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An Axially Symmetric Analogue for General Three-Dimensional Boundary Layers

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Summary.—On the assumption that the flow normal to the external streamlines is small compared with that along the external streamlines it is shown that the streamwise momentum equation, the energy equation and the equation of continuity reduce to equations identical with those for flow over an axisymmetrical body, whose radius is determined from the local external flow. By solving these equations it is possible to obtain approximate values for the skin friction and the displacement surface in the case of wings at small angles of incidence, having no separations except at the edges.

The theory applies directly for laminar compressible or incompressible flow, and the suggestion is made that it may apply also for turbulent flow.

1. *Introduction*.—The problem of calculating three-dimensional boundary layers in general is so intractable that additional assumptions over and above the usual assumptions of boundary-layer theory need to be made in order to make any progress. One which has been made by Eichelbrenner and Oudart[1] and by Zaat[2] is that the flow in the boundary layer normal to the external streamlines is small. After this assumption is made it is possible to devise momentum equation methods to complete a solution. The lack of exact solutions makes it difficult to assess the accuracy of these approximate methods, though in the case of laminar boundary layers one comparison is being made, and it appears that the assumptions may lead to acceptable results provided the angle between the external streamlines and the limiting streamlines is not greater than about 15 deg. In the case of turbulent flow a comparison with experiment has been made which shows fair agreement in the case of a thin swept wing at a low angle of incidence[3].

It is necessary to use streamline co-ordinates, and the solution proceeds by following an external streamline. There is thus a series of differential equations to be solved, three for each streamline.

The chief result of this paper is that the streamwise momentum equation, the energy equation and the equation of continuity reduce to equations identical with those for flow over an axisymmetrical body, and so they may be solved by well-known methods or reduced to two-dimensional equations after the Mangler transformation has been made[4]. There is of course one more equation (that for the cross-flow component) to be solved, which will require a new technique, but the

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solution of the first three may be sufficient to give some idea of the skin friction and the displacement thickness of a wing at low lift coefficients. The last equation will certainly be needed to determine separation. We do not consider it here.

The fact that under the present assumption the streamwise equations reduce to axisymmetric equations in laminar incompressible flow was noted by Eichelbrenner and Oudart[1] who indeed solved the streamwise equations by means of the Mangler transformation.

In a recent paper Nickel[5] writes down the incompressible equations of motion in a general co-ordinate system, not completely orthogonal. He then obtains the conditions that an 'independence principle' exists; that is, he obtains conditions that two of the equations can be solved independently of the third, or as he puts it, the equations 'reduce to 2+1'. Having done this he uses a generalised Mangler transformation to obtain the standard equations of two-dimensional flow. Nickel only deals with incompressible flow, and his conditions that the equations of motion reduce approximately to 2+1, are that certain expressions are small. These expressions reduce to those given below, namely, (20) and (22). In this paper the reduction is made in all cases to axisymmetric flow, for which calculation methods have been devised for all circumstances of flow, even turbulent and compressible, and so can be taken over in the three-dimensional problem. The co-ordinate system is more practical than the general one of Nickel.

2. *The Equations of Motion.*—We choose a set of orthogonal co-ordinates ξ, η, ζ in which ζ is distance measured along the outward normal to the surface. The line element can then be written in the form

$$ds^2 = h_1^2\,d\xi^2 + h_2^2\,d\eta^2 + d\zeta^2, \tag{1}$$

where h_1 and h_2 are supposed to be functions of ξ and η only. This implies that in the region concerned, which is the thin boundary layer close to the surface, the radius of curvature of any normal section of the surface is large compared with the boundary-layer thickness.

The general equations of motion are[6]

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} - \boldsymbol{V}\times\operatorname{curl}\boldsymbol{V} + \tfrac{1}{2}\operatorname{grad}V^2\right) = \rho\boldsymbol{F} - \operatorname{grad}p + \tfrac{4}{3}\operatorname{grad}(\mu\Delta) + \operatorname{grad}(\boldsymbol{V}\,.\,\operatorname{grad}\mu) - \boldsymbol{V}\,\nabla^2\mu +$$

$$+\operatorname{grad}\mu\times\operatorname{curl}\boldsymbol{V} - \Delta\operatorname{grad}\mu - \operatorname{curl}\operatorname{curl}\mu\boldsymbol{V},$$

$$\rho c_p\left(\frac{\partial T}{dt} + \boldsymbol{V}\,.\,\operatorname{grad}T\right) = \Phi + \frac{\partial p}{\partial t} + \boldsymbol{V}\,.\,\operatorname{grad}p + \operatorname{div}(k\operatorname{grad}T),$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{V}) = 0,$$

where $\boldsymbol{V}$ is the vector fluid velocity, ρ the density, μ the coefficient of viscosity, p the pressure, c_p the specific heat at constant pressure, k the thermal conductivity, Φ the dissipation function, T the temperature, $\boldsymbol{F}$ the body force vector and

$$\Delta = \operatorname{div}\boldsymbol{V}.$$

Written out in full the dissipation function is given by

$$\Phi = \mu(e_{12}^2 + e_{23}^2 + e_{31}^2) + \tfrac{1}{6}\mu\{(e_{11}-e_{22})^2 + (e_{22}-e_{33})^2 + (e_{33}-e_{11})^2\},$$

where $e_{\alpha\beta}$ is a rate of strain component. General expressions for $e_{\alpha\beta}$ in curvilinear co-ordinates are given by Goldstein[7].

We assume that the motion is steady and make the usual boundary-layer approximations. Taking the three velocity components as u, v, w, assuming $F = 0$ and using equation (1) we have

$$\rho\left\{\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right\} = -\frac{1}{h_1}\frac{\partial p}{\partial \xi}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right), \tag{2}$$

$$\rho\left\{\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi}-\frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}\right\} = -\frac{1}{h_2}\frac{\partial p}{\partial \eta}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right), \tag{3}$$

$$0 = \frac{\partial p}{\partial \zeta}, \tag{4}$$

$$\rho c_p\left\{\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{v}{h_2}\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right\} = \Phi+\frac{u}{h_1}\frac{\partial p}{\partial \xi}+\frac{v}{h_2}\frac{\partial p}{\partial \eta}+w\frac{\partial p}{\partial \zeta}+\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right), \tag{5}$$

$$\frac{1}{h_2 h_1}\left\{\frac{\partial}{\partial \xi}(\rho h_2 u)+\frac{\partial}{\partial \eta}(\rho h_1 v)+\frac{\partial}{\partial \zeta}(\rho h_1 h_2 w)\right\} = 0. \tag{6}$$

We now suppose that ξ and η are such that the curves of intersection of the surfaces $\eta =$ const. with the given surface $\zeta = 0$ are streamlines on the surface, and that the intersections of $\xi =$ const. with the surface are their orthogonal trajectories on the surface. For such co-ordinates (usually called 'streamline co-ordinates') we have $v_1 = 0$.

In equations (2) and (3) we determine $\partial p/\partial \xi$ and $\partial p/\partial \eta$ from conditions in the main stream, since (4) shows that p does not change in passing through the boundary layer. On writing $\partial/\partial \zeta = 0$, $u = u_1$, $v = v_1 = 0$ in equations (2) and (3) we obtain

$$-\frac{1}{h_1}\frac{\partial p}{\partial \xi} = \frac{\rho_1 u_1}{h_1}\frac{\partial u_1}{\partial \xi}, \qquad -\frac{1}{h_2}\frac{\partial p}{\partial \eta} = -\frac{\rho_1 u_1^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}.$$

A simplification is achieved if the element of length along a streamline $\eta =$ const., namely, $h_1\,d\xi$, is written ds, so that

$$\frac{1}{h_1}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial s}.$$

We shall also put $h_2 = r$.

Finally we make the assumption that v and its derivatives are small. We shall consider the nature of this assumption later.

Equations (2), (5) and (6) now become

$$\rho\left(u\frac{\partial u}{\partial s}+w\frac{\partial u}{\partial \zeta}\right) = \rho_1 u_1\frac{\partial u_1}{\partial s}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right), \tag{7}$$

$$\rho c_p\left(u\frac{\partial T}{\partial s}+w\frac{\partial T}{\partial \zeta}\right)+\rho_1 u u_1\frac{\partial u_1}{\partial s} = \frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)+\mu\left(\frac{\partial u}{\partial \zeta}\right)^2, \tag{8}$$

$$\frac{1}{r}\frac{\partial}{\partial s}(\rho r u)+\frac{\partial}{\partial \zeta}(\rho w) = 0. \tag{9}$$

These are the standard equations of motion in the boundary layer over an axisymmetric body[6] of radius r. r has of course a very different meaning here; it must also be understood that equations (7), (8) and (9) are to be taken along a given external streamline, and that s is the arc along this streamline, ζ being measured normal to the surface.

Once u and w are found from equations (7), (8) and (9) v can be found from equation (3) which becomes linear in v and may be written

$$\rho\left(u\frac{\partial v}{\partial s}+w\frac{\partial v}{\partial \zeta}+\frac{uv}{r}\frac{\partial r}{\partial s}+\kappa u^2\right) = \rho_1 \kappa u_1{}^2+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right), \tag{10}$$

where

$$\kappa = \frac{1}{h_1 r}\frac{\partial h_1}{\partial \eta}.$$

3. *The Value of r.*—We show in Appendix I that r is determined from the equation

$$u_1\frac{\partial}{\partial s}\left(\log\frac{u_1{}^2 r^2}{g}\right) = 2\left(\frac{\delta \bar{U}}{\delta x}+\frac{\delta \bar{V}}{\delta y}\right), \tag{11}$$

where $\bar{U}$ and $\bar{V}$ are velocity components parallel to fixed Cartesian axes, x, y, z and

$$u_1\frac{\partial}{\partial s} = \bar{U}\frac{\delta}{\delta x}+\bar{V}\frac{\delta}{\delta y}, \tag{12}$$

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x}+z_x\frac{\partial}{\partial z}, \qquad \frac{\delta}{\delta y} = \frac{\partial}{\partial y}+z_y\frac{\partial}{\partial z},$$

$$g = 1+z_x{}^2+z_y{}^2,$$

the equation of the surface being $z = z(x, y)$, and suffixes attached to z denoting partial derivatives. In many cases z_x and z_y are small and in this case the equations (11) and (12) simplify to

$$\left.\begin{aligned} u_1\frac{\partial}{\partial s}(\log u_1 r) &= \frac{\partial \bar{U}}{\partial x}+\frac{\partial \bar{V}}{\partial y} \\ u_1\frac{\partial}{\partial s} &= \bar{U}\frac{\partial}{\partial x}+\bar{V}\frac{\partial}{\partial y} \end{aligned}\right\}. \tag{13}$$

The geodesic curvature of the curve ξ = const., namely,

$$\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = \frac{1}{r}\frac{\partial r}{\partial s},$$

is a measure of the amount that the streamlines converge or diverge. Thus if r increases downstream, two neighbouring streamlines have moved further apart, and so if the streamlines diverge $\partial r/\partial s$ is positive. Conversely, if the streamlines converge $\partial r/\partial s$ is negative (*see* Fig. 1).

The form of equation (11) shows that r is not completely determinate. It may in fact be multiplied by any function of η. Changing r will also change η. It is convenient to choose r and η in such a way that r is non-dimensional; η will then have the dimensions of a length.

4. *The Mangler Transformation.*—We make the following substitutions in laminar compressible or incompressible flow:

$$\left.\begin{gathered} s' = \int r^2\,ds, \qquad z' = r\zeta, \qquad w' = \frac{1}{r}\left(w+\frac{\zeta u}{r}\frac{\partial r}{\partial s}\right), \\ v' = rv, \qquad u' = u, \qquad \rho' = \rho, \qquad u_1{}' = u_1, \qquad \mu' = \mu, \\ \kappa' = \frac{1}{r}\kappa. \end{gathered}\right\} \tag{14}$$

We find that equations (7), (8), (9) and (10) take the form

$$\rho'\left(u'\frac{\partial u'}{\partial s'}+w'\frac{\partial u'}{\partial z'}\right)=\rho_1' u_1'\frac{\partial u_1'}{\partial s'}+\frac{\partial}{\partial z'}\left(\mu'\frac{\partial u'}{\partial z'}\right); \tag{15}$$

$$\rho' c_p\left(u'\frac{\partial T}{\partial s'}+w'\frac{\partial T}{\partial z'}\right)+\rho_1' u' u_1'\frac{\partial u_1'}{\partial s'}=\frac{\partial}{\partial z'}\left(k\frac{\partial T}{\partial z'}\right)+\mu'\left(\frac{\partial u'}{\partial z'}\right)^2 \tag{16}$$

$$\frac{\partial}{\partial s'}(\rho' u')+\frac{\partial}{\partial z'}(\rho' w')=0, \tag{17}$$

$$\rho'\left(u'\frac{\partial v'}{\partial s'}+w'\frac{\partial v'}{\partial z'}\right)=\kappa'(\rho_1' u_1'^2-\rho' u'^2)+\frac{\partial}{\partial z'}\left(\mu'\frac{\partial v'}{\partial z'}\right). \tag{18}$$

The first three of these equations are standard two-dimensional equations, but the fourth is new.

5. *Examination of the Approximations Made.*—It will be seen that if v and its derivatives are to be small equation (10) implies that κ must be small. Now κ is the geodesic curvature of the external streamlines η = const.[17], and so we deduce that in order that the basic assumption may hold the external streamlines must have small geodesic curvature.

The geodesic curvature at a point P of a curve S in the surface is the curvature of the orthogonal projection of S on the tangent plane at P. Thus we may say loosely that the external streamlines must not bend 'sideways' very much, though 'up-and-down' motion is permitted.

It will be noted from equation (10) that it is possible to have identically zero cross-flow if κ is zero, that is, if the streamlines are geodesics in the surface. This was proved by Squire[18].

We write

$$\frac{1}{r}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial n}; \tag{19}$$

the operator $\partial/\partial n$ thus denotes differentiation normal to the external streamlines.

It is not possible to make a complete examination of the approximations made, except *a posteriori*. However, we may note that the terms missed out in equations (7), (8), (9) and (10) are

$$\rho v\left\{\frac{\partial u}{\partial n}-\frac{u}{u_1}\frac{\partial u_1}{\partial n}-\frac{v}{r}\frac{\partial r}{\partial s}\right\}, \tag{20}$$

$$v\left\{\rho c_p\frac{\partial T}{\partial n}-\rho u_1\frac{\partial u_1}{\partial n}\right\}-\mu\left(\frac{\partial v}{\partial\zeta}\right)^2, \tag{21}$$

$$u_1\frac{\partial}{\partial n}\left(\frac{\rho v}{u_1}\right), \tag{22}$$

$$\rho v\frac{\partial v}{\partial n}. \tag{23}$$

We assume first that v is small compared with u. In addition we suppose that $\partial v/\partial n$ is small compared with $\partial u/\partial s$. Unless $\partial u/\partial n$, $\partial u_1/\partial n$ and $\partial T/\partial n$ are large this will be in general sufficient for the rejection of the expressions (20), (21), (22) and (23). It seems that these conditions are likely to hold for thin wings at low incidences, but there is more doubt in the case of slender wings. This is because derivatives with respect to n may in some places become large if the wing is slender; in fact, they may be multiplied by a factor of order $\frac{\text{root chord}}{\text{maximum semi-span}}$ as compared with their non-slender values.

It is unfortunately not possible to be more specific about the errors introduced by the present procedure. One may say that in general the boundary-layer flow direction does not depart greatly from that in the external flow except near separation, and so the method may be applied so long as separation is not approached too closely.

6. *Some Solutions.*—6.1. *Laminar Incompressible Flow.*—One method of attack on equations (14) and (16) is that of Thwaites[9]. In particular if we denote by θ_{11}' the momentum thickness in the variables of Section 4

$$\theta_{11}' = \int_0^\infty \frac{u'}{u_1'}\left(1-\frac{u'}{u_1'}\right) dz',$$

then according to Thwaites in laminar flow

$$\theta_{11}'^2 = \frac{0{\cdot}45\nu'}{u_1'^6}\int_{s_0'}^{s'} u_1'^5\, ds' + \frac{(\theta_{11}' u_1'^6)_0}{u_1'^6},$$

where the suffix 0 refers to some starting position.

Transforming back to our original variables this becomes

$$\theta_{11}^2 = \frac{0{\cdot}45\nu}{r^2 u_1^6}\int_{s_0}^{s} u_1^5 r^2\, ds + \frac{(\theta_{11}^2 r^2 u_1^6)_0}{r^2 u_1^6}.$$

If, following Zaat[2] we write $r^2 = 1/\bar{\rho} u_1^2$, this equation becomes,

$$\theta_{11}^2 = \frac{0{\cdot}45\bar{\rho}\nu}{u_1^4}\int_{s_0}^{s} \frac{u_1^3}{\bar{\rho}}\, ds + \left(\frac{\theta_{11}^2 u_1^4}{\bar{\rho}}\right)_0 \frac{\bar{\rho}}{u_1^4}.$$

Zaat[2] gives an equation which can be reduced to this one, but with 0·45 replaced by 0·436.

Alternatively the method of Blasius[7] or Görtler[10] could be used on equations (15) and (17).

6.2. *Turbulent Flow.*—We may integrate equation (7) with respect to ζ between the limits 0 and ∞, and make use of (9) in integrating by parts the second term of equation (7). We obtain, as may easily be verified, though the algebra is tedious, the momentum integral equation

$$\theta_{11}' + \theta_{11}\left\{(H+2)\frac{u_1'}{u_1} + \frac{r'}{r} + \frac{\rho'}{\rho}\right\} = \frac{\tau_{01}}{\rho_1 u_1^2}, \tag{24}$$

primes denoting differentiation with respect to s.

In this equation

$$\theta_{11} = \int_0^\infty \frac{\rho u}{\rho_1 u_1}\left(1-\frac{u}{u_1}\right) d\zeta, \qquad \delta_{11} = \int_0^\infty \left(1-\frac{\rho u}{\rho_1 u_1}\right) d\zeta,$$

$$H = \frac{\delta_{11}}{\theta_{11}}, \qquad \tau_{01} = \left(\mu\frac{\partial u}{\partial \zeta}\right)_{\zeta=0}.$$

It is usual in two dimensions, though considerable justification is required for this, to take the momentum integral equation over into turbulent flow, replacing τ_{01} by an empirical formula for skin friction. If we may also do this for three-dimensional flow then our governing equation for streamwise flow is equation (24) which is exactly that for flow past an axisymmetric body of radius r.

It does not necessarily follow that the empirical formula for τ_{01} will be the same. Nevertheless, Cooke[3] took the same formula as in two-dimensional flow and this led to results giving fair agreement with experiment. Even if the same formula cannot be used this does not invalidate the axially symmetric analogy for the turbulent case.

The momentum equation (24) could be the starting point in streamwise calculations for the general case similar to those of Young[11] for the axisymmetric case, using for r the value determined in this paper.

If, as is done by Spence[12] in two-dimensional incompressible turbulent flow, we may take in solving equation (23)

$$\frac{\tau_{01}}{\rho u_1^2} = 0{\cdot}00885\left(\frac{u_1\,\theta_{11}}{\nu}\right)^{-1/5}, \qquad H = \text{const.} = 1{\cdot}5,$$

which is due to Young[11], that equation reduces to

$$\frac{\partial}{\partial s}(\Theta u_1^4 r^{6/5}) = 0{\cdot}0106 u_1^4 r^{6/5},$$

where

$$\Theta = \theta_{11}\left(\frac{u_1\,\theta_{11}}{\nu}\right)^{1/5}.$$

If we write $r^2 = 1/\bar{\rho}u_1^2$ we obtain

$$\frac{\partial}{\partial s}(\Theta u_1^{14/5}\,\bar{\rho}^{-3/5}) = 0{\cdot}0106 u_1^{14/5}\,\bar{\rho}^{-3/5},$$

as given by Cooke[3].

Other versions such as those of Truckenbrodt[13] for axially symmetric flow may also be used.

7. *Information Obtainable from the Streamwise Solution.*—7.1. *Skin-Friction Drag.*—The limiting streamlines are by definition lines making an angle β with the streamlines, where

$$\tan\beta = \lim_{\zeta=0}(v/u)$$
$$= \left(\frac{\partial v}{\partial \zeta}\right)_0 \Big/ \left(\frac{\partial u}{\partial \zeta}\right)_0.$$

These lines are in the direction of the resultant skin friction. Our assumption is that β is small so that the resultant skin friction does not differ much in direction from that of the external streamlines, and its magnitude is approximately equal to the streamwise component of skin friction. Hence for a wing at a small angle of incidence, whose streamlines are nearly parallel to the velocity at infinity, an approximation to the skin-friction drag may be found by integrating the local streamwise skin friction over the wing. This will be an upper limit to the skin-friction drag, but should give an improvement on the usual method of considering an equivalent flat plate at zero incidence.

7.2. *Displacement Surface.*—If the equation of the given surface is $\zeta = 0$, then there is a displacement surface $\zeta = \delta^*$. If one wishes to know the flow outside the boundary layer of the given body, one must replace the body by this displacement surface and calculate the inviscid flow about the body so deformed.

Lighthill[14] shows that the displacement thickness δ^* at any point is given by

$$\delta^* = \delta_{11} - \frac{1}{\rho_1 u_1 r}\frac{\partial}{\partial \eta}\int_0^s \rho_1 u_1\,\delta_{22}\,ds,$$

where

$$\delta_{11} = \frac{1}{\rho_1 u_1}\int_0^\infty (\rho_1 u_1 - \rho u)\,d\zeta, \qquad \delta_{22} = \frac{1}{\rho_1 u_1}\int_0^\infty \rho v\,d\zeta.$$

If we continue to suppose that v is small then we may say that the displacement thickness δ^* is approximately equal to the streamwise displacement thickness δ_{11}. Thus streamwise calculations will be sufficient to give a first approximation to the value of the displacement thickness. This may be sufficient to determine the effect of boundary-layer thickness on drag.

8. *Separation.*—In two dimensions and in axially symmetric flow separation is determined by the vanishing of the skin friction, and one is tempted to use the vanishing of the streamwise skin friction as the criterion here, as has sometimes been done by other workers in the field. This is incorrect. A full discussion of the matter is given by Maskell and Weber[15] and by Maskell[16]. Separation depends closely upon the direction of the limiting streamlines, and for this it is necessary to solve the cross-flow equation. However, separation usually involves considerable divergence of the limiting streamline direction from that of the external streamlines, so that β is not small and our fundamental assumption is violated, except perhaps in the case of very slender wings at low incidences.

At any rate separation cannot be found without solving the cross-flow equation, and little information about separation is obtainable from the streamwise solution.

9. *Conclusions.*—If the main assumption of small cross-flow can be justified, it will provide great simplification in the calculation of three-dimensional boundary-layer flow, because at least in laminar flow, the streamwise equation of motion, the energy equation and the equation of continuity reduce to standard equations in axially symmetric flow.

This result applies to compressible flow as well as incompressible. It seems also that the streamwise momentum integral equation in turbulent flow may reduce to that of axially symmetric turbulent flow.

No details of the equations for the cross-flow are given here, though the main assumption does simplify them somewhat.

Nevertheless, useful information may be gained from the streamwise solution, since skin friction and displacement thickness may be determined approximately from the streamwise equations.

It is necessary to solve the cross-flow equation in order to obtain information about separation; although the basic assumption will become less valid as separation is approached, sometimes the onset of separation is so rapid that there may be little error in determining separation using the basic assumption.

LIST OF SYMBOLS

$e_{\alpha\beta}$		Rate of strain components
$\boldsymbol{F}$		Vector body force
h_1, h_2		Curvilinear co-ordinate coefficients
H		Form parameter $= \delta_{11}/\theta_{11}$
dn	$=$	$r\,d\eta$
p		Pressure
r		Defined by equation (11)
ds		Arc length
t		Time
T		Temperature
u_1		Value of u just outside the boundary layer
u, v, w		Velocity components in the directions ξ, η, ζ
U_0		Velocity at infinity
$\bar{U}, \bar{V}, \bar{W}$		Velocity components in the directions x, y, z
$\boldsymbol{V}$		Velocity vector
z, y, z		Cartesian co-ordinates
β		Angle between streamlines and limiting streamlines
δ_{11}		Streamwise displacement thickness
δ_{22}		Cross-flow displacement thickness
θ_{11}		Streamwise momentum thickness
Θ	$=$	$\theta_{11}(u_1\,\theta_{11}/\nu)^{1/5}$
Δ	$=$	$\operatorname{div}\boldsymbol{V}$
δ^*		Displacement thickness
κ	$=$	$\dfrac{1}{h_1 r}\dfrac{\partial h_1}{\partial \eta}$
ξ, η, ζ		Curvilinear co-ordinates
μ		Coefficient of viscosity
ρ		Density
$\bar{\rho}$		Given by $r^2 = 1/\bar{\rho}u_1^2$
τ_{01}		Streamwise skin friction
suffix $_1$		Refers to conditions in the main stream
$'$		A prime attached to any quantity denotes its value by the transformation of equations (14)

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APPENDIX I

The Determination of r

If the surface is defined in a Cartesian co-ordinate system, x, y, z by the equation $z = z(x, y)$, with components $\bar{U}$, $\bar{V}$, $\bar{W}$, we shall have

$$\bar{W} = \bar{U} z_x + \bar{V} z_y, \tag{25}$$

suffixes denoting partial derivatives, since the surface is supposed impermeable.

We have

$$u_1^2 = \bar{U}^2 + \bar{V}^2 + \bar{W}^2,$$

and the direction cosines of the velocity vector, V, are

$$\frac{\bar{U}}{u_1}, \quad \frac{\bar{V}}{u_1}, \quad \frac{\bar{W}}{u_1}.$$

Hence we have

$$\frac{\partial}{\partial s} = \frac{\bar{U}}{u_1}\frac{\partial}{\partial x} + \frac{\bar{V}}{u_1}\frac{\partial}{\partial y} + \frac{\bar{W}}{u_1}\frac{\partial}{\partial z}$$

and so, using equation (25) we have

$$u_1 \frac{\partial}{\partial s} = \bar{U}\frac{\delta}{\delta x} + \bar{V}\frac{\delta}{\delta y},$$

where

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + z_x \frac{\partial}{\partial z}, \qquad \frac{\delta}{\delta y} = \frac{\partial}{\partial y} + z_y \frac{\partial}{\partial z}.$$

In order to find r, we evaluate $\operatorname{div} V$ in two ways, supposing that we are confining ourselves to the surface $z = z(x, y)$.

According to Weatherburn[17], for a vector which is everywhere tangential to the surface, as V is here

$$\operatorname{div} V = \sqrt{(g)}\, \operatorname{div}\left(\frac{V}{\sqrt{g}}\right), \tag{26}$$

where div represents the 'two-parameter' divergence in the co-ordinate system s, η, and

$$g = 1 + z_x^2 + z_y^2.$$

In the surface system the components of V are u_1, 0, and so

$$\sqrt{(g)}\, \operatorname{div}\left(\frac{V}{\sqrt{g}}\right) = \frac{\sqrt{g}}{r}\frac{\partial}{\partial s}\left(\frac{u_1 r}{\sqrt{g}}\right).$$

In the Cartesian system we have

$$\operatorname{div} V = \frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} + \frac{\partial \bar{W}}{\partial z}.$$

Bearing in mind that we are confining ourselves to the surface, for which equation (25) holds everywhere, we have

$$\operatorname{div} V = \frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{V}}{\delta y}, \tag{27}$$

and so the equation for r by (26) and (27) is

$$\frac{\sqrt{g}}{r}\frac{\partial}{\partial s}\left(\frac{u_1 r}{\sqrt{g}}\right) = \frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{V}}{\delta y},$$

or

$$u_1 \frac{\partial}{\partial s} \log\left(\frac{r^2 u_1^2}{g}\right) = 2\left(\frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{V}}{\delta y}\right).$$

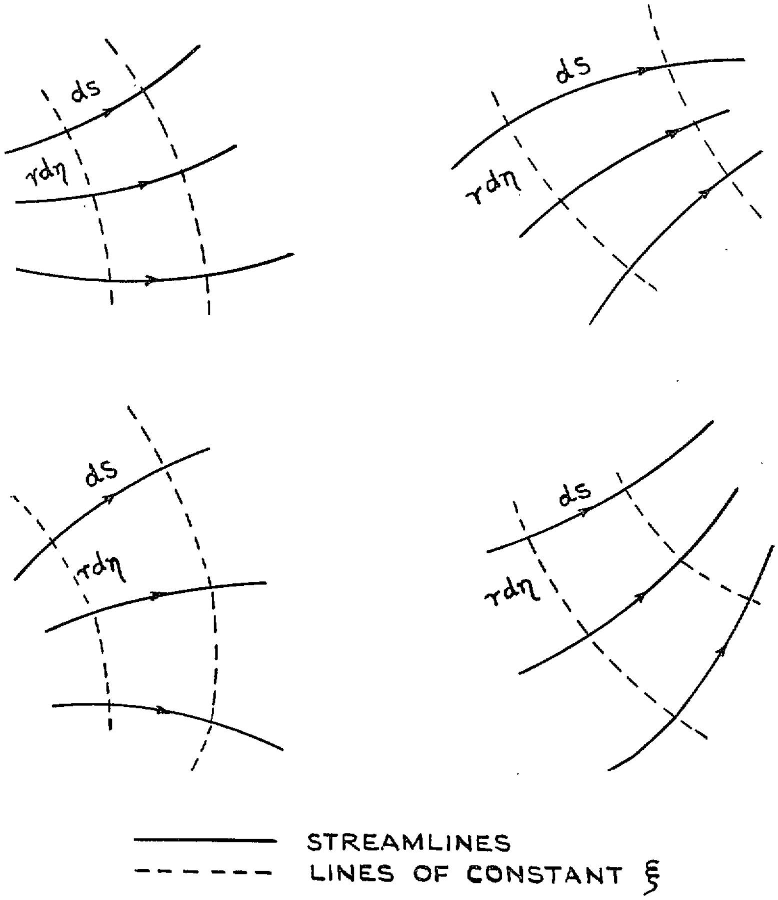

FIG. 1. Diverging and converging streamlines.

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