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Accurate Solutions of the Laminar-Boundary-Layer Equations, for Flows having a Stagnation Point and Separation

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Summary. An accurate method of solution is developed for steady incompressible laminar boundary layers whose main-stream velocity $U(x)$ is expressible as an odd polynomial in distance x measured along the wall,

$$U(x) = \sum_{0}^{\infty} u_{2n+1} (x/c)^{2n+1} = \sum_{0}^{\infty} u_{2n+1} \xi^{2n+1} .$$

The velocity distribution within the boundary layer is expanded in a similar series,

$$u = u_1 \sum_{0}^{\infty} F_{2n+1}'(\eta) \xi^{2n+1} ,$$

the coefficients of the first six terms being given as sums of multiples of known universal functions. The relatively small contribution of the subsequent terms is estimated by using an idea of Howarth, whereby the coefficients of the seventh and subsequent terms are assumed to have the same dependence upon the distance normal to the wall as does the sixth term. With this approximation the equation for the non-dimensional skin-friction is reduced to a very simple first-order non-linear ordinary differential equation.

Details are worked out for the case of a main-stream velocity $U = U_0(\xi - \xi^3)$, boundary layer separation being estimated to occur at $\xi = 0 \cdot 655_1$. The boundary-layer thicknesses and the skin-friction distribution are tabulated.

Consideration is also given to main-stream velocities $U = U_0(\xi - \xi^3 + \alpha\xi^5)$ for two values of α for which the analysis takes a particularly simple form. Similar results are given for these two cases. Separation is predicted at $\xi = 0 \cdot 664_7$ when $\alpha = 0 \cdot 078_9$, and at $\xi = 0 \cdot 624_5$ when $\alpha = -0 \cdot 121_6$.

1. *Introduction.* It is often necessary to predict the development of the laminar boundary layer on a two-dimensional body. In particular one needs to predict whether or not boundary-layer separation will occur for a given pressure distribution. Accordingly, a large number of approximate methods have been devised for dealing with this problem, and these vary in speed, simplicity and accuracy, and in the amount of information obtained. Most of the best approximate methods are

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empirical in that they make use of the available exact solutions of the laminar-boundary-layer equations; exact in this context means that the solution has an accuracy of 1 per cent or thereabouts, that is sufficient for empirical fitting in an approximate method.

At the time of writing there are probably only six such solutions of the laminar-boundary-layer equations*:

(i) Falkner and Skan[1] (1930) have given exact solutions for the family of similarity solutions corresponding to an external velocity

$$U = U_0 \xi^m , \tag{1.1}$$

where ξ will everywhere equal x/c, a non-dimensional distance.

(ii) Howarth[2] (1938) considers the case

$$U = U_0(1 - \xi) . \tag{1.2}$$

This solution was obtained by expanding the stream function ψ in a power series in ξ, the coefficients being given by solution of ordinary differential equations. A few of the relevant functions were obtained, and the series converged well when $\xi < 0{\cdot}1$. A specially devised method was used to continue the solution accurately to higher values of ξ, and separation was found at $\xi = 0{\cdot}120$, a value which has since been verified by other workers.

(iii) Tani[3] (1949) considered the three cases

$$U = U_0(1 - \xi^n), \qquad n = 2, 4, 8 . \tag{1.3}$$

His solutions were obtained by evaluating a few terms of a series in powers of ξ^n, and then using Howarth's procedure to continue the solution beyond the range of ξ for which the convergence is rapid enough. He finds separation at $\xi = 0{\cdot}271$, $0{\cdot}462$ and $0{\cdot}640$ respectively.

(iv) Hartree[4] (1939) has used a differential analyser to obtain a precise numerical solution for Schubauer's experimentally observed pressure distribution.

These solutions, valuable though they are, do not cover exhaustively the situations which arise in practice. For example, Thwaites[5] (1949) has pointed out that the similarity solutions are not necessarily a significant criterion for the accuracy of a one-parameter approximate method. Equally, the solutions obtained by Howarth and Tani are all for boundary layers where there is a sharp leading edge and the flow is everywhere retarded. The only exact solution for a flow with a forward stagnation point and a separating boundary layer is that for Schubauer's ellipse, in which the velocity is given numerically. Although in some ways this is not a serious drawback, there is nevertheless real need of an exact solution for an analytically defined flow in which there is both a stagnation point and boundary-layer separation. It is this need which has led to the work described in this paper.

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* Since writing this paper a further solution has been obtained by R. M. Terrill[12] (1960) on the Manchester University computer for the case $U = U_0 \sin \xi$. In attempting to calculate the solution for this case by the present method, for purposes of comparison, the author met considerable numerical difficulties, mainly because the constant k in equation (1.4) was large, being 232. A numerical and analytical investigation of these difficulties is being made by P. G. Williams[13] (1960).

The general theory is developed in Section 2 for cases where the main-stream velocity may be expressed as an odd polynomial in the distance x measured from the forward stagnation point. In such a boundary-layer flow the stream function may also be expressed as an odd polynomial in ξ, the coefficients being functions of the distance z normal to the wall. These functions have been expressed as linear multiples of certain universal functions (Howarth[6] (1934)), which have been calculated by various workers. The most accurate calculations are those due to Tifford[7] (1954) who used a modern high-speed computing machine, and as a result of his work the coefficients of all terms up to and including ξ^{11} are known with sufficient accuracy. The basic idea of the present work, which must be examined in each individual case, is that the small contribution of subsequent terms may be evaluated by assuming that their dependence upon z is similar to that of the coefficient of ξ^{11}. This will often be a reasonable approximation, one imagines, for Howarth[2], in considering the case (1.2) found that the coefficients of ξ^n were remarkably similar in shape when $n = 5, 6, 7, 8$. Subject to this approximation an equation for the non-dimensional skin-friction T, defined later, may be reduced to the simple form

$$T^2 = Q(\xi) - k \int_0^\xi T \, d\xi \, . \tag{1.4}$$

Here $Q(\xi)$ is a known polynomial and k is a known constant, which are easily obtained from a knowledge of the main-stream velocity. The equation is easily integrated by a numerical procedure, having the same accuracy as Simpson's integration formula. Accurate results are accordingly obtained quite quickly.

The method is applied in Section 3 to the case of an external velocity

$$U = U_0(\xi - \xi^3) \, . \tag{1.5}$$

The basic approximation, that the coefficient of ξ^{2n+1} in the power series for the stream function tends to a uniform shape as n increases, is examined and appears to be reasonable. The momentum and displacement thicknesses are evaluated, and are shown in Table 2, along with the non-dimensional skin-friction.

In Section 4 results are derived for two cases with an external velocity of the form

$$U = U_0(\xi - \xi^3 + \quad \xi^5) \, . \tag{1.6}$$

The parameter α is chosen so that the constant k in (1.4) becomes zero, with a considerable simplification of the resulting analysis. Again the basic approximation appears to be reasonable.

2. *General Theory.* It is well known that, when the external velocity is given as a power series in ξ, a formal solution of the boundary-layer equations may be obtained by expanding the stream function as a power series whose coefficients are functions of the distance normal to the wall. Thus for a main stream

$$U = \sum_0^\infty u_{2n+1} \, \xi^{2n+1} \, , \tag{2.1}$$

the stream function ψ may be written as

$$\psi = (u_1 \nu c)^{1/2} \sum_0^\infty \xi^{2n+1} F_{2n+1}(\eta) \, . \tag{2.2}$$

where

$$\eta = \left(\frac{u_1}{\nu c}\right)^{1/2} z \, . \tag{2.3}$$

Howarth[6] (1934) showed that the functions F_{2n+1} may be expressed as linear combinations of a sequence of universal functions, which can be tabulated once and for all. Thus

$$\left.\begin{aligned}
F_1 &= f_1 \\
F_3 &= 4\frac{u_3}{u_1}f_3 \\
F_5 &= 6\left\{\frac{u_5}{u_1}g_5 + \frac{u_3^2}{u_1^2}h_5\right\} \\
F_7 &= 8\left\{\frac{u_7}{u_1}g_7 + \frac{u_3u_5}{u_1^2}h_7 + \frac{u_3^3}{u_1^3}k_7\right\} \\
F_9 &= 10\left\{\frac{u_9}{u_1}g_9 + \frac{u^3u_7}{u_1^2}h_9 + \frac{u_5^2}{u_1^2}k_9 + \frac{u_3^2u_5}{u_1^3}j_9 + \frac{u_3^4}{u_1^4}q_9\right\} \\
F_{11} &= 12\left\{\frac{u_{11}}{u_1}g_{11} + \frac{u_3u_9}{u_1^2}h_{11} + \frac{u_5u_7}{u_1^2}k_{11} + \frac{u_3^2u_7}{u_1^3}j_{11} + \frac{u_3u_5^2}{u_1^3}q_{11} + \right. \\
&\qquad\qquad \left. + \frac{u_3^3u_5}{u_1^4}m_{11} + \frac{u_3^5}{u_1^5}n_{11}\right\} .
\end{aligned}\right\} . \tag{2.4}$$

Several of these universal functions were calculated to a reasonable accuracy by Howarth, and his calculations were later improved and extended by Frössling[8] (1940) and Ulrich[9] (1949). Quite recently Tifford[7] (1954), using modern high-speed computing machinery, has evaluated all the universal functions in (2.4) to much greater accuracy, so that even the least reliable of the results, namely, those for the function n_{11}, may be correct to 3 figures and almost certainly are correct to 2 figures. From (2.2) it follows by differentiation that the velocity u in the boundary layer is

$$u = u_1 \sum_0^\infty \xi^{2n+1} F_{2n+1}{}'(\eta) , \tag{2.5}$$

and that the skin-friction at the wall is derived from

$$\left(\frac{\partial u}{\partial z}\right)_w = \left(\frac{u_1{}^3}{\nu c}\right)^{1/2} \sum_0^\infty \xi^{2n+1} F_{2n+1}{}''(0) . \tag{2.6}$$

In the series expansions (2.2), (2.5) and (2.6), the first six terms are known. To determine the influence of the subsequent terms use will be made of an idea of Howarth[2] (1938). In the series expansion for the case $U = U_0(1 - \xi)$, Howarth noted that the coefficients of ξ^n were of similar shape when $n = 5, 6, 7, 8$, and reasoned that, as the differential equations to be satisfied by these functions were of similar form, it would be plausible to assume that the similarity in shape would persist to higher values of n. Accordingly, two related basic ideas will be used in what follows. (i) It will be assumed that the convergence of the series is such that the seventh and subsequent terms may be treated as a relatively small correction to the first six. The validity of this approximation must be examined *a posteriori*, of course, but in the examples considered in this paper the series appear to converge reasonably almost all the way to separation. (ii) It is further assumed that the contribution of the seventh and subsequent terms may be adequately estimated by assuming that the $F_{2n+1}(\eta)$, $\eta \geqslant 6$, are similar in shape to $F_{11}(\eta)$. Again, nothing general can be said about the validity of this assumption. By examining F_7, F_9 and F_{11}, one can get some idea as to whether there is a tendency towards a universal shape, and in many cases the approximation will be a good one, particularly as it is being used only to obtain a relatively small correction term.

On the basis of these approximations we may write the velocity as

$$\frac{u}{u_1} = \sum_0^4 \xi^{2n+1} F_{2n+1}'(\eta) + A(\xi) F_{11}'(\eta), \tag{2.7}$$

and the skin-friction as

$$\left(\frac{\nu c}{u_1^3}\right)^{1/2} \left(\frac{\partial u}{\partial z}\right)_w = \sum_0^4 \xi^{2n+1} F_{2n+1}''(0) + A(\xi) F_{11}''(0), \tag{2.8}$$

where $A(\xi)$ is to be determined. Howarth suggested two possible methods, which would yield identical results if (2.7) were exactly true. One is to choose $A(\xi)$ so that (2.7) satisfies the momentum integral equation, whilst the other is to satisfy a boundary condition at the wall. Now the first condition at the wall which is not identically satisfied by (2.7) is obtained by differentiating the momentum equation twice with respect to z, and is (Howarth[2], 1938)

$$\left(\frac{\partial u}{\partial z}\right)_w \frac{d}{dx}\left\{\left(\frac{\partial u}{\partial z}\right)_w\right\} = \nu\left(\frac{\partial^4 u}{\partial z^4}\right)_w . \tag{2.9}$$

Provided the basic approximations above are reasonable, either of these methods should lead to a satisfactory prediction of $A(\xi)$. Accordingly, we shall use (2.9), since the subsequent analysis can be reduced to a particularly simple form.

From (2.7), by four-fold differentiation with respect to η, we find

$$\frac{\nu^2 c^2}{u_1^3}\left(\frac{\partial^4 u}{\partial z^4}\right)_w = \sum_0^4 \xi^{2n+1} F_{2n+1}^{\mathrm{v}}(0) + A(\xi) F_{11}^{\mathrm{v}}(0). \tag{2.10}$$

By eliminating $A(\xi)$ between (2.8) and (2.10) we obtain the simple result

$$\nu\left(\frac{\partial^4 u}{\partial z^4}\right)_w = \frac{u_1^3}{\nu c^2}\left\{\sum_0^4 b_{2n+1}\xi^{2n+1} + \frac{F_{11}^{\mathrm{v}}(0)}{F_{11}''(0)}\left(\frac{\nu c}{u_1^3}\right)^{1/2}\left(\frac{\partial u}{\partial z}\right)_w\right\}, \tag{2.11}$$

where

$$b_{2n+1} = F_{2n+1}^{\mathrm{v}}(0) - \frac{F_{11}^{\mathrm{v}}(0)}{F_{11}''(0)} F_{2n+1}''(0). \tag{2.12}$$

From (2.9) and (2.11) we have

$$\left(\frac{\partial u}{\partial z}\right)_w \frac{d}{d\xi}\left\{\left(\frac{\partial u}{\partial z}\right)_w\right\} = \frac{u_1^3}{\nu c}\left\{\sum_0^4 b_{2n+1}\xi^{2n+1} + \frac{F_{11}^{\mathrm{v}}(0)}{F_{11}''(0)}\left(\frac{\nu c}{u_1^3}\right)^{1/2}\left(\frac{\partial u}{\partial z}\right)_w\right\}, \tag{2.13}$$

an equation for the skin-friction. If we write T as the non-dimensional skin-friction,

$$T = \left(\frac{\nu c}{u_1^3}\right)^{1/2}\left(\frac{\partial u}{\partial z}\right)_w, \tag{2.14}$$

and

$$P = \sum_0^4 b_{2n+1}\xi^{2n+1}, \tag{2.15}$$

then (2.13) becomes

$$T\frac{dT}{d\xi} = P + \frac{F_{11}^{\mathrm{v}}(0)}{F_{11}''(0)} T, \tag{2.16}$$

which may be integrated to give

$$T^2 = Q + \frac{2F_{11}{}^{\mathrm{v}}(0)}{F_{11}{}''(0)} \int_0^{\xi} T\, d\xi\,, \qquad (2.17)$$

where

$$Q = 2\int_0^{\xi/xi} P\, d\xi = \sum_0^4 \frac{b_{2n+1}}{n+1}\,\xi^{2n+2}\,. \qquad (2.18)$$

A convenient and simple solution of (2.17) may easily be derived by a procedure due to Thwaites[10] (1949), in which $\int_0^{\xi} T\, d\xi$ is replaced by its Simpson's rule equivalent. Hence if $T(\xi)$ and $T(\xi + h)$ are known, the value of $T(\xi + 2h)$ may be determined by solution of a quadratic equation.

Accordingly, the procedure to be adopted in the general case is as follows. The main-stream velocity being known, the constants u_{2n+1} in (2.1) are known, and hence the F_{2n+1} are known by (2.4). By examination of the numerical values of these functions one can see whether say F_7, F_9 and F_{11} are similar in shape, and if so, reasonably hope that the subsequent functions may not be too different in shape from F_{11}. The skin-friction T is then obtained by solving (2.17) in the manner described above, the starting conditions being determined from the series (2.6). It is necessary to know the values of $F_{2n+1}{}^{\mathrm{v}}(0)$, which may be deduced in terms of the second derivatives at the wall by examination of the equations satisfied by the universal functions. The second derivatives, given by Tifford, are reproduced here in Table 1 together with the fifth derivatives which have been deduced from them. Having obtained T, the relevant value of $A(x/c)$ is deduced from (2.8) by subtracting the first five terms, and the velocity profile is then given by (2.7). An *a posteriori* check on the accuracy can be made by using the momentum integral equation.

It is perhaps worth noting that in certain cases, if great accuracy is not required, it may be possible to integrate (2.17) directly. It is an essential requirement of the method that the series expansion (2.6) for T should converge reasonably, so that the first six terms give some indication of the flow right up to separation. It follows that upon term by term integration of this series, the result

$$\int_0^{\xi/xi} T\, d\left(\frac{x}{c}\right) = \sum_0^{\infty} \frac{F_{2n+1}{}''(0)}{2(n+1)}\,\xi^{2n+2} \qquad (2.19)$$

should converge even more rapidly. If it converges rapidly enough to terminate at the sixth term, then upon combining with the series for Q (2.17) yields

$$T^2 = \sum_0^5 \frac{F_{2n+1}{}^{\mathrm{v}}(0)}{n+1}\,\xi^{2n+2}\,. \qquad (2.20)$$

It is of course rigorously true that

$$T^2 = \sum_0^{\infty} \frac{F_{2n+1}{}^{\mathrm{v}}(0)}{n+1}\,\xi^{2n+2}\,, \qquad (2.21)$$

so (2.20) seems to indicate that the series (2.21) may often converge much more rapidly than the associated series (2.6) for T. This is certainly borne out by calculations for the case (1.5). The first six terms of the series for T^2 indicate separation at $\xi = 0{\cdot}647$, whereas the first six terms of the series for T indicate separation at $\xi = 0{\cdot}684$. The accurate integration of (2.17), which is described in Section 3, predicts separation at $\xi = 0{\cdot}655$, so the series for T^2 leads to an error less than 30 per cent of that obtained by the series for T.

Whether the first six terms of the T^2 series will give an accurate enough prediction of separation depends purely on the accuracy desired. In what follows the equation (2.17) will be integrated by the accurate procedure described earlier in this Section.

It is possible also to obtain a simple general formula for the displacement thickness. We have, by definition,

$$U\delta_1 = \int_0^\infty (U - u)\, dz\,, \tag{2.22}$$

and by (2.3) and (2.7) this yields

$$\begin{aligned} U\delta_1 &= \left(\frac{\nu c}{u_1}\right)^{1/2} \int_0^\infty (U - u)\, d\eta \\ &= \lim_{\eta \to \infty} \left(\frac{\nu c}{u_1}\right)^{1/2} \left\{ U\eta - u_1 \sum_0^4 \xi^{2n+1} F_{2n+1}(\eta) - u_1 A(\xi) F_{11}(\eta) \right\}, \end{aligned}$$

so that

$$\frac{U}{(u_1 \nu c)^{1/2}}\, \delta_1 = \lim_{\eta \to \infty} \left\{ \frac{U}{u_1}\eta - \sum_0^4 \xi^{2n+1} F_{2n+1}(\eta) - A(\xi) F_{11}(\eta) \right\}. \tag{2.23}$$

There is not a simple general expression for δ_2, as far as the author can determine.

3. *Solution for the Main-stream Velocity* $U = U_0(\xi - \xi^3)$. As this example will be considered in some detail, it will be useful to know roughly where separation occurs, before the full analysis is attempted. Solutions to this problem had earlier been obtained (Curle and Skan[11] (1957)) by the approximate methods of Görtler, Stratford and Thwaites, the predicted positions of separation being respectively $\xi = 0{\cdot}685$, $0{\cdot}659$, $0{\cdot}648$. Bearing in mind that typical errors for Stratford's and Thwaites' method are 1 per cent and 3 per cent respectively, and that Görtler's method usually overestimates the distance to separation by up to 5 per cent, it appears that the true separation position may well be $\xi = 0{\cdot}66$ to two figures.

For this particular problem the coefficients in the series (2.1) are

$$u_1 = U_0,\ u_3 = -U_0,\ u_5 = u_7 = u_9 = \ldots. = 0\,. \tag{3.1}$$

Thus, from (2.4), the relevant functions are

$$\left.\begin{array}{lll} F_1 = f_1, & F_3 = -4f_3, & F_5 = 6h_5, \\ F_7 = -8k_7, & F_9 = 10q_9, & F_{11} = -12n_{11} \end{array}\right\}. \tag{3.2}$$

Accordingly the series expansion (2.6) for the skin-friction yields

$$\begin{aligned} \left(\frac{\nu c}{U_0^3}\right)^{1/2} \left(\frac{\partial u}{\partial z}\right)_w = T &= f_1''(0)\xi - 4f_3''(0)\xi^3 + 6h_5''(0)\xi^5 - 8k_7''(0)\xi^7 + \\ &+ 10q_9''(0)\xi^9 - 12n_{11}''(0)\xi^{11} \ldots.\,, \end{aligned} \tag{3.3}$$

and upon substitution for these second derivatives (Table 1) we have

$$T = 1{\cdot}232588\xi - 2{\cdot}89779\xi^3 + 0{\cdot}71509\xi^5 - 0{\cdot}06111\xi^7 - 0{\cdot}3079\xi^9 - 0{\cdot}6187\xi^{11} \ldots. \tag{3.4}$$

The accuracy to which the various coefficients are known is such that they should be correct at least to one figure less than those quoted, except that the coefficient of ξ'' may only be correct to two figures. To examine the rate of convergence of the series we note the following examples. When $\xi = 0{\cdot}35$ the series becomes

$$T = 0{\cdot}431406 - 0{\cdot}124243 + 0{\cdot}003756 - 0{\cdot}000039 - 0{\cdot}000024 - 0{\cdot}000006 \ldots, \tag{3.5}$$

so the first six terms appear to give the series to 5 decimal places. At the pressure minimum, $\xi = 1/\sqrt{3}$, the series becomes

$$T = 0{\cdot}7116 - 0{\cdot}5577 + 0{\cdot}0459 - 0{\cdot}0013 - 0{\cdot}0022 - 0{\cdot}0015 \ldots, \tag{3.6}$$

the convergence being insufficient to yield T to 3 decimal places. When $\xi = 0\cdot66$, in the vicinity of separation, the series is

$$T = 0\cdot8135 - 0\cdot8331 + 0\cdot0896 - 0\cdot0033 - 0\cdot0073 - 0\cdot0064 \ldots, \tag{3.7}$$

so that the six terms cannot give an accurate estimation of the separation position. However, since the 4th to 6th terms are of order 10^{-2} times the first three, presumably the series will give some idea as the flow near to separation, and it is therefore reasonable to regard the subsequent terms as a correction to the first six.

We now examine the functions $F_{2n+1}'(\eta)/F_{2n+1}(\infty)$, $n = 3, 4, 5$, to test whether there is any indication of similarity in the shapes of coefficients of higher powers of ξ. By (3.2) these three functions are simply $k_7'(\eta)/k_7(\infty)$, $q_9'(\eta)/q_9(\infty)$, $n_{11}'(\eta)/n_{11}(\infty)$. Fig. 1 shows that they have the same general shape, and that the latter two are much closer to each other than the first is to either of them. It seems possible, therefore, that the shapes of the subsequent functions might well differ from that of n_{11} by even less than q_9 does. Accordingly, the use of the approximation (2.7) seems to be most reasonable, particularly when it is remembered that it is being used only to obtain a relatively small correction term.

We therefore solve the equation (2.17) for T. The relevant coefficients are easily obtained from (2.12), (2.4) and (3.2), and when this is done (2.17) becomes

$$T^2 = Q - 3\cdot328 \int_0^{\xi} T\, d\xi\,, \tag{3.8}$$

$$Q = 3\cdot570193\xi^2 - 9\cdot554395\xi^4 + 10\cdot55662\xi^6 - 4\cdot3204\xi^8 + 0\cdot0041\xi^{10}\,, \tag{3.9}$$

the coefficients again being at least correct to one figure less than the number quoted. This equation was integrated by the step-by-step procedure outlined in Section 2. The series expansion (3.4) yields

$$\left.\begin{aligned} T(0\cdot275) &= 0\cdot338962 - 0\cdot060265 + 0\cdot001125 - 0\cdot000007 - 0\cdot000003 - \\ &\qquad - 0\cdot000000 = 0\cdot279812 \\ T(0\cdot300) &= 0\cdot369776 - 0\cdot078240 + 0\cdot001738 - 0\cdot000013 - 0\cdot000006 - \\ &\qquad - 0\cdot000001 = 0\cdot293254 \end{aligned}\right\}. \tag{3.10}$$

With these starting values the equation was integrated in steps of $\Delta\xi = 0\cdot025$. A second integration, using the starting values at $\xi = 0\cdot25$ and $0\cdot30$, was also used, Richardson's h^2 extrapolation then being applied to obtain improved values. Smaller steps were used as separation was approached, and separation was predicted at $\xi = 0\cdot655_1$. The values of T being known, it was a simple matter to subtract the first five terms of the series expansion, and then use (2.8) to obtain $A(\xi)$.

The displacement thickness, (2.23), is easily obtained. Upon substitution from (3.2) we have

$$\left(\frac{U_0}{\nu c}\right)^{1/2} \delta_1 = (\xi - \xi^3)^{-1} \lim_{\eta\to\infty} \{\xi(\eta - f_1) - \xi^3(\eta - 4f_3) - 6\xi^5 h_5 + 8\xi^7 k_7 - \\ - 10\xi^9 q_9 + 12 A n_{11}\}\,, \tag{3.11}$$

and upon substitution from Tifford's Tables this becomes

$$\left(\frac{U_0}{\nu c}\right)^{1/2} \delta_1 = (\xi - \xi^3)^{-1}\{0\cdot647900\xi + 0\cdot113896\xi^3 + 0\cdot44341\xi^5 + \\ + 0\cdot79612\xi^7 + 1\cdot3967\xi^9 + 2\cdot490A\}\,. \tag{3.12}$$

Similarly, by integration one can obtain δ_2, and after a considerable amount of work this yields

$$\left(\frac{U_0}{\nu c}\right)^{1/2} \delta_2 = \int_0^\infty \frac{Uu - u^2}{U^2}\, d\eta$$
$$= (\xi - \xi^3)^{-2}\{0\cdot292344\xi^2 - 0\cdot266996\xi^4 + 0\cdot10223\xi^6 + $$
$$+ 0\cdot0591\xi^8 + 0\cdot0684\xi^{10} + 4\cdot163\xi^{12} - 0\cdot709\xi^{14} - 0\cdot765\xi^{16} -$$
$$- 0\cdot628\xi^{18} - 1\cdot82A^2 + A[2\cdot07\xi - 2\cdot95\xi^3 - 1\cdot55\xi^6 -$$
$$- 2\cdot56\xi^7 - 2\cdot13\xi^9]\}\,. \qquad (3.13)$$

In both (3.12) and (3.13) the accuracy of the coefficients is such that the last figure quoted is in many cases doubtful, if not meaningless. For small values of ξ this is not important, but near separation it is a serious drawback, particularly as regards δ_2. For example, at the predicted separation position (3.12) and (3.13) yield

$$\left.\begin{aligned} \left(\frac{U_0}{\nu c}\right)^{1/2} \delta_1 &= 2\cdot2986 \pm 0\cdot001\,, \\ \left(\frac{U_0}{\nu c}\right)^{1/2} \delta_2 &= 0\cdot6072 \pm 0\cdot03 \end{aligned}\right\}. \qquad (3.14)$$

We note that the proportionate error in δ_2 is much greater than that in δ_1. Accordingly more reliable values of δ_2 are obtainable by accepting the values of δ_1 from (3.12) and T from (3.8), and then deriving δ_2 from the momentum integral equation. Thus, in the general case, we have

$$\frac{d}{d\xi}\{U^2\delta_2\} = \nu c\left(\frac{\partial u}{\partial z}\right)_w - \delta_1 U \frac{dU}{d\xi}\,,$$

from which one can derive

$$U^2\delta_2\left(\frac{U_0}{\nu c}\right)^{1/2} = U_0^{\,2}\int_0^{\xi/xi}\left\{T - \frac{U}{U_0}\frac{d(U/U_0)}{d\xi}\,\delta_1\left(\frac{U_0}{\nu c}\right)^{1/2}\right\} d\xi\,. \qquad (3.15)$$

This also yields δ_2 more rapidly, as it is not necessary to determine all the coefficients of (3.13). It must be stressed, however, that the use of the basic approximation, (2.7), cannot be avoided, and any errors due to this remain. Use of (3.15) in determining δ_2 merely ensures that the available information is used to the best effect.

In Table 2 are shown the results for the present case. The momentum thickness δ_2 has been calculated both by (3.13) and (3.15). Those values derived from (3.13) show a physically unrealistic decrease in $(U_0/\nu c)^{1/2}\delta_2$ just prior to separation. This decrease, however, is of magnitude $0\cdot01$, which is considerably less than the uncertainty near separation. The values derived from (3.15), on the other hand, agree remarkably well for values of ξ less than or equal to $0\cdot64$, the random difference in the fourth decimal place being clearly a rounding-off error. When $\xi \geqslant 0\cdot64$, however, the value of $(U_0/\nu c)^{1/2}\delta_2$ continues to increase in a reasonable manner. The fact that the values of δ_2 derived from (3.13) and (3.15) agree so well when $\xi \leqslant 0\cdot64$ confirms the accuracy of the basic approximation in this region. When $\xi \geqslant 0\cdot64$ the discrepancy is nowhere greater than the known uncertainty, so one might reasonably hope that (3.15) is quite accurate.

4. *Solutions for Mainstream Velocities* $U = U_0(\xi - \xi^3 + \alpha\xi^5)$. For main-stream velocities of this form, with the coefficients in (2.1) becoming

$$u_1 = U_0,\ u_3 = -U_0,\ u_5 = \alpha U_0,\ u_7 = u_9 \ldots = 0\,, \qquad (4.1)$$

the functions which appear in the series for the stream function are

$$\left.\begin{aligned} F_1 &= f_1 \\ F_3 &= -4f_3 \\ F_5 &= 6(\alpha g_5 + h_5) \\ F_7 &= -8(\alpha h_7 + k_7) \\ F_9 &= 10(\alpha^2 k_9 + \alpha j_9 + q_9) \\ F_{11} &= -12(\alpha^2 q_{11} + \alpha m_{11} + n_{11}) \end{aligned}\right\} . \tag{4.2}$$

Now the non-dimensional skin-friction T, defined in (2.14), is, by (4.1) simply

$$T = \left(\frac{\nu c}{U_0^3}\right)^{1/2} \left(\frac{\partial u}{\partial z}\right)_w , \tag{4.3}$$

and by the method of the present paper must satisfy the equation

$$T^2 = Q + \frac{2F_{11}{}^{\mathrm{v}}(0)}{F_{11}{}''(0)} \int_0^\xi T\, d\xi , \tag{4.4}$$

where Q is defined by (2.12) and (2.18). Now $F_{11}{}^{\mathrm{v}}(0)$ is, by (4.2), a quadratic in α, namely,

$$\begin{aligned} F_{11}{}^{\mathrm{v}}(0) &= -12\{\alpha^2 q_{11}{}^{\mathrm{v}}(0) + \alpha m_{11}{}^{\mathrm{v}}(0) + n_{11}{}^{\mathrm{v}}(0)\} \\ &= 1\cdot02966 - 4\cdot58707\alpha - 107\cdot422\alpha^2 . \end{aligned} \tag{4.5}$$

It is interesting to notice that $F_{11}{}^{\mathrm{v}}(0)$ is zero for each of two real values of α, namely,

$$\alpha = 0\cdot07885 \text{ and } \alpha = -0\cdot12156 . \tag{4.6}$$

We shall here develop the solution for each of these values of α, since (4.4) reduces to

$$T^2 = Q . \tag{4.7}$$

In other words, T^2 is known without any numerical integration being required. The positions of separation, for which $T^2 = 0$, are easily obtained, and are

$$\left.\begin{aligned} \xi &= 0\cdot664_7 \text{ when } \alpha = 0\cdot078_9 \\ \xi &= 0\cdot624_5 \text{ when } \alpha = -0\cdot121_6 \end{aligned}\right\} . \tag{4.8}$$

Since T^2, and hence T, is known, it is easy to determine $A(\xi)$ from (2.8), whence δ_1 follows from (2.26) and then δ_2 from (3.15). The results are shown in Tables 3 and 4.

Finally, one should check whether the basic approximation, (2.7), is reasonable. Accordingly the convergence of the series (2.6) for T, namely,

$$T = \sum_0^\infty F_{2n+1}{}''(0)\xi^{2n+1} , \tag{4.9}$$

has been examined. Substituting for $F_{2n+1}{}''(0)$ from (4.2) and Table 1, we find that the series becomes

$$T = 1\cdot232588\xi - 2\cdot89779\xi^3 + 1\cdot01537\xi^5 - 0\cdot1765\xi^7 - 0\cdot2564\xi^9 - 0\cdot457\xi^{11} \ldots \quad \text{when } \alpha = 0\cdot078_9, \tag{4.10}$$

$$T = 1\cdot232588\xi - 2\cdot89779\xi^3 + 0\cdot25217\xi^5 + 0\cdot1168\xi^7 - 0\cdot3733\xi^9 - 0\cdot901\xi^{11} \ldots \quad \text{when } \alpha = -0\cdot121_6 . \tag{4.11}$$

Hence at the predicted separation positions, (4.8), the series become

$$T = 0\cdot 8193 - 0\cdot 8511 + 0\cdot 1318 - 0\cdot 0101 - 0\cdot 0065 - 0\cdot 0051 \ldots \quad \text{when } \alpha = 0\cdot 078_9\,, \tag{4.12}$$

$$T = 0\cdot 7697 - 0\cdot 7057 + 0\cdot 0239 + 0\cdot 0043 - 0\cdot 0054 - 0\cdot 0051 \ldots \quad \text{when } \alpha = -\,0\cdot 121_6\,. \tag{4.13}$$

Since the fifth and sixth terms are of order 10^{-2} times the largest terms of the series, one would conclude that the series is converging sufficiently rapidly to justify using an approximate form for the subsequent terms. Further, to justify the form chosen for the seventh and subsequent terms, the functions $F_7'(\eta)/F_7(\infty)$, $F_9'(\eta)/F_9(\infty)$ and $F_{11}'(\eta)/F_{11}(\infty)$ have been examined, and are shown in Figs. 2 and 3 for $\alpha = 0\cdot 078_9$ and $\alpha = -\,0\cdot 121_6$ respectively. In each case it will be noted that the curves are similar, and that $F_9'(\eta)/F_9(\infty)$ and $F_{11}'(\eta)/F_{11}(\infty)$ are particularly alike in the crucial region near to the wall. Accordingly the use of the present method seems reasonably justified.

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TABLE 1

Second and Fifth Derivatives of the Universal Functions

$F(\eta)$	$F''(0)$	$F^{v}(0)$
f_1	$+1\cdot232588$	$+1\cdot519273$
f_3	$0\cdot724447$	$3\cdot571779$
g_5	$0\cdot634702$	$4\cdot693956$
h_5	$0\cdot119182$	$5\cdot080001$
g_7	$0\cdot579202$	$5\cdot711339$
h_7	$0\cdot182948$	$12\cdot839387$
k_7	$0\cdot007638$	$2\cdot147501$
g_9	$0\cdot539932$	$6\cdot655137$
h_9	$0\cdot151970$	$15\cdot300401$
k_9	$0\cdot057185$	$7\cdot956095$
j_9	$+0\cdot060741$	$7\cdot713064$
q_9	$-0\cdot030787$	$0\cdot053268$
g_{11}	$+0\cdot509986$	$7\cdot543231$
h_{11}	$0\cdot132290$	$17\cdot602794$
k_{11}	$0\cdot074200$	$18\cdot743288$
j_{11}	$0\cdot080554$	$8\cdot908709$
q_{11}	$+0\cdot116361$	$8\cdot951840$
m_{11}	$-0\cdot179648$	$+0\cdot382256$
n_{11}	$+0\cdot051561$	$-0\cdot085805$

TABLE 2

Values of T, δ_1 and δ_2 for the Case $U = U_0(\xi - \xi^3)$

ξ	T	$\delta_1\left(\frac{U_0}{\nu c}\right)^{1/2}$	$\delta_2\left(\frac{U_0}{\nu c}\right)^{1/2}$		A
			From (3.13)	From (3.15)	
0·00	0	0·6479	0·2923	0·2923	0
0·10	0·12037	0·6556	0·2956	0·2956	0
0·20	0·22356	0·6804	0·3058	0·3059	0
0·30	0·29325	0·7279	0·3251	0·3251	0
0·40	0·31468	0·8119	0·3579	0·3580	0·0001
0·50	0·27476	0·9688	0·4149	0·4149	0·0009
0·55	0·22739	1·1064	0·4597	0·4596	0·0033
0·60	0·15691	1·3396	0·5243	0·5238	0·0121
0·61	0·13911	1·4092	0·5406	0·5401	0·0160
0·62	0·11953	1·4926	0·5583	0·5580	0·0214
0·63	0·09761	1·5966	0·5773	0·5776	0·0292
0·64	0·07206	1·7461	0·5998	0·5993	0·0432
0·65	0·03854	1·9568	0·6153	0·6235	0·0653
0·652	0·02926	2·0286	0·6174	0·6286	0·0741
0·654	0·01690	2·1336	0·6168	0·6340	0·0878
0·655	0·00580	2·2390	0·6120	0·6367	0·1026
$0\cdot655_1$	0	2·2986	0·6072	0·6371	0·1115

TABLE 3

Values of T, δ_1 and δ_2 for the Case $U = U_0(\xi - \xi^3 + \alpha\xi^5)$, $\alpha = 0{\cdot}078_9$

ξ	T	$\delta_1\left(\frac{U_0}{\nu c}\right)^{1/2}$	$\delta_2\left(\frac{U_0}{\nu c}\right)^{1/2}$	A
0	0	0·6479	0·2923	0
0·2	0·22366	0·6803	0·3057	0
0·4	0·31759	0·8081	0·3566	0·0001
0·5	0·28349	0·9546	0·4101	0·0010
0·54	0·25134	1·0492	0·4417	0·0027
0·56	0·23081	1·1103	0·4607	0·0044
0·58	0·20696	1·1852	0·4825	0·0074
0·60	0·17929	1·2802	0·5075	0·0126
0·62	0·14675	1·4079	0·5367	0·0222
0·64	0·10663	1·5997	0·5710	0·0420
0·66	0·04509	2·0090	0·6121	0·1022
0·664	0·01755	2·2547	0·6214	0·1467
$0{\cdot}664_7$	0	2·4352	0·6231	0·1823

TABLE 4

Values of T, δ_1 and δ_2 for the Case $U = U_0(\xi - \xi^3 + \alpha\xi^5)$, $\alpha = -0{\cdot}121_6$

ξ	T	$\delta_1\left(\frac{U_0}{\nu c}\right)^{1/2}$	$\delta_2\left(\frac{U_0}{\nu c}\right)^{1/2}$	A
0	0	0·6479	0·2923	0
0·2	0·22342	0·6807	0·3059	0
0·4	0·31019	0·8180	0·3601	0·0001
0·5	0·26112	0·9936	0·4222	0·0011
0·54	0·21795	1·1213	0·4614	0·0034
0·56	0·18989	1·2124	0·4860	0·0059
0·58	0·15615	1·3370	0·5152	0·0108
0·60	0·11372	1·5288	0·5501	0·0211
0·62	0·04722	1·9556	0·5931	0·0538
0·624	0·01517	2·2396	0·6030	0·0806
$0{\cdot}624_5$	0	2·3947	0·6042	0·0964

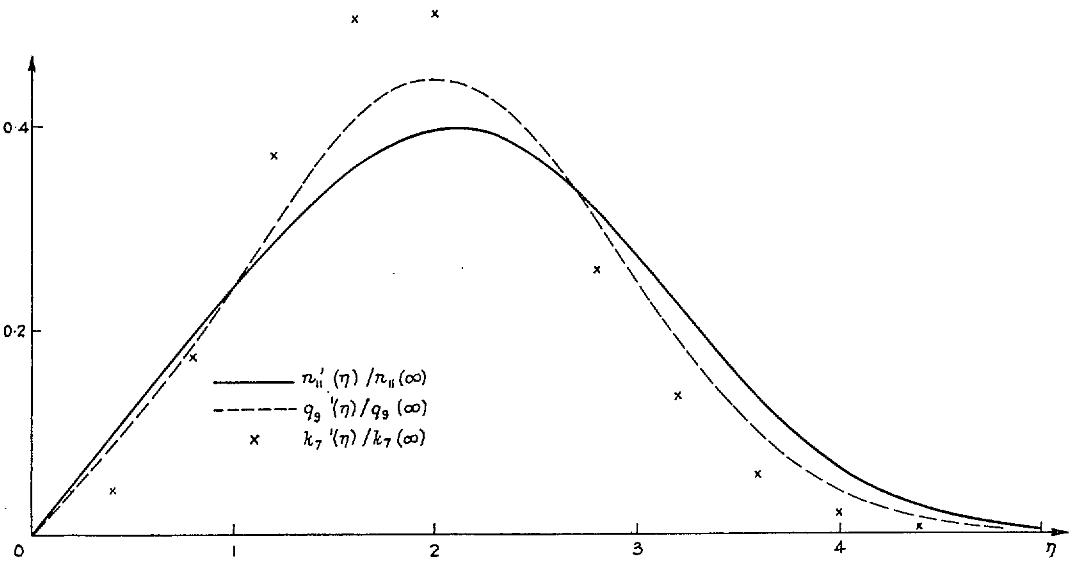

FIG. 1. Some functions which appear in Section 3.

(78648) Wt. 54/8210 K.5 8/60 Hw.

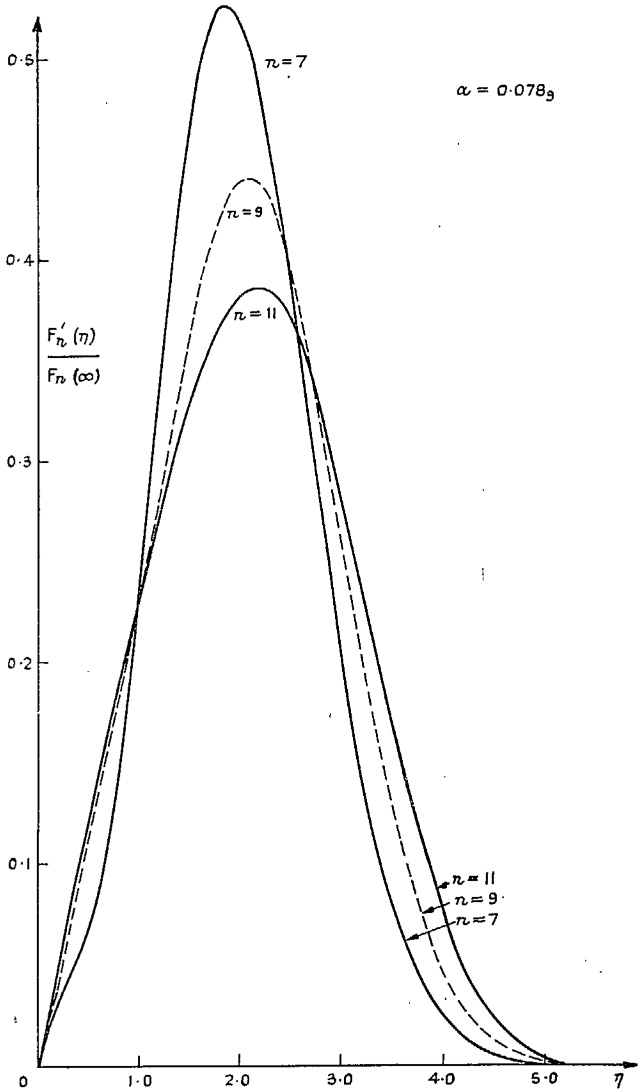

FIG. 2. Some functions which appear in Section 4.

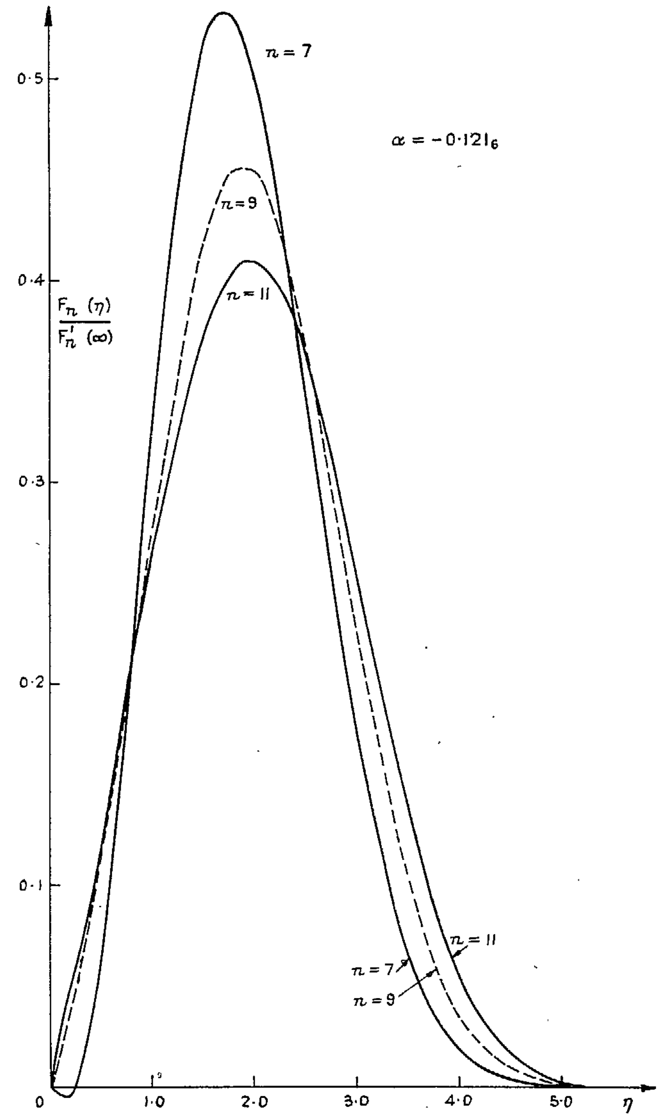

FIG. 3. Some functions which appear in Section 4.

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