, however, must remain unchanged since the loss of mass is symmetrical about the origin.

By making $\dot{m}$ dependent on θ_j and $\dot{\theta}_j$ the equations take the form of damped harmonic motion, modified somewhat by the presence of small effects representing components of change of angular momentum due to rotation of the satellite body axes. For small displacements the attitude rates will be small and the jet system can be regarded as providing pure couples about each of the three body axes.

A unidirectional perturbation torque can be catered for by the jet-reaction attitude control, since this simply requires continuous operation of the jet source in question.

4. *On the Elimination of Initial Angular Momentum of the Satellite.* The placing of a satellite into an orbit may result in the satellite having an initial angular velocity relative to its reference frame. The first task of the attitude-control system will be to remove this angular momentum and orientate the satellite axes along the desired reference directions. The control torque necessary for the satellite to maintain a desired attitude once the initial angular momentum has been removed will in general be of a much smaller magnitude than that required to eliminate the initial angular momentum. Consequently, the attitude-control system must be capable of developing a relatively large control torque for a short time duration, and a relatively small torque over a much longer period of time. The initial angular momentum is probably best eliminated by the use of reaction-jets, but subsequent attitude control may be by reaction-flywheels, reaction-jets or any other device suitable as an attitude control element.

It will be convenient to refer to the elimination of any initial angular momentum of the satellite as phase I of the attitude control, and subsequent attitude control as phase II. In the following theory, the satellite is assumed to be spinning about only one of its principal axes of inertia, which is assumed to remain constant throughout. The small effect representing change of angular momentum due to loss of mass at the satellite's angular velocity is also neglected.

4.1. *Space-Stabilised Satellite.* Suppose that the axis of spin is the Y-axis of the satellite and is assumed to be coincident with the y-axis of the reference system. The attitude-deviation angle θ' is then measured in the X-Z plane. In the absence of perturbation torques the uncontrolled rotational motion of the satellite relative to its reference axes may be expressed simply as:

$$I_Y \ddot{\theta}' = 0 \; . \tag{42}$$

By applying a control torque of the form $\Gamma_1 = -k\dot{\theta}'$, the equation of motion becomes:

$$I_Y \ddot{\theta}' + k\dot{\theta}' = 0 \, . \tag{43}$$

For the initial condition $\dot{\theta}' = \dot{\theta}_i'$ when $t = 0$, Equation (43) has the solution for $\dot{\theta}'$:

$$\dot{\theta}' = \dot{\theta}_i' e^{-(k/I_Y)t} \tag{44}$$

and

$$\Gamma_1 = -k\dot{\theta}_i' e^{-(k/I_Y)t} \, . \tag{45}$$

Assuming the control torque to be supplied by reaction-jets, the jet-reaction angular impulse necessary to eliminate the initial angular momentum is:

$$J_1 = \int_0^\infty \Gamma_1 dt \tag{46}$$

$$= -k\dot{\theta}_i' \int_0^\infty e^{-(k/I_Y)t} dt \, ,$$

i.e.,

$$J_1 = -I_Y \dot{\theta}_i' \, . \tag{47}$$

Having eliminated the angular momentum relative to the reference frame, the X and Z satellite axes will in general be at some angle θ_0' relative to the x and z reference axes respectively. This residual angle can now be removed by cutting out phase I control and letting phase II control take over. In order that $\theta' = 0$ shall be a position of equilibrium the phase II control torque must be of the form $\Gamma_2 = -(k_{11}\theta' + k_{12}\dot{\theta}')$, where k_{11} and k_{12} are control parameters. The equation of motion now takes the form:

$$I_Y\ddot{\theta}' + k_{12}\dot{\theta}' + k_{11}\theta' = 0. \tag{48}$$

If the motion is taken to be critically damped the solution of Equation (48) is

$$\theta' = (a+bt)e^{-Kt_1} \tag{49}$$

where a and b are constants and $K = \sqrt{\dfrac{k_{11}}{I_Y}} = \dfrac{k_{12}}{2I_Y}$.

The initial conditions for the phase II control are $\theta' = \theta_0'$, $\dot{\theta}' = 0$, $t_1 = 0$, where t_1 is measured from the end of phase I control. (Assuming phase II to commence immediately phase I is cut out.) These initial conditions give $a = \theta_0'$ and $b = K\theta_0'$.

$$\therefore \qquad \theta' = \theta_0'(1+Kt_1)e^{-Kt_1} \tag{50}$$

and

$$\dot{\theta}' = -K^2\theta_0' t_1 e^{-Kt_1}. \tag{51}$$

Hence

$$\begin{aligned}\Gamma_2 &= -[k_{11}\theta_0'(1+Kt_1) - k_{12}K^2\theta_0' t_1]\, e^{-Kt_1} \\ &= -[K^2 I_Y(1+Kt_1) - 2I_Y K^3 t_1]\,\theta_0' e^{-Kt_1},\end{aligned}$$

i.e.,

$$\Gamma_2 = -[1 - Kt_1]\,K^2 I_Y \theta_0' e^{-Kt_1}. \tag{52}$$

Equation (52) indicates that Γ_2 changes direction at some time $t_0 = 1/K$.

If phase II control is also by reaction-jets, the jet-reaction angular impulse necessary to correct the attitude deviation angle θ_0' under the above conditions is:

$$J_2 = \int_{t_1=0}^{t_0} \Gamma_2 dt_1 - \int_{t_1=t_0}^{\infty} \Gamma_2 dt_1 \tag{53}$$

Now

$$\begin{aligned}\int_{t_1=0}^{t_0} \Gamma_2 dt_1 &= -I_Y K^2\theta_0' \int_{t_1=0}^{t_0} (1-Kt_1)e^{-Kt_1}dt_1 \\ &= -I_Y K^2 \theta_0' (t_0 e^{-Kt_0}), \quad \text{and if } t_0 = \frac{1}{K}:\end{aligned}$$

$$\int_{t_1=0}^{t_0} \Gamma_2 dt_1 = -\frac{I_Y K\theta_0'}{e}. \tag{54}$$

Since there is no initial angular momentum for phase II,

$$\int_{t_1=0}^{t_0} \Gamma_2 dt_1 = -\int_{t_1=t_0}^{\infty} \Gamma_2 dt_1.$$

Hence,

$$J_2 = -\frac{2I_Y K\theta_0'}{e}. \tag{55}$$

Using this method of control, the total jet-reaction angular impulse necessary to remove the initial angular momentum and then line up the satellite in the desired direction is:

$$J' = |J_1| + |J_2| \tag{56}$$

$$J' = |I_Y\dot{\theta}_i| + \left|\frac{2I_Y K\theta_0'}{e}\right|. \tag{57}$$

If, after attitude control is complete, a total amount M of gaseous matter has been expelled from the orifices in the form of a mass flow at a constant exhaust velocity V_E, then the change of angular momentum about the centre of mass of the satellite as a result of this is $2r'MV_E$, if the torque arm is of length $2r'$. This overall change of angular momentum must be equal to the jet-reaction angular impulse J' given by Equation (57):
i.e.,

$$2r'MV_E = |I_Y\dot{\theta}_i'| + \left|\frac{2I_YK\theta_0'}{e}\right| \tag{58}$$

$$\therefore \qquad M = \frac{|I_Y\dot{\theta}_i'| + \left|\frac{2I_YK\theta_0'}{e}\right|}{2r'V_E}. \tag{59}$$

Equation (59) is a theoretical estimate of the mass expelled at velocity V_E in order that attitude control be complete. In a practical system the efficiency will not be 100 per cent, and in addition, the weight of the gas containers and feed pipes must be taken into account for an overall estimate of weight.

Alternatively, of course, phase I control may be designed to provide a torque of the form $\Gamma_1 = -(k'\theta' + k\dot{\theta}')$, so as to produce the desired result (*i.e.*, satellite and reference axes coincident) before phase II is brought into operation. In particular, this would be a necessity if the phase II control was designed to cope only with small angular deviations. On the other hand, if phase II has been designed to deal with arbitrarily large angles, then $\Gamma_1 = -k\dot{\theta}'$ is a simpler system for phase I.

4.2. *Earth-Pointing Satellite.* If the satellite is dependent solely on the earth's gravitational field for providing stabilisation, a control torque of the form $\Gamma_1 = -(k'\theta' + k\dot{\theta}')$ is desirable during phase I control. For example, correction by the torque arising from the earth's gravitational field of a residual angle of 90 deg or more of the 'long axis' from the radius vector would result in the satellite pointing the 'wrong end' towards the earth.

5. *Conclusions.* The problem of attitude control has been discussed for an earth-pointing satellite and a satellite stabilised to a space-fixed reference frame. The equations of attitude motion have been derived using small-angle approximations. However, choice of optimum control parameters may result in a given control system providing satisfactory control for angles in excess of the values for which the small-angle approximation is valid.

An earth-pointing satellite using the angular acceleration of reaction-flywheels to provide damping of attitude motion would seem to be a suitable starting point for the design of an attitude-control system. Such an orbiting satellite could be used for making observations of the earth and its surroundings, or for making astronomical observations. In the absence of damping, any small disturbance producing a nett turning effect about the centre of mass of the satellite will cause the vehicle to execute oscillations about its equilibrium position. Additional disturbances could increase the amplitude of oscillation, and eventually result in a stable attitude being lost over the period of time for which the satellite was intended to be used. Damping of the first disturbance would eliminate the oscillatory motion and return the satellite to its equilibrium position ready for any future perturbation.

The major torque disturbance for a space-stabilised satellite orbiting relatively close to the earth will in general be due to the gravitational-field gradient due to the earth; depending on the orientation of the satellite relative to the earth, this perturbation can be expected to cause either an oscillatory torque of zero mean, or the more troublesome oscillatory torque with finite mean.

For the space-stabilised satellite position stabilisation has to be provided in order that the desired reference directions may be maintained. In general reaction-jets are probably more suitable than flywheels for this purpose as the flywheel system is more susceptible to angular-momentum saturation. The more logical choice for a space-stabilised satellite is the configuration having its principal moments of inertia all equal since the gravitational torque then disappears.

In general, the design and nature of the attitude-control system will be determined by the type of mission to which the satellite has been assigned, different missions being characterised essentially by the attitude tolerance which may be allowed, in order that useful data may be obtained from observations and recordings made by the satellite's instruments.

Any angular momentum possessed by the satellite at orbit injection may be eliminated by reaction-jets. The magnitudes of the control torques used in this process will in general be far greater than those required to counter the attitude perturbations once the satellite has settled down in its orbit. The attitude-control designer may therefore have the task of supplying two control mechanisms, not necessarily different in principle, to satisfy both needs.

Attitude stabilisation is necessary for a great many satellite applications and must therefore demand detailed studies of suitable control systems. The account presented in this Report is intended only as an introduction to the subject, written (it is hoped) in sufficiently simple mathematical language to enable the non-specialist to appreciate the principles involved.

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NOTATION

$\boldsymbol{\omega}$	Spin vector of satellite
$\boldsymbol{\Omega}$	Spin vector of reference axes
$\boldsymbol{\omega}_i$	Spin vector of a typical flywheel relative to satellite axes
$\mathbf{H}_T$	Total angular-momentum vector of satellite
$\mathbf{L}$	Resultant external torque acting on the satellite from all sources
$\boldsymbol{\Gamma}$	Gravitational torque on the satellite configuration
$\mathbf{r}_0$	Radius vector from centre of mass of satellite to centre of the earth
$\mathbf{r}$	Position vector of a typical satellite particle relative to the centre of mass
$\mathbf{J}_R$	Resultant angular impulse per orbit on space-stabilised satellite due to the earth's gravitational torque
(X, Y, Z)	Orthogonal system of right-handed axes, origin at the centre of mass of the satellite and defining the directions of the principal axes of inertia of the satellite
(x, y, z)	Orthogonal system of right-handed axes, origin at the centre of mass of the satellite and defining the attitude reference directions
(X′, Y′, Z′)	System of axes having Z′ and X′ parallel to the major and minor axes of an elliptical orbit and Y′ normal to the orbital plane
$(\mathbf{i}, \mathbf{j}, \mathbf{k})$	System of unit vectors along the (X, Y, Z) satellite axes
$(\mathbf{i}'', \mathbf{j}'', \mathbf{k}'')$	System of unit vectors along the (x, y, z) reference axes
$(\mathbf{i}', \mathbf{j}', \mathbf{k}')$	System of unit vectors along axes (X′, Y′, Z′)
(I_X, I_Y, I_Z)	Principal moments of inertia of the satellite
$[(l_1, m_1, n_1); (l_2, m_2, n_2); (l_3, m_3, n_3)]$	Direction cosines specifying relative spatial orientations of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$
$(\omega_X, \omega_Y, \omega_Z)$	Spin components of satellite along satellite axes
$(\omega_X', \omega_Y', \omega_Z')$	Spin components along satellite axes of satellite relative to reference axes
$(\omega_1, \omega_2, \omega_3)$	Angular velocities of reaction-flywheels along the X, Y and Z axes respectively (relative to satellite axes)
(L_X, L_Y, L_Z)	Components of $\mathbf{L}$ along satellite axes
(L_X', L_Y', L_Z')	Components of $\mathbf{L}$ after the gravitational torque components have been omitted
$(\Omega_x, \Omega_y, \Omega_z)$	Components of $\boldsymbol{\Omega}$ along reference axes
(r_{X0}, r_{Y0}, r_{Z0})	Components of $\mathbf{r}_0$ along satellite axes
G	Gravitational constant
M_E	Mass of the earth

I	Polar moment of inertia of typical flywheel
H_Y	Angular momentum of constant-speed flywheel relative to satellite axes
ω_H	Angular velocity of constant-speed flywheel relative to satellite axes
h	Angular momentum per unit mass of satellite in orbit
k_1, k_2, k_3	Control parameters
e	Eccentricity of elliptic orbit
l	Semi-latus rectum of elliptic orbit
$(\theta+\beta)$	Angle which radius vector from satellite to earth's centre makes with the major axis of elliptic orbit
J	Magnitude of total angular impulse per orbit due to earth's gravitational torque acting on space-stabilised satellite in a non-precessing orbit
V_E	Exhaust velocity of jet-stream relative to satellite axes
r	Radius of spherical satellite
M_s	Mass of cylindrical satellite
r_s	Radius of cylindrical satellite
$(\dot{m}_X, \dot{m}_Y, \dot{m}_Z)$	Rate of loss of mass parallel to the X, Y, Z satellite axes respectively
U^*	Unit tensor
$\frac{\partial}{\partial t}\mathbf{A}$	Denotes differentiation of **A** with respect to satellite axes
$\frac{d}{dt}\mathbf{A}$	Denotes differentiation of **A** with respect to space axes

Section 4 *of text*

θ'	Angular displacement from attitude equilibrium position for space-stabilised satellite
$\dot{\theta}_i'$	Initial angular velocity of satellite about an axis having moment of inertia I_Y
θ_0'	Residual angle from attitude equilibrium position
Γ_1, Γ_2	Control torques
k, k', K	Control parameters
J_1, J_2, J'	Jet-reaction angular impulses
M	Total amount of matter expelled from the satellite in order to give complete attitude correction for an arbitrary initial spin $\dot{\theta}_i'$
V_E	Exhaust velocity of jet-stream relative to satellite axes
$2r'$	Length of torque arm

APPENDIX

Gravitational Torque on the Satellite Configuration

Let O be the centre of mass of the satellite and P the position of any particle of the satellite characterised by the vector $\mathbf{r}$ relative to the centre of mass. Let $\mathbf{r}_0$ be the position vector of the centre of the earth relative to the satellite's centre of mass, *see* Fig. 5. If δm is the mass of the particle at P and ρ the density, then $\delta m = \rho\ \delta v$ where δv is the volume element it occupies.

The gravitational force of attraction on the particle by the earth is

$$\mathbf{F} = \frac{GM_E}{|\mathbf{r}_0 - \mathbf{r}|^3}(\mathbf{r}_0 - \mathbf{r})\rho\delta v \tag{60}$$

where M_E is the mass of the earth and G the gravitational constant.

The torque about the centre of mass of the satellite due to the force exerted on this particle is:

$$\delta\mathbf{\Gamma} = \frac{GM_E}{|\mathbf{r}_0 - \mathbf{r}|^3}\mathbf{r} \times (\mathbf{r}_0 - \mathbf{r})\rho\delta v, \tag{61}$$

i.e.,

$$\delta\mathbf{\Gamma} = \frac{GM_E}{|\mathbf{r}_0 - \mathbf{r}|^3}\mathbf{r} \times \mathbf{r}_0\rho\delta v. \tag{62}$$

The resultant torque about the centre of mass due to all particles of the satellite is therefore:

$$\mathbf{\Gamma} = GM_E \iiint_S \frac{\mathbf{r} \times \mathbf{r}_0}{|\mathbf{r}_0 - \mathbf{r}|^3}\rho dv, \tag{63}$$

where the triple integral indicates integration over the whole configuration.

Since the satellite is small compared with its distance from the centre of the earth, we may expand $|\mathbf{r}_0 - \mathbf{r}|^{-3}$ as follows:

$$\begin{aligned}
|\mathbf{r}_0 - \mathbf{r}|^{-3} &= [(\mathbf{r}_0 - \mathbf{r}) . (\mathbf{r}_0 - \mathbf{r})]^{-3/2} \\
&= |r_0|^{-3}\left[1 - \frac{2\mathbf{r} . \mathbf{r}_0}{|r_0|^2} + \frac{|\mathbf{r}|^2}{|r_0|^2}\right]^{-3/2} \\
&= |r_0|^{-3}\left[1 - \frac{3}{2}\left(-\frac{2\mathbf{r} . \mathbf{r}_0}{|r_0|^2} + \frac{|\mathbf{r}|^2}{|r_0|^2}\right) + \right. \\
&\qquad \left. + \frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{|r_0|^4}(-2\mathbf{r} . \mathbf{r}_0 + |r|^2)^2 + \dots\right] \\
&= |r_0|^{-3}\left[1 + \frac{3\mathbf{r} . \mathbf{r}_0}{|r_0|^2} + \text{terms of higher order in } \left(\frac{r}{r_0}\right)\right].
\end{aligned}$$

Thus

$$\mathbf{\Gamma} = \frac{GM_E}{|r_0|^3}\iiint (\mathbf{r} \times \mathbf{r}_0)\left(1 + \frac{3\mathbf{r} . \mathbf{r}_0}{|r_0|^2}\right)\rho\, dv. \tag{64}$$

Since $\mathbf{r}$ is the position vector referred to the centre of mass,

$$\iiint\limits_S \mathbf{r}\rho dv = 0, \text{ and if } \mathbf{a} = \mathbf{r} \times \mathbf{r}_0 \text{ then } \iiint\limits_S \mathbf{a}\rho dv = 0.$$

Hence,

$$\boldsymbol{\Gamma} = \frac{3GM_E}{|r_0|^5} \iiint\limits_S (\mathbf{r} \times \mathbf{r}_0)(\mathbf{r} \cdot \mathbf{r}_0)\rho dv, \tag{65}$$

i.e.,

$$\boldsymbol{\Gamma} = \frac{3GM_E}{|r_0|^5} \left[\left(\iiint\limits_S \mathbf{rr}\rho dv\right) \cdot \mathbf{r}_0\right] \times \mathbf{r}_0. \tag{66}$$

The inertia tensor about the centre of mass of the satellite is defined by:

$$I^* = \iiint\limits_S (\mathbf{r}^2 U^* - \mathbf{rr})\rho dv,$$

where U^* is the unit tensor, and since

$$(U^* \cdot \mathbf{r}_0) \times \mathbf{r}_0 = 0$$

we have

$$\left[\left(\iiint\limits_S \mathbf{rr}\rho dv\right) \cdot \mathbf{r}_0\right] \times \mathbf{r}_0 = -(I^* \cdot \mathbf{r}_0) \times \mathbf{r}_0.$$

Hence

$$\boldsymbol{\Gamma} = -\frac{3GM_E}{|r_0|^5}(I^* \cdot \mathbf{r}_0) \times \mathbf{r}_0. \tag{67}$$

For components along the principal axes of inertia of the satellite we have:

$$I^* = I_X\mathbf{ii} + I_Y\mathbf{jj} + I_Z\mathbf{kk} \text{ and } \mathbf{r}_0 = r_{X0}\mathbf{i} + r_{Y0}\mathbf{j} + r_{Z0}\mathbf{k},$$

$$I^* \cdot \mathbf{r}_0 = I_X r_{X0}\mathbf{i} + I_Y r_{Y0}\mathbf{j} + I_Z r_{Z0}\mathbf{k},$$

and

$$(I^* \cdot \mathbf{r}_0) \times \mathbf{r}_0 = (I_Y - I_Z) r_{Y0} r_{Z0}\mathbf{i} + (I_Z - I_X) r_{X0} r_{Z0}\mathbf{j} + (I_X - I_Y) r_{X0} r_{Y0}\mathbf{k}.$$

So,

$$\boldsymbol{\Gamma} = -\frac{3GM_E}{|r_0|^5}[(I_Y - I_Z) r_{Y0} r_{Z0}\mathbf{i} + (I_Z - I_X) r_{X0} r_{Z0}\mathbf{j} + (I_X - I_Y) r_{X0} r_{Y0}\mathbf{k}]. \tag{68}$$

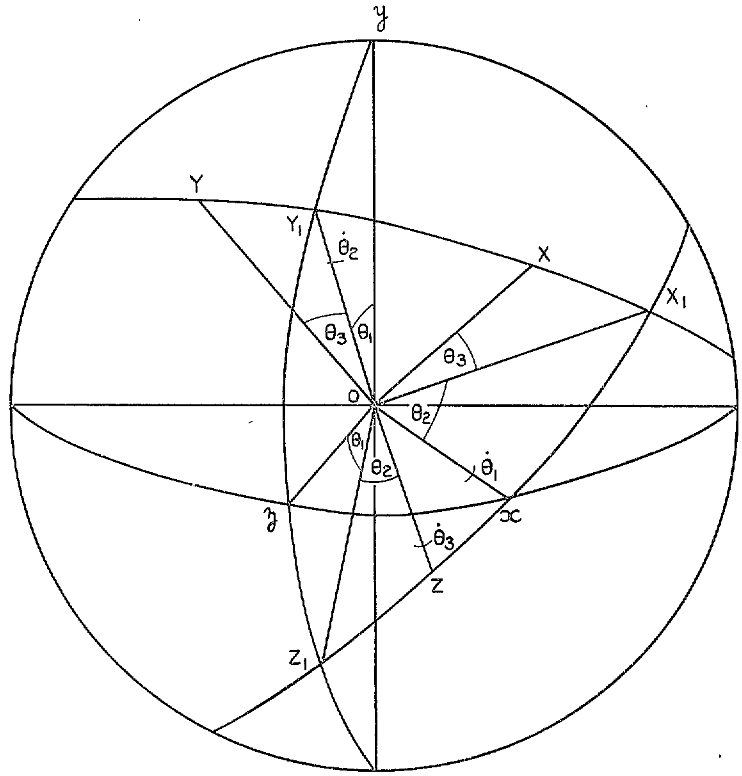

FIG. 1. Orientation of satellite axes relative to reference axes.

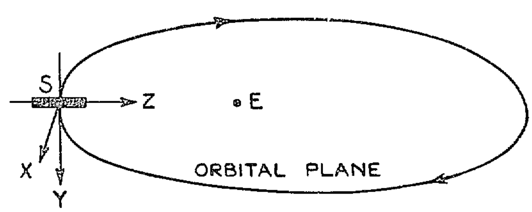

FIG. 2. Earth-pointing cylindrical satellite. (Satellite axes and reference axes coincident.)

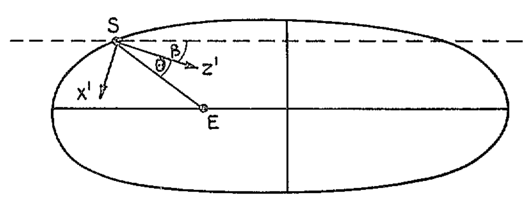

FIG. 3. Space-stabilised satellite in elliptic orbit.

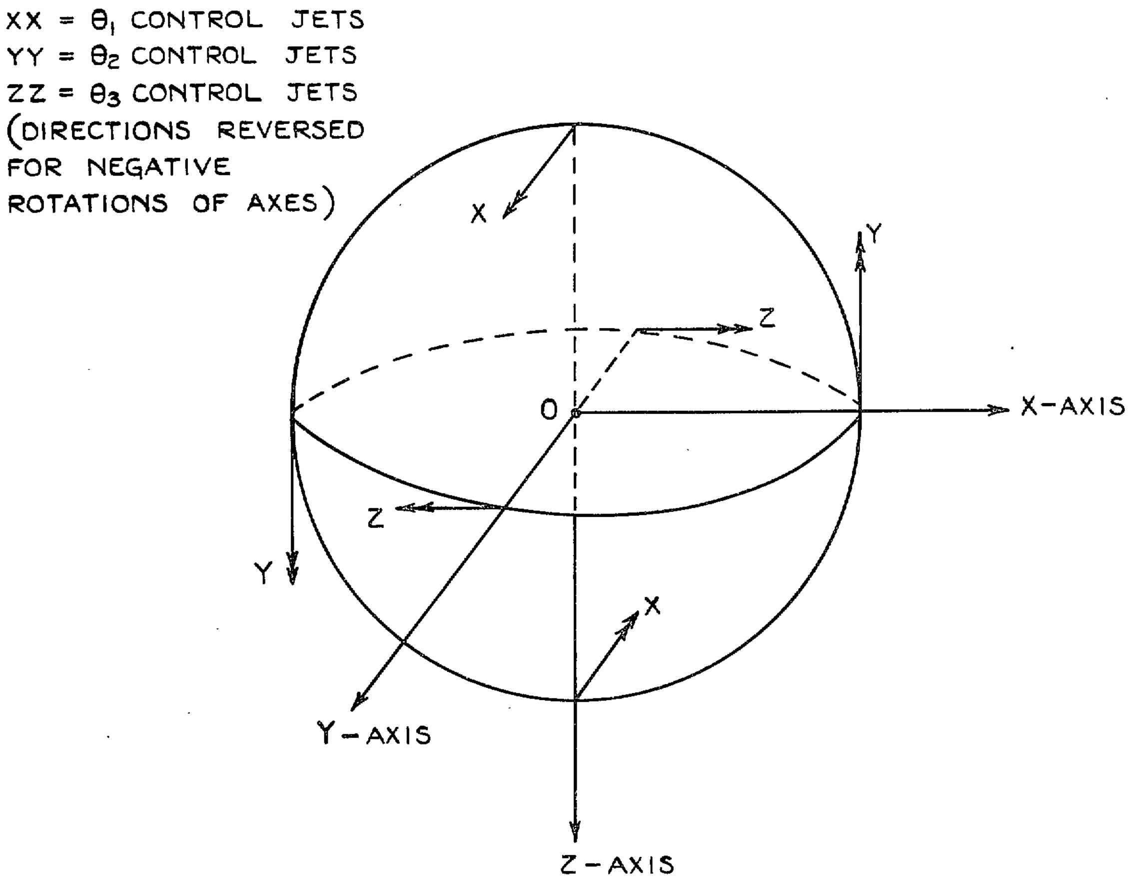

FIG. 4. Spherical satellite with control jets positioned at the surface.

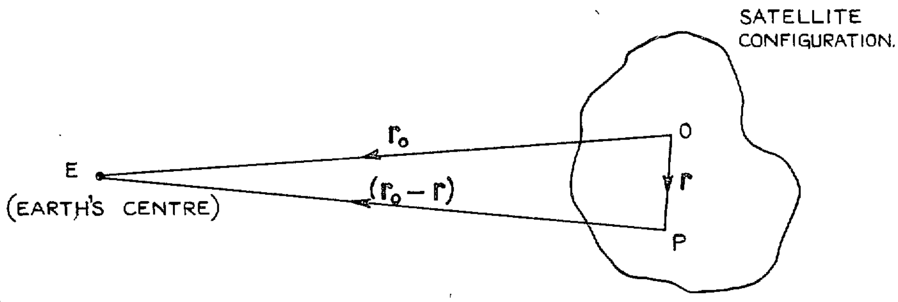

FIG. 5. Position vectors from satellite configuration to centre of the earth.

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