R. & M. No. 3182
(19,732)
A.R.C. Technical Report

R. & M. No. 3182

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

Vortex-Lattice Treatment of Rectangular Wings with Oscillating Control Surfaces

By

DORIS E. LEHRIAN, B.Sc.,
of the Aerodynamics Division, N.P.L.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1960
PRICE 9*s*. 6*d*. NET

Vortex-Lattice Treatment of Rectangular Wings with Oscillating Control Surfaces

By

DORIS E. LEHRIAN, B.SC.,
of the Aerodynamics Division, N.P.L.

Reports and Memoranda No. 3182*

December, 1957

Summary.—The vortex-lattice method for simple harmonic motion of general frequency (R. & M. 2961) is used to calculate the derivatives for rectangular wings with oscillating constant-chord flaps. The discontinuous chordwise boundary condition associated with full-span flaps, is replaced by a continuous equivalent downwash which is determined on the basis of two-dimensional oscillatory theory. In the particular case when the frequency tends to zero, the equivalent downwash is obtained on a distinct quasi-steady basis; stability derivatives are then evaluated by using an alternative form of the vortex-lattice method for low frequency (R. & M. 2922). To allow for the spanwise discontinuity due to outboard flaps, a further adjustment is made to the boundary condition by the use of partial-span downwash factors.

Comparison of the stability derivatives with values obtained by the Multhopp-Garner method, indicates that the present treatment for low frequency is satisfactory for full-span and outboard flaps on plan-forms of aspect ratio 2 and 4. For general frequencies, results for aspect ratio 2 with full-span flaps compare well with the values for lift and pitching-moment derivatives obtained by Lawrence and Gerber.

1. *Introduction.*—The development of vortex-lattice theory for wings in simple harmonic motion has provided simple routine methods which can be applied to general plan-forms in incompressible flow[1, 2]. In this report a vortex-lattice treatment for a wing with oscillating flaps is investigated, and the method is used to calculate derivatives for a rectangular plan-form with symmetrical full-span and outboard flaps. Apart from the limitations common to any method which is based on linearised theory, strict application of lifting-surface methods to the problem of deflected control surfaces is precluded by the discontinuities occurring in the boundary condition.

In the case of steady flow, various devices have been sought to overcome this difficulty. One procedure is to replace the discontinuous boundary condition by theoretically determined equivalent slopes. Falkner[3] and Multhopp[4] treat chordwise discontinuity at the hinge on a two-dimensional basis: Multhopp[4] then fairs the spanwise discontinuity whereas Falkner[5] and DeYoung[6] represent it as an equivalent continuous function with the aid of special spanwise loadings. Another method, developed by Brebner and Lemaire[7], is based on an analysis of electrolytic tank tests on swept-wings with flaps: this analysis provides three-dimensional data for the equivalent incidence and the spanwise loading.

---

* Published with the permission of the Director, National Physical Laboratory.

Since the vortex-lattice method economizes in collocation points, it can only be expected to give values of the overall forces on a wing with flaps. It would appear that the treatment should be kept relatively simple. The lift distribution is therefore represented by the usual Fourier series which is independent of the control-surface geometry, and the discontinuous boundary condition is replaced by a continuous one. Consideration is first given to a wing with full-span oscillating control (Section 3), and an adjustment is then made to allow for the spanwise discontinuity in the case of partial-span flaps. The discontinuities are thus treated independently and relate to the chordwise and spanwise disposition of the collocation points.

For a full-span control oscillating at general frequencies, the chordwise discontinuity in the boundary condition is replaced by a continuous function which is determined, on the basis of Jones's[8, 9] two-dimensional oscillatory theory, to give the same overall forces as an aerofoil with oscillating control (Section 3). The loading used in this analysis involves the oscillatory lift function $C(\omega)$; it would therefore be more appropriate to use the corresponding form of the vortex-lattice method due to Jones[10] rather than the method of Ref. 1. However, the latter is recommended since it leads to a more general routine with simpler computation. This should be satisfactory for non-zero up to moderately large frequencies, but it is thought that the method of Ref. 10 might give better results for the higher values of the frequency parameter, say $\nu \ll 1$.

When the frequency tends to zero, it is not possible to adapt the general treatment of the chordwise discontinuity because of the limiting behaviour of the function $C(\omega)$. Moreover, the use of two-dimensional oscillatory theory in conjunction with any chordwise loading (*e.g.*, Ref. 2), is found to be unsuitable. A distinct treatment using a quasi-steady approach is therefore suggested for the determination of the equivalent slopes.

For a rectangular wing with a partial-span constant-chord control surface, the spanwise discontinuity is treated on a steady theoretical basis. Downwash factors independent of frequency, are evaluated from integrated spanwise loadings which can conveniently be determined from low-aspect-ratio theory[6]. It is important to note that identical factors would be obtained by the use of classical lifting-line theory as suggested by Garner[11].

There is very little information relating to control-surface oscillations at general frequencies. Reissner and Stevens[12] calculated values of the derivatives for elliptic plan-forms only, although their method seems applicable to rectangular wings of moderate aspect ratio. Derivatives for low-aspect-ratio rectangular plan-forms with full-span flaps have been calculated by Lawrence and Gerber[13]. For low-frequency oscillations, the Multhopp-Garner method[14] has been applied to the control problem by using chordwise and spanwise equivalent slopes based on Refs. 4 and 6 respectively ; this work has not been published but values obtained by Garner for rectangular plan-forms are quoted for comparison. These results for low frequency and those of Ref. 13 for general frequency support the present vortex-lattice treatment.

2. *General Theory*.—The lift distribution $\rho V \Gamma \, e^{ipt}$ over the plan-form is represented by the usual finite series (Ref. 1)

$$\Gamma = V \sum_n \sum_m \Gamma_n C_{nm} A_m \, , \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

in which the distributions Γ_n are functions of the chordwise parameter θ and the frequency parameter ω, and the spanwise distributions A_m are defined by

$$cA_m = s\eta^{m-1} \sqrt{(1 - \eta^2)} \, , \qquad m = 1, 2 \ldots .$$

The downwash $W \, e^{ipt}$ at any point on the plan-form is then

$$W = V \sum_n \sum_m W_{nm} C_{nm} \, , \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

where the downwash $VW_{nm} \, e^{ipt}$ corresponds to a lift distribution $\rho V^2 \Gamma_n A_m \, e^{ipt}$ and is independent of the control-surface geometry.

For general frequencies, the vortex-lattice method of Ref. 1 can be used to calculate values of W_{nm}. In the present application to rectangular wings, the distribution Γ is limited to two chordwise distributions

$$\left.\begin{aligned} \Gamma_0 &= 2\cot\tfrac{1}{2}\theta \\ \Gamma_1 &= (-2\sin\theta + \cot\tfrac{1}{2}\theta) + i\omega(\tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta) \end{aligned}\right\} , \quad \ldots \quad \ldots \quad \ldots \qquad (3)$$

and to three symmetrical spanwise distributions A_1, A_3 and A_5. The arbitrary coefficients C_{nm} in equation (1) are then to be determined from equation (2) by collocation at six points which are placed on the 1/2 and 5/6 chord at spanwise positions $\eta_1 = 0{\cdot}2$, $0{\cdot}6$ and $0{\cdot}8$.

For a wing with partial-span controls describing symmetrical oscillations, the normal downward displacement of any point on the lifting surface is

$$\left.\begin{aligned} z &= 0 && \text{off the controls} \\ z &= (x - x_h)\xi\, e^{ipt} && \text{on the controls} \end{aligned}\right\} , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

where $x = x_h$ is the position of the control hinge-line and $\xi\, e^{ipt}$ is the angular displacement of the control. The tangential flow condition is

$$W e^{ipt} = \frac{\partial z}{\partial t} + V\frac{\partial z}{\partial x} ,$$

so that by equation (4), the downwash distribution in (2) is required to satisfy the boundary condition

$$\left.\begin{aligned} W &= 0 && \text{off the control} \\ \left.\begin{aligned} W &= \xi[V + ip(x - x_h)] \\ &= V\xi[1 + \tfrac{1}{2}i\omega(\cos\psi - \cos\theta)] \end{aligned}\right\} && \text{on the control} \end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots \qquad (5)$$

In order to obtain an adequate solution for a partial-span control surface by collocation, it is necessary to replace the discontinuous boundary condition (5) by a continuous one. The discontinuities in the chordwise and spanwise directions will be treated independently.

3. *Full-span Control Oscillating at General Frequency.*—The boundary condition (5) for a full-span control is discontinuous only in the chordwise direction. Furthermore, in the case of a constant-chord wing and control the condition is identical for all spanwise positions ; the same is true for the continuous boundary condition which is to replace (5). As already mentioned in Section 2, the vortex-lattice method is to be used with two chordwise terms in the lift distribution and therefore two chordwise positions for the collocation points. In such a solution the continuous boundary condition along each chord may be written as

$$W = V\xi W_E , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6)$$

where $$W_E = [a_0 + a_1(\tfrac{1}{2} + \cos\theta)], \qquad 0 \leqslant \theta \leqslant \pi .$$

The problem of replacing discontinuous chordwise boundary conditions due to deflected controls by continuous functions, has been considered for steady flow by Falkner[3] and Multhopp[4]. In both cases, equivalent slopes were determined on a two-dimensional basis to give the same overall characteristics, such as lift and pitching moment, as an aerofoil with deflected control. By an analogous treatment based on two-dimensional oscillatory theory (Ref. 8), a continuous equivalent downwash W_E may be determined for general frequencies.

It follows from the two-dimensional theory for an oscillating aerofoil, that the lift distribution $\rho V\Gamma\, e^{ipt}$ corresponding to the continuous downwash W_E of equation (6) is

$$\Gamma = V[a_0\Gamma_0 + a_1\Gamma_1] , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7)$$

where
$$\begin{aligned} \Gamma_0 &= 2C(\omega)\cot\tfrac{1}{2}\theta + i\omega\sin\theta \\ \Gamma_1 &= (-2\sin\theta + \cot\tfrac{1}{2}\theta) + i\omega(\tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta) . \end{aligned}$$

Then, the lift $-Z_E\,e^{ipt}$, pitching-moment about half-chord $M_E\,e^{ipt}$ and hinge-moment $H_E\,e^{ipt}$, which correspond to the continuous downwash W_E, are given by

$$-\frac{Z_E}{\pi\rho c V^2} = [C(\omega) + \tfrac{1}{4}i\omega]a_0 + [\tfrac{1}{8}i\omega]a_1\,, \quad \ldots \qquad (8a)$$

$$-\frac{M_E}{\pi\rho c^2 V^2} = -\tfrac{1}{4}[C(\omega)a_0 + \tfrac{1}{2}(1 + \tfrac{1}{8}i\omega)a_1]\,, \quad \ldots \qquad (8b)$$

$$-\frac{H_E}{\pi\rho c^2 V^2} = [C(\omega)X_{12} + i\omega X_1]a_0 + \tfrac{1}{2}[(X_{12} - 4X_1) + i\omega(X_1 - 4X_2)]a_1\,, \quad \ldots \qquad (8c)$$

where the functions X_1, $X_2 \ldots X_{12}$ are defined in Appendix I and depend only on the control parameter ψ ; values of these functions and of the oscillatory lift function $C(\omega)$ are tabulated in Ref. 9.

The aerodynamic forces on a two-dimensional aerofoil with a flap describing oscillations of unit amplitude are determined in Ref. 8, but for present purposes it is more convenient to use the following formulae from Ref. 9 :

$$-\frac{Z}{\pi\rho c V^2} = C(\omega)[X_{10} + i\omega X_{11}] + i\omega X_4 - \omega^2 X_1\,, \quad \ldots \qquad (9a)$$

$$-\frac{M}{\pi\rho c^2 V^2} = -\tfrac{1}{4}C(\omega)[X_{10} + i\omega X_{11}] + X_8 + i\omega X_5 - \omega^2 X_2\,, \quad \ldots \qquad (9b)$$

$$-\frac{H}{\pi\rho c^2 V^2} = C(\omega)X_{12}[X_{10} + i\omega X_{11}] + X_9 + i\omega X_7 - \omega^2 X_3\,, \quad \ldots \qquad (9c)$$

where the functions X_1, X_2, . . . X_{12} are defined in Appendix I.

The unknown coefficients a_0 and a_1 of the continuous equivalent downwash W_E in (6), can be determined for particular values of the frequency parameter ω and the flap ratio $E = \frac{1}{2}(1 + \cos\psi)$ by equating any two of the equations (8) to the corresponding two equations (9). It is suggested that the equivalent downwash W_E obtained by satisfying the lift and pitching-moment equations of (8) and (9), should be used in the finite-wing solution in order to evaluate the derivatives for lift and pitching-moment, while the equivalent downwash $W_E \equiv W_E^*$ obtained from the lift and hinge-moment equations of (8) and (9) should be used to evaluate the hinge-moment derivatives. Some sets of values of W_E and W_E^* are given in Table 1.

4. *Full-span Control Oscillating at Low Frequency.*—In the case of low-frequency oscillations, it is not possible to use equations (8) and (9) of the previous Section because of the $\omega \log \omega$ term inherent in the two-dimensional oscillatory lift function $C(\omega)$. However, since only first-order terms in frequency are retained in the finite-wing solution for $\nu \to 0$, the continuous boundary condition (6) may be expressed as

$$W = V\xi W_E = V\xi[\alpha_{1E} + i\omega\alpha_{2E}]\,, \quad \ldots \qquad (10)$$

where

$$\left.\begin{aligned} \alpha_{1E} &= a_0' + a_1'(\tfrac{1}{2} + \cos\theta) \\ \alpha_{2E} &= a_0'' + a_1''(\tfrac{1}{2} + \cos\theta) \end{aligned}\right\} \quad 0 \leqslant \theta \leqslant \pi\,.$$

Now, the continuous functions α_{1E} and α_{2E} can be determined on a quasi-steady basis by treating the real and imaginary parts of the discontinuous boundary condition (5) as independent conditions. Thus equation (5) is written as

$$W = V\xi[\alpha_1 + i\omega\alpha_2]\,, \quad \ldots \qquad (11)$$

where

$$\left.\begin{aligned} \alpha_1 &= 0 \qquad & 0 \leqslant \theta \leqslant \psi \\ &= 1 \qquad & \psi \leqslant \theta \leqslant \pi \end{aligned}\right\} \quad \ldots \qquad (12)$$

and
$$\left.\begin{aligned} \alpha_2 &= 0 & 0 \leqslant \theta \leqslant \psi \\ &= \tfrac{1}{2}(\cos\psi - \cos\theta) & \psi \leqslant \theta \leqslant \pi \end{aligned}\right\} \tag{13}$$

The continuous equivalent downwashes α_{1E} and α_{2E} are determined independently to give the same overall characteristics as the discontinuous boundary conditions α_1 and α_2 in two-dimensional steady flow.

The quantities Z_E, M_E, H_E corresponding to the continuous downwash α_{1E} in (10) are obtained by substituting a_0' and a_1' for a_0 and a_1 in equations (8) and putting $\omega = 0$. Hence

$$Z_E = -\pi\rho c V^2[a_0'] \,, \tag{14a}$$

$$M_E = \tfrac{1}{4}\pi\rho c^2 V^2[a_0' + \tfrac{1}{2}a_1'] \,, \tag{14b}$$

$$H_E = -\pi\rho c^2 V^2[X_{12}a_0' + \tfrac{1}{2}(X_{12} - 4X_1)a_1'] \,. \tag{14c}$$

Equation (12) expresses the boundary condition α_1 for an aerofoil with deflected flap in steady flow. The corresponding aerodynamic forces, obtained by substituting $\omega = 0$ in equations (9), are

$$Z = -\pi\rho c V^2[X_{10}] \,, \tag{15a}$$

$$M = \tfrac{1}{4}\pi\rho c^2 V^2[X_{10} - 4X_8] \,, \tag{15b}$$

$$H = -\pi\rho c^2 V^2[X_{10}X_{12} + X_9] \,. \tag{15c}$$

Then the equivalent downwash α_{1E} obtained by satisfying the lift and pitching-moment equations of (14) and (15) is

$$\alpha_{1E} = X_{10} - 8X_8(\tfrac{1}{2} + \cos\theta) \,, \tag{16}$$

while satisfying the lift and hinge-moment equations of (14) and (15) gives

$$\alpha_{1E}^{*} = X_{10} + \left(\frac{2X_9}{X_{12} - 4X_1}\right)(\tfrac{1}{2} + \cos\theta) \,. \tag{17}$$

The values of α_{1E} and α_{1E}^{*} evaluated from (16) and (17) for any particular value of the control ratio E, will be the same as the values of the equivalent slopes which are given in Ref. 3 for two chordwise terms.

The continuous downwash α_{2E} defined in (10) is of the same form as α_{1E}; therefore the corresponding quantities Z_E, M_E and H_E are given by equations (14) with $a_0' = a_0''$ and $a_1' = a_1''$. The lift distribution $\rho V\Gamma$ which corresponds to the discontinuous boundary condition α_2 of equation (13) is determined in Appendix II by two-dimensional steady theory; then integration of equation (39) gives the aerodynamic forces

$$Z = -\pi\rho c V^2[X_{11}] \,, \tag{18a}$$

$$M = \tfrac{1}{4}\pi\rho c^2 V^2[X_{11} - \tfrac{1}{2}X_4 - 2X_5] \,, \tag{18b}$$

$$H = -\pi\rho c^2 V^2[X_{11}\,X_{12} + X_4^2] \,. \tag{18c}$$

It follows from equations (14) and (18) that the equivalent downwash

$$\alpha_{2E} = X_{11} - (X_4 + 4X_5)(\tfrac{1}{2} + \cos\theta) \tag{19}$$

gives the same lift and pitching moment as α_2, whereas the equivalent downwash

$$\alpha_{2E}^{*} = X_{11} + \left(\frac{2X_4^2}{X_{12} - 4X_1}\right)(\tfrac{1}{2} + \cos\theta) \tag{20}$$

gives the same lift and hinge moment as α_2.

The equivalent downwashes for $\omega \to 0$ as defined by equations (16), (17), (19) and (20) can be evaluated by using the values of the functions X_1, X_2 . . . X_{12} which are tabulated in Ref. 9. However, when the flap ratio $E = \frac{1}{2}(1 + \cos \psi)$ is small, there may not be enough significant figures and it is then better to work with the formulae given in Appendix I. Some sets of α_{1E}, α_{2E} and α^*_{1E}, α^*_{2E} are given in Table 2. Their application is discussed in detail in Section 7.

For low frequency, the vortex-lattice method of Ref. 2 may be applied but the choice of chordwise distributions Γ_0 and Γ_1, as in equation (3), is not consistent with the quasi-steady basis on which W_E of equation (10) has been determined. Therefore the alternative method described in Appendix III is used, so that all the chordwise lift distributions Γ_n are independent of frequency. This alternate distribution facilitates the application of the method to hinge-moment derivatives (Section 7). Thus, in the present application to rectangular wings, the chordwise distributions $\Gamma_0 = 2 \cot \frac{1}{2}\theta$ and $\Gamma_1 = (-2 \sin \theta + \cot \frac{1}{2}\theta)$ are used in the solutions for $\nu \to 0$; the spanwise loading and the position of the collocation points are the same as those given in Section 2 for the general frequency solutions.

5. *Partial-span Controls.*—The spanwise discontinuity in the boundary condition for a partial-span control is treated by steady-flow theory so as to give a continuous spanwise function which produces the same overall forces such as lift and rolling moment. The chordwise discontinuity has already been dealt with, by either Sections 3 or 4, so that in the spanwise direction

$$\left.\begin{aligned} W &= 0 \qquad & 0 \leqslant |\eta| \leqslant \eta_a \\ &= V\xi W_E \qquad & \eta_a \leqslant |\eta| \leqslant 1 \end{aligned}\right\} . \tag{21}$$

Provided that both the plan-form and the control are of constant chord, the equivalent downwash $W_E(\theta)$ is independent of the spanwise parameter η. Then the continuous boundary condition will be of the form

$$W = V\xi W_E(\theta)F(\eta) , \tag{22}$$

where the downwash factor $F(\eta)$ will now be determined for symmetrically oscillating partial-span controls.

Since only three collocation positions η_1 are to be used in the present finite-wing solutions, the factor F is taken as

$$F(\eta) = (b_0 + b_2\eta^2 + b_4\eta^4) . \tag{23}$$

The arbitrary coefficients b_0, b_2, b_4 are to be chosen so that three selected integrals are numerically exact. Garner[11] has shown that the application of either classical lifting-line theory or DeYoung's low-aspect-ratio theory[6] will lead to identical downwash factors. For convenience the latter method will be used in the following analysis.

The spanwise load distribution due to the downwash $F(\eta)$ of equation (23) can be expressed as $2\rho V^2 s\gamma(\eta)$, where

$$\gamma(\eta) = 2\pi[d_0 + d_2\eta^2 + d_4\eta^4] \sqrt{(1 - \eta^2)} . \tag{24}$$

By Ref. 6, the downwash is

$$\frac{W}{V} = F(\eta) = \frac{1}{\pi}\int_{-1}^{1} \frac{d\gamma(\eta')}{d\eta'} \left(\frac{1}{\eta - \eta'}\right) d\eta' , \tag{25}$$

so that the downwash corresponding to (24) is

$$F(\eta) = \pi[2d_0 + d_2(6\eta^2 - 1) + d_4(10\eta^4 - 3\eta^2 - \tfrac{1}{4})] . \tag{26}$$

The lift, first-moment, second-moment and partial-span integrals corresponding to (24) are respectively

$$I_0 = \int_0^1 \gamma \, d\eta = \frac{\pi^2}{2}\left[d_0 + \frac{1}{4}d_2 + \frac{1}{8}d_4\right], \tag{27a}$$

$$I_1 = \int_0^1 \gamma\eta \, d\eta = \frac{2\pi}{3}\left[d_0 + \frac{2}{5}d_2 + \frac{8}{35}d_4\right], \tag{27b}$$

$$I_2 = \int_0^1 \gamma\eta^2 \, d\eta = \frac{\pi^2}{8}\left[d_0 + \frac{1}{2}d_2 + \frac{5}{16}d_4\right], \tag{27c}$$

and

$$I_a = \int_{\eta_a}^1 \gamma \, d\eta = 2\pi[d_0 J_0 + d_2 J_2 + d_4 J_4], \tag{27d}$$

where

$$J_r = \int_{\eta_a}^1 \eta^r \sqrt{(1 - \eta^2)} \, d\eta .$$

It remains to evaluate the integrals, as defined in (27), which correspond to the exact solution for the discontinuous boundary condition

$$\left.\begin{aligned} W &= 0 , \qquad & 0 \leqslant |\eta| \leqslant \eta_a \\ &= V , \qquad & \eta_a \leqslant |\eta| \leqslant 1 \end{aligned}\right\} . \tag{28}$$

By Ref. 6, the load distribution corresponding to (28) is given by

$$\gamma = f(\phi, \phi_a) + f(\pi - \phi, \phi_a) , \tag{29}$$

where

$$f(\phi, \phi_a) = \frac{1}{\pi}\left[\phi_a \sin\phi + (\cos\phi - \cos\phi_a) \ln\left(\frac{\sin\frac{1}{2}(\phi + \phi_a)}{\sin\frac{1}{2}|\phi - \phi_a|}\right)\right],$$

$$\eta = \cos\phi \text{ and } \eta_a = \cos\phi_a .$$

Substitution of γ from (29) into the integrals (27) gives

$$2I_0 = [\phi_a - \sin\phi_a \cos\phi_a] , \tag{30a}$$

$$3\pi I_1 = \left[2\phi_a - \sin\phi_a \cos\phi_a - \tfrac{1}{2}\cos^3\phi_a \ln\left(\frac{1 + \sin\phi_a}{1 - \sin\phi_a}\right)\right], \tag{30b}$$

$$8I_2 = [\phi_a - \tfrac{1}{3}\sin\phi_a \cos\phi_a(1 + 2\cos^2\phi_a)] , \tag{30c}$$

$$\pi I_a = [\phi_a{}^2 - 2\phi_a \sin\phi_a \cos\phi_a - 2\cos^2\phi_a \ln\cos\phi_a] . \tag{30d}$$

The downwash factor $F(\eta)$ as defined by (26) can therefore be determined for any value of η_a, by equating three of the integrals which are given in equations (27) to the corresponding integrals of equation (30). For the particular values $\eta_a = 0\cdot342020$, $0\cdot5$ and $0\cdot766044$, the arbitrary coefficients d_0, d_2, d_4 in Table 3 (*a*) and d_0^*, d_2^*, d_4^* in Table 3(*b*) are obtained by satisfying respectively

(*a*) the equations for I_0, I_1, I_2,

(*b*) the equations for I_0, I_1, I_a.

The use of the three equations (*b*) leads to a singular matrix and no solution for the particular value $\eta_a \simeq 0\cdot535$, and gives ill-conditioned solutions in the neighbourhood of this value ; the solution (*b*) for $\eta_a = 0\cdot5$ tends to be ill-conditioned. It seems advisable to avoid this limitation by using the three equations (a) which are independent of η_a. The solution of equations (a) can be expressed generally in matrix notation as

$$\begin{Bmatrix} d_0 \\ d_2 \\ d_4 \end{Bmatrix} = \frac{1}{\pi^2}\begin{bmatrix} 4\cdot8 & -3\cdot5 & 3\cdot2 \\ -37\cdot6 & 42\cdot0 & -46\cdot4 \\ 44\cdot8 & -56\cdot0 & 67\cdot2 \end{bmatrix}\begin{Bmatrix} 2I_0 \\ 3\pi I_1 \\ 8I_2 \end{Bmatrix}, \qquad \ldots \quad (31)$$

where { } denotes a column matrix and I_0, I_1 and I_2 are obtained from equations (30).

The downwash factors are required in the finite-wing solution at the collocation positions $\eta_1 = 0\cdot2$, $0\cdot6$ and $0\cdot8$. The values $F(\eta_1)$ and $F^*(\eta_1)$ corresponding to solutions (a) and (b) are evaluated not from equation (26), but from the formulae given in Table 3 which are obtained from a 21×1 vortex-lattice integration of equation (25). Thus the downwash factors in Tables 3(a) and 3(b) incorporate a correction which is consistent with the use of vortex-lattice theory for the finite-wing solution. The values $F(\eta_1)$ and $F^*(\eta_1)$ will be referred to as partial-span factors.

6. *Results.*—The treatment described in Sections 2 to 5 is applied to rectangular plan-forms of aspect ratio $A = 4$ and $A = 2$ with oscillating full-span and outboard flaps. Values of the derivatives for lift, pitching moment about the leading edge and hinge-moment are given in Tables 4 and 5. These derivatives are calculated by the vortex-lattice method with a 21×6 lattice and six collocation points as defined in Section 2. The solutions for particular values of the frequency parameter ν are obtained by using the equivalent downwashes $W_E(\theta_1)$ and $W_E^*(\theta_1)$ given in Table 1 for flap-chord ratios $E = 0\cdot08$ and $0\cdot25$. To obtain solutions for low frequency $\nu \rightarrow 0$, the quantities α_{1E}, α_{2E} and α_{1E}^*, α_{2E}^* from Table 2 are used as discussed in Section 7. For all frequencies, partial-span flaps are represented by partial-span factors $F(\eta_1)$ which are tabulated in Table 3(a).

The rectangular wing $A = 4$ is considered with full-span flaps ($E = 0\cdot08$ and $0\cdot25$) oscillating at low frequency and $\nu = 0\cdot2$ and $0\cdot6$. Derivatives are also obtained for this wing with outboard flaps ($E = 0\cdot25$, $\eta_a = 0\cdot5$) oscillating at the same frequencies. These results are given in Table 4 together with derivatives for the rectangular wing $A = 2$ with full-span flaps ($E = 0\cdot25$) oscillating at low frequency and $\nu = 0\cdot2$ and $1\cdot2$. Derivatives for $A = 4$ at low frequency are also tabulated for different values of η_a in Table 5.

The lift, pitching moment and hinge-moment derivatives for the flap-chord ratio $E = 0\cdot25$ are plotted against ν in Figs. 1 and 2. No general conclusions can be drawn from so few results. Nevertheless, the effect of frequency is not large and appears to diminish with decreasing flap-span (Fig. 1) and with decreasing aspect ratio (Fig. 2). For low frequency $\nu \rightarrow 0$, the derivatives for the wing $A = 4$ with outboard flaps are plotted against η_a in Fig. 3 ; similar curves are obtained for the flap-chord ratios $E = 0\cdot08$ and $E = 0\cdot25$.

Molyneux and Ruddlesden[15] have measured the forces on a rectangular wing $A = 4\cdot05$ with full-span control $E = 0\cdot2$; over the frequency-parameter range $0\cdot2 < \nu < 1\cdot3$, Fig. 16 of Ref. 15 gives the hinge-moment derivative values $-h_{\dot\xi} = 0\cdot22$† and $-h_{\xi} = 0\cdot12$. These are respectively 40 per cent and 12 per cent below the values obtained by interpolation from the vortex-lattice results for $\nu \rightarrow 0$ in Table 5. Such differences may be expected due to wing thickness and effects of Reynolds number.

7. *Accuracy and Application of the Method.*—As an initial investigation it seemed advisable to compare the result of using partial-span factors $F(\eta_1)$ based on lift, first and second moment instead of the factors $F^*(\eta_1)$ based on lift, first moment and hinge-moment. Of the derivatives thus evaluated for the plan-form $A = 4$ with outboard flaps in steady flow, the lift and pitching-moment values are in good agreement, but the hinge-moment values show progressively larger

---

† This value does not include the aerodynamic inertia term.

differences as η_a increases. These two sets of results are given in Table 6 together with values calculated by an extension of the Multhopp-Garner theory[14] with 15 spanwise and 2 chordwise terms. Comparison with the latter results indicates that the solutions using the factors $F^*(\eta_1)$ are more reliable. Thus the discrepancies in h_ξ in Fig. 3, for outboard flaps of $\eta_a = 0{\cdot}766$, are halved if $F^*(\eta_1)$ is used in place of $F(\eta)_1$. This is to be expected, since substitution of the values of the coefficients d_0, d_2, d_4 into the hinge-moment equation (27*d*), does not give a good approximation to the exact hinge-moment equation (30*d*) for the larger values of η_a. For most practical values of η_a, however, the partial-span factors $F(\eta_1)$ are preferable since the factors $F^*(\eta_1)$ cannot be obtained in the neighbourhood of $\eta_a = 0{\cdot}535$ (Section 5). In view of this initial investigation the factors $F(\eta_1)$ were used for all the solutions given in Tables 4 and 5.

The accuracy of the derivatives for non-zero values of the frequency parameter ν cannot be fully assessed. The only results available for comparison are the lift and pitching-moment derivatives obtained by Lawrence and Gerber[13] for the wing $A = 2$ with full-span flaps. However, Fig. 2 shows good agreement between these values and the present vortex-lattice results.

The application of the method for low frequency $\nu \rightarrow 0$ is now considered in some detail. Initially, the lift and pitching-moment derivatives were obtained by using the equivalent slopes α_{1E} and α_{2E} based on lift and moment, whilst the hinge-moment derivatives were calculated by using throughout the slopes α^*_{1E} and α^*_{2E} based on lift and hinge-moment. As a check, the latter solution was also used to calculate the lift derivatives for the wing $A = 4$ with full-span flaps. Although satisfactory values were obtained for the chord ratio $E = 0{\cdot}25$, in the case of $E = 0{\cdot}08$ the two values for $-z_{\dot\xi}$ differed by a factor of $2\frac{1}{2}$. Furthermore, for $E = 0{\cdot}08$ the hinge-moment derivative $-h_{\dot\xi} = 0{\cdot}016$ was appreciably different from the value $-h_{\dot\xi} = 0{\cdot}051_5$ obtained by means of Ref. 14. However, the hinge-moment derivative $-h_\xi = 0{\cdot}385$ compared satisfactorily with the Multhopp-Garner value $-h_\xi = 0{\cdot}390$. Solutions for the wing $A = 4$ with half-span outboard flaps showed similar differences for $E = 0{\cdot}08$, but were again satisfactory for $E = 0{\cdot}25$.

It is useful here to state the form which the $\nu \rightarrow 0$ solution takes in the case of a constant-chord wing and control. The lift distribution is given by equation (1) with distributions Γ_n as defined in Appendix III, and the arbitrary coefficients C_{nm} are determined by solving the matrix equation

$$[A + i\nu B]\,\{C_{nm}\} = \{(\alpha_{1E} + i\nu\alpha_{2E})F(\eta_1)\}\,,$$

where $[A + i\nu B]$ is the matrix of downwash values W_{nm} at the collocation points, and the right-hand column matrix corresponds to the general case of partial-span flaps. Then, a solution to first order in frequency is given by

$$\{C_{nm}\} = A^{-1}\{\alpha_{1E}F(\eta_1) + i\nu(\alpha_{2E}F(\eta_1) - \alpha_3)\}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where A^{-1} is the inverse matrix of A, and

$$\{\alpha_3\} = BA^{-1}\{\alpha_{1E}F(\eta_1)\}\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

The use of equivalent chordwise downwashes α_{1E} and α_{2E}, as defined by equations (16) and (19), in the solution for the lift and pitching-moment derivatives is supported by the Multhopp-Garner results in Table 5.

The solutions for the hinge-moment derivatives which are discussed above, were obtained by using the values α^*_{1E} and α^*_{2E} from (17) and (20) in equations (32) and (33). In view of the large discrepancies in the damping derivatives for $E = 0{\cdot}08$, some modification to the imaginary part of the solution was then considered. Even though, α_1 is discontinuous, the column matrix $A^{-1}\{\alpha_1\}$ represents a continuous loading ; it can therefore be argued that $\{\alpha_3\}$ in equations (32) and (33) should be independent of the forces and moments to be evaluated. It is relevant to note that for the particular value $E = 0{\cdot}25$, α^*_{1E} is numerically equal to α_{1E} and the hinge-moment solutions are satisfactory. Therefore, in the hinge-moment solutions for $A = 4$ with full-span flaps $E = 0{\cdot}08$, the equivalent downwash α_{1E} was used in equation (33) instead of α^*_{1E}. Thus modified, the solution both checks the accepted value of $-z_{\dot\xi}$ and gives $-h_{\dot\xi} = 0{\cdot}054$ which is in satisfactory agreement with the Multhopp-Garner value $-h_{\xi} = 0 \cdot 051_5$. Similar improvements are obtained in the case of half-span flaps. Hence, the solution

$$\left.\begin{aligned} \{C_{nm}\} &= A^{-1}\{\alpha^*_{1E}F(\eta_1) + i\nu(\alpha^*_{2E}F(\eta_1) - \alpha_3)\} \\ \{\alpha_3\} &= BA^{-1}\{\alpha_{1E}F(\eta_1)\} \end{aligned}\right\} \quad \ldots \quad (34)$$

is adopted for the calculation of all the hinge-moment derivatives for low frequency in Tables 4 and 5.

In conclusion, the present application of the vortex-lattice treatment to rectangular wings with symmetrically oscillating constant-chord flaps appears satisfactory for general frequencies and gives results for low frequency in reasonable agreement with the Multhopp-Garner values. The method can be applied directly to constant-chord swept wings with flaps of constant E. Extension to the general case of a swept tapered wing with controls of arbitrary shape oscillating at any frequency, would present considerable difficulty. For low frequency, however, the treatment can be extended readily to a swept tapered wing with flaps of constant E; by further modifications to the partial-span factors, it should be possible to treat the case of E variable along the span. It would generally be advisable to use three chordwise and extra spanwise collocation points, and the equivalent downwashes $W_E(\theta)$ and the partial-span factors $F(\eta_1)$ may easily be determined for an arbitrary number of collocation positions by an extension of the procedures used in Sections 3, 4 and 5. Compressibility effects for oscillations of general frequency cannot be determined by the vortex-lattice method (Ref. 16), but it would be possible to obtain derivatives for low frequency at subsonic Mach number by applying the present treatment to a wing and control surface of reduced plan-form.

Acknowledgements.—The numerical results given in this report were calculated by Mrs. S. Lucas and Miss B. Burnham of the Aerodynamics Division.

NOTATION

A	Aspect ratio $[= 2s/c]$
c	Chord of rectangular plan-form
$C(\omega)$	Two-dimensional oscillatory lift function (Ref. 9)
E	Control chord/chord c, $[= \frac{1}{2}(1 + \cos\psi)]$
$F(\eta_1)$, $F^*(\eta_1)$	Partial-span downwash factors (Section 5)
$p/2\pi$	Frequency of oscillation of control-surface
s	Semi-span of plan-form
V	Velocity of undisturbed flow
$W\,e^{ipt}$	Downward velocity at the plan-form
$W_E(\theta)$, $W_E^*(\theta)$	Continuous equivalent downwashes (Section 3)
x, y, z	Rectangular co-ordinates: x in the stream direction with $x = 0$ at leading edge; y in the spanwise direction, positive to starboard; z positive downwards
x_h	Value of x at the control hinge
$X_1, X_2 \ldots X_{12}$	Functions of ψ (Appendix I)
y_a	Value of y at inboard edge of partial-span control
α_{1E}, α_{2E}	Defined by $W_E(\theta) = \alpha_{1E} + i\omega\alpha_{2E}$, for $\omega \to 0$ (Section 4)
α_{1E}^*, α_{2E}^*	Defined by $W_E^*(\theta) = \alpha_{1E}^* + i\omega\alpha_{2E}^*$, for $\omega \to 0$ (Section 4)
$\Gamma\,e^{ipt}$	Lift distribution/ρV
η	Spanwise parameter $[= y/s]$
η_a	Value of η at inboard edge of partial-span control
θ	Local chordwise parameter defined as $x = \frac{1}{2}c(1 - \cos\theta)$, $(0 \leqslant \theta \leqslant \pi)$
ν	Frequency parameter of plan-form $[= pc/V]$
$\xi\,e^{ipt}$	Angular displacement of control in a plane $y =$ const
ϕ	Spanwise parameter $[= \cos^{-1}\eta]$
ψ	Value of θ at the control hinge
ω	Local frequency parameter $[= pc/V]$

Definitions used in the two-dimensional analysis for $W_E(\theta)$:

$$\text{Lift} = -Z\,e^{ipt} = \tfrac{1}{2}\rho Vc\int_0^{\pi} \Gamma\,e^{ipt}\sin\theta\,d\theta$$

$$\text{Pitching moment} = M\,e^{ipt} = \tfrac{1}{4}\rho Vc^2\int_0^{\pi} \Gamma\,e^{ipt}\cos\theta\sin\theta\,d\theta \qquad \text{(about mid-chord)}$$

$$\text{Hinge moment} = H\,e^{ipt} = -\tfrac{1}{4}\rho Vc^2\int_{\psi}^{\pi} \Gamma\,e^{ipt}(\cos\psi - \cos\theta)\sin\theta\,d\theta$$

NOTATION—*continued*

Definitions used in the spanwise analysis for $F(\eta_1)$:

Load distribution $= 2\rho V^2 s\gamma$

Lift integral $= I_0 = \int_0^1 \gamma \, d\eta$

First-moment integral $= I_1 = \int_0^1 \gamma\eta \, d\eta$

Second-moment integral $= I_2 = \int_0^1 \gamma\eta^2 \, d\eta$

Partial-span integral $= I_a = \int_{\eta_a}^1 \gamma \, d\eta$

$F^*(\eta_1)$ replaces $F(\eta_1)$ when I_a is used instead of I_2.

Definition of derivatives for rectangular plan-form with symmetrical constant-chord outboard controls :

$$\frac{Z}{\rho V^2 S} = (z_\xi + i\nu z_{\dot{\xi}})\xi$$

$$\frac{M}{\rho V^2 Sc} = (m_\xi + i\nu m_{\dot{\xi}})\xi$$

$$\frac{H}{\rho V^2 S_f c_f} = (h_\xi + i\nu h_{\dot{\xi}})\xi$$

S	Area of plan-form $[= 2sc]$
c_f	Chord of control $[= Ec]$
S_f	Area of one control $[= c_f(1 - \eta_a)s]$
$-Z\,\mathrm{e}^{ipt}$	Lift $= \int_{-s}^{s} \int_0^c \rho V \Gamma \,\mathrm{e}^{ipt}\, dx\, dy$
$M\,\mathrm{e}^{ipt}$	Pitching moment about leading edge $= -\int_{-s}^{s} \int_0^c \rho V \Gamma \,\mathrm{e}^{ipt}\, x\, dx\, dy$
$H\,\mathrm{e}^{ipt}$	Hinge moment on one control $= -\int_{y_a}^{s} \int_{x_h}^{c} \rho V \Gamma \,\mathrm{e}^{ipt}\, (x - x_h)\, dx\, dy$

REFERENCES

No.	*Author*	*Title, etc.*
1	D. E. Lehrian	Calculation of flutter derivatives for wings of general plan-form. R. & M. 2961. January, 1954.
2	D. E. Lehrian	Calculation of stability derivatives for oscillating wings. R. & M. 2922. February, 1953.
3	V. M. Falkner	The use of equivalent slopes in vortex-lattice theory. R. & M. 2293. March, 1946.
4	H. Multhopp	Methods for calculating the lift distribution of wings (subsonic lifting-surface theory). R. & M. 2884. January, 1950.
5	V. M. Falkner	Tables of Multhopp and other functions for use in lifting-line and lifting-plane theory. With an Appendix by E. J. Watson. R. & M. 2593. February, 1948.
6	J. DeYoung	Spanwise loading for wings and control surfaces of low aspect ratio. N.A.C.A. Tech. Note 2011. January, 1950.
7	G. G. Brebner and D. A. Lemaire	The calculation of the spanwise loading of sweptback wings with flaps or all-moving tips at subsonic speeds. A.R.C. 18,273. September, 1955.
8	W. P. Jones	Aerodynamic forces on an oscillating aerofoil-aileron-tab combination. R. & M. 1948. September, 1941.
9	W. P. Jones	Summary of formulae and notations used in two-dimensional derivative theory. R. & M. 1958. August, 1941.
10	W. P. Jones	The calculation of aerodynamic derivative coefficients for wings of any plan-form in non-uniform motion. R. & M. 2470. December, 1946.
11	H. C. Garner	Note on the theoretical treatment of partial-span control surfaces in subsonic flow. (Unpublished.)
12	E. Reissner and J. E. Stevens	Effect of finite span on the airload distributions for oscillating wings. II.—Methods of calculation and examples of application. N.A.C.A. Tech. Note 1195. October, 1947.
13	H. R. Lawrence and E. H. Gerber	The aerodynamic forces on low-aspect-ratio wings oscillating in an incompressible flow. *J. Ae. Sci.* Vol. 19. p. 769. November, 1952.
14	H. C. Garner	Multhopp's subsonic lifting-surface theory of wings in slow pitching oscillations. R. & M. 2885. July, 1952.
15	W. G. Molyneux and F. Ruddlesden	Derivative measurements and flutter tests on a rectangular wing with a full-span control surface, oscillating in modes of wing roll and aileron rotation. R. & M. 3010. February, 1955.
16	D. E. Lehrian	Calculated derivatives for rectangular wings oscillating in compressible subsonic flow. R. & M. 3068. July, 1956.

APPENDIX I

Trigonometrical Relations for $X_n(\psi)$

The functions X_n, $n = 1, 2 \ldots 5$; $7, 8 \ldots 12$, which are used in Sections 3 and 4, are defined in Ref. 9 and can be expressed as follows in terms of the control parameter ψ :

$$24\pi X_1 = 3(\pi - \psi + \sin\psi\cos\psi)\cos\psi + 2\sin^3\psi$$

$$384\pi X_2 = 3(\pi - \psi + \sin\psi\cos\psi) + 2\sin^3\psi\cos\psi$$

$$128\pi^2 X_3 = 9(\pi - \psi + \sin\psi\cos\psi)^2 - 4\sin^2\psi[2(\pi - \psi)^2 + (\pi - \psi)\sin\psi\cos\psi - \sin^2\psi]$$

$$4\pi X_4 = \pi - \psi + \sin\psi\cos\psi$$

$$48\pi X_5 = 3(\pi - \psi + \sin\psi\cos\psi) + 4\sin^3\psi$$

$$4\pi X_7 = (\pi - \psi + \sin\psi\cos\psi)X_{11}$$

$$4\pi X_8 = \sin\psi(1 - \cos\psi)$$

$$4\pi^2 X_9 = \sin\psi(1 - \cos\psi)(\pi - \psi - \sin\psi)$$

$$\pi X_{10} = \pi - \psi + \sin\psi$$

$$4\pi X_{11} = (\pi - \psi + \sin\psi)(1 + 2\cos\psi) + \sin\psi(1 - \cos\psi)$$

$$4\pi X_{12} = (\pi - \psi + \sin\psi)(2\cos\psi - 1) + 3\sin\psi(1 - \cos\psi)$$

APPENDIX II

Lift Distribution corresponding to α_2

The discontinuous boundary condition α_2 of equation (13) may be satisfied in two-dimensional steady flow by a lift distribution $\rho V \Gamma$ with

$$\Gamma = V\left[2C_0 \cot \tfrac{1}{2}\theta + C_1(-\,2 \sin \theta + \cot \tfrac{1}{2}\theta) - \sum_{n=2}^{\infty} 2C_n \sin n\theta\right] . \qquad \ldots \quad \ldots \quad (35)$$

Since the downwash corresponding to (35) is

$$W = V\left[C_0 + C_1(\tfrac{1}{2} + \cos \theta) + \sum_{n=2}^{\infty} C_n \cos n\theta\right] , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

it follows that

$$W/V = \alpha_2$$

when
$$\left.\begin{aligned} \pi C_0 &= \int_0^{\pi} (1 - \cos \theta)\alpha_2 \, d\theta \\ \pi C_n &= \int_0^{\pi} 2 \cos n\theta \, \alpha_2 \, d\theta \, , \quad n \geqslant 1 \end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

Then, for α_2 given by equation (13),

$$\left.\begin{aligned} 4\pi C_0 &= (\pi - \psi)(1 + 2 \cos \psi) + \sin \psi(2 + \cos \psi) \\ 2\pi C_1 &= -\,\pi + \psi - \sin \psi \cos \psi \\ 2\pi C_n &= \frac{\sin{(n - 1)\psi}}{n(n - 1)} - \frac{\sin{(n + 1)\psi}}{n(n + 1)} \, , \quad n \geqslant 2 \end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Therefore the required lift distribution is given by (35) and (38) and this may be expressed as

$$\begin{aligned} \Gamma = \frac{V}{\pi}\Big[&\{(\pi - \psi) \cos \psi + \sin \psi\} \cot \tfrac{1}{2}\theta + (\pi - \psi) \sin \theta \\ &- (\cos \theta - \cos \psi) \ln \left(\frac{\sin \frac{1}{2}(\theta + \psi)}{\sin \frac{1}{2}|\theta - \psi|}\right)\Big] . \end{aligned} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

APPENDIX III

Alternative Lift Distribution for Low-Frequency Method

As indicated in Section 4, it is appropriate that the chordwise lift distributions Γ_n should be independent of frequency. Accordingly, the lift distribution over the plan-form is represented by

$$\left.\begin{aligned} \Gamma &= V \sum_n \sum_m \Gamma_n C_{nm} A_m \\ \text{where} \quad \Gamma_0 &= 2 \cot \tfrac{1}{2}\theta \\ \Gamma_1 &= -2 \sin \theta + \cot \tfrac{1}{2}\theta \\ \Gamma_n &= -2 \sin n\theta \, , \quad n \geqslant 2 \\ \text{and} \quad cA_m &= s\eta^{m-1} \sqrt{(1-\eta^2)} \end{aligned}\right\} \qquad (40)$$

The corresponding doublet distribution over the wing and wake is

$$K = V \sum_n \sum_m K_n C_{nm} A_m \, , \qquad (41)$$

and it follows from Ref. 2 that for low frequency, $\omega \to 0$, the chordwise distribution K_n can be expressed as

$$\left.\begin{aligned} K_n &= K_n' + K_n'' \, , \\ \text{where} \quad K_n' &= K_n(\theta) = c[P_n(\theta) + i\omega Q_n(\theta) + 0(\omega^2)] && \text{on the wing,} \quad 0 \leqslant \theta \leqslant \pi \, , \\ &= K_n(\pi) && \text{over the wake,} \quad x \geqslant x_t \\ K_n'' &= 0 && \text{on the wing,} \quad 0 \leqslant \theta \leqslant \pi \, , \\ &= P_n(\pi)[-i\omega(x - x_t)] + 0(\omega^2) && \text{over the wake,} \quad x \geqslant x_t \end{aligned}\right\} \qquad (42)$$

The downwash W induced at any point on the plan-form by the distribution K is then obtained for low frequency as

$$W = V \sum_n \sum_m (W_{nm}' + W_{nm}'') C_{nm} \, , \qquad (43)$$

where W_{nm}' and W_{nm}'' are the downwashes corresponding to the distributions $K_n' A_m$ and $K_n'' A_m$ respectively, with only first-order terms in frequency retained.

For the calculation of W_{nm}' by the vortex-lattice method, each chordwise distribution K_n' is replaced by N discrete vortices of strength $cL_n'(k)$, $k = 1, 2 \ldots N$, which are chosen on the usual two-dimensional basis to give the exact downwash W_n' at selected points on the chord. Since K_n' is constant in the wake, W_n' is given by

$$W_n'(x_1) = -\frac{1}{2\pi} \int_0^c \frac{1}{x - x_1} \frac{\partial K_n'}{\partial x} \, dx \, ,$$

and this can be expressed as

$$W_n'(\theta_1) = \frac{1}{2\pi} \int_0^\pi \frac{[\Gamma_n - i\omega P_n + 0(\omega^2)]}{(\cos \theta - \cos \theta_1)} \sin \theta \, d\theta \, , \qquad (44)$$

$$\begin{aligned} \text{where} \quad P_0 &= \theta + \sin \theta \\ P_1 &= \tfrac{1}{2}(\sin \theta + \tfrac{1}{2} \sin 2\theta) \\ P_n &= \frac{\sin (n+1)\theta}{2(n+1)} - \frac{\sin (n-1)\theta}{2(n-1)} \, , \quad n \geqslant 2 \, . \end{aligned}$$

Therefore, to first order in frequency

$$\left.\begin{aligned} W_0' &= 1 + i\omega[\tfrac{1}{2}\cos\theta_1 + \tfrac{1}{2}\ln(2 + 2\cos\theta_1)] \\ W_1' &= \tfrac{1}{2} + \cos\theta_1 + i\omega[\tfrac{1}{4}\cos\theta_1 + \tfrac{1}{8}\cos 2\theta_1] \\ W_n' &= \cos n\theta_1 + i\omega\left[\frac{\cos(n+1)\theta_1}{4(n+1)} - \frac{\cos(n-1)\theta_1}{4(n-1)}\right], \quad n \geqslant 2 \end{aligned}\right\} . \quad .. \quad .. \qquad (45)$$

Furthermore, the vortices are chosen so that $\sum_{k=1}^{N} cL_n'(k)$ is equal to K_n' over the wake ; to first order in frequency

$$\left.\begin{aligned} K_0' &= c\pi[1 - \tfrac{3}{4}i\omega] \\ K_1' &= c\pi[-\tfrac{1}{8}i\omega] \\ K_2' &= c\pi[\tfrac{1}{8}i\omega] \\ K_n' &= 0\,, \qquad n \geqslant 3 \end{aligned}\right\}\text{over the wake.} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (46)$$

Values of $L_n'(k)/\pi$, $k = 1, 2 \ldots N = 6$, are given below for $n = 0$ and $n = 1$, together with the values for $N = 2$ which are required for the reduced lattice :

k	$L_0'(k)/\pi$	$L_1'(k)/\pi$	Position x/c
1	$0{\cdot}45117 - i\omega\, 0{\cdot}05714$	$0{\cdot}17090 - i\omega\, 0{\cdot}02507$	$\frac{1}{12}$
2	$0{\cdot}20508 - i\omega\, 0{\cdot}10124$	$0{\cdot}01139 - i\omega\, 0{\cdot}03418$	$\frac{3}{12}$
3	$0{\cdot}13672 - i\omega\, 0{\cdot}12683$	$-0{\cdot}03581 - i\omega\, 0{\cdot}03038$	$\frac{5}{12}$
4	$0{\cdot}09765 - i\omega\, 0{\cdot}14477$	$-0{\cdot}05534 - i\omega\, 0{\cdot}02170$	$\frac{7}{12}$
5	$0{\cdot}06836 - i\omega\, 0{\cdot}15747$	$-0{\cdot}05696 - i\omega\, 0{\cdot}01139$	$\frac{9}{12}$
6	$0{\cdot}04102 - i\omega\, 0{\cdot}16255$	$-0{\cdot}03418 - i\omega\, 0{\cdot}00228$	$\frac{11}{12}$
1	$0{\cdot}75000 - i\omega\, 0{\cdot}28836$	$0{\cdot}12500 - i\omega\, 0{\cdot}09375$	$\frac{1}{4}$
2	$0{\cdot}25000 - i\omega\, 0{\cdot}46164$	$-0{\cdot}12500 - i\omega\, 0{\cdot}03125$	$\frac{3}{4}$

The calculation of W_{0m}'' is fully treated in Ref. 2: from the definition of K_n'' given by (42) and (44), it follows that to first order in frequency the downwash W_{nm}'' is zero for $n \geqslant 1$.

TABLE 1

Values of the Equivalent Downwash $W_E(\theta) = a_0 + a_1(\frac{1}{2} + \cos\theta)$

E	ω	$\cos\theta$	Correct lift and moment	Correct lift and hinge moment
			$W_E(\theta)$	$W_E^*(\theta)$
0·08	0·2	0	0·03760 + i 0·00657	−0·52547 + i 0·02610
		$-\frac{2}{3}$	0·46164 + i 0·00298	0·64187 + i 0·04005
0·08	0·6	0	0·03953 + i 0·02065	−0·54935 + i 0·10649
		$-\frac{2}{3}$	0·46625 + i 0·00929	0·64209 + i 0·15227
0·25	0·2	0	0·19535 + i 0·00750	0·19415 − i 0·01332
		$-\frac{2}{3}$	0·74714 + i 0·02513	0·74592 + i 0·03183
0·25	0·6	0	0·19458 + i 0·02330	0·18037 − i 0·04098
		$-\frac{2}{3}$	0·75023 + i 0·07604	0·73587 + i 0·09659
0·25	1·2	0	0·19437 + i 0·05003	0·12079 − i 0·06863
		$-\frac{2}{3}$	0·76402 + i 0·15350	0·69949 + i 0·22271

TABLE 2

Equivalent Downwash $W_E(\theta) \equiv \alpha_{1E} + i\omega\alpha_{2E}$, *for* $\omega \rightarrow 0$

E	$\cos\theta$	Correct lift and moment		Correct lift and hinge moment	
		α_{1E}	α_{2E}	α_{1E}^*	α_{2E}^*
0·08	0	0·037478	0·001208	−0·518430	−0·079018
	$-\frac{2}{3}$	0·461195	0·025000	0·646498	0·051742
0·25	0	0·195501	0·020041	0·195501	−0·062316
	$-\frac{2}{3}$	0·746830	0·131152	0·746830	0·158604

TABLE 3

Values of d_0, d_2 and d_4 [equations (27) and (30)], and Partial-Span Factors $F(\eta_1)$

For vortex-lattice theory,

$$F(\eta_1 = 0{\cdot}2) = 2\pi[1{\cdot}00155d_0 - 0{\cdot}37572d_2 - 0{\cdot}17526d_4]$$

$$F(\eta_1 = 0{\cdot}6) = 2\pi[1{\cdot}00359d_0 + 0{\cdot}58694d_2 - 0{\cdot}00406d_4]$$

$$F(\eta_1 = 0{\cdot}8) = 2\pi[1{\cdot}00377d_0 + 1{\cdot}43118d_2 + 0{\cdot}99073d_4]$$

3(*a*) : *Correct* I_0, I_1, I_2

η_a	d_0	d_2	d_4
0·342020	+0·063216	+0·182726	−0·141395
0·5	0·037243	+0·124354	−0·048813
0·766044	+0·011822	−0·001663	+0·075510

η_a	$F(0{\cdot}2)$	$F(0{\cdot}6)$	$F(0{\cdot}8)$
0·342020	+0·1222	+1·0761	+1·1617
0·5	−0·0054	0·6947	1·0493
0·766044	−0·0048	+0·0665	+0·5297

3(*b*) : *Correct* I_0, I_1, I_a

η_a	d_0^*	d_2^*	d_4^*
0·342020	+0·061323	+0·210180	−0·181156
0·5	0·022356	+0·340215	−0·361438
0·766044	+0·012870	−0·016873	+0·097539

η_a	$F^*(0{\cdot}2)$	$F^*(0{\cdot}6)$	$F^*(0{\cdot}8)$
0·342020	+0·0892	+1·1664	+1·1491
0·5	−0·2644	1·4049	0·9504
0·766044	+0·0134	+0·0164	+0·5366

TABLE 4

Rectangular Wings of Aspect Ratio A with Outboard Flaps (E, η_a) *Oscillating at a Frequency Parameter Value* ν

A	E	η_a	ν	$-z_\xi$	$-z_{\dot\xi}$	$-m_\xi$	$-m_{\dot\xi}$	$-h_\xi$	$-h_{\dot\xi}$
4	0·08	0	→0	0·677	−0·296	0·389	−0·0106	0·385	0·054
			0·2	0·659	−0·190	0·385	−0·0177	0·384	0·068
			0·6	0·599	−0·089	0·372	+0·0065	0·378	0·074
4	0·25	0	→0	1·144	−0·259	0·567	+0·086	0·363	0·172
			0·2	1·114	−0·097	0·560	0·099	0·359	0·214
			0·6	1·019	+0·060	0·537	+0·137	0·344	0·225
4	0·25	0·5	→0	0·483	−0·086	0·249	+0·047	0·251	0·149
			0·2	0·471	−0·018	0·246	0·052	0·250	0·177
			0·6	0·435	+0·042	0·237	+0·067	0·242	0·180
2	0·25	0	→0	0·830	+0·062	0·456	+0·148	0·304	0·180
			0·2	0·823	0·117	0·455	0·140	0·303	0·212
			1·2	0·748	+0·172	0·433	+0·154	0·266	0·211

TABLE 5

Rectangular Wing $A = 4$ *with Outboard Flaps* (E, η_a) *Oscillating at Low Frequency* $\nu \to 0$

E	η_a	$-z_\xi$	$-z_{\dot\xi}$	$-m_\xi$	$-m_{\dot\xi}$	$-h_\xi$	$-h_{\dot\xi}$	Solution
0·08	0	0·677	−0·296	0·389	−0·0106	0·385	0·054	(1)
		0·678	−0·260	0·391	−0·0110	0·390	0·051	(2)
0·08	0·5	0·288	−0·113	0·173	−0·0008	0·295	0·049	(1)
		0·290	−0·098	0·174	+0·0003	0·315	0·049	(2)
0·25	0	1·144	−0·259	0·567	+0·086	0·363	0·172	(1)
		1·142	−0·214	0·566	+0·086	0·361	0·166	(2)
0·25	0·342	0·689	−0·135	0·350	+0·062	0·291	0·162	(1)
		0·687	−0·108	0·347	+0·060	0·292	0·157	(2)
0·25	0·500	0·483	−0·086	0·249	+0·047	0·251	0·149	(1)
		0·484	−0·067	0·248	+0·047	0·264	0·152	(2)
0·25	0·766	0·170	−0·025	0·090	+0·019	0·159	0·104	(1)
		0·174	−0·016	0·093	+0·020	0·180	0·115	(2)

(1) Vortex-lattice solutions are calculated as discussed in Section 7.

(2) Multhopp-Garner solution with 15 spanwise and 2 chordwise terms.

TABLE 6

Rectangular Wing $A = 4$ with Outboard Flaps (E, η_a) in Steady Flow

E	η_a	$-z_\xi$			$-m_\xi$			$-h_\xi$		
		(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
0·08	0	0·677	0·677	0·678	0·389	0·389	0·391	0·385	0·385	0·390
	0·342	0·410	0·410	0·410	0·242	0·242	0·242	0·332	0·337	0·336
	0·500	0·288	0·287	0·290	0·173	0·171	0·174	0·295	0·306	0·315
	0·766	0·102	0·102	0·106	0·063	0·063	0·066	0·197	0·209	0·229
0·25	0	1·144	1·144	1·142	0·567	0·567	0·566	0·363	0·363	0·361
	0·342	0·689	0·689	0·687	0·350	0·350	0·347	0·291	0·295	0·292
	0·500	0·483	0·482	0·484	0·249	0·247	0·248	0·251	0·260	0·264
	0·766	0·170	0·170	0·174	0·090	0·090	0·093	0·159	0·168	0·180

(1) Vortex-lattice solution using partial-span factors $F(\eta_1)$.
(2) Vortex-lattice solution using partial-span factors $F^*(\eta_1)$.
(3) Multhopp-Garner solution with 15 spanwise and 2 chordwise terms.

(a) Values of lift derivatives against ν.

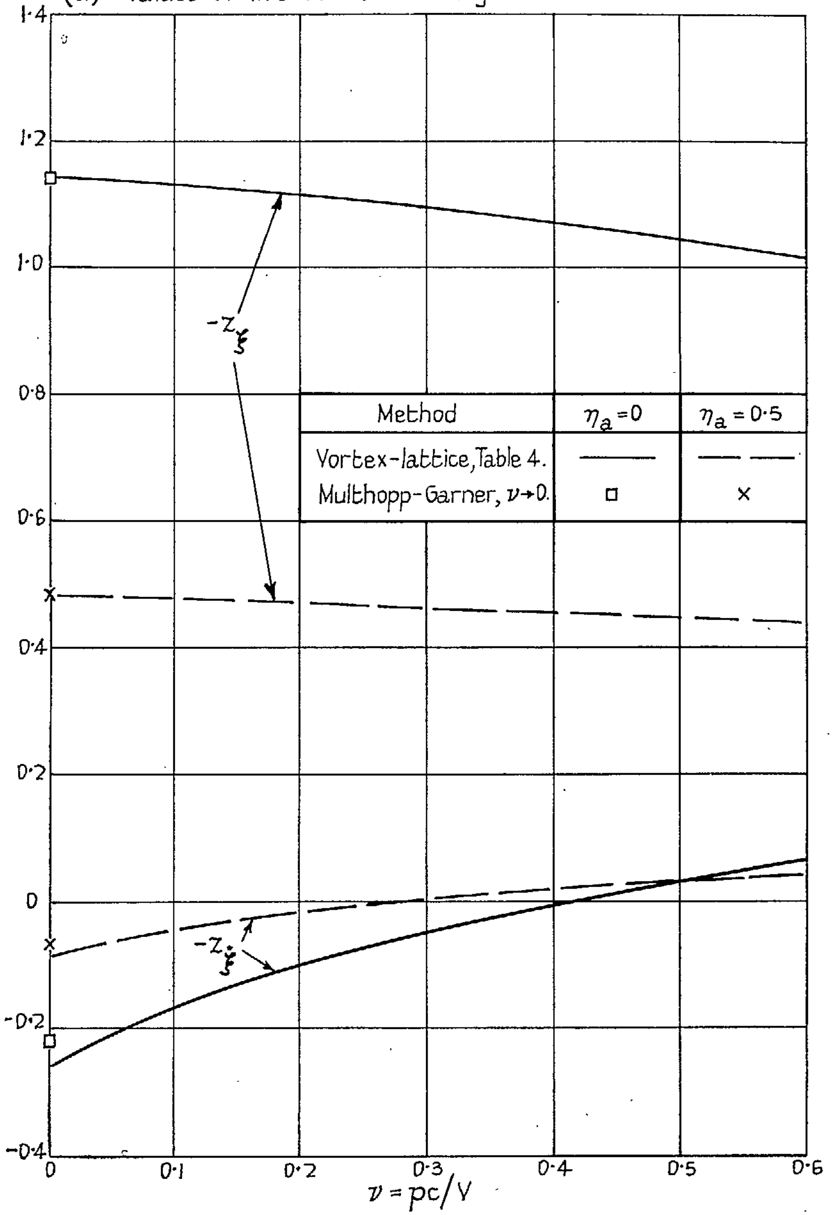
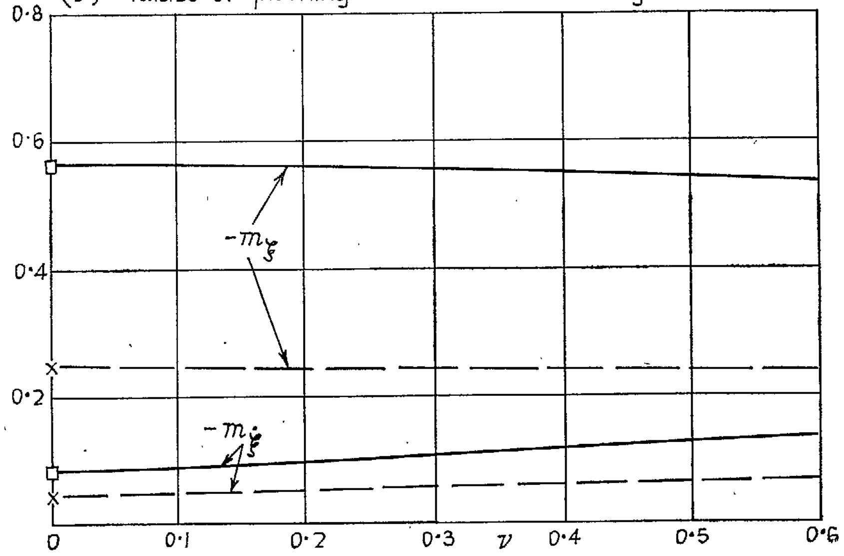
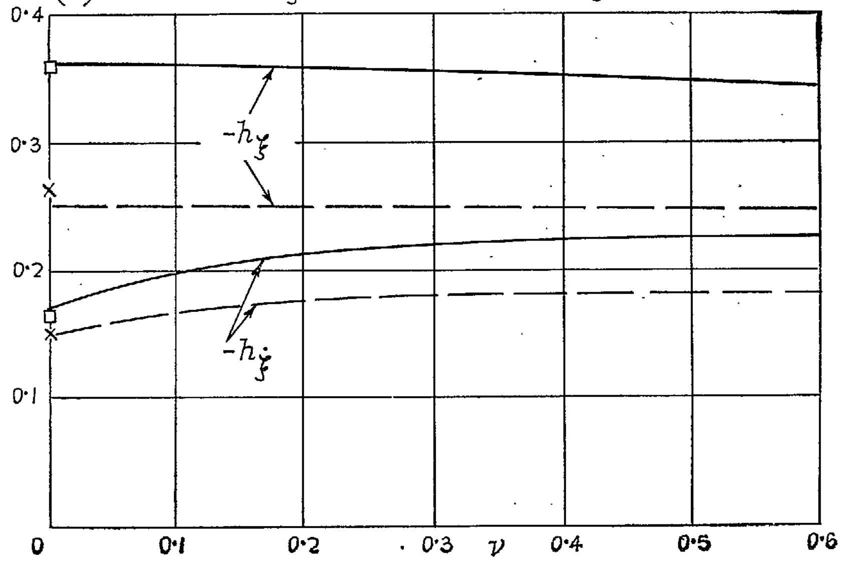


FIG. 1a. Rectangular wing $A = 4$ with outboard flaps ($E = 0 \cdot 25$, η_a).

(b) Values of pitching-moment derivatives against ν.



(c) Values of hinge-moment derivatives against ν.



FIGS. 1b and 1c. Rectangular wing $A = 4$ with outboard flaps ($E = 0 \cdot 25$, η_a).

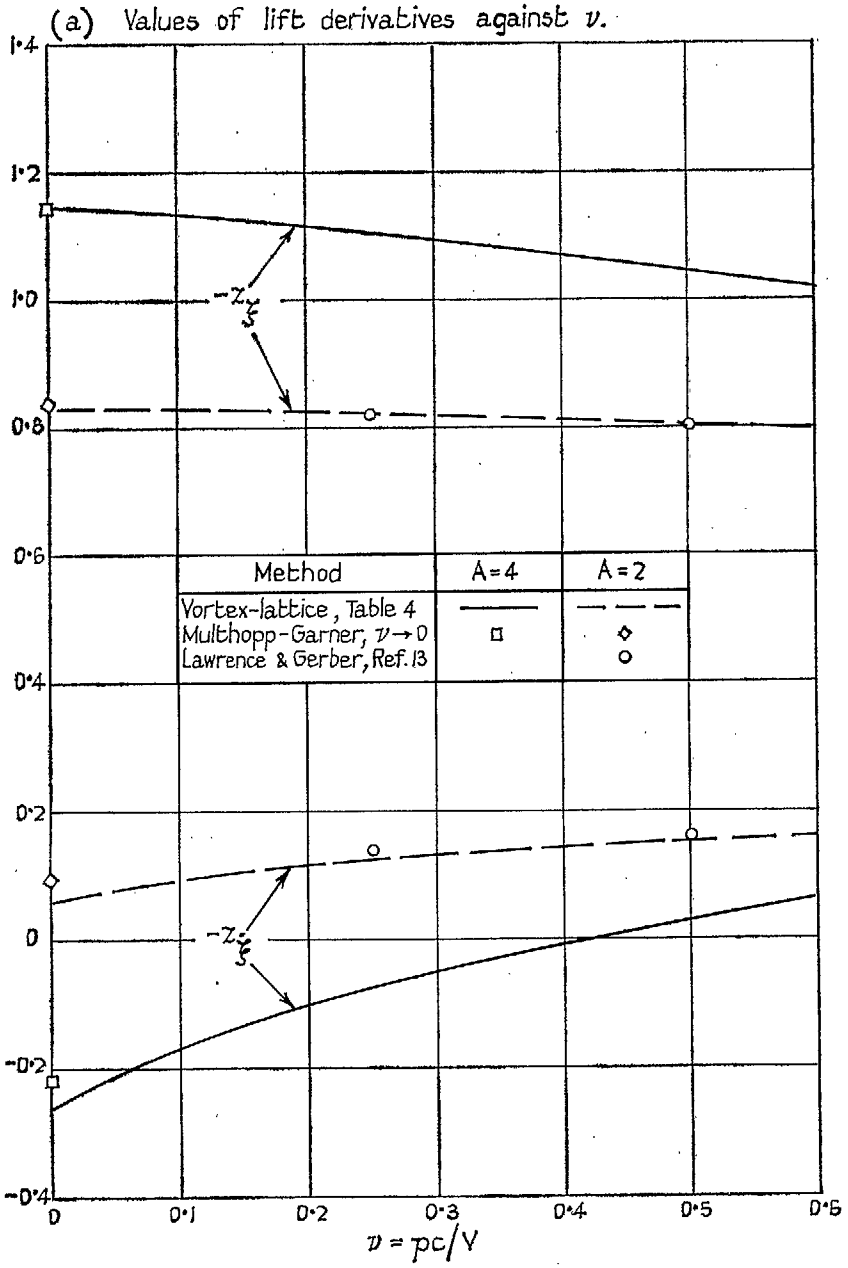
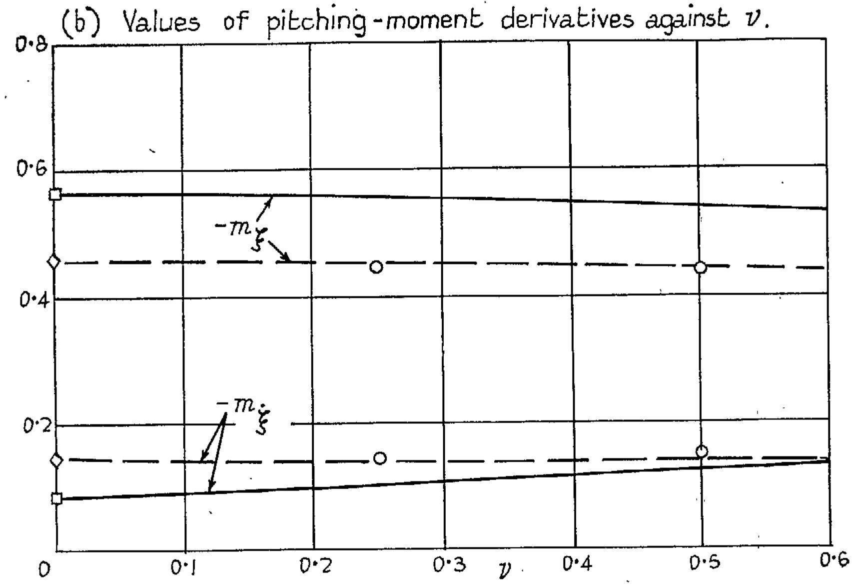
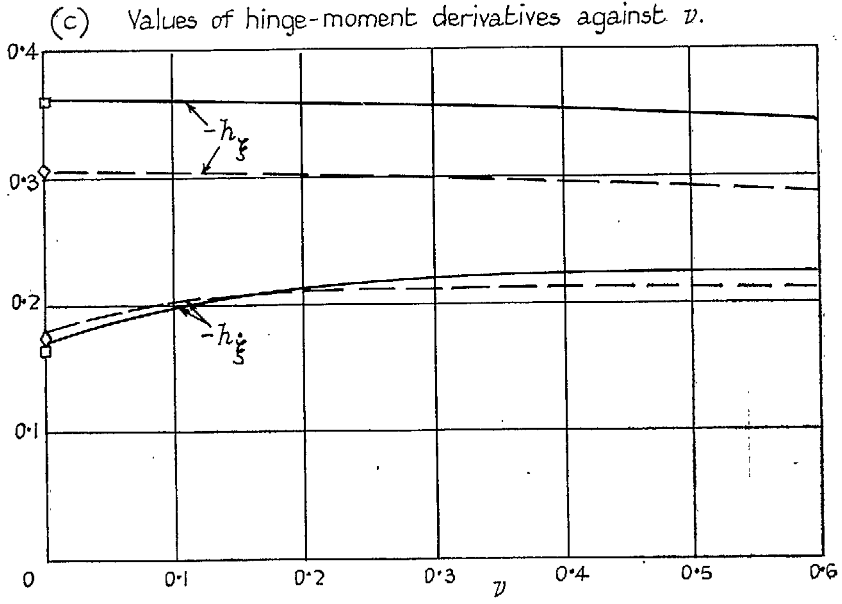


FIG. 2a. Rectangular wings $A = 2$ and $A = 4$ with full-span flaps, $E = 0 \cdot 25$.





FIGS. 2b and 2c. Rectangular wings $A = 2$ and $A = 4$ with full-span flaps, $E = 0 \cdot 25$.

(79481) Wt. 3523/82 10 K7 11/60 Hw.

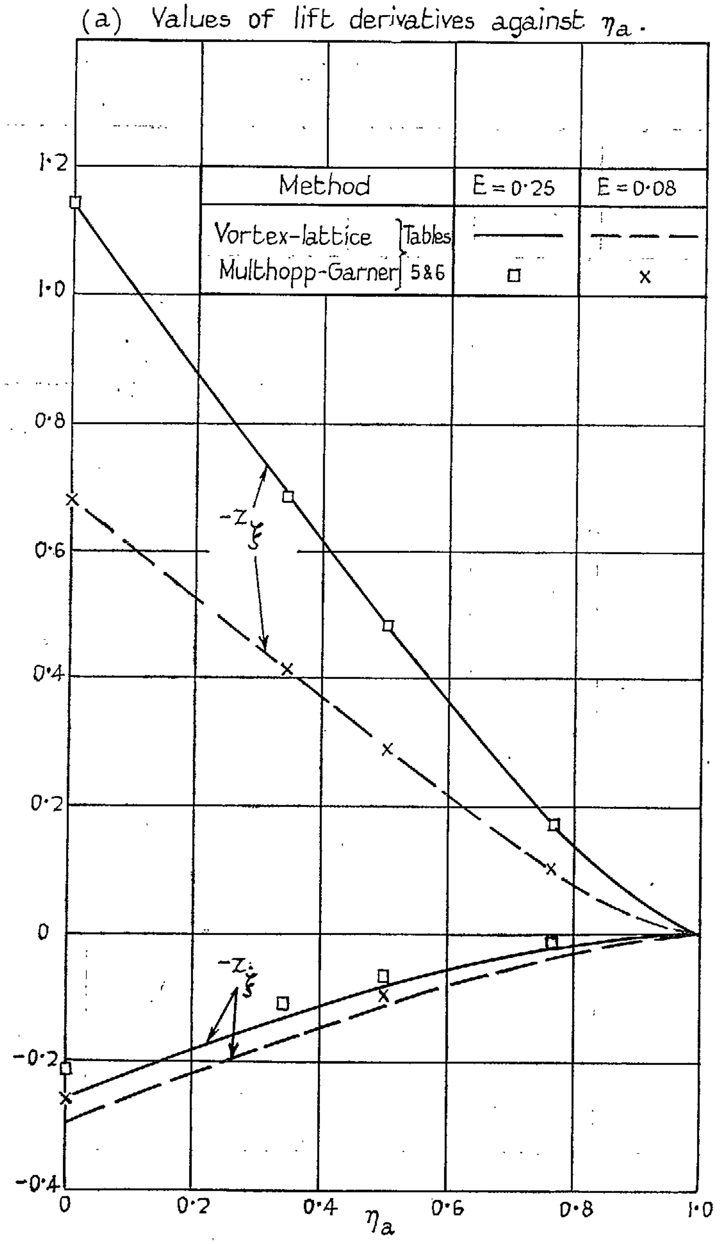
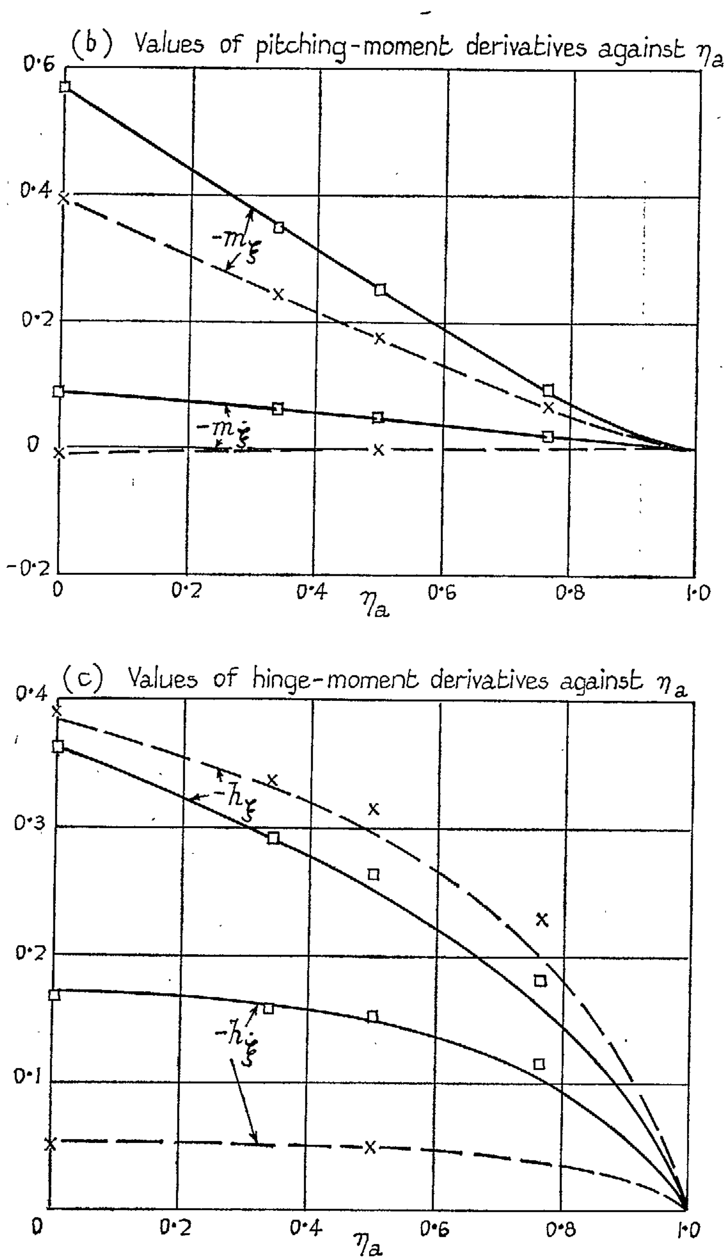


FIG. 3a. Rectangular wing $A = 4$ with flaps (E, η_a) oscillating at low frequency $\nu \to 0$.



FIGS. 3b and 3c. Rectangular wing $A = 4$ with flaps (E, η_a) oscillating at low frequency $\nu \to 0$.

PRINTED IN ENGLAND

R. & M. No. 3182

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50s. (52s.)
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63s. (65s.)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50s. (52s.)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control, Structures. 63s. (65s. 3d.)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75s. (77s. 3d.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47s. 6d. (49s. 3d.)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80s. (82s.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90s. (92s. 3d.)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84s. (86s. 6d.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84s. (86s. 6d.)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130s. (133s.)
Vol. II. Aircraft, Airscrews, Controls. 130s. (133s.)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130s. (132s. 9d.)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130s. (132s. 9d.)

1947 Vol. I. Aerodynamics, Aerofoils, Aircraft. 168s. (171s. 3d.)

Annual Reports of the Aeronautical Research Council—

1939–48 3s. (3s. 5d.) 1949–54 5s. (5s. 5d.)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - R. & M. 2600 6s. (6s. 2d.)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2351–2449	R. & M. No. 2450	2s. (2s. 2d.)
Between Nos. 2451–2549	R. & M. No. 2550	2s. 6d. (2s. 8d.)
Between Nos. 2551–2649	R. & M. No. 2650	2s. 6d. (2s. 8d.)
Between Nos. 2651–2749	R. & M. No. 2750	2s. 6d. (2s. 8d.)
Between Nos. 2751–2849	R. & M. No. 2850	2s. 6d. (2s. 8d.)
Between Nos. 2851–2949	R. & M. No. 2950	3s. (3s. 2d.)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE
York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1; 13a Castle Street, Edinburgh 2;
39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; 50 Fairfax Street, Bristol 1;
80 Chichester Street, Belfast 1, *or through any bookseller.*

S.O. Code No. 23-3182

R. & M. No. 3182