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Thermal Buckling of Circular Plates

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Summary. The post-buckling behaviour of a circular plate is investigated for two conditions—heating over the centre and heating at the edge. In the former case the plate buckles into the form of a saucer ; in the latter into a saddle shape. The plate may be free or it may be subjected to a variety of edge restraints. For centre heating, edge restraint has little effect on the behaviour ; for edge heating it is extremely important.

General non-dimensional curves are presented for the deflected form and bending stresses. Experiments carried out on plates under centre heating generally support the analyses, but precise confirmation was impossible owing to inherent experimental difficulties.

Introduction. The following report on the post-buckling behaviour of circular plates subjected to radially-symmetric heating is based on a Cambridge Ph.D. thesis by B. Cotterell of the same title. The original thesis, which is mainly analytical, is in two volumes each of some 200 pages. No attempt will be made in the present paper to provide a comprehensive summary of the original mathematics —nor indeed would it be easy to do so. The physical and mathematical arguments involved in the problems of convergence, choice of roots, and so on (due to the non-linearity of the problem) are extremely complicated, and anyone wishing to undertake research in this field should consult the thesis. The object of this paper is simply to present the results of Cotterell's investigation in a form most readily understandable and usable by a structural designer.

The paper is divided into five parts: first, a description of the problem and the mathematical approach; second and third, a discussion of the analytical results for the two types of heating considered; fourth, the experimental results; and fifth, the general conclusions of the investigation.

1. *The Problem and the Analysis.* The problem considered is that of a circular plate subjected to heating at its centre or at its edge (Fig. 1). In the former case the plate buckles into a saucer shape, in the latter into the form of a saddle. The plate may be free or it may have edge restraints of the kind shown in Fig. 2.

The radiative heat flux which is applied to each side of the plate in the case of centre heating is supposed to have a maximum value at the centre of q_0 and to vary between the centre and non-dimensional radius $r = r_1$ according to the equation

$$q = q_0 \exp(-kr^2). \tag{1}$$

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Outside the radius $r = r_1$ the heat flux is zero. In all the numerical calculations, the value of k was taken as 30 and r_1 as 0·3: the total flux was thus

$$Q = \int_0^{0\cdot 3} q_0 \exp(-30r^2) 4\pi r R^2 dr$$

$$= 0\cdot 1955 q_0 R^2 \tag{2}$$

where R is the radius of the plate. The distribution (shown in Fig. 3) was chosen to correspond with the experimental conditions, but clearly the exact form of Q is not significant in the problem.

For edge heating, the heat input is q_0 per unit area, so that the total flux is $Q = 2\pi q_0 Rt$, where t is the thickness of the plate.

The heat loss from the surface of the plate is taken as $h_0(1+\beta T^2)T$ per unit area where h_0 is the heat-transfer coefficient, β is a small modifying parameter and T is the temperature above ambient. If the conductivity of the plate material is K, then for steady-state conditions, assuming negligible variation of temperature through the plate

$$\frac{d^2T'}{dr^2} + \frac{1}{r}\frac{dT'}{dr} + \lambda^2 \frac{q}{q_0} - \lambda^2(1+\beta' T'^2)T' = 0 \tag{3}$$

where

$$\left.\begin{aligned} \lambda^2 &= \frac{2R^2 h_0}{Kt} \\ \beta' &= \left(\frac{q_0}{h_0}\right)^2 \beta \\ T' &= \frac{h_0 T}{q_0} \end{aligned}\right\} \tag{4}$$

For the case of centre heating, the boundary conditions on Eqn. (3) are symmetry at $r = 0$, and $dT'/dr = CT'$ at $r = 1$ where C is a constant depending on the edge conditions. For edge heating, we have $dT'/dr = h_0R/K$ at $r = 1$. Eqn. (3) can be solved numerically and some of the results are quoted below.

For centre heating, variation of C and β' has but slight effect on the temperature distribution in the plate, as may be seen from Figs. 4 and 5. The distribution of q is also unimportant and the temperature distribution is primarily dependent on λ (Fig. 6).

For edge heating, the distribution of temperature is again insensitive to variation of β'. For small values of λ^2 the distribution is parabolic: for larger values the temperature becomes more nearly constant over the centre of the plate, but unlike the previous case, the temperature variation is very insensitive to changes in λ^2 (Fig. 7).

The large-deflection equations for the equilibrium and compatibility of the circular plate are derived by Cotterell in a manner closely following that of Timoshenko. From them he obtained for the case of centre heating the set of equivalent first-order equations in terms of five variables $y_0 \ldots y_4$ shown in Table 1. In these equations, ϕ is an Airy stress function from which the mean stresses can be obtained, w is the deflection, expressed as a multiple of the plate thickness, E is Young's modulus, α the coefficient of expansion, D the plate flexural rigidity per unit width and s is the step length in the integration process.

TABLE 1

Variables of the first-order equations	Equivalent function	Initial value at $y_0 = 0$	First derivative	Values of first derivative at $y_0 = 0$
y_0	$\frac{r}{s}$	0	$\frac{dy_0}{dr} = \frac{1}{s}$	$\frac{1}{s}$
y_1	$r\frac{d\phi}{dr}$	0	$\frac{dy_1}{dr} = sy_0 y_2$	0
y_2	$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$	?	$\frac{dy_2}{dr} = -Ea\frac{dT}{dr} - \frac{1}{2}E\left(\frac{t}{r}\right)^2 \frac{y_3^2}{s^3 y_0^3}$	0
y_3	$r\frac{dw}{dr}$	0	$\frac{dy_3}{dr} = sy_0 y_4$	0
y_4	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$	?	$\frac{dy_4}{dr} = \frac{R^2 t}{D}\frac{y_1 y_3}{s^3 y_0^3}$	0

A permanent subroutine of EDSAC 2 (the digital computer used) makes it possible to advance by one step the integration of a set of first-order differential equations of the type $dy_0/dr = f(y_0, y_1, y_2 \ldots)$. By starting at the centre of the plate ($y_0 = 0$) it is thus possible to integrate the equations of Table 1 step-by-step, provided one assumes initial values for y_2 and y_4, which are unknown. The problem is to guess values for y_2 and y_4 at $y_0 = 0$ such that when the integration process is complete the boundary conditions at the edge of the plate ($y_0 = 1/s$) are satisfied. These conditions are

$$\left.\begin{aligned} y_1 + \frac{S_r}{E}\Big[y_2 + (1+\nu)y_1\Big] - \alpha T &= 0 \\ y_4 - (1-\nu)y_3 + \frac{S_\psi R^2}{Dt} y_3 &= 0 \end{aligned}\right\} \tag{5}$$

where S_r and S_ψ are stiffness coefficients for the resistance of the edge members to displacement and rotation.

The correct determination of the initial values of y_2 and y_4 and the problems of convergence occupy a major part of Cotterell's thesis—we shall not discuss this here, but will merely present the results of the analysis.

For the case of edge heating we no longer have radial symmetry, since the plate buckles into the form of a saddle. The deflection after buckling may be written in the form

$$w = \sum_{m=2,6,10\ldots} \rho_m \sin m\theta \tag{6}$$

where θ is an angular co-ordinate and ρ_m is a function of r only. The complexity of the solution depends on the number of terms taken in the series. Using the first term only, Cotterell obtained eight equivalent first-order equations for equilibrium and compatibility. There are three unknown values initially at the centre of the plate and three corresponding boundary conditions at the edge. By taking further terms in the series, Cotterell was able to show that the one-term solution is of adequate accuracy. Again, we shall not consider the details of his analysis, but only the results.

2. *The Plate Heated at the Centre.* The heat flux and temperature distributions used in this case have been discussed in Section 1. The results which follow are mainly non-dimensional, but where quantitative values are given they are those appropriate to a chrome steel having the characteristics shown in Table 2.

TABLE 2

Characteristics of Chrome Steel

$E = 31 \times 10^6$ lb/in.^{2}

$K = 0{\cdot}70$ watts/in. deg C

$\alpha = 12 \times 10^{-6}$/deg C

$h_0 = 2{\cdot}41 \times 10^{-3}$ watts/in.^{2} deg C

$\beta = 1{\cdot}67 \times 10^{-5}$/deg C^2

$\nu = 0{\cdot}26$

The important parameter in determining the temperature distribution in the plate is, as we have previously noted, $\lambda^2 = 2R^2h_0/Kt$, which is a measure of the ratio of heat loss from the surface of the plate to heat flow radially through the plate. The intensity of heat input, which will determine the amount of buckling, depends on q_0 and may conveniently be expressed non-dimensionally in terms of $\mu = (\alpha q_0/h_0)(R/t)^2$. λ^2 and μ determine the problem—the effects of varying the other parameters β' and C are negligible. In Fig. 8 we show how the deflected form of a free plate changes with varying heat input. It will be seen that the 'saucer' has a point of inflexion, and bends down at the edge. As the buckle develops, the point of inflexion moves inwards. In Fig. 9 we show, for a given heat input, the effect of varying the characteristics of the plate. Fig. 10 shows the growth of the maximum deflection with heat input μ for different values of λ^2.

In Fig. 11 we show for a particular plate the effect of the various types of edge restraint shown in Fig. 2. It will be seen that edge restraint has only a very local effect and that edge members cannot inhibit buckling significantly.

The mean stress in the plate rises to a certain value at buckling and thereafter (like the end load in a strut) remains nearly constant. This is shown for a particular plate in Fig. 12 where it will be seen that the mean stress is very insensitive to heat input.

Since the mean stresses after buckling remain constant and fairly small, the important stresses are those due to bending. The maximum bending stress occurs at the centre of the plate, whatever edge restraints may be applied and its value is not significantly affected by the edge conditions. It is convenient to express the bending stress non-dimensionally as

$$\sigma' = \left(\frac{t}{R}\right)^2 \times \frac{\text{bending stress}}{E}. \tag{7}$$

The variation of $\sigma'_{\max}$ with λ^2 and μ is shown in Fig. 13 and the distribution of σ' for a particular value of heat input μ in Fig. 14.

3. *The Plate Heated at the Edge.* The temperature distribution for this case has been previously considered in Section 1. As may be seen from Fig. 7, the variation of temperature is insensitive to change of λ^2 and the behaviour of the plate is determined almost entirely by the heat input parameter μ. Prior to buckling, the largest stresses in the plate are the tangential compressive stresses near the edge. As might be expected, the edge behaves like a strut and buckles into four lobes with two perpendicular nodal lines, the whole plate forming a saddle (Fig. 1). The variation of deflection with radius is closely parabolic, and that with angular position is closely sinusoidal (*i.e.*, in Eqn. (6) r^2 predominates in ρ_2 and terms beyond the first are almost negligible). The variation of maximum deflection with μ and λ^2 is shown in Fig. 15—it will be seen that unlike the previous case, λ^2 is not an important parameter.

Radial restraints on the edge of the plate have only a slight effect on deflection, as would be expected since the buckle is primarily circumferential. Restraints perpendicular to the plane of the plate, on the other hand, have a very profound effect. Preventing deflection completely (*e.g.*, by building the plate into a cylinder) increases the heat input for buckling tenfold. Figs. 16 and 17 show the effect of making dw/dr zero at the edge of the plate and of attaching a small circular edge member: for an edge member of area greater than $3 \cdot 8Rt$ the plate cannot buckle, since the principal stresses are nowhere compressive.

Prior to buckling, as we have previously noted, the outer part of the plate is subjected to compressive tangential stresses and the inner part of the plate is in tension. After buckling the stress pattern changes and the amount of the outer part of the plate which is subjected to compression is much reduced (Fig. 18). The mean stresses after buckling remain small. The bending stresses are not greatly affected by variations of λ^2: the distribution of bending stresses for different values of μ is shown in Fig. 19.

4. *Experimental Results.* Tests were carried out under centre heating conditions on two 10 in. radius ground flat chrome steel plates having the material properties shown in Table 2. One plate was of $\frac{1}{8}$ in. and the other of $\frac{1}{16}$ in. thickness. Heat was applied to each side of the centre of the plate under test by means of an infra-red lamp. Strains were recorded using Phillips high-temperature gauges and temperatures close to the gauges by copper-constantan thermocouples. The strain gauges were calibrated individually for temperature dependence by placing the complete plate in a hot-air oven.

The readings of the thermocouples confirmed that the temperature distributions used in the analytical work were closely similar to those obtained in the tests. The correspondence of the theoretical and experimental stress distributions was, however, less satisfactory. Figs. 20 and 21 show the theoretical and experimental bending stresses in the two plates for various values of the temperature difference (ΔT) between the centre and edge of the plate. When buckling is well established (ΔT large) the experimental bending stresses seem generally to be rather larger nearer the centre of the plate and smaller near the edge than those predicted theoretically. This is also true in the thinner ($\frac{1}{16}$ in.) plate for smaller values of ΔT, but in the $\frac{1}{8}$ in. plate the stresses at small values of ΔT are everywhere smaller than predicted.

It must be realised in evaluating the significance of the experimental results that in obtaining the experimental stresses in this problem one is always concerned with the small difference of two large quantities (*i.e.*, the difference of the recorded strain reading and the strain due to temperature alone). The strain due to bending stress is always less than one third of the total strain. That due to the mean stress is less than one fifteenth of the total strain: we have not shown the experimental mean stresses—in the case of the $\frac{1}{8}$ in. plate they confirm the analysis fairly well; those in the $\frac{1}{16}$ in. plate are too small to have much significance.

Bearing in mind the difficulties of the experimental work (even a small difference in temperature between the gauge and the plate has a marked effect on the derived experimental stress) the results shown in Figs. 20 and 21 may be said to show that the general form and magnitude of the analytical results are correct, without providing a precise confirmation.

5. *Conclusions.* Analytical solutions have been obtained for the temperature distributions and post-buckling stress distributions in a circular plate subjected to heating either at its centre or at its edge: the plate may be free or it may be subjected to various kinds of edge restraint.

For centre heating the plate buckles into a saucer shape. Edge restraints have very little effect on the behaviour. The mean stresses remain constant and small after buckling and the important stresses are those due to bending. The bending stresses are essentially determined by two parameters only: one, λ^2, defines the characteristics of the plate, and the other, μ, determines the level of heat input.

For edge heating the plate buckles into a saddle shape. Edge members can have a profound effect on the buckling and may inhibit it altogether. In the post-buckled condition the bending stresses are again dominant. Unlike the previous case, λ^2 is not an important parameter, and the bending stresses are largely determined by μ alone.

Experiments on plates under centre heating provided some measure of confirmation of the analytical work, but precise agreement was not obtained, largely due to the inherent experimental difficulties.

NOTATION

C		Heat-transfer coefficient at edge of plate
D		Flexural rigidity of plate per unit width
E		Young's modulus
h_0		Heat-transfer coefficient at surface of plate
K		Thermal conductivity
k		Constant determining distribution of radiant heat flux
m		Number defining term in deflection equation
Q		Total heat flux
q		Heat flux
q_0		Maximum heat flux
R		Radius of plate
r		Non-dimensional co-ordinate = radius/R
r_1		Value of r defining extent of radiant heat flux
S_r		Stiffness coefficient for resistance of edge member to displacement
S_ψ		Stiffness coefficient for resistance of edge member to rotation
s		Step length in integration process
T		Temperature above ambient
T'	=	$h_0 T/q_0$
ΔT		Temperature difference between centre and edge of plate
t		Thickness of plate
w		Non-dimensional deflection = deflection/t
$y_0 \ldots y_4$		Variables in first-order equations
α		Coefficient of linear expansion
β		Secondary heat-transfer coefficient
β'	=	$q_0{}^2\beta/h_0{}^2$
ϕ		Airy stress function
ν		Poisson's ratio
λ^2		$2R^2h_0/Kt$
ρ_m		Coefficient in deflection equation
σ'		Non-dimensional bending stress = bending stress $\times\, t^2/R^2E$
θ		Angular co-ordinate
μ		$\alpha q_0 R^2/h_0 t^2$

(83532) A**

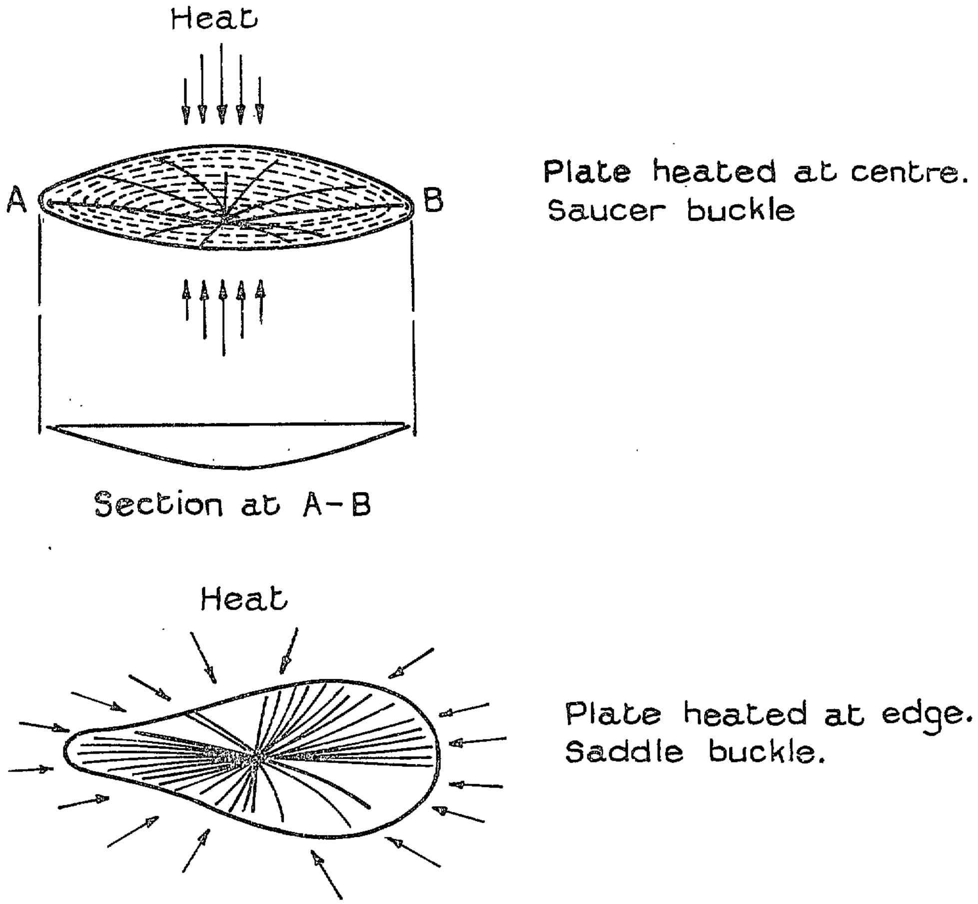

FIG. 1. Types of heating.

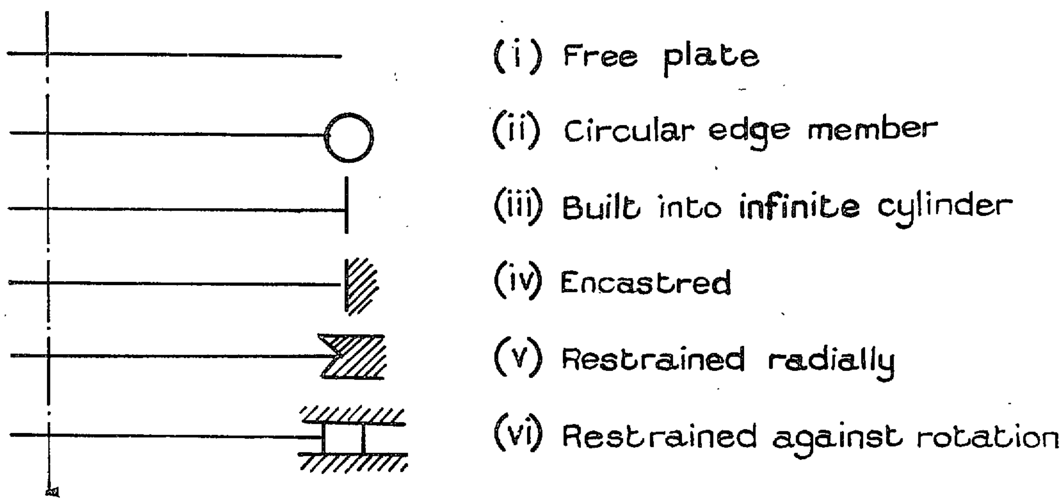

FIG. 2. Edge restraints.

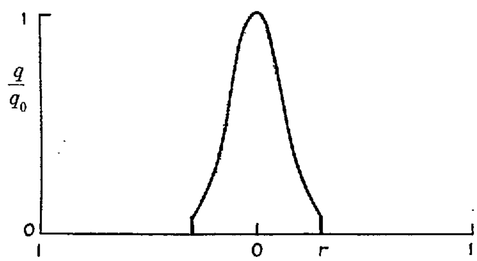

FIG. 3. Distribution of radiant heat flux. (Centre heating.)

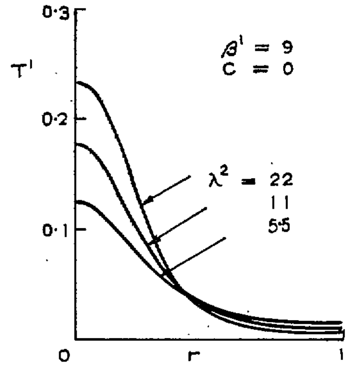

FIG. 6. Temperature variation with λ^2. (Centre heating.)

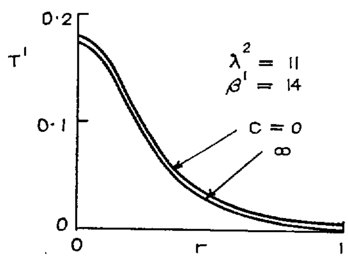

FIG. 4. Temperature variation with C. (Centre heating.)

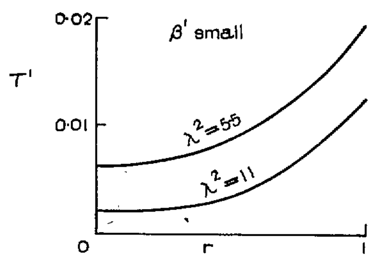

FIG. 7. Temperature variation with λ^2. (Edge heating.)

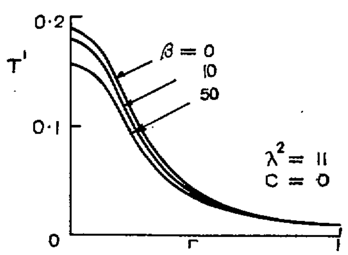

FIG. 5. Temperature variation with β'. (Centre heating.)

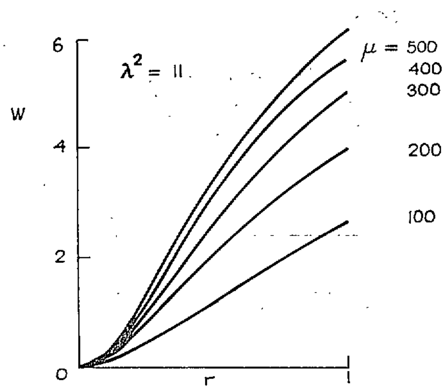

FIG. 8. Variation of deflection with μ. (Centre heating.)

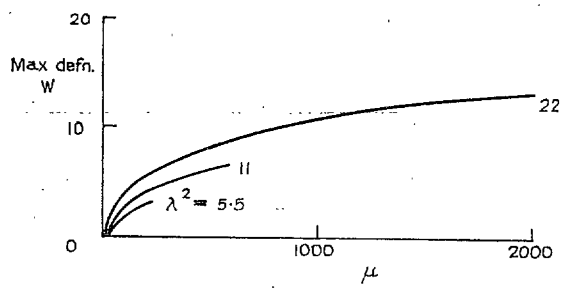

FIG. 10. Growth of maximum deflection with heat input. (Centre heating.)

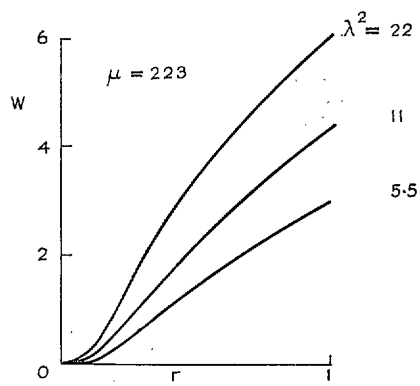

FIG. 9. Variation of deflection with λ^2. (Centre heating.)

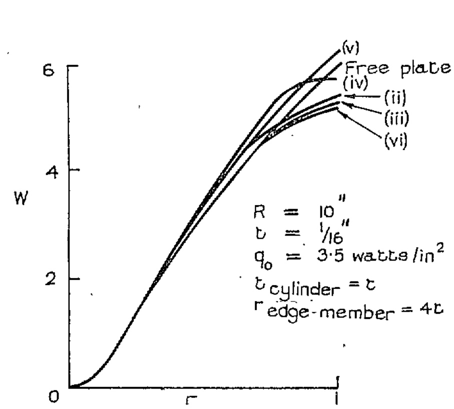

FIG. 11. Effect of edge restraint (*see* Fig. 2). (Centre heating.)

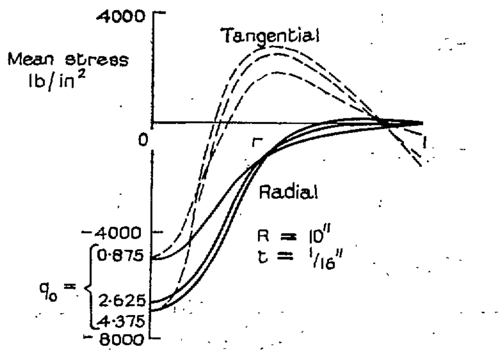

FIG. 12. Variation of mean stresses with heat input. (Centre heating.)

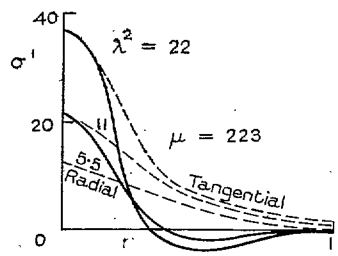

FIG. 14. Distribution of bending stress. (Centre heating.)

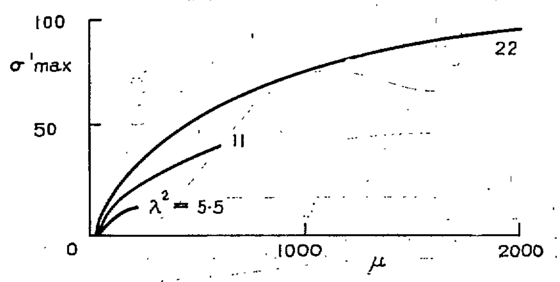

FIG. 13. Maximum bending stress. (Centre heating.)

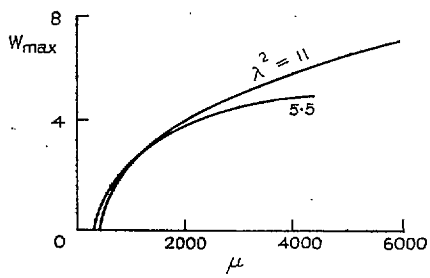

FIG. 15. Growth of maximum deflection with heat input. (Edge heating.)

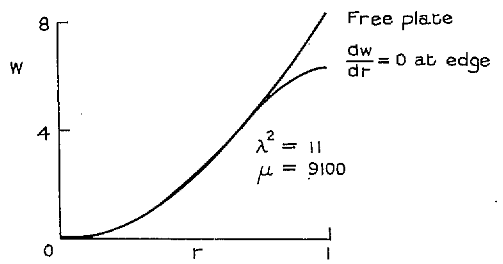

FIG. 16. Effect on deflected form of inhibiting edge slope. (Edge heating.)

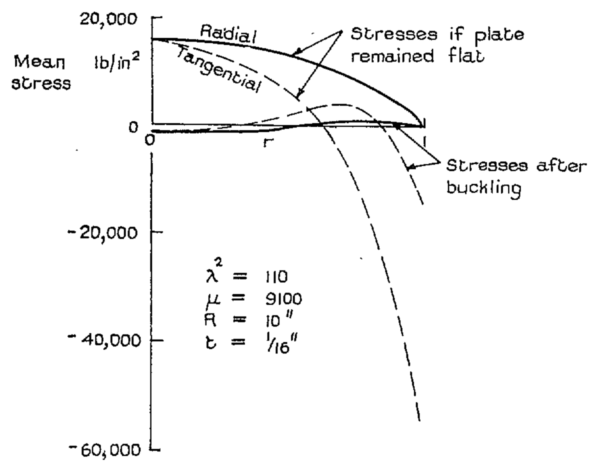

FIG. 18. Change in mean stress distribution on buckling. (Edge heating.)

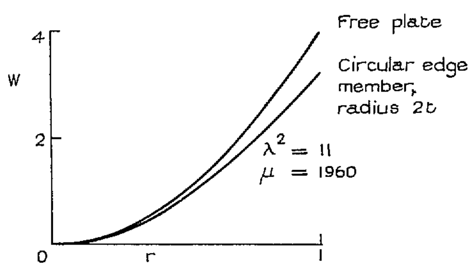

FIG. 17. Effect on deflected form of small edge member. (Edge heating.)

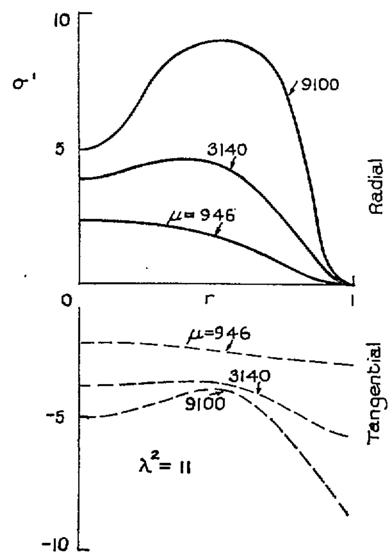

FIG. 19. Distribution of bending stress. (Edge heating.)

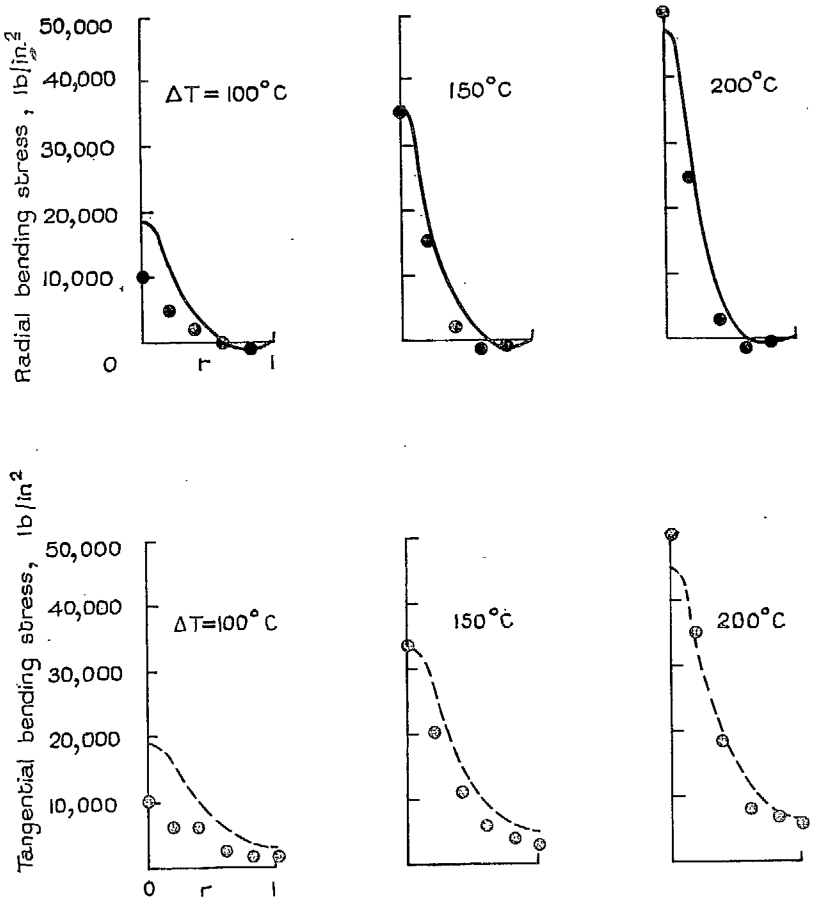

FIG. 20. Theoretical and experimental bending stresses for ⅛ in. plate.

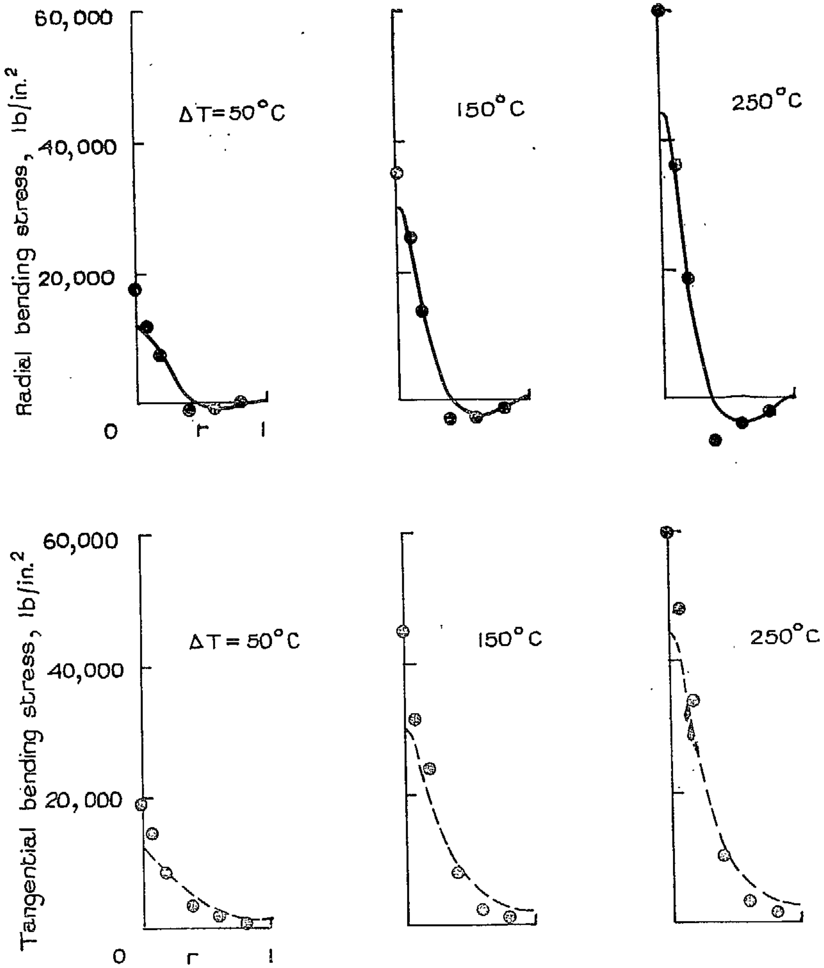

FIG. 21. Theoretical and experimental bending stresses for $\frac{1}{16}$ in. plate.

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