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Summary. This paper is concerned with the design of annular supersonic nozzles to produce uniform flow in supersonic wind tunnels which are axi-symmetrical and which have an internal coaxial circular cylinder throughout. Symmetrical two-dimensional and conventional axi-symmetrical nozzles are special cases of annular nozzles.

Proposals are made for design criteria sufficient to ensure that the flow inside a nozzle is free from limit lines and shock waves; the criteria for (symmetrical) two-dimensional and (conventional) axi-symmetrical nozzles are new. The two outstanding procedures for designing two-dimensional and axi-symmetrical nozzles are generalised to apply to annular nozzles. One of the design procedures is mainly analytical and the other is mainly numerical; the analytical expressions in both procedures are made much more complicated by the presence of the internal cylinder but the numerical process (the method of characteristics) is not. The design criteria and the mainly numerical design procedure are successfully applied to the design of a particular annular nozzle.

1. *Introduction.* This paper refers particularly to nozzles for producing uniform flow in the working sections of supersonic wind tunnels. The two most common types of wind tunnel working section are rectangular and circular in cross-section, the former type requiring two-dimensional nozzles and the latter requiring axi-symmetrical nozzles. Recently, there has been introduced[1] a new type of wind tunnel which is axi-symmetrical but which has an internal coaxial circular cylinder throughout its length from reservoir to working section so that its cross-sections are annular. This type of tunnel is especially suitable for base pressure and jet exhaust tests[2, 3], and may become more prominent in future experimental investigations of rocket jets[4]. The nozzles required for such a tunnel are called annular nozzles. This paper describes a theoretical investigation of annular supersonic nozzles, with the emphasis on the problem of design.

Symmetrical two-dimensional and conventional axi-symmetrical nozzles are given by the extreme cases of annular nozzles when the radius of the internal cylinder is infinite and zero respectively. There exists a vast amount of information about these extreme cases, and although there are various review papers describing the extreme cases separately[5, 6] there is no adequate review covering simultaneously both types of nozzle. Nevertheless, the scattered information gives a qualitative idea of the flow through annular nozzles and suggests calculation procedures by which they may be designed.

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Although the heyday of the design of two-dimensional and axi-symmetrical nozzles is long since past, there still remains[7] a very unsatisfactory feature of all the existing design methods: there are no criteria which ensure that the design requirements do not lead to a resultant nozzle flow that has limit lines and shock waves. Therefore, the establishing of a satisfactory design method for annular nozzles requires first of all the derivation of design criteria and then the generalisation to the annular case of the design procedures for the extreme cases.

The flow through a nozzle is considered here to be a theoretical steady irrotational homentropic flow of an inviscid non-conducting perfect gas with constant specific heats[8]. In practice, it is necessary to modify the contour of a nozzle to allow for boundary-layer development[9, 10, 11], and to make further modifications if the flow in the working section does not turn out to be sufficiently close to a uniform flow[12]; the problems of calculating such modifications are not considered in this paper.

A meridian section of a typical annular supersonic nozzle is shown in Fig. 1. (With a few exceptions the terminology employed henceforth is that appropriate to a meridian section.) A nozzle is essentially convergent-divergent, with the rate of divergence decreasing until the nozzle contour becomes parallel to the internal cylinder at the beginning of the working section. Sonic speed is attained in the vicinity of the narrowest cross-section of the nozzle, and the subsonic part of a nozzle has little effect on the supersonic part provided the approach to the throat is smooth[13]. In the present discussion, the flow through a nozzle is considered from the sonic line, along which the Mach number is unity, to the straight characteristic along which the Mach number has its working section value.

It is helpful to divide the flow through a nozzle into three regions by the forward characteristic from the sonic point on the cylinder and the backward characteristic from the point on the cylinder at which the working section Mach number is first attained. These characteristics are subsequently referred to as the first and second dividing characteristics respectively. The region between the sonic line and the first dividing characteristic is called the throat region, that between the first and second dividing characteristics is called the expansion region, and the region between the second dividing characteristic and the straight characteristic marking the beginning of the working section is called the transition region. In the throat region the flow changes from converging to diverging; in the expansion region the flow accelerates until the working section Mach number is reached on the cylinder; in the transition region the flow is turned into a uniform supersonic stream.

For a given gas, the basic design parameters of an annular supersonic nozzle may be considered to be the working section Mach number, the radius of the internal cylinder, the radius (in a cross section) of the nozzle contour at the working section, and the distance along the cylinder between the sonic point and the beginning of the working section. However, prescribing values for these parameters does not define a nozzle uniquely. An excellent way of making the design of a nozzle unique is to prescribe in addition the distribution of the velocity along the cylinder. In this paper attention is concentrated on velocity distributions in which the velocity increases and the velocity gradient decreases from the sonic point to the beginning of the working section.

The flow in the expansion region depends only on the velocity distribution between the sonic point and the beginning of the working section, whereas that in the throat region depends also on the velocity distribution upstream of the sonic point and that in the transition region is determined by the condition that the velocity is constant in the working section. The velocity may be chosen to have all its derivatives continuous at the sonic point. However, there is inevitably a discontinuity in some derivative of the velocity at the beginning of the working section, and for this reason the transition region must be regarded as an essentially separate region.

Prescribing the velocity distribution along the cylinder leads to a limit line in the flow at some distance from the cylinder and the flow is physically significant only in a certain neighbourhood of the cylinder. It is therefore important to be able to choose a velocity distribution for which the limit line is not likely to be situated within the throat or expansion regions of the desired nozzle. It is important also to ensure that there are no shock waves in the transition region of the desired nozzle.

The basic design parameters and prescribed velocity distribution are arbitrary provided they satisfy criteria which are sufficient to ensure that the desired nozzle is free from both limit lines and shock waves. An investigation (described in an unpublished note) has been made of design criteria for annular nozzles and the results are presented in Section 2. It is assumed that, for given values of the design parameters, the velocity gradient at the sonic point is the most significant property of a velocity distribution and may be regarded as a parameter which typifies the prescribed velocity distribution. The working section Mach number, the radius of the internal cylinder and the cross-sectional radius of the nozzle contour at the working section are regarded as fixed, and the criteria then take the form of an expression for the distance along the cylinder from the sonic point to the beginning of the working section together with formulae which define a permissible range of values for the velocity gradient at the sonic point. It is suggested that the criteria are adequate for any velocity distribution which has an appropriate gradient at the sonic point and decreasing gradient throughout. The criteria apply for any design Mach number. The results for the extreme cases of two-dimensional and axi-symmetrical nozzles constitute a new contribution to the theory of such nozzles.

When the design parameters and velocity distribution have been chosen, in conformity with the design criteria, the next problem is how best to calculate the shape of the desired nozzle contour. The problem for an annular nozzle is to consider how to generalise the existing design procedures for two-dimensional and axi-symmetrical nozzles, and to select a suitable one. The most elegant method of design would be one which would enable the nozzle contour to be calculated directly from theoretical formulae. However, since the transition region is a fundamentally separate region, it is not possible for a simple formula to provide the whole of the contour of the nozzle; the portion in the transition region must be calculated separately using knowledge of the shape of, and the properties of the flow on, the second dividing characteristic and the straight characteristic. For two-dimensional nozzles the transition region can be calculated analytically because it is a simple wave[14]. In axi-symmetrical flow the transition region is not a simple wave, and although there exists a rough method of construction[15] and an approximate analysis[16] of the flow through an axi-symmetrical transition region, such a flow is best calculated numerically by the method of characteristics[17,18]. Likewise the transition region of an annular nozzle is best calculated by the method of characteristics. Hence, the most that can be hoped for, from an analytical point of view, in the design of an annular nozzle is to be able to calculate the shape of the nozzle contour in the throat and expansion regions and the shape of, and the flow properties on, the second dividing characteristic.

The calculations in the throat and expansion regions can be performed for two-dimensional and axi-symmetrical nozzles from an analytical solution, holding also in the subsonic region upstream of the throat region, which is essentially a series expansion near to the axis. This idea of expanding in series from the prescribed velocity distribution was first suggested and worked out for axi-symmetrical nozzles[19,20], and has since been adapted for two-dimensional flow[21,22]. The generalisation of this work to the case of annular nozzles involves expanding in series near to the cylinder. The presence of the internal cylinder makes it more convenient not to follow the precise lines of the original approach, and the solution has been developed by an equivalent but physically more illuminating method. The development has been carried far enough for an idea of the fundamental form of the infinite series expansion to be obtained. It appears that the solution is so complicated by the presence of the internal cylinder that the use of terms of higher order than the first is scarcely practicable. Since the first-order solution has only a very limited practical application it follows that the possibility of an analytical calculation of the necessary information in both the throat and expansion regions of an annular nozzle must be rejected. For this reason the details of the analysis of the flow near the cylinder are not given here, but the salient features are described in the Appendix.

An alternative, mainly numerical, procedure for designing two-dimensional and axi-symmetrical nozzles consists of the calculation of the flow in the throat region from an analytical solution which is essentially a series expansion near to the sonic point on the axis, followed by the numerical calculation of the flow in the expansion region by the method of characteristics. In fact, the analytical solution holds outside the precise throat region and so overlaps part of the expansion region. Because the method of characteristics is difficult to use for Mach numbers near unity it is convenient to use the throat solution to calculate a line of constant and sufficiently high Mach number from which to start the numerical calculation. Since the original treatment[23] of the two-dimensional throat flow, the first-order solutions in two-dimensional and axi-symmetrical nozzles have been derived and developed by many authors, of whom only a few[24 to 30] are referred to here, and recently the use of a high speed automatic computer has enabled the solutions to be extended[31] to a high number of terms. The corresponding solution for the transonic flow in the throat region of an annular nozzle is given in detail in Section 3. The method of solution involves a series expansion near the sonic point on the cylinder. The solution is given as far as the second-order terms and it is clear that the internal cylinder complicates the solution enormously. A suggestion is made for avoiding in the annular case the derivation of terms of high order by interpolating between the corresponding high order terms in the extreme cases. Since the method of characteristics applies directly to annular flow, it follows that this mainly numerical design procedure is a reasonable one for annular nozzles.

It therefore appears that the best way of designing an annular nozzle is to use the old procedure of calculating the throat flow from the analytical solution up to a certain line of constant Mach number and then to use the method of characteristics for the remainder of the expansion region and for the transition region. It may be remarked that, whereas formerly the method of characteristics was applied manually, nowadays calculations are usually performed on an automatic computer and so the necessity of using a numerical procedure for designing annular nozzles is no longer a great disadvantage.

Experience has shown that, for a given design accuracy, nozzles with continuous curvature are likely to induce a more nearly uniform flow in the working section than nozzles with discontinuous curvature. It has been proved[32] that, in two-dimensional flow, prescribed velocity distributions for which the gradient is zero at the end of the expansion region give rise to nozzles with continuous curvature. Although the proof cannot be taken over directly to axi-symmetrical flow, it is usual to assume[31, 33] that the result is still true. Therefore, it is assumed here that a sufficient condition for the design of annular nozzles of continuous curvature is that the velocity gradient along the cylinder is zero at the end of the expansion region.

An example of the successful design of an annular nozzle with continuous curvature, using the design criteria and the mainly numerical design procedure proposed here, is described in Section 4.

2. *Design Criteria.* For a given gas of specific heat ratio γ, the basic design parameters of an annular supersonic nozzle may be considered to be the working section Mach number $\bar{M}$, the radius a of the internal cylinder, the cross-sectional radius $(a+\bar{h})$ of the nozzle contour at the working section, and the distance l along the cylinder from the sonic point to the beginning of the working section. For brevity, the parameter $\bar{h}$ is referred to henceforth as the working section width of an annular nozzle; for a symmetrical two-dimensional nozzle it is half the total working section width and for a conventional axi-symmetrical nozzle it is the working section radius.

The prescribed velocity distribution along the cylinder is taken to be the distribution of the specific speed κ, defined as the ratio of the local speed to the critical speed. Attention is concentrated on velocity distributions for which the velocity increases and the velocity gradient decreases from the sonic point to the beginning of the working section. It is assumed that the velocity gradient at the sonic point is the most significant property of a prescribed velocity distribution. The gradient at the sonic point is represented by the non-dimensional parameter χ defined by

$$\chi = l/(\bar{\kappa}-1)b\,, \tag{1}$$

where $1/b$ is the gradient of the specific speed along the cylinder at the sonic point, and $\bar{\kappa}$, the value of the specific speed corresponding to $\bar{M}$, is given by the relations

$$\bar{\kappa} = \bar{M}[(1-\delta) + \delta\bar{M}^2]^{-1/2}, \tag{2}$$

$$\delta = (\gamma-1)/(\gamma+1)\,. \tag{3}$$

Of the five parameters $\bar{M}$, a, $\bar{h}$, l and χ it is probable that in practice $\bar{M}$, a and $\bar{h}$ are fixed at the outset. The effect of design criteria, to ensure that the flow through the nozzle is free from limit lines and shock waves, is therefore to impose restrictions on the values of l and χ.

Now any streamline of the flow through a given nozzle may be taken to be the contour of another nozzle, and the converse that any nozzle contour may be considered as a streamline of the flow through some other nozzle is also true within the limit set by the outermost nozzle contour for which there is no risk that shock waves may occur in the transition region. It therefore follows that the transition region of a desired nozzle will be free from shock waves if it can be arranged that the desired nozzle is within this limiting nozzle. It has been proved[19] that in two-dimensional flow the contour of the limiting nozzle is the streamline through the intersection of the first and second dividing characteristics. In axi-symmetrical flow the precise determination of the limiting nozzle is more difficult and there is only an approximate criterion[16] for the existence of a shock-free transition region. However, there is evidence[15, 34] that the flow bounded by the streamline through the first and second dividing characteristics is shock-free in some important particular cases of axi-symmetrical flow. Therefore, it is conjectured here that, in annular flow, there are no shock waves within the flow bounded by the streamline through the intersection of the first and second dividing characteristics. This streamline is here called the principal streamline, and the corresponding nozzle is called the principal nozzle and is shown in Fig. 2. If $\bar{h}_{\max}$ denotes the value of h associated with a principal nozzle, then the desired nozzle may be arranged to be within the principal nozzle by first choosing a value for $\bar{h}_{max}$ which is greater than the given value of $\bar{h}$ and then working with $\bar{h}_{max}$ instead of $\bar{h}$.

If the desired nozzle is arranged to be within the principal nozzle, then its throat and expansion regions will be free from limit lines if the prescribed velocity distribution can be chosen so that there are no limit lines within the throat and expansion regions of the principal nozzle. It then becomes necessary to relate l and χ to $\bar{M}$, a and $\bar{h}_{max}$ in such a way that the throat and expansion regions of the principal nozzle are free from limit lines.

One way in which this can be done is by considering a velocity distribution for the case of $\bar{M} = \infty$, infinite Mach number being attained at an infinite distance along the cylinder from the sonic point. Such a distribution contains what may be regarded as special velocity distributions for all $\bar{M} < \infty$. If the velocity distribution for $\bar{M} = \infty$ has increasing velocity and decreasing velocity gradient throughout then the special velocity distribution for any $\bar{M}$ possesses the same properties. Also, provided the principal streamline for $\bar{M} = \infty$ is outside the principal streamlines for all $\bar{M} < \infty$, the throat and expansion regions for the principal nozzle for $\bar{M} = \infty$ contain what may be called special throat and expansion regions for all $\bar{M} < \infty$, as shown in Fig. 3. If there are no limit lines in the throat and expansion regions for $\bar{M} = \infty$ then there can be no limit lines in the special throat and expansion regions for all $\bar{M}$. If the principal streamline for any value of $\bar{M}$ can be located, that is, if a relation giving $\bar{h}_{max}$ in terms of $\bar{M}$, a and the special velocity distribution can be found, then criteria can be established for the special velocity distributions for all $\bar{M}$. By assuming that the velocity gradient at the sonic point is the dominant property of a velocity distribution it then becomes possible to obtain criteria for any distribution with the appropriate value of velocity gradient at the sonic point.

A heuristic argument using the foregoing ideas has resulted in the establishment of criteria which are helpful in preventing the occurrence of limit lines in the throat and expansion regions of the principal nozzles for any value of $\bar{M}$. Because some of the information on which they are based was obtained by numerical calculation with $\gamma = 1{\cdot}4$ the criteria apply only for this single value of the specific heat ratio.

The criteria may be expressed most conveniently as follows. Let $\bar{h}_{max}$ be chosen greater than or equal to $\bar{h}$. Let $(a+\bar{g}_{max})$ be the radius of the sonic stream with the same mass flow as the supersonic stream of Mach number $\bar{M}$ and working section width $\bar{h}_{max}$; then $\bar{g}_{max}$, which is referred to as the effective throat width of the principal nozzle, is calculated from the exact relation

$$[(a+\bar{g}_{max})^2 - a^2] = \bar{\lambda}\,[(a+\bar{h}_{max})^2 - a^2]\,, \tag{4}$$

where $\bar{\lambda}$, the value of the area-ratio corresponding to $\bar{M}$, is given by

$$\bar{\lambda} = \bar{M}\,[(1-\delta) + \delta\bar{M}^2]^{-1/2\delta}. \tag{5}$$

Then l is given by the expression

$$l = (\bar{h}_{max} - \bar{g}_{max})(a+\bar{h}_{max})/\bar{\nu}\,(\bar{H}_{\infty}a + \bar{H}_0\bar{h}_{max})\,, \tag{6}$$

where $\bar{H}_{\infty}$ and $\bar{H}_0$ are functions of $\bar{M}$ known numerically and tabulated in Table 1, and $\bar{\nu}$, the value of the Prandtl-Meyer angle corresponding to $\bar{M}$, is given by

$$\bar{\nu} = \delta^{-1/2}\cot^{-1}[\delta^{-1/2}(\bar{M}^2-1)^{-1/2}] + \cot^{-1}[(\bar{M}^2-1)^{1/2}] - \frac{\pi}{2}. \tag{7}$$

This result for l is independent of the parameter χ and it holds with sufficient accuracy provided that χ is chosen within the range $\chi_l \leqslant \chi \leqslant \chi_u$ where the end values χ_l and χ_u are given by the equations:

$$\chi_{l,u} = T_{l,u}/\{(\bar{\kappa}-1)(2U)^{-1}[W_{l,u}+(W_{l,u}^2-4UV_{l,u})^{1/2}]\}, \tag{8}$$

$$T_{l,u} = 0\cdot9228t_{l,u}(\bar{h}_{\max}-\bar{g}_{\max})(a+\bar{h}_{\max})/(\bar{\kappa}-1)^2\bar{g}_{\max}(\bar{H}_\infty a+\bar{H}_0\bar{h}_{\max}), \tag{9}$$

$$U = (\bar{\kappa}-1)+0\cdot3812(\bar{\kappa}-1)^{5/2}(6^{1/2}-\bar{\kappa})\bar{g}_{\max}/\bar{\nu}a, \tag{10}$$

$$V_{l,u} = (\bar{\kappa}-1)+1\cdot269t_{l,u}(6^{1/2}-\bar{\kappa}), \tag{11}$$

$$W_{l,u} = U+V_{l,u}+0\cdot4520t_{l,u}(\bar{\kappa}-1)^{3/2}(6^{1/2}-\bar{\kappa})^2\bar{g}_{\max}/\bar{\nu}a, \tag{12}$$

$$t_{l,u} = 0\cdot8044,\ 1\cdot576; \tag{13}$$

the accuracy is greatest when $\chi = \chi_l$ and diminishes somewhat as χ increases.

The corresponding results for the extreme cases, which are themselves new, are as follows: for two-dimensional nozzles ($a = \infty$)

$$\bar{g}_{\max} = \bar{\lambda}\bar{h}_{\max}, \tag{14}$$

$$l = (\bar{h}_{\max}-\bar{g}_{\max})/\bar{\nu}\bar{H}_\infty, \tag{15}$$

$$\chi_{l,u} = 0\cdot9228t_{l,u}(\bar{\lambda}^{-1}-1)(\bar{\kappa}-1)^{-2}\bar{H}_\infty^{-1}/\{(\bar{\kappa}-1)+1\cdot269t_{l,u}(6^{1/2}-\bar{\kappa})\}; \tag{16}$$

for axi-symmetrical nozzles ($a = 0$)

$$\bar{g}_{\max} = \bar{\lambda}^{1/2}\bar{h}_{\max}, \tag{17}$$

$$l = (\bar{h}_{\max}-\bar{g}_{\max})/\bar{\nu}\bar{H}_0, \tag{18}$$

$$\chi_{l,u} = 0\cdot9228t_{l,u}(\bar{\lambda}^{-1/2}-1)(\bar{\kappa}-1)^{-2}\bar{H}_0^{-1}/\{(\bar{\kappa}-1)+1\cdot186t_{l,u}(6^{1/2}-\bar{\kappa})\}. \tag{19}$$

The variations of $l/\bar{h}_{\max}$ and $\chi_{l,u}$ with $\bar{M}$, for the cases $a = \infty$, $a = \bar{h}_{\max}$ and $a = 0$, are shown in Figs. 4 and 5 respectively. The ratio $l/\bar{h}_{\max}$ may be used, by putting $\bar{h}_{\max} = \bar{h}$, to give an estimate of the minimum permissible value of l for given values of $\bar{M}$, a and $\bar{h}$ and an appropriate value of χ. It is noted that $\chi_{l,u} = 1$ when $\bar{M} = 1$ and $\chi_{l,u} = \infty$ when $\bar{M} = \infty$.

It is suggested that a nozzle free from limit lines and shock waves may be obtained by making a calculation (following the procedure established in this paper) using a prescribed velocity distribution which is any increasing function having decreasing gradient and satisfying the above design criteria. If the calculation is taken as far as the principal nozzle, the calculated working section width of the principal nozzle is not likely to be exactly the same as that originally assumed for $\bar{h}_{\max}$ in the evaluation of the design criteria. The actual working section width of the principal nozzle will be different from that assumed for two reasons: the formulae on which the criteria for the special velocity distribution are based are fairly crude, and the velocity distribution for which the calculation is made may be slightly different from the special velocity distribution. The difference is of no consequence provided that the working section width of the desired nozzle is obtained in the calculation before the actual principal nozzle is reached. It is therefore important to choose the assumed working section width of the principal nozzle to be sufficiently greater than the working section width of the desired nozzle to eliminate the possibility of adverse effects arising from the crudeness of the criteria. It is suggested that it is sufficient to take $\bar{h}/\bar{h}_{\max} \leqslant 3/4$. The smaller the value of $\bar{h}/\bar{h}_{\max}$ the bigger the value of l and hence the longer the nozzle. In some cases it may be convenient to choose the precise value of $\bar{h}/\bar{h}_{\max}$ so that the nozzle has a given length. The actual value of χ may be chosen, preferably near to χ_l, in such a way that the distribution of κ along the cylinder is given by as simple a formula as possible.

The justification for the criteria rests ultimately on their efficacy in practice, and this has been well demonstrated for a certain group of typical cases. The criteria are used in Section 4 in the design of an annular nozzle and in this case they are entirely successful.

3. *The Flow near the Sonic Point on the Cylinder.* The flows through the throat regions of two-dimensional and axi-symmetrical nozzles have been investigated extensively[23 to 31]. This background information provides a qualitative picture of the likely flow pattern in the throat region of an annular nozzle. Along a streamline the velocity increases while the flow direction angle first decreases to zero and then increases. There are four lines traversing the streamlines which are of special importance. They are the sonic line, the limiting characteristic[29] which bounds that part of the supersonic region which can influence the subsonic region, the line on which the flow direction is zero, and the first dividing characteristic. Near the axis, the sonic line, limiting characteristic and zero flow direction line are concave to the oncoming flow whereas the first dividing characteristic is convex, as illustrated in Fig. 6 for a particular nozzle.

In two-dimensional and axi-symmetrical flows the solution for the velocity potential may be found by expanding near to the sonic point on the axis and assuming a series in powers of the distance from the axis, and then it turns out that an exact solution of the first-order equation for the velocity potential is obtained[27, 29]. In the annular case the flow may be found by a series expansion near to the sonic point on the cylinder, but expansion in powers of the distance from the cylinder does not yield a solution. The solution is found by recognising[28] that the solutions in the extreme cases are polynomials in the axial co-ordinate with the sonic point as origin, and then assuming that the solution in the annular case is of the same form. The solution in the annular case is much more complicated than in the extreme cases, but in the throat region the first-order solution is of definite practical value.

3.1. *The First-Order Solution.* 3.1.1. *The velocity potential.* Take rectangular co-ordinates x, y with the origin at the sonic point on the cylinder and x measured parallel to the cylinder. The velocity potential Φ satisfies the partial differential equation

$$(q^{*2}-\Phi_x^{\,2}-\delta\Phi_y^{\,2})\,\Phi_{xx}-2\,(1-\delta)\,\Phi_x\Phi_y\Phi_{xy}+(q^{*2}-\delta\Phi_x^{\,2}-\Phi_y^{\,2})\,\Phi_{yy}+ \\ +(q^{*2}-\delta\Phi_x^{\,2}-\delta\Phi_y^{\,2})\,\Phi_y\,(a+y)^{-1}=0\,, \tag{20}$$

where q^* is the critical speed and the suffix notation of partial differentiation is used. In the throat region the fundamental length is b and x is small compared with b. The form of the boundary condition to be satisfied by Φ depends on the order of the desired solution. To the first order the boundary conditions for Φ are

$$\Phi_x = q^*\,[1 + x/b + \,.\,.],\qquad \Phi_y = 0 \text{ on } y = 0\,. \tag{21}$$

To solve for Φ to the first order the following substitutions are made:

$$x = \epsilon^2 Xb\,, \tag{22}$$

$$y = \epsilon Yb\,, \tag{23}$$

$$a = \epsilon Ab\,, \tag{24}$$

$$\Phi = q^*\,[x + \epsilon^4\phi_4\,(X,\ Y;\ A)\,b + \,.\,.]\,; \tag{25}$$

here ϵ is a dimensionless quantity which is small compared with unity; the independent variables X, Y and the dependent variable ϕ_4 are $0(1)$; the parameter A varies in the range $\infty \geqslant A \geqslant 0$ but the functions which involve A in the terms of a specified order in ϵ are always $0(1)$. The introduction of ϵ is a device to enable a solution which is valid uniformly in a to be obtained. By substituting the Relations (22) to (25) into Equation (20) and Boundary Conditions (21) and equating to zero the coefficient of ϵ^2 it follows that the equation and boundary conditions satisfied by ϕ_4 are

$$\phi_{4x}\phi_{4xx} - \tfrac{1}{2}(1-\delta)[\phi_{4YY} + \phi_{4Y}(A+Y)^{-1}] = 0\,; \tag{26}$$

$$\phi_{4X} = X, \qquad \phi_{4Y} = 0 \text{ on } Y = 0\,. \tag{27}$$

The equation satisfied by ϕ_4 is the non-linear equation of transonic small perturbation theory.

An exact solution of Equation (26) satisfying the Boundary Conditions (27) may be found by assuming ϕ_4 to be a quadratic in X with coefficients depending on Y and A. If ϕ_4 is written as

$$\phi_4 = \tfrac{1}{2}X^2 + \phi_{21}X + \phi_{22}\,, \tag{28}$$

then substituting in Equation (26) and Boundary Conditions (27) and equating to zero the coefficients of each power of X leads to the following ordinary differential equations and boundary conditions for ϕ_{21} and ϕ_{22}:

$$\phi_{21YY} + \phi_{21Y}(A+Y)^{-1} = 2(1-\delta)^{-1}, \tag{29}$$

$$\phi_{21} = 0, \qquad \phi_{21Y} = 0 \text{ on } Y = 0\,; \tag{30}$$

$$\phi_{22YY} + \phi_{22Y}(A+Y)^{-1} = 2(1-\delta)^{-1}\phi_{21}\,, \tag{31}$$

$$\phi_{22} = 0\,, \qquad \phi_{22Y} = 0 \text{ on } Y = 0\,. \tag{32}$$

These equations may be integrated directly, and the results may be expressed most compactly in terms of the variable ω defined by

$$\omega = (1+Y/A)^2\,. \tag{33}$$

The results are

$$\phi_{21} = cA^2[\omega - 1 - \log\omega]\,, \tag{34}$$

$$\phi_{22} = c^2A^4[\tfrac{1}{4}\omega^2 + \omega - \tfrac{5}{4} - (\omega+\tfrac{1}{2})\log\omega]\,, \tag{35}$$

where the constant c is given by

$$c = 2^{-1}(1-\delta)^{-1} \tag{36}$$

and is used henceforth when it produces a simplification in the expression of a result. The corresponding results for two-dimensional flow ($A = \infty$) are

$$\phi_{21} = 2cY^2\,, \tag{37}$$

$$\phi_{22} = \tfrac{2}{3}c^2Y^4\,, \tag{38}$$

while for axi-symmetrical flow ($A = 0$)

$$\phi_{21} = cY^2\,, \tag{39}$$

$$\phi_{22} = \tfrac{1}{4}c^2Y^4\,. \tag{40}$$

3.1.2. *The lines of constant velocity magnitude.* The specific speed κ is given by

$$\kappa = q^{*-1}[\Phi_x^{\,2} + \Phi_y^{\,2}]^{1/2}\,. \tag{41}$$

From Equations (22), (23), (25), (28) and (33) it follows that

$$\Phi_x = q^*[1 + \epsilon^2(X+\phi_{21}) + \ldots], \tag{42}$$

$$\Phi_y = q^*[\epsilon^3 2A^{-1}\omega^{1/2}(\phi_{21}'X+\phi_{22}') + \ldots], \tag{43}$$

where a dash here denotes complete differentiation with respect to ω, and where from Equations (34) and (35)

$$\phi_{21}' = cA^2[1 - \omega^{-1}], \tag{44}$$

$$\phi_{22}' = c^2A^4[\tfrac{1}{2}(\omega-\omega^{-1}) - \log \omega]. \tag{45}$$

Therefore the first order solution for κ is simply

$$\kappa = 1 + \epsilon^2 (X+\phi_{21}) + \ldots \tag{46}$$

It follows that the first order results for the Mach angle μ and the Prandtl-Meyer angle ν, which will be required subsequently, are

$$\cot \mu = \epsilon 2^{1/2} (1-\delta)^{-1/2} (X+\phi_{21})^{1/2} + \ldots, \tag{47}$$

$$\nu = \epsilon^3 2^{3/2} 3^{-1} (1-\delta)^{-1/2} (X+\phi_{21})^{3/2} + \ldots \tag{48}$$

It is convenient to take the lines of constant velocity magnitude to be represented by the lines of constant ν and these are given to the first order by

$$X = \epsilon^{-2}2^{-1}3^{2/3} (1-\delta)^{1/3}\nu^{2/3} - \phi_{21}. \tag{49}$$

In particular the sonic line $\nu = 0$ is

$$X = -\phi_{21}, \tag{50}$$

which explicitly in terms of dimensional co-ordinates is

$$x/b = -c(a/b)^2 [\omega - 1 - \log \omega], \tag{51}$$

where ω is here interpreted, using Equations (23), (24) and (33), as

$$\omega = (1+y/a)^2. \tag{52}$$

The results for the sonic line in two-dimensional flow ($a = \infty$) and axi-symmetrical flow ($a = 0$) are respectively

$$x/b = -2c(y/b)^2, \tag{53}$$

$$x/b = -c(y/b)^2. \tag{54}$$

3.1.3. *The lines of constant flow direction.* The flow direction angle θ is given by

$$\theta = \tan^{-1} [\Phi_y/\Phi_x]. \tag{55}$$

Using Equations (42) and (43) it follows that to the first order

$$\theta = \epsilon^3 2A^{-1}\omega^{1/2}(\phi_{21}'X+\phi_{22}') + \ldots \tag{56}$$

The lines of constant flow direction are given by

$$X = (\phi_{21}')^{-1} [\epsilon^{-3}2^{-1}A\omega^{-1/2}\theta - \phi_{22}']. \tag{57}$$

In particular the zero flow direction line is

$$X = -(\phi_{21}')^{-1}\phi_{22}', \tag{58}$$

which explicitly in terms of dimensional co-ordinates is

$$x/b = -c(a/b)^2[\tfrac{1}{2}(\omega+1) - \omega(\omega-1)^{-1}\log\omega], \tag{59}$$

with ω interpreted as in Equation (52). The corresponding results for two-dimensional flow ($a = \infty$) and axi-symmetrical flow ($a = 0$) are respectively

$$x/b = -\tfrac{2}{3}c(y/b)^2, \tag{60}$$

$$x/b = -\tfrac{1}{2}c(y/b)^2. \tag{61}$$

3.1.4. *The characteristics through the sonic point on the cylinder.* The differential equations of the characteristics may be written

$$dx/dy = \cot(\theta \pm \mu) = \pm(\cot\mu \mp \tan\theta)(1 \pm \tan\theta\cot\mu)^{-1}. \tag{62}$$

From Equations (22), (23), (47) and (56) it follows that to the first order

$$dX/dY = \pm 2^{1/2}(1-\delta)^{-1/2}(X+\phi_{21})^{1/2} + \ldots, \tag{63}$$

which on squaring and rearranging becomes

$$(dX/dY)^2 - 2(1-\delta)^{-1}X = 2(1-\delta)^{-1}\phi_{21}. \tag{64}$$

Using Equations (33), (34) and (36) it follows that

$$(2A^{-1}\omega^{1/2}dX/d\omega)^2 - 2(1-\delta)^{-1}X = (1-\delta)^{-2}A^2[\omega - 1 - \log\omega]. \tag{65}$$

Now let X be replaced by the variable Z defined by

$$X = 2^{-1}(1-\delta)^{-1}A^2Z. \tag{66}$$

Then the differential equation for the characteristics becomes

$$\omega(dZ/d\omega)^2 - Z = \omega - 1 - \log\omega, \tag{67}$$

where Z is a function of ω only. The boundary conditions are $Z = 0$, $dZ/d\omega = 0$ on $\omega = 1$. If the two solutions for Z are denoted by $-Z_-(\omega)$, for the limiting characteristic, and $+Z_+(\omega)$, for the first dividing characteristic, then it may be shown that for ω near 1

$$Z_- = \tfrac{1}{4}(\omega-1)^2 + \ldots, \qquad Z_+ = \tfrac{1}{2}(\omega-1)^2 + \ldots, \tag{68}$$

while for large ω

$$Z_- = \tfrac{1}{2}(5^{1/2}-1)\omega + \ldots, \qquad Z_+ = \tfrac{1}{2}(5^{1/2}+1)\omega + \ldots \tag{69}$$

Closed form solutions for Z_- and Z_+ have not been found, but some values calculated numerically* are given in Table 2. The limiting and first dividing characteristics are then given in dimensional form by

$$x/b = -c(a/b)^2Z_-(\omega), \tag{70}$$

$$x/b = c(a/b)^2Z_+(\omega). \tag{71}$$

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* The solutions were calculated, using a method of wide applicability, by A. Gibbons at the Computing Machine Laboratory, Manchester University.

The corresponding results for two-dimensional flow ($a = \infty$) are

$$x/b = -c(y/b)^2, \tag{72}$$

$$x/b = 2c(y/b)^2, \tag{73}$$

while for axi-symmetrical flow ($a = 0$)

$$x/b = -\tfrac{1}{2}(5^{1/2}-1)c(y/b)^2 \tag{74}$$

$$x/b = \tfrac{1}{2}(5^{1/2}+1)c(y/b)^2. \tag{75}$$

The compatibility relations along the characteristics in the annular case may be written

$$d\theta \mp d\nu + \tan\theta(1 \pm \tan\theta \cot\mu)^{-1}(a+y)^{-1}dy = 0. \tag{76}$$

From Equations (24), (47), (52) and (56) it follows that

$$d\theta \mp d\nu + \epsilon^3 A^{-1}\omega^{-1/2}(\phi_{21}'X + \phi_{22}')d\omega = 0, \tag{77}$$

so that on substituting for X along the characteristics through the sonic point on the cylinder it follows that

$$d\theta \mp d\nu + c^2(a/b)^3\omega^{-1/2}\{[1-\omega^{-1}](\pm Z_{\pm}) + [\tfrac{1}{2}(\omega-\omega^{-1}) - \log\omega]\}d\omega = 0, \tag{78}$$

where the upper sign corresponds to the first dividing characteristic and the lower sign to the limiting characteristic. There is no possibility of integrating this relation without explicit solutions for Z_- and Z_+. However, the separate results for θ and ν along the characteristics are given by Equations (56), (48) and (66), in terms of the two solutions for Z, as

$$\theta = 2c^2(a/b)^3\omega^{1/2}\{[1-\omega^{-1}](\pm Z_{\pm}) + [\tfrac{1}{2}(\omega-\omega^{-1}) - \log\omega]\}, \tag{79}$$

$$\nu = 2^2 3^{-1}c^2(a/b)^3\{\pm Z_{\pm} + [\omega - 1 - \log\omega]\}^{3/2}; \tag{80}$$

it is unlikely that, even if Z_- and Z_+ were known explicitly, it would be possible to eliminate ω from Equations (79) and (80) and so obtain explicit compatibility relations involving θ and ν only. The compatibility relations in two-dimensional flow ($a = \infty$) are exactly

$$\theta \mp \nu = 0, \tag{81}$$

while in axi-symmetrical flow ($a = 0$), from the first order throat solution,

$$\theta \mp \tfrac{3}{4}\nu = 0. \tag{82}$$

This latter result compares with the well-known relation $\theta \mp \frac{1}{2}\nu$ = constant near to the axis in axi-symmetrical flow away from the sonic point.

3.1.5. *The streamlines.* The differential equation of the streamlines is

$$dy/dx = \tan\theta. \tag{83}$$

Therefore, from Equations (22), (23) and (56) to the first order

$$dY/dX = \epsilon^4 2A^{-1}\omega^{1/2}(\phi_{21}'X + \phi_{22}') + \ldots, \tag{84}$$

and so to the first order the streamlines are quadratic in X. Hence, in dimensional variables, to the first order the streamlines may be written

$$y = y_t + \tfrac{1}{2}(x-x_t)^2/r_t + \ldots, \tag{85}$$

where x_t and y_t are the co-ordinates of the point at which the flow direction is zero, and r_t is the radius of curvature there. The first-order results for x_t, y_t and r_t may be conveniently expressed in terms of the effective throat width g of the streamline, defined so that $(a+g)$ is the radius of the sonic stream with the same mass flow as that contained between the streamline and the cylinder. The results are

$$x_t/b = -\,c(a/b)^2\,[\tfrac{1}{2}(\tau+1) - \tau(\tau-1)^{-1}\log\tau]\,, \tag{86}$$

$$y_t/b = g/b + c^3(a/b)^5\,[\tau^{-1/2}\{\frac{1}{12}\tau^3 - \frac{3}{4}\tau^2 + \frac{5}{4}\tau - \frac{7}{12} + \tau\log\tau - \tau(\tau-1)^{-1}\log^2\tau\}]\,, \tag{87}$$

$$r_t/b = (1-\delta)\,(1+\tau^{-1/2})^{-1}b/g\,, \tag{88}$$

where the quantity τ is defined by

$$\tau = (1+g/a)^2. \tag{89}$$

The result for r_t may be written

$$r_t/b = (1-\delta)\,[(a+g)/(2a+g)]b/g\,. \tag{90}$$

The corresponding results for two-dimensional flow ($a = \infty$) are

$$x_t/b = -\,\tfrac{2}{3}c(g/b)^2, \tag{91}$$

$$y_t/b = g/b + \frac{32}{45}c^3(g/b)^5, \tag{92}$$

$$r_t/b = \tfrac{1}{2}(1-\delta)b/g\,, \tag{93}$$

while for axi-symmetrical flow ($a = 0$)

$$x_t/b = -\,\tfrac{1}{2}c(g/b)^2, \tag{94}$$

$$y_t/b = g/b + \frac{1}{12}c^3(g/b)^5, \tag{95}$$

$$r_t/b = (1-\delta)b/g\,. \tag{96}$$

3.2. *Higher-Order Terms.* The flow in the throat region may, in principle, be found to any desired accuracy by including further terms in the assumed expansion for Φ in Equation (25). In two-dimensional and axi-symmetrical flow this procedure is feasible up to very high orders[31] because the expansion for Φ is a double power series in X and Y. However, the first-order terms in the general annular case show quite clearly that the introduction of the internal cylinder complicates the solution enormously. The complete second-order solution in the annular case has been obtained and the expressions involved are so lengthy as to discourage any attempt to calculate terms of higher order. Here, some selected results from the second-order solution are given. They comprise the result for the velocity potential Φ, from which any other quantity can be derived, and the result for the throat radius of curvature r_t, which is useful and not unduly complicated.

If the second-order boundary condition on Φ_x is written as

$$\Phi_x = q^*[1 + x/b + j(x/b)^2 + . .], \tag{97}$$

and then Φ is expanded as

$$\Phi = q^*[x + \{\epsilon^4\phi_4(X, Y; A) + \epsilon^6\phi_6(X, Y; A) + . .\}b], \tag{98}$$

the equation and boundary conditions to be satisfied by ϕ_6 are

$$\begin{aligned}\phi_{6\,X}\phi_{4\,XX} + \phi_{6\,XX}\phi_{4\,X} - \tfrac{1}{2}(1-\delta)[\phi_{6\,XY} + \phi_{6\,Y}(A+Y)^{-1}] &= \\ = -\tfrac{1}{2}(1-\delta)^{-1}(1+3\delta)\phi_{4\,XX}\phi_4{}^2{}_X - (1-\delta)\phi_{4\,Y}\phi_{4\,XY}&,\end{aligned} \tag{99}$$

$$\phi_{6\,X} = jX^2, \qquad \phi_{6\,X} = 0 \text{ on } Y = 0, \tag{100}$$

where ϕ_4 is the first order term calculated earlier. It appears that all further terms in the series for Φ satisfy linear equations of the same form as ϕ_6. The solution for ϕ_6 may be found by writing

$$\phi_6 = \tfrac{1}{3}jX^3 + \phi_{31}X^2 + \phi_{32}X + \phi_{33}; \tag{101}$$

substituting into Equation (99) and equating to zero the coefficients of each power of X then leads to simple ordinary differential equations for ϕ_{31}, ϕ_{32} and ϕ_{33} which can be integrated directly. The results may be expressed in the following way

$$\phi_{3s} = 2^{-2}(1-\delta)^{-s}A^{2s}\sum_{p=0}^{s}\sum_{q=0}^{s-p}\phi_{3spq}\omega^q \log{}^p\omega, \qquad s = 1, 2, 3 \tag{102}$$

where the coefficients ϕ_{3spq} are given by

$$\phi_{3spq} = (1-\delta)^{-1}Q_{3spq} + 4/3j\,R_{3spq} \tag{103}$$

with the numbers Q_{3spq} and R_{3spq} given as follows

$$Q_{3100} = -Q_{3101} = Q_{3110} = -(1+3\delta), \tag{104}$$

$$R_{3100} = -R_{3101} = R_{3110} = -9/2, \tag{105}$$

$$\left.\begin{aligned}&Q_{3200} = -\tfrac{1}{4}(1+59\delta),\ Q_{3201} = -(1-13\delta),\ Q_{3202} = \tfrac{1}{4}(5+7\delta),\\ &Q_{3210} = \frac{3}{2}(1-5\delta),\ Q_{3211} = -3(1+3\delta),\ Q_{3220} = (1-\delta);\end{aligned}\right\} \tag{106}$$

$$\left.\begin{aligned}&R_{3200} = -105/8,\ R_{3201} = 21/2,\ R_{3202} = 21/8,\ R_{3210} = -21/4,\ R_{3211} = -21/2,\\ &R_{3220} = 0;\end{aligned}\right\} \tag{107}$$

$$\left.\begin{aligned}&Q_{3300} = -\frac{1}{24}(41+223\delta),\ Q_{3301} = \frac{3}{8}(3+17\delta),\ Q_{3302} = \frac{3}{8}(1+7\delta),\\ &Q_{3303} = \frac{1}{24}(5+7\delta),\ Q_{3310} = \tfrac{1}{4}(1-13\delta),\ Q_{3311} = -\tfrac{1}{4}(7+29\delta),\\ &Q_{3312} = -(1+2\delta),\ Q_{3320} = \tfrac{1}{4}(1-\delta),\ Q_{3321} = \tfrac{1}{4}(5+7\delta),\ Q_{3330} = 0;\end{aligned}\right\} \tag{108}$$

$$\left.\begin{aligned}&R_{3300} = -203/24,\ R_{3301} = 87/16,\ R_{3302} = 21/8,\ R_{3303} = 19/48,\\ &R_{3310} = -37/16,\ R_{3311} = -57/8,\ R_{3312} = -39/16,\ R_{3320} = 0,\\ &R_{3321} = 9/4,\ R_{3330} = 0\end{aligned}\right\} \tag{109}$$

It may be shown that the second-order solution for the throat radius of curvature is

$$
\begin{aligned}
r_t/b = {} & (1-\delta)(1+\tau^{-1/2})^{-1}b/g \\
& - (1-\tau^{-1/2})^{-1}(\tau-1)^{-2}\{\tfrac{1}{4}[(1-\delta)^{-1}(3+5\delta)+8j]\tau^2 - \\
& - \tau - \tfrac{1}{4}[(1-\delta)^{-1}(9\delta-1)+8j] - \\
& - ([(1-\delta)^{-1}(1+3\delta)+4j]\tau - \tfrac{1}{2})\log\tau\}g/b\,. \qquad (110)
\end{aligned}
$$

The corresponding result for two-dimensional flow ($\tau = 1$) is

$$r_t/b = \tfrac{1}{2}(1-\delta)\,b/g - \frac{2}{3}[(1-\delta)^{-1}(1+\delta)+2j]g/b\,, \qquad (111)$$

while for axi-symmetrical flow ($\tau = \infty$) the result is

$$r_t/b = (1-\delta)\,b/g - \tfrac{1}{4}[(1-\delta)^{-1}(3+5\delta)+8j]g/b\,. \qquad (112)$$

Although in the annular case high order terms are prohibitively lengthy, in the extreme cases[31] they are not. It is suggested now that it may be satisfactory, for designing annular nozzles which are not subject to very exacting requirements on the uniformity of the flows they produce, to deduce estimates of the higher-order terms in the annular case by interpolating between the higher-order terms in the extreme cases. The interpolation may be based on the first-order solution, and if desired the second-order solution may be used to decide a satisfactory form of interpolation.

4. *The Design of a Particular Annular Nozzle.* A brief description is now given of the design, along the lines suggested in the previous sections, of a particular annular nozzle. This nozzle has proved to be successful in its practical application[2, 3]. It was designed some years ago when the study described in this paper was in its earlier stages. At that time the design criteria given here had not been finally established; nevertheless, this nozzle fits very conveniently into the present design proposals. For this reason the design is presented as though it followed the present design method.

The fixed basic design parameters have the values $\bar{M} = 2\cdot023$ (corresponding to $\bar{\nu} = 27$ deg exactly), $a = 1\cdot950$ inches and $\bar{h} = 3\cdot600$ inches. The ratio $\bar{h}/\bar{h}_{\max}$ is chosen to be $0\cdot7380$, giving an assumed value of $\bar{h}_{\max} = 4\cdot878$ inches. From Equation (4) it follows that $\bar{g}_{\max} = 3\cdot407$ inches, and then from Equation (6) it follows that $l = 8\cdot122$ inches. The end values of the permitted range of χ are obtained from Equations (8) to (13) as $\chi_l = 1\cdot40$, $\chi_u = 1\cdot79$. The design value of χ is then chosen to be $\chi = 3/2$, so that the prescribed velocity distribution along the cylinder may be chosen to take the simple form $\kappa = 1 + (\bar{\kappa}-1)[\tfrac{1}{2}\xi(3-\xi^2)]$, where $\xi = x/l$; this satisfies the condition $d\kappa/dx = 0$ at $x = l$ so that it is expected to produce a nozzle of continuous curvature.

The solution of the throat flow is then used to provide a starting line for the application of the method of characteristics*. The line on which $\nu = 3$ deg is chosen; its shape is calculated from Equation (49) and the variation of θ along it is calculated from Equations (56) and (49). The starting line is not used beyond the position at which θ has a maximum in y. Then by selecting points at equal intervals of y along the $\nu = 3$ deg line, and at equal intervals of ν along the cylinder, a pattern of characteristics is constructed. The nozzle contour within the pattern of characteristics is located by calculating the point on each characteristic with which is associated the mass flow appropriate to the desired nozzle. The calculated points are joined by a line of

* The characteristic calculations were performed by J. Reid and R. C. Hastings, to whom credit for the design of this nozzle belongs.

continuous curvature. The termination of the $\nu = 3$ deg line at the position of the maximum in θ leaves a small gap between the streamline calculated from the characteristics pattern and the throat streamline up to the $\nu = 3$ deg line given by Equations (85) to (89). The gap is bridged by ignoring the precise form of Equation (85) and fitting a polynomial which preserves continuity of slope and curvature, between the point in the throat region with zero flow direction and the first point on the calculated streamline; in order to achieve as high accuracy as possible, use is made of the second-order result for the throat radius of curvature given by Equation (110). The resultant shape of the nozzle contour from the sonic point to the beginning of the working section is shown in Fig. 6.

The first-order solutions for the shapes of the sonic line, the limiting characteristic, the zero flow direction line, the first dividing characteristic and the line on which $\nu = 3$ deg are also shown in Fig. 6. In addition, the second dividing characteristic and the straight characteristic are displayed. The first and second dividing characteristics intersect outside the desired nozzle, and the associated value of $\bar{h}_{\max}$ is 4·832 inches which compares extraordinarily well with the assumed value of 4·878 inches.

It is noticed that the starting line intersects the first dividing characteristic within the nozzle, so that a characteristic calculation including the whole starting line involves a calculation in a part of the throat region. A similar situation was found in the design of a two-dimensional nozzle[17], in which case the intersection occurred at the position of the maximum of θ along the starting line, and a fine characteristic net was employed to complete the calculation beyond the intersection. In the present case the analytical complications of annular flow only allow it to be stated that the intersection occurs close to the position of the maximum of θ along the starting line. Thus a characteristic calculation in the throat region is avoided in this case, at the expense of leaving a gap between the two parts of the nozzle contour which are calculated respectively, analytically and numerically. It appears that, unless special care is to be given to the characteristic calculation in the throat region, it is preferable to avoid its use there by choosing the constant value of ν on the starting line to be large enough for the analytical throat solution to cover the entire throat region.

Therefore, it is desirable to be able to arrange that the constant value of ν on the starting line is greater or equal to the value of ν, say $\tilde{\nu}$, at the intersection of the desired streamline and the first dividing characteristic. The value of $\tilde{\nu}$, which is related to the value assumed for the ratio $\bar{h}/\bar{h}_{\max}$, is a measure of the number of terms of the throat solution which are needed to enable the starting line to be calculated with sufficient accuracy. The question of the number of terms required in a given case has been considered recently[31] for two-dimensional and axi-symmetrical nozzles, but no precise answer suitable for generalisation to annular nozzles is available. The indications are that for supersonic nozzles with $\bar{M}$ less than about 2 the first-order throat solution may be adequate for nozzles of any length, but for hypersonic nozzles with $\bar{M}$ greater than about 5 the first-order solution applies only to nozzles of excessive length, and for hypersonic nozzles of near minimum length many terms in the throat solution may be required. For hypersonic nozzles, however, boundary-layer development is very significant and leads to a suggested method[35] for shortening such nozzles. Nevertheless, it is impossible to escape the conclusion that there is, even at this late stage in the history of nozzle design, scope for further detailed investigation.

LIST OF SYMBOLS

a		Radius of internal cylinder
b		Length parameter representing velocity gradient at sonic point on cylinder
$b_{npq}(\xi)$		Coefficients in formal expansion of $\frac{1}{2}(a+y)\sin\theta/l$ near cylinder
c	$=$	$2^{-1}(1-\delta)^{-1} = (\gamma+1)/4$
$c_{npq}(\xi)$		Coefficients in formal expansion of $\cos\theta$ near cylinder
$f_{npq}(\xi)$		Coefficients in formal expansion of Φ/q^*l near cylinder
g		Effective throat width of typical streamline
$\bar{g}_{\max}$		Effective throat width of principal nozzle for design Mach number $\bar{M}$
$g_{npq}(\xi)$		Coefficients in formal expansion of $\Psi/4\pi\rho^*q^*l^2$ near cylinder
$\bar{h}$		Working section width of desired nozzle for design Mach number $\bar{M}$
$\bar{h}_{\max}$		Working section width of principal nozzle for design Mach number $\bar{M}$
j		Coefficient of $(x/b)^2$ in expansion near sonic point of distribution of κ along cylinder
$k_{npq}(\xi)$		Coefficients in formal expansion of κ near cylinder
l		Distance along cylinder from sonic point to beginning of working section
$l_{npq}(\xi)$		Coefficients in formal expansion of λ near cylinder
q		Velocity magnitude
q^*		Critical speed
r_t		Radius of curvature at point of zero flow direction on typical streamline in throat region
$t_{l,u}$		Two constants used in definition of $\chi_{l,u}$
x		Axial co-ordinate, measured from sonic point on cylinder
x_t		Value of x at point of zero flow direction on typical streamline in throat region
y		Radial co-ordinate, measured from cylinder
y_t		Value of y at point of zero flow direction on typical streamline in throat region
A	$=$	$a/\epsilon b$
$\bar{H}_\infty$, $\bar{H}_0$		Functions of $\bar{M}$ tabulated in Table 1
$\bar{M}$		Design Mach number in working section
Q_{3spq}		Coefficients of $(1-\delta)^{-1}$ in expression of ϕ_{3spq}
R_{3spq}		Constant coefficients in term involving j in expression of ϕ_{3spq}
$T_{l,u}$, U, $V_{l,u}$, $W_{l,u}$		Functions used in expression of $\chi_{l,u}$

X	=	$x/\epsilon^2 b$
Y	=	$y/\epsilon b$
Z	=	$X/2^{-1}(1-\delta)^{-1}A^2$
$-Z_-$, $+Z_+$		Solutions for Z along limiting characteristic and first dividing characteristic respectively
α	=	a/l
γ		Ratio of specific heats of gas
δ	=	$(\gamma-1)/(\gamma+1)$
ϵ		Dimensionless quantity small compared with unity
θ		Inclination of flow direction to axis
κ		Specific speed; $= q/q^*$
$\bar{\kappa}$		Value of κ corresponding to $M = \bar{M}$
$\kappa_0(\xi)$		Distribution of κ along cylinder
λ		Area ratio; $= \rho q/\rho^* q^*$
$\bar{\lambda}$		Value of λ corresponding to $M = \bar{M}$
$\lambda_0(\xi)$		Distribution of λ along cylinder
μ		Mach angle
ν		Prandtl-Meyer angle
$\bar{\nu}$		Value of ν corresponding to $M = \bar{M}$
$\tilde{\nu}$		Value of ν at intersection of desired nozzle and first dividing characteristic
ξ	=	x/l
ρ		Density
ρ^*		Critical density
τ	=	$(1+g/a)^2$
$\phi_4(X, Y; A)$		Term giving first-order solution for Φ in throat region
ϕ_{21}, ϕ_{22}		Functions of Y, A used in expression of ϕ_4
$\phi_6(X, Y; A)$		Term giving second-order solution for Φ in throat region
$\phi_{31}, \phi_{32}, \phi_{33}$		Functions of Y, A used in expression of ϕ_6
ϕ_{3spq}		Coefficients used in expression of ϕ_{3s}
χ		Dimensionless parameter representing velocity gradient at sonic point on cylinder
$\chi_{l,u}$		Lower and upper values of χ in permissible range defined by design criteria
ω	=	$(1+y/a)^2 = (1+Y/A)^2$
Φ		Dimensional velocity potential
Ψ		Dimensional stream function; $= \pi\rho^* q^* [(a+g)^2 - a^2]$

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APPENDIX

The Flow near the Cylinder

The fundamental aim is to find an analytical solution of the flow through both the throat and expansion regions (and also through the subsonic region upstream of the sonic line). Although this aim has been achieved*, the solution does not yield the streamlines and characteristics in a usable form, and therefore only an outline of the investigation and some of the principal results are given.

The velocity potential and the stream function have proved to be very useful independent variables for dealing with two-dimensional flow[21, 22] and axi-symmetrical flow[19, 20]. However, they can no longer conveniently be used in the annular case, and the independent variables of the physical plane are a more satisfactory choice.

In the physical plane the basic technique is to seek a series expansion solution near to the cylinder along which the flow is prescribed. The analogous technique in two-dimensional and axi-symmetrical flows is to expand in series near to the axis, and the solution for any dependent variable can be found by assuming an expansion in appropriate powers of the distance from the axis. In the annular case the corresponding expansion in powers of the distance from the cylinder does not yield a solution, and the basic form of expansion is not known at the outset.

Although knowledge of a single dependent variable is sufficient to enable all other desired quantities to be found, a quantity acceptable as the single dependent variable has not been found. The stream function cannot be used because the exact equation that it must satisfy cannot be written down explicitly. Also, it may be shown[36] that, even in the extreme cases of two-dimensional and axi-symmetrical nozzles, the velocity potential is not satisfactory in the present problem because it is exceptionally difficult to derive the streamlines from it. However, if the stream function and velocity potential are treated simultaneously with certain other dependent variables then a satisfactory series solution may be obtained.

Take rectangular co-ordinates x, y with the origin at the sonic point on the cylinder. Consider the following dependent variables: ρ the density; q the velocity magnitude; θ the flow direction angle; Φ the velocity potential; Ψ the stream function; g the effective throat width of a streamline, which is connected with Ψ by the relation

$$\Psi = \pi\rho^* q^* [(a+g)^2 - a^2], \tag{113}$$

where ρ^* is the critical density and q^* is the critical speed; κ the specific speed defined by

$$\kappa = q/q^*; \tag{114}$$

λ the area ratio defined by

$$\lambda = \rho q/\rho^* q^* \tag{115}$$

and expressed in terms of κ by

$$\lambda = \kappa[(1-\delta)^{-1}(1-\delta\kappa^2)]^{(1-\delta)/2\delta}. \tag{116}$$

The fundamental length is taken to be l, the distance along the cylinder from the sonic point to the beginning of the working section.

* Acknowledgement is due to R. C. Hastings for his assistance in the determination of the solution near the cylinder.

It may be shown that the six quantities g/l, $\sin\theta$, $\cos\theta$, Φ/q^*l, κ, λ form a set of simultaneous dependent variables for which a convergent series solution of the appropriate partial differential equations and algebraic equations may be obtained (it is advantageous to retain $\sin\theta$ and $\cos\theta$ as separate variables). It is assumed that x/l is $0(1)$ and y/l is small compared with unity, the effect of a/l being included by the same device as that employed in obtaining the solution near to the sonic point on the cylinder. The solution develops in a progressive manner, the first-order terms being obtained for each quantity in the sequence quoted, and then the second-order terms in the same sequence and so on. The first-order solutions which satisfy the boundary conditions of a prescribed velocity distribution along, and zero velocity normal to, the internal cylinder may be expressed in the form

$$g/l = \alpha\{[1 + \lambda_0(\omega-1)]^{1/2} - 1\}, \tag{117}$$

$$\sin\theta = -\tfrac{1}{2}\lambda_0^{-1}\lambda_0'\alpha\omega^{-1/2}(\omega-1), \tag{118}$$

$$\cos\theta = 1 - \tfrac{1}{2}\sin^2\theta, \tag{119}$$

$$\Phi/q^*l = \int\kappa_0 d\xi - \tfrac{1}{4}\kappa_0\lambda_0^{-1}\lambda_0'\alpha^2\{\omega - 1 - \log\omega\}, \tag{120}$$

$$\kappa = \kappa_0 + \tfrac{1}{4}\alpha^2\begin{bmatrix}\kappa_0\lambda_0^{-2}(\lambda_0')^2(\omega^{-1}-1) - \\ -\{\kappa_0'\lambda_0^{-1}\lambda_0' - \tfrac{3}{2}\kappa_0\lambda_0^{-2}(\lambda_0')^2 + \kappa_0\lambda_0^{-1}\lambda_0''\}(\omega-1) + \\ +\{\kappa_0'\lambda_0^{-1}\lambda_0' - \kappa_0\lambda_0^{-2}(\lambda_0')^2 + \kappa_0\lambda_0^{-1}\lambda_0''\}\log\omega\end{bmatrix}, \tag{121}$$

$$\lambda = \lambda_0 - \kappa_0^{-1}(\kappa_0^2-1)(1-\delta\kappa_0^2)^{-1}\lambda_0(\kappa-\kappa_0), \tag{122}$$

where

$$\xi = x/l \tag{123}$$

and κ_0 and λ_0 are functions of ξ, $\kappa_0(\xi)$ denoting the prescribed distribution of κ along the cylinder and $\lambda_0(\xi)$ the corresponding distribution of λ; the independent variable ω is equal to $(1+y/a)^2$ here, and the parameter α is defined by

$$\alpha = a/l; \tag{124}$$

a dash denotes complete differentiation with respect to ξ.

Although throughout this paper it is assumed that the internal boundary is both a cylinder and a streamline, the solution is easily extended to cover the case when the boundary is not necessarily either a cylinder or a streamline. When the boundary is a cylinder but not a streamline the solution applies to the flow through an annular nozzle which has distributed suction or blowing through the internal cylinder. When the boundary is a streamline but not a cylinder the solution, to the first order, applies to the flow past a quasi-cylindrical body of revolution. The application of the first-order solution to the flow past such quasi-cylinders leads[37] to an illustration of how differences arise between the linearised pressure coefficient for a thin aerofoil and for a slender body of revolution.

The first-order solution of the flow near the cylinder has only a limited application in nozzle design and for most purposes higher-order terms are needed. These may be obtained by the convergent procedure already outlined for the dependent variables g/l, $\sin\theta$, $\cos\theta$, Φ/q^*l, κ, λ but the analysis is extremely tedious. However, the derivation of the higher-order terms can be simplified by using the set of variables $\Psi/4\pi\rho^*q^*l^2$, $(a+y)\sin\theta/2l$, $\cos\theta$, Φ/q^*l, κ, λ. A formal solution of the appropriate differential and algebraic equations can be found by assuming expansions in even powers of α for $\infty \geqslant \alpha \geqslant 0$, and the formal solution can be shown[37] to be equivalent to the previous convergent solution. The first-, second- and third-order terms of the formal solution may be written

$$\Psi/4\pi\rho^* q^* l^2 = \sum_{n=1}^{3} (\tfrac{1}{2}\alpha)^{2n} \left[\sum_{p=0}^{(n-1)} \sum_{q=-(n-p-2)}^{(n-p)} g_{npq}\omega^q \log^p \omega + g_{n(n-1)0} \log^{n-1}\omega \right], \tag{125}$$

$$(a+y) \sin \theta/2l = \sum_{n=1}^{3} (\tfrac{1}{2}\alpha)^{2n} \sum_{p=0}^{(n-1)} \sum_{q=-(n-p-1)}^{(n-p)} b_{npq}\omega^q \log^p \omega , \tag{126}$$

$$\cos \theta = c_{000} + \sum_{n=1}^{3} (\tfrac{1}{2}\alpha)^{2n} \sum_{p=0}^{(n-1)} \sum_{q=-(n-p)}^{(n-p)} c_{npq}\omega^q \log^p \omega , \tag{127}$$

$$\Phi/q^* l = f_{000} + \sum_{n=1}^{3} (\tfrac{1}{2}\alpha)^{2n} \left[\sum_{p=0}^{(n-1)} \sum_{q=-(n-p-1)}^{(n-p)} f_{npq}\omega^q \log^p \omega + f_{nn0} \log^n \omega \right], \tag{128}$$

$$\kappa = k_{000} + \sum_{n=1}^{3} (\tfrac{1}{2}\alpha)^{2n} \sum_{p=0}^{n} \sum_{q=-(n-p)}^{(n-p)} k_{npq}\omega^q \log^p \omega , \tag{129}$$

$$\lambda = l_{000} + \sum_{n=1}^{3} (\tfrac{1}{2}\alpha)^{2n} \sum_{p=0}^{n} \sum_{q=-(n-p)}^{(n-p)} l_{npq}\omega^q \log^p \omega , \tag{130}$$

where the coefficients g_{npq}, b_{npq}, c_{npq}, f_{npq}, k_{npq}, l_{npq} are functions of ξ and

$$c_{000} = 1 , \qquad f_{000} = \int \kappa_0 d\xi , \qquad k_{000} = \kappa_0 , \qquad l_{000} = \lambda_0 ; \tag{131}$$

the coefficients f_{n00}, g_{n00} for $n \geqslant 1$ are arbitrary and allow the boundary condition to be satisfied. It is likely that the expressions are of the same form for $n > 3$ also, in which case they constitute the basis of an exact infinite series solution. They are the counterpart in the annular case of the expansions in even powers of y which are the fundamental forms in two-dimensional and axi-symmetrical flows.

It is clear that the fundamental form of the series expansion near the cylinder in the annular case is enormously complicated by comparison with the series expansions near the axis in the two-dimensional and axi-symmetrical cases. The conclusion must therefore be that, except for the first-order solution which may be of some use for providing limiting results, the analytical solution for the flow near the cylinder is too complicated to be of practical use in the design of annular supersonic nozzles.

There is a connection between the solution near the cylinder outlined here and the solution near the sonic point on the cylinder given in Section 3, but the connection is not a simple one. If the solution for a certain quantity near the cylinder is suitably expanded near the sonic point the usefulness of the results obtained depends on the particular quantity. Strictly, the result is not valid because the original series expansion near the cylinder does not incorporate the essential non-linearity of the equations of motion near the sonic point, but it happens that some of the results of the throat solution can be deduced in this way. In particular, the first-order solution for the throat radius of curvature can be obtained from the first-order solution of the streamlines near the cylinder.

TABLE 1

The Functions $\bar{H}_\infty$, $\bar{H}_0$

$\bar{M}$	$\bar{H}_\infty$	$\bar{H}_0$
1	0·6021	0·3221
1·1	0·5945	0·3198
1·2	0·5895	0·3180
1·3	0·5858	0·3165
1·4	0·5822	0·3150
1·5	0·5792	0·3138
1·6	0·5768	0·3128
1·7	0·5748	0·3120
1·8	0·5730	0·3112
1·9	0·5712	0·3108
2·0	0·5698	0·3102
2·5	0·5635	0·3088
3·0	0·5588	0·3078
3·5	0·5545	0·3070
4·0	0·5515	0·3062
4·5	0·5485	0·3055
5·0	0·5460	0·3048
6	0·5418	0·3040
7	0·5370	0·3032
8	0·5325	0·3022
9	0·5285	0·3012
10	0·5250	0·2998
∞	0·5000	0·2500

Note. Graphical interpolation is permissible for $1 \leqslant \bar{M} \leqslant 10$.

TABLE 2

The Functions Z_-/ω, Z_+/ω

$1/\omega$	Z_-/ω	Z_+/ω
1	0	0
0·95	0·0006	0·0013
0·90	0·0026	0·0053
0·85	0·0060	0·0122
0·80	0·0109	0·0222
0·75	0·0175	0·0357
0·70	0·0259	0·0528
0·65	0·0362	0·0741
0·60	0·0487	0·1000
0·55	0·0635	0·1310
0·50	0·0809	0·1678
0·45	0·1013	0·2111
0·40	0·1250	0·2621
0·35	0·1526	0·3221
0·30	0·1847	0·3930
0·25	0·2222	0·4773
0·20	0·2666	0·5790
0·15	0·3198	0·7046
0·10	0·3854	0·8662
0·05	0·4715	1·0933
0	0·6180	1·6180

Note. Graphical interpolation is permissible for $1 \geqslant 1/\omega \geqslant 0{\cdot}05$ but is not recommended for $0{\cdot}05 \geqslant 1/\omega \geqslant 0$.

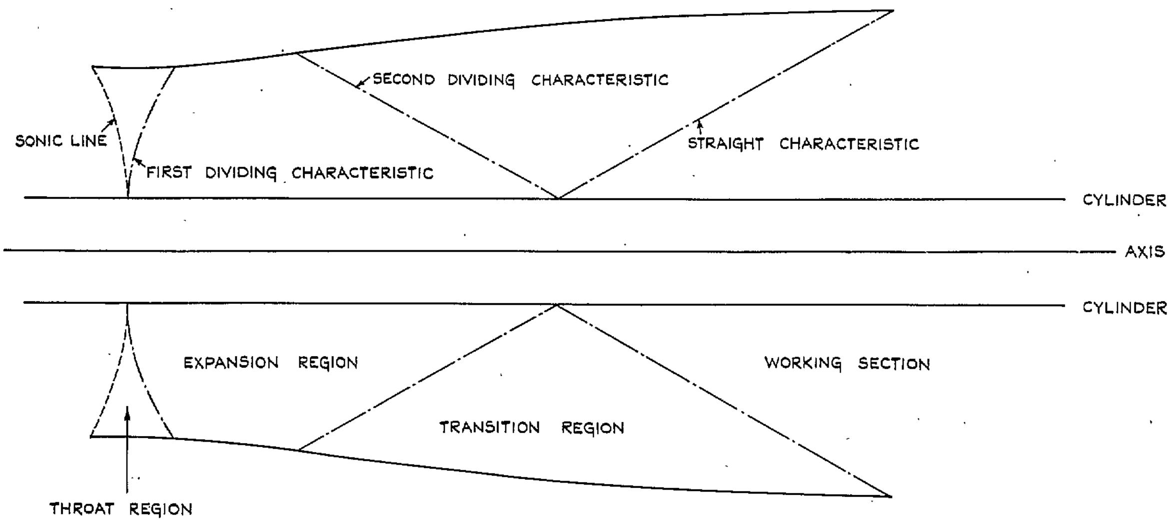

FIG. 1. Meridian section of a typical annular supersonic nozzle.

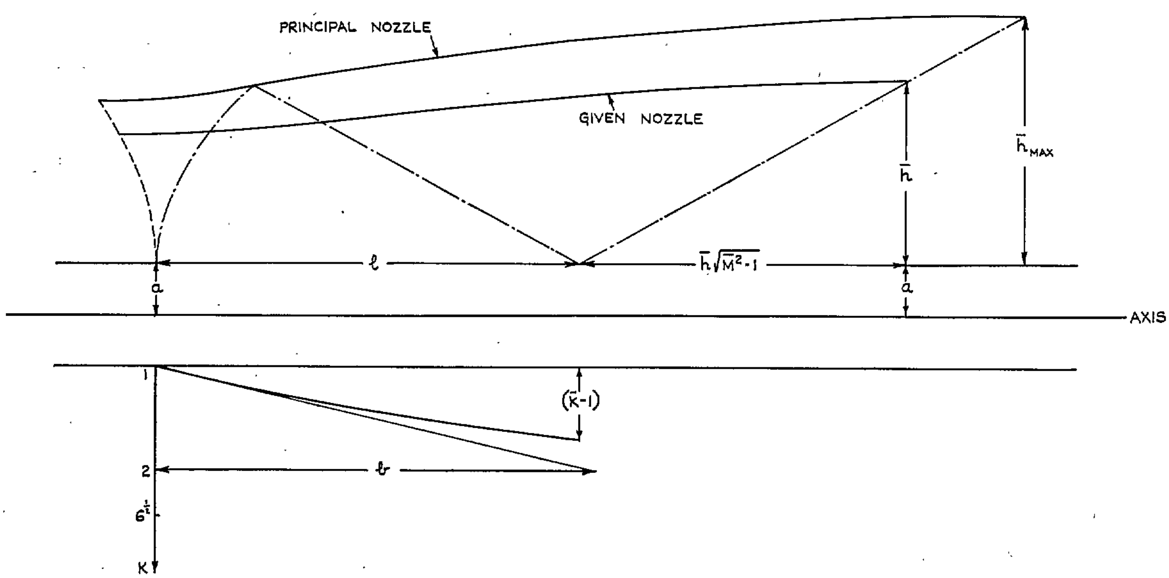

FIG. 2. Principal nozzle corresponding to a given nozzle.

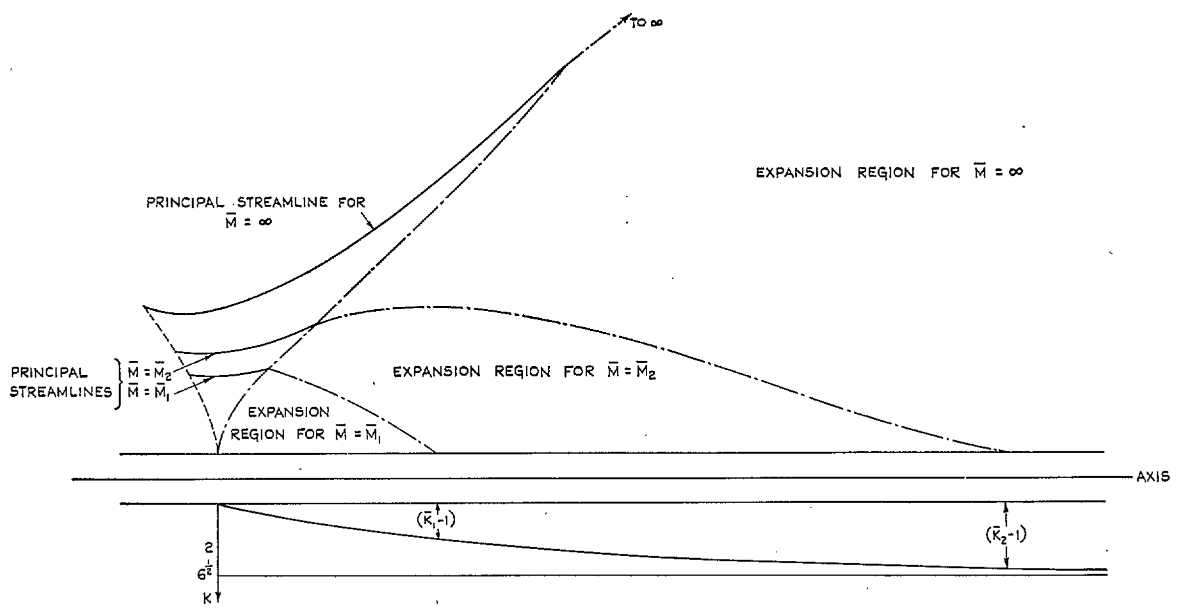

FIG. 3. Throat and expansion regions for infinite design Mach number.

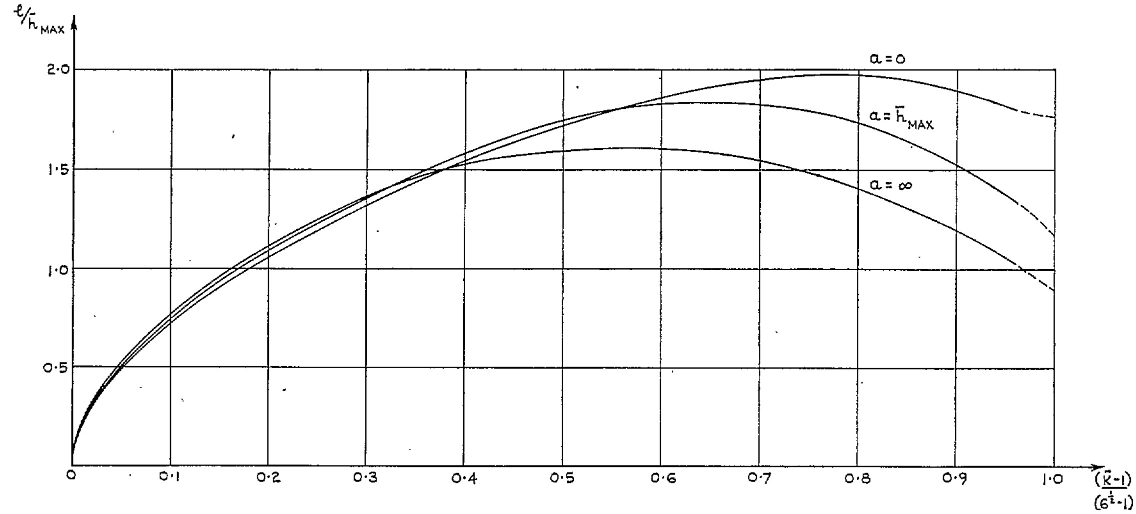

FIG. 4. Design criteria : some values of l.

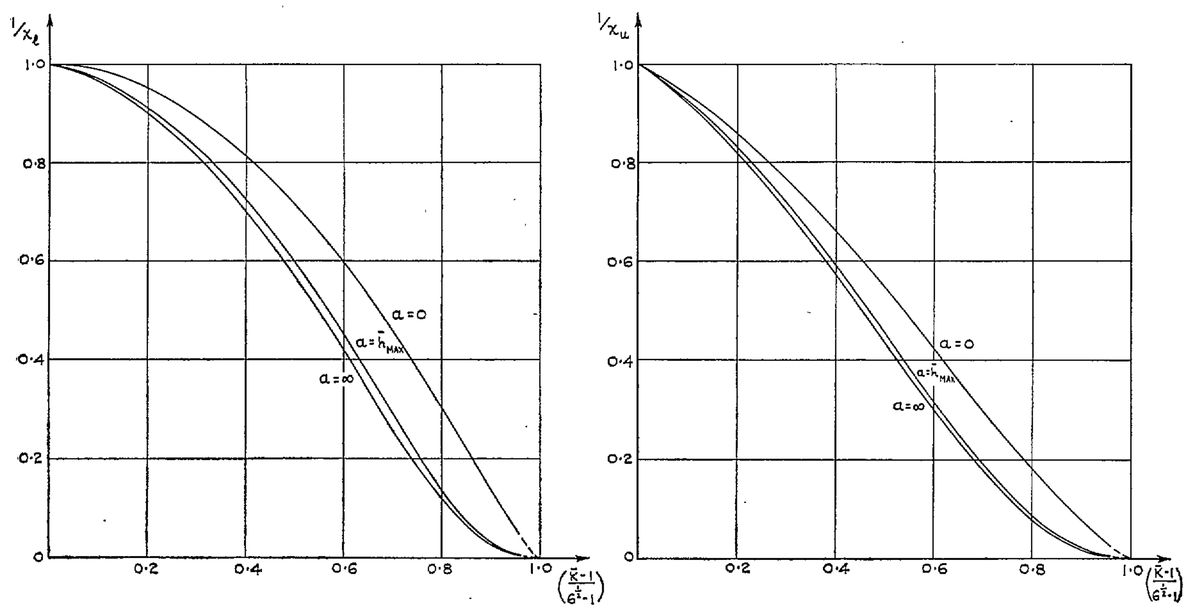

FIG. 5. Design criteria : some values of $\chi_{l,\,u}$.

(82493) Wt. 66/991 K.5 11/61 Hw.

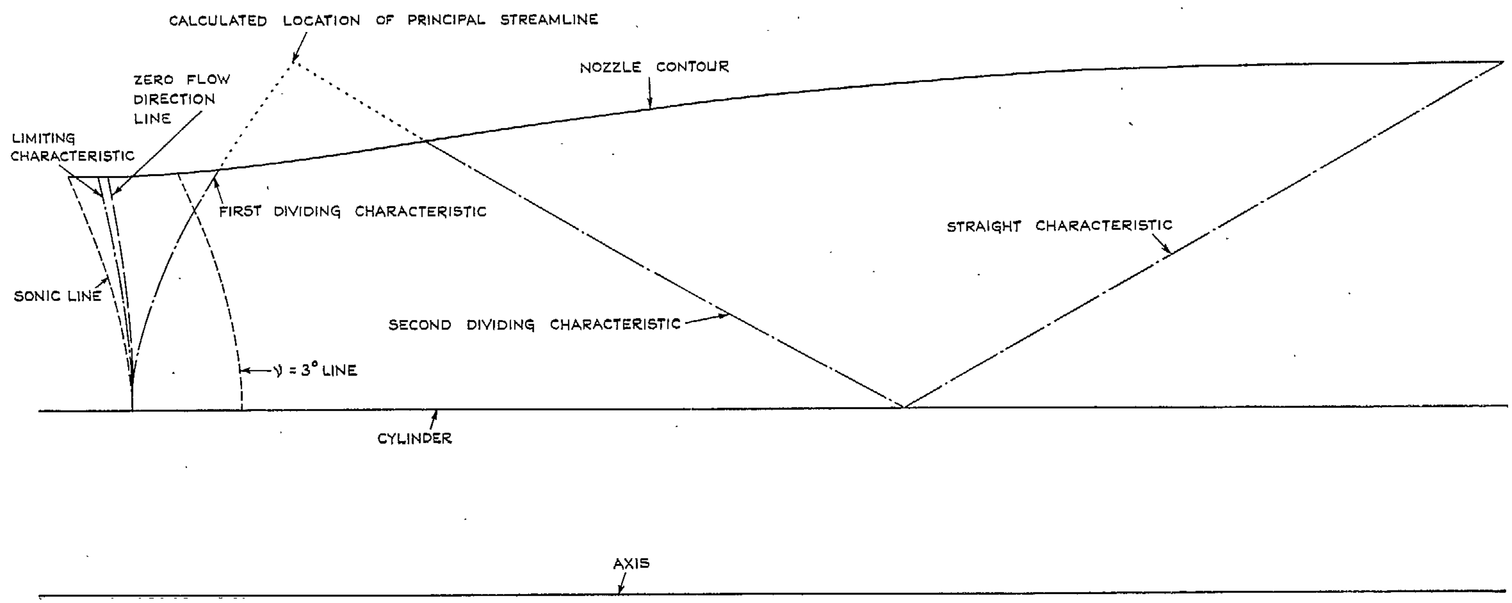

FIG. 6. Some details of an annular nozzle for $\overline{M} = 2 \cdot 023$.

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