R. & M. No. 3154
(19,338)
A.R.C. Technical Report

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

Free Vibrations of a Stiffened Cylindrical Shell

By

P. R. Miller, B.Sc.,
Department of Aeronautical Engineering,
University of Southampton

LONDON: HER MAJESTY'S STATIONERY OFFICE
1960
PRICE 13*s*. 6*d*. NET

Free Vibrations of a Stiffened Cylindrical Shell

By

P. R. Miller, B.Sc.,

Department of Aeronautical Engineering, University of Southampton

---

Reports and Memoranda No. 3154

May, 1957

Summary.—This report presents the first stage of an investigation of the response to random noise of a stiffened cylinder, representing an aircraft fuselage, including an analysis of the lower modes of such a cylinder.

Preliminary investigations suggested that assumptions and approximations which are valid for the uniform, *i.e.*, unstiffened, shell are not necessarily valid for a shell with heavy stiffening, and this report therefore starts with a review of existing theories in which the assumptions are examined critically, and, it is hoped, somewhat rationalised. There is very little literature on stiffened shells and this review deals mainly with uniform ones, including the general analysis of strain in a thin shell and the vibrations of a uniform cylindrical shell *in vacuo*, but includes some comments on the effects of an acoustic medium round the shell.

The energy approach is then extended to give the resonant frequencies and natural modes of a circular cylindrical shell uniformly stiffened with closely spaced longerons and frames, ' close spaced ' implying that stiffener spacing is much less than the spacing of nodal lines. The effects of rotary inertia have had to be included owing to the lack of symmetry of the section about the skin, but shear deflections are still neglected. Further numerical work is required before much comment can be made on the results, but they seem to be similar in nature to those obtained for the uniform cylinder.

Finally, some indication is given of how the theory can be extended to cover higher-order modes where shear deflection and stiffener spacing become important.

---

1. *Introduction.*—To understand either the effects of noise on a structure or the transmission of noise through a structure it is necessary first to understand the response of the structure. Whatever the nature of the ' noise ', whether random ' white ' noise, pure tones, or even excitation which is not acoustic, this is most conveniently done after first finding the nature of the natural modes of the structure.[20]. At sonic frequencies these modes, and the total response, are appreciably affected by the surrounding acoustic medium, and in turn the pressures in that medium are dependent on the incident noise field, but the overall problem can be broken down, firstly into separate mechanical and acoustic problems, and then further to give the steps :

(*a*) The determination of the ' undamped ' normal modes *in vacuo*, *i.e.*, determination of mode shapes, frequencies and generalised masses

(*b*) The estimation of the mechanical damping

(*c*) The modification of these modes by the acoustic medium, *i.e.*, allowance for the pressure field due to radiated sound ; adding damping and mass and modifying the frequencies and possibly mode shapes

(*d*) The determination of the response in each modified mode due to the incident noise field, including the effects of the reflected sound, *i.e.*, due to the forcing pressure as it would be at a rigid surface

(*e*) The summation of the responses in individual modes to give the total response.

Each of these steps is independent of those following, or at least very nearly so ; certainly (*a*), which is considered in this report, can be treated independently[3]. It is convenient to do so since, for the order of mechanical damping usually encountered, (*b*) will have little effect on the quantities derived from (*a*), that is (*b*) will have little effect on frequencies and does not introduce much coupling of the ' undamped ' normal modes. Similarly the effects of (*c*), which are discussed in Section 2.1, will involve little coupling of these modes. Thus the ' undamped ' modes can be carried through independently until the final summation in (*e*). This, at least, is the conventional argument, but, as shown below, it will need close examination when dealing with a cylinder.

The uniformly stiffened circular cylinder, approximating to an aircraft fuselage, seems to be a suitable model for response calculations. It can be considered representative of a fair proportion of the practical structures involved in noise problems, having sufficient complication, and yet still being amenable to theoretical analysis : albeit rather more involved than for a simple beam. One of the attractions theoretically is its lack of dependence on boundary conditions : there is no longitudinal boundary, and the end conditions are not critical (*see* Section 2.4.3).

The comparative complication of the stiffened cylinder gives it some interesting properties. In common with the uniform cylinder its natural modes are often very closely spaced in frequency[5], so that certain coupling terms, from (*b*) and (*c*) above, and cross-product terms, from (*e*)[20], must be examined carefully. The stiffening itself introduces further effects when the structural half-wavelength approaches the stiffener spacing, the natural modes of the complete cylinder taking on rather unexpected shapes, similar to ' local panel modes '.

Although the vibrations of uniform circular cylindrical shells have been fairly extensively investigated (especially Refs. 5 and 6) there is no satisfactory treatment of stiffened cylinders: in particular, no consideration at all has been given to stiffeners which are not closely spaced relative to structural wavelengths, or of their shear and rotational energies. This report represents the first stage of an investigation to consider these effects, covering the uniform cylinder and the closely stiffened cylinder. It is hoped to cover the effects of stiffener spacing and shear strain energy in a later report and the work here has been presented with those extensions in mind. It must be remembered that in a typical aircraft fuselage the stiffening is much too heavy to be treated merely as a modification to a uniform shell.

A critical review of the relevant literature is presented, as one result of which it was found necessary to include a review of general shell theory. The method of analysis used herein for the stiffened cylinder follows on existing theory for the uniform cylinder[5], a summary of which is included in an Appendix.

A general description of cylinder modes is given together with a general discussion of the effects of stiffening. A solution for the cylinder with closely spaced stiffeners is then presented, the detailed analysis being included in the Appendices. There are some preliminary discussions of the extension of the theory to higher-order modes.

2. *Survey of Previous Work.*—2.1. *The Shell in an Acoustic Medium.*—The general problem of the vibrations of a cylindrical shell in an acoustic medium has been considered by Junger[1,2] and by Baron and Bleich[3,4]. To simplify the acoustic problem all these papers consider infinitely long cylinders but this need not concern the mechanical problem since the infinite cylinder can be considered as a series of finite ones placed end to end. In fact, Junger[2] does effectively do this by introducing rigid septa which isolate the sections of cylinder mechanically. All these papers seem to be mainly concerned with a steel shell submerged in water but filled with air. Only Ref. 2 takes any account of stiffening. Further details of these papers follow.

Ref. 1 considers only the unstiffened cylinder with no nodal rings so that the problem is reduced to two dimensions. Ref. 2 is an extension to allow for nodal rings but also introduces the complication of having the cylinder rigidly built in at regular intervals by rigid septa. It is necessary to break the cylinder up into finite lengths so as to restrict the choice of distance between nodal rings to integral fractions of that finite length, but this can be done more simply by thin diaphragms which are free to warp so that the normal modes are still sinusoidal along the length of the cylinder. Junger still expands his deflections in series of circular functions which do not individually satisfy his end conditions at the rigid septa. Consequently, instead of finishing with three coupled equations corresponding to the three modes with the same nodal configuration, he ends with an infinite set of coupled equations of which a large number must be solved together. The present writer feels that the problem of the cylinder with fixed ends is best solved in the manner used by Arnold and Warburton[6], who used a mixture of circular and exponential functions similar to the mode shapes of an encastre beam.

Junger also considers the effects of closely spaced stiffening rings, including their strain energy in extension and torsion but omitting their bending energy which is the most important contribution. He does not attempt to isolate the mechanical and acoustical problems and his results are in a cumbersome form.

Baron and Bleich do separate the acoustic and structural problems, dealing with them in separate papers. They are concerned only with uniform cylinders, although they did remark that the approach would be the same for a stiffened shell provided the distances between nodes were much greater than the stiffener spacings. This remark presumably refers only to the contents of Ref. 3, since Ref. 4, dealing with the cylinder *in vacuo*, cannot be extended to cover stiffening. In Ref. 4 they first analyse the vibrations ignoring bending stiffness and then add the bending effects as a modification to frequencies only, using Rayleigh's principle : this would hardly do for a stiffened shell where the relative magnitudes of bending and extensional stiffnesses are very different from those in the uniform shell. The advantage of their approach is that it leads to a fairly compact tabulation of mode shapes and frequencies, all the factors tabulated being independent of the physical dimensions of the cylinder. It is not likely that any such 'once only' tabulation will be possible for stiffened shells due to the large number of possible variables. Even for the uniform shell Baron and Bleich's method is restricted to low-order modes ($n \leqslant 6$, $\lambda \leqslant \pi$).

In Ref. 3 Baron and Bleich treat the cylinder as a mechanical system with three degrees of freedom corresponding to three modes with the same nodal configuration. These modes become coupled acoustically through the radial components of their displacements. Their final frequency equation (in Ω) is of the form

$$\frac{1}{m_1(\omega_1^2 - \Omega^2)} + \frac{1}{m_2(\omega_2^2 - \Omega^2)} + \frac{1}{m_3(\omega_3^2 - \Omega^2)} = F(\Omega) \,, \qquad \ldots \quad \ldots \quad (2.1)$$

where m_1, m_2, m_3 are the generalised masses of the three modes, referred to a radial reference displacement.

ω_1, ω_2, ω_3 are the resonant frequencies of the three modes *in vacuo*, $\omega_1 < \omega_2$, $\omega_1 < \omega_3$.

$F(\Omega)$ depends on the nodal configuration.

Now ω_2 and ω_3 are very much greater than ω_1, so that when Ω is near ω_1, $(\omega_2^2 - \Omega^2)$ and $(\omega_3^2 - \Omega^2)$ must be large. Also m_2 and m_3 are much greater than m_1 since they correspond to modes with predominantly tangential displacements. Thus near the lowest resonant frequency the second and third terms in equation (2.1) will be very small, *i.e.*, the acoustic coupling between the modes will be very small. The present writer therefore suggests that the acoustic analysis can be done on single degree of freedom systems. In fact, usually the lowest mode of each set of three is the only one of interest since, firstly, the frequencies of the other two are generally too high and, secondly, being mainly tangential modes they will not be excited very much by radial forces, *i.e.*, they have large generalised masses. These points have yet to be verified for the stiffened cylinder.

2.2. *The Shell in Vacuo*.—The most complete treatment of the vibrations of a uniform circular cylinder *in vacuo* is that of Arnold and Warburton (Ref. 5, simply supported ends and Ref. 6, fixed ends. Appendix II, para. 3 of this paper follows their method closely, arriving at essentially the same results as their first paper which differs from equations (II.22) only in the addition of insignificant bending terms, similar to those in equations (II.20) but not the same. Appendix II differs from Ref. 5 in that : slightly different strain expressions, equation (II.9), are used ; in the energy integrals the element $rd\phi$ is not shortened to $ad\phi$; some note is taken of the effect of rotary inertia, indicating the range of validity of the theory which ignores shear deflections and rotary inertia ; the approximate equation (II.24) is obtained for the lowest mode and this equation is used to explain the variations of frequency with nodal patterns.

Arnold and Warburton use an energy approach, deriving expressions for the kinetic and potential energies and then substituting into Lagrange's equations, but equations similar to equation (II.20) can be obtained *via* the equations of equilibrium of an element. The stresses are obtained from equations (II.9), or their equivalent, and equations (II.14) are still used, these forms satisfying the differential equations. This approach is used by Kennard[7] in a ' paraphrase ' of Epstein[8], by Naghdi and Berry[9] and by Yi-Yuan Yu[10], but none of these investigate their results to the extent that Arnold and Warburton do. At the expense of increasing the order of the equation, the w equation can be de-coupled from the u and v equations : if a form such as equations (II.14) is going to be assumed for the deflections, this merely amounts to obtaining the terms of the frequency equation (II.19). On the whole the energy approach is rather simpler and is the only practical one for the stiffened shell with its discontinuities.

Further comments on these other papers are given later (at the end of Section 2.3).

2.3. *The General Theory of Thin Shells.*—One point which arises from a study of these papers is that there is considerable confusion as to what the form of the final equations should be. This arises through various workers applying approximations in rather haphazard manners. The general conclusion reached is that the discrepancies are of little practical importance, *e.g.*, Ref. 11 and discussion on Ref. 12, but in the reply to the discussion on Ref. 13, Joseph points out that ' one does not know in general whether the modifications and corrections are of practical importance except by considering each special case on its own merits '. Since they have only been demonstrated to be small for the special case of the thin uniform shell where extensional strains are generally small and in particular

$$(\text{extensional strain}) \times \frac{z}{a} \ll \text{bending strains} \quad \ldots \quad (2.2)$$

and since the relative magnitudes of bending and extensional stiffnesses are entirely altered by the addition of heavy stiffening, the present writer decided that an investigation of the discrepancies was necessary.

When making use of the ' thinness ' of the shell two types of approximation may be made which I will term ' physical ' and ' mathematical ' approximations. Examples of the first are the neglect of shear strains e_{23} and e_{13} and of normal stress σ_3. The second may take the form of assuming

$$\frac{r}{a} = \left(1 + \frac{z}{a}\right) \simeq 1 \quad \ldots \quad (2.3)$$

as was done by Love, or of curtailing series expansions at particular powers of z/a. It is these latter assumptions which give rise to the discrepancies, since they may be applied in various ways. For instance (by using (2.3) in the derivation) Love (Ref. 14, page 543) obtains the following strain expressions :

$$e_1 = u_x - zw_{xx} \,, \quad \ldots \quad (2.4a)$$

$$e_2 = \frac{1}{a} v_\phi + \frac{w}{a} - \frac{z}{a^2}(w_{\phi\phi} + v_\phi), \quad \ldots \quad (2.4b)$$

$$e_{12} = v_x + \frac{1}{a} u_\phi - \frac{2z}{a}(w_{x\phi} + v_x) \,, \quad \ldots \quad (2.4c)$$

whilst two different linearised forms may be obtained from equations (II.9) by firstly putting $r/a = 1$ to find

$$e_1 = u_x - zw_{xx} , \qquad (2.5a)$$

$$e_2 = \frac{1}{a} v_\phi + \frac{w}{a} - \frac{z}{a^2} w_{\phi\phi} , \qquad (2.5b)$$

$$e_{12} = \frac{1}{a} u_\phi + v_x - \frac{z}{a} 2w_{x\phi} , \qquad (2.5c)$$

or by expanding in powers of z to find

$$e_1 = u_x - zw_{xx} , \qquad (2.6a)$$

$$e_2 = \frac{1}{a} v_\phi + \frac{w}{a} - \frac{z}{a^2} (w_{\phi\phi} + w) , \qquad (2.6b)$$

$$e_{12} = \frac{1}{a} u_\phi + v_x - \frac{z}{a} \left(2w_{x\phi} - \frac{1}{a} u_\phi + v_x\right) . \qquad (2.6c)$$

Thus apparently the same assumption, that z/a is small, can lead to different results. Equations such as (2.4), (2.5) and (2.6) have all be used by various workers. For instance, Arnold and Warburton started by assuming Love's expressions, equations (2.4). One advantage of equations (II.9) over equations (2.4) is that every term in equations (II.9) can be given a direct physical interpretation.

The difficulty with mathematical approximations is that the power of z/a is not a true guide to the importance of any particular term ; for instance, $(z/a)^2 w_{\phi\phi\phi\phi}$ may not be very small compared with $w_{\phi\phi}$. Also, if such approximations are introduced at an early stage, many of the 'large' terms used for comparison may cancel out, thus completely invalidating the assumption. For instance, in Appendix II, the mathematical approximation (II.21) (which it will be noted is less severe than equation (2.3)), has been applied at the last possible stage : if it were applied one stage earlier it would eliminate two of the terms in equations (II.23) and thus eliminate the n^6 and n^4 terms in equation (II.22a) with consequent error when λ and n are small.

It is not necessary to make these assumptions during the derivations and the present writer feels that once the initial physical assumptions have been made there should be no more curtailment until the last possible stage. The principle is the same as carrying an extra figure or two during lengthy numerical calculations in order to avoid rounding-off errors : the extra figures may have no significance in themselves but they do contribute to the final accuracy. Consequently in the Notation such assumptions have been carefully avoided and the physical assumptions made in the most logical manner possible. There is nothing new about the strain expressions obtained (equations (II.8) and (II.9)) but the derivation from equations (II.2) has been presented in a slightly different and, it is hoped, more logical form. It is appreciated that the modifications to Arnold and Warburton's results[5] are of no consequence in that context but they will be important when the theory is extended to cover shear deflections and/or stiffening.

Provided unnecessary 'mathematical' assumptions are avoided, the energy and equilibrium approaches will yield identical results from the same physical assumptions, spurious anomalies being eliminated.

A number of authors have made comments on these discrepancies. Bleich and Di Maggio[11] obtain a strain-energy expression for a cylindrical shell, pointing out that functions of r can be integrated without first expanding them in powers of z : the only expansion they use being for the logarithmic terms. They quote strain expressions identical with equations (II.9) and their result is identical with equation (II.13a).

Langhaar and Carver[15] agree with Ref. 11 and point out that the procedure is possible for any shell. They say that the linearisation (in z) of the strain equations is of doubtful validity and after giving a statical example suggest that it may be justified when calculating the displacements only but not when these are used to calculate strains and then stresses.

Osgood and Joseph[13] re-derive the strain expressions, following Love's method (Ref. 14, Chap. XXIV) in general, but they still linearise the equations, their results for the general shell reducing to equations (2.7) for the cylinder. They then modified the equation of equilibrium by including terms due to the differences between the strained and unstrained axes. This is of little value since the extra terms are non-linear and, under the assumptions of small strains and displacements, are neglected in the linearised theory of elasticity.

Kennard[7] uses a power-series approach, including the effects of shear deflections and normal stresses, to obtain equations of equilibrium expanded up to $(h/a)^2$, i.e., to first bending term only. However, his method involved cutting off series at practically every stage, which I consider to be dangerous. It should be possible to use a similar approach in which the series for the stresses are curtailed (*e.g.*, assume transverse shear stresses to be distributed parabolically) and thereafter all terms are retained. The resulting equations, although not cut off cleanly at any power of t/a, would represent a consistent approximation. For a stiffened cylinder the approach would have to be somewhat different because of the discontinuities. In a further paper (Ref. 16), Kennard modifies his equations by adding multiples of terms which he considers small. This would only be valid for low-order modes for which shear deflections would probably be unimportant anyway.

The final equations resulting from a power-series approach are of successively higher order as more terms are retained. This is not a serious drawback if assumed deflection forms are substituted in the usual way.

Naghdi and Berry[9] start from linearised strain equations and derive equations of equilibrium in the usual way. They then derive an equation, of increased order, in w only : if assumed deflection forms are used this is no advantage. Their equations are no more logical than Love's, and they conclude themselves that they offer no practical improvement.

Yi-Yuan Yu[10] again obtains an equation in w alone, and then makes simplifying assumptions, valid for long axial wavelength only, which reduce it to the fourth order so that it can be readily solved. He obtains solutions for free ends, clamped ends, and one free and one clamped end.

Byrne[17] derives strain expressions which are actually identical with equations (II.8) although algebraically more complex. He has still neglected shear strains and contraction of normals so that much of the algebraic complexity of his subsequent work is of little value.

2.4. *Summary of Conclusions from this Survey*.—2.4.1. *Approximations*.—Some physical assumptions must be made at the outset in order to make the problem tractable. The usual ones are that normals to the middle surface remain normals after straining, that extensions of these normals are negligible, and that normal stresses are negligible. These are adequate for thin uniform shells up to quite high-order modes so that no great effort has been made to improve on them.

Most authors have then made further approximations during their analyses, the most common being that of linearisation (in z) of the strain expressions : these are of little real value and introduce unnecessary errors. Again they are of little practical consequence for uniform shells, provided they are thin enough : the physical assumptions would probably break down before these errors became significant.

Some authors have then simplified their results by making assumptions about the magnitudes of the strains of the middle surface or of such operators as $(h/a)^2\,(\partial/\partial x^2)$. This may be legitimate for low-order modes of uniform shells but it is easy to go astray. In Appendix II the frequency equation has been simplified to equations (II.22) without loss of accuracy but as is pointed out there the approximations (II.21) cannot be applied any earlier without loss of accuracy due to terms cancelling out. Thus manipulations on the equations of equilibrium (equivalent to equations (II.18)) are suspect.

All these assumptions become even more doubtful for the stiffened shell where the bending stiffness is higher compared with the extensional stiffness than in the uniform shell : in particular, assumptions about the smallness of extensional strains fall down. Also the order of modes possible in an aircraft fuselage is much higher than seems to have been envisaged in any of the papers quoted. It is therefore concluded that all approximations, other than the initial physical assumptions, should be avoided in the analysis of stiffened shells.

2.4.2. *Method of analysis*.—It seems that if the same physical assumptions are made and all other approximations avoided, then the same frequency equation is obtained by energy methods (Lagrange's equations) and *via* the differential equation. The displacement functions which satisfy the differential equations also give rise to Lagrange equations which are uncoupled so far as different nodal patterns are concerned and after substitution the sets of equations are identical.

Even for the uniform shell the energy approach seems to be rather simpler as it avoids bringing in ' stress resultants ', *i.e.*, net loads and couples on shell elements. These could be avoided by writing the equations of equilibrium for a general three-dimensional element, substituting the strain expressions and then integrating with respect to z. This is in effect done in some papers but the ' stress resultants ' are still introduced. They are as unnecessary to the theory as the so-called ' changes of curvature '.

For stiffened shells the energy approach is definitely simpler and is used exclusively in this work.

2.4.3. *End conditions*.—The analysis is greatly simpler for ' freely supported ' ends than for any other end conditions because only those conditions give purely sinusoidal normal modes (for the uniform cylinder). The effects of the end supports are small provided the modes are described by λ(= mean circumference/longitudinal wavelength) defined at some part of the cylinder remote from the ends. The discrete values of λ which are allowable will, however, be dependent on end conditions, and if it is desired to identify individual modes the end conditions must be allowed for. However, interest is usually centred on the overall response to an excitation which covers an appreciable frequency band, so that a considerable number of the closely packed cylinder modes are excited. It then makes little difference what end conditions are assumed : the total number of modes excited will be very nearly the same for all end conditions so that the total response will be virtually the same. This applies to displacements and to quantities derived from displacements such as loads and stresses, except, of course, in the immediate vicinity of the ends.

The present work therefore assumes ' simply supported ' ends for the cylinder, *i.e.*, ends closed by thin diaphragms. The extra work involved in allowing for other end conditions does not seem justified, but if it is required it is recommended that coupling of the dynamical equations should be avoided by choosing displacement functions which satisfy the new end conditions, at least in the more important respects, but are still orthogonal. This was done by Arnold and Warburton[6] for the uniform cylinder with fixed ends.

2.4.4. *Application of results*.—It is possible that the coupling between the radial mode and the two tangential modes with the same nodal pattern which arises from the acoustic medium will be small. If so, the information required from this analysis will be simplified. The results must be examined along with the acoustical problem to see whether or not this is so.

3. *The General Character of Cylinder Vibrations*.—3.1. *The Uniform Cylinder*.—The natural modes of the uniform cylinder have been described fully in the literature (*e.g.*, Ref. 5). A typical pattern of the radial deflections is shown in Fig. 2. There are a number of orthogonal nodal lines, longitudinal and circumferential ; each nodal pattern can be described by the number (n) of full waves round the circumference, and the number (m) of half-waves in the length (l). When the ends of the cylinder are simply supported both circumferential and axial mode shapes are sinusoidal so that we can write the radial deflection in the form

$$w = W \cos n\phi \, \sin (m\pi x/l) \, \cos \omega t \, ,$$

where ϕ and x are co-ordinates as shown in Fig. 1, ω is the circular frequency of the vibration and n, m are integers ; $n \geqslant 0$; $m \geqslant 1$.

In general there is also an appreciable amount of tangential movement of the shell walls ; to be consistent with the radial displacement this can be expressed by

$$u = U \cos n\phi \cos (m\pi x/l) \cos \omega t ,$$

$$v = V \sin n\phi \sin (m\pi x/l) \sin \omega t .$$

Clearly, a mode with m half-waves in a length l is exactly equivalent to one with $2m$ half-waves in a length $2l$ so that it is convenient to eliminate l by writing

$$\lambda = \frac{m\pi a}{l} , \qquad \text{so that } \frac{m\pi x}{l} = \frac{\lambda x}{a} .$$

So far as the general theory is concerned λ may take any positive value, but for a particular cylinder it is restricted to a series of values dependent on l. Modes where the generators remain straight (*i.e.*, $\lambda = 0$) can be looked on as the limiting case of an infinitely long cylinder with $m = 1$ (*see* Section 4.8).

It has already been shown (*e.g.*, Refs. 19 and 5) that, corresponding to each nodal configuration (n, m) there are three normal modes with different proportions of U, V, W, and that the resonant frequencies of these modes are of different orders of magnitude. The lowest mode has predominantly radial movement involving both bending and stretching of the cylinder walls : the other two modes have predominantly tangential movement and involve very little bending. The lower, radial, modes are of the most interest generally since the excitation is usually radial and also because the other modes are usually outside the frequency range considered. Both extensional and flexural strain energies are important with these modes and these energies vary in opposite senses with n at constant m, giving the curious situation in which an apparent increase in complexity of the mode does not necessarily mean an increase in resonant frequency[5]. In Appendix II, para. 5 it is shown that this behaviour can be predicted easily, and accurately, from an approximate expression for the lowest root of the frequency equation.

Any system with normal modes which can be ordered by a pair of numbers such as (n, m) would be expected to have its resonant frequencies comparatively closely packed ; much more so than, say, a simple beam with a single modal parameter. For the radial modes of a cylinder over part of the frequency range they are even more closely packed due to the effect mentioned above. Thus, given a random excitation covering a given frequency band, not only will there be an increased number of modes in that band but they will also show a greater diversity of mode shapes, with consequent increase in the chances of finding spatial correspondence of acoustic and structural waves[20, 21].

3.2. *The Effects of Adding Periodic Stiffening.*—In a typical aircraft fuselage the stiffening is fairly heavy compared with the skin so that it can by no means be considered as a minor modification and is of fundamental importance in determining the vibrations of the shell. We will now discuss, in general terms, the effects of this stiffening.

Firstly, the relative magnitudes of the extensional and bending stiffnesses are completely altered : the extensional stiffness and the mass are comparatively little changed, but the bending stiffness is of an altogether different order. The general remarks of Section 3.1 still apply : the quantitative results are changed, of course, but qualitatively the modes are the same (but *see* the third point, below). We can no longer employ order-of-magnitude arguments depending on the smallness of extensional strains, as used by many authors dealing with uniform cylinders, in order to make simplifying approximations (*see* Sections 2.3 and 2.4.1).

Secondly, rotary inertia and shear deflections become important for modes of comparatively low order. Remembering that we are concerned with n and up to at least 50, which is much higher than seems to be considered by workers investigating uniform shells, it is seen that some allowance must eventually be made for these effects. Although they will be of quantitative importance they are not likely to alter the modes qualitatively.

Thirdly, as shown in Appendix IV, the sinusoidal modes described in Section 3.1 are not completely independent, the periodicity of the stiffeners introducing coupling terms between certain of them. A sinusoidal mode with distances between nodal lines considerably larger than the stiffener spacing will be only lightly coupled to other modes, so that a good approximation to the corresponding normal mode will be obtained by ignoring the coupling. However, as the order of the mode increases the coupling becomes more important and will be very large for two modes of which one has nodal intervals just less than the stiffener spacing and the other just greater. The resulting normal modes will be far from sinusoidal, having the appearance of modulated sinusoidal waves with considerably varying amplitude. Thus we can imagine a natural mode in which one side of the cylinder vibrates with large amplitude whilst the other is hardly moving. The so-called panel and stringer modes must arise in this way. It is hoped to make these modes the subject of further investigations ; some preliminary thoughts are included in the Appendices here.

The present paper is by way of clearing the ground for a fuller investigation and, in the main, is confined to natural modes of long wavelength for which the second and third points above are not important, but the possible extensions have been kept in mind and explain the comparative rigour of the treatment. The results arrived at by the summations of Appendix IV can also be obtained by imagining the various stiffnesses of the frames and longerons to be spread uniformly over the shell, but that approach gives no idea of the range of validity and, of course, cannot be extended to higher modes.

4. *The Analysis of the Stiffened Cylinder.—Closely Spaced Stiffeners.*—4.1. *Summary of Method.*— This is basically the method used by Arnold and Warburton[5] for the uniform cylinder, and which is set out in Appendix II with minor modifications. It is an energy method, using Lagrange's equations, the main steps being as follows :

(i) The thinness of the shell enables its entire deflected form, and hence the components of strain at any point, to be expressed in terms of the displacements of a reference surface, taken at the middle surface of the skin. This depends mainly on the neglect of shear strains through the thickness of the shell wall, corresponding to engineers' bending theory.

(ii) Neglecting certain components of the stresses, the strain energy is expressed in integral and summation form in terms of the deflections of the reference surface and their derivatives. The kinetic energy can be similarly expressed in terms of time derivatives of the deflections.

(iii) Sinusoidal forms are assumed for the deflections and the integrations and summations carried out.

(iv) The final energy expressions are substituted into a set of three Lagrange equations yielding three simultaneous equations with the maximum deflection components as unknowns. These equations are homogeneous and their eliminant gives a cubic frequency equation and hence three resonant frequencies, as described in the previous section.

These steps are now covered in more detail, the mathematics being given in full in the Appendices.

4.2. *Strain Components.*—In Appendix II, para. 1, it is assumed that the co-ordinate system is so chosen that $z = 0$ defines the middle surface of the shell. This is not essential : the whole of para. 1 is still valid if $z = 0$ is any surface parallel to the middle surface. $z = 0$ is simply a reference surface at which the displacements are specified, the ' thin-shell ' assumptions enabling the deflected form of the entire shell to be defined by the deflected form of the reference surface. All members of the family of surfaces $z =$ constant are parallel and remain approximately parallel after straining, so that assumption II.5, that normals to $z = 0$ become normals to the strained reference surface with negligible extensions, have the same physical significance for any choice of reference surface from that family.

The most convenient reference surface for the stiffened shell is the middle surface of the skin, there being no unique neutral surface for bending, and the complication of using different reference radii for the u and r displacements hardly seeming worth while. If then, we neglect shear deflections of the longerons we find that, in the longerons, normals to the skin remain normal to it after straining so that, if we also neglect the extensions of those normals, the radial and longitudinal components of displacement at any point in the longerons are given by equations (II.10*a*) and (II.10*c*). For the cylinder equation (II.4*a*) becomes simply

$$e_1 = \frac{\partial}{\partial x}(\delta_1)$$

and substituting from (II.10*a*) we find the longitudinal strain, e_1, in the longerons to be given by equation (II.9*a*). Equations (II.9*b*) and (II.9*c*) are irrelevant for the longerons, but equation (II.10*b*) still applies if we neglect lateral shearing, e_{23}.

Similarly, equations (II.10) and equation (II.9*b*) apply to the frames.

These arguments can be generalised for any shell with stiffeners lying along its lines of curvature, the appropriate equations from (II.7) and (II.8) applying.

4.3. *Strain Energy*.—For the longerons, only the strain energy due to longitudinal strains is relevant and that is given by the integral of $\frac{1}{2}Ee_1^2$ over the volume of the longerons. Equation (II.9*a*) gives e_1 in terms of z and the displacements so that we can carry out the z integration and obtain the strain energy of one longeron as a line integral, equation (III.3), in terms of u, w and the section constants. After substituting the sinusoidal deflection forms (II.14) we can carry out the x integration to give equation (III.4).

Provided the stringers are equally spaced round the circumference we can sum equation (III.4) over the stringers, the result being exactly the same as if we imagined the various section constants to be spread uniformly round the circumference. For instance, if the cross-sectional area A_l were spread uniformly it would give $pA_l/2\pi a$ per unit circumference and we have

$$\int_0^{2\pi} (pA_l/2\pi a) \cos^2 (n\phi)\, ad\phi = p/2 \, .$$

But from Appendix IV we see that, within the limits of the assumption of close stiffener spacing,

$$\sum_{r=0}^{p-1} A_l \cos^2 (r2\pi n/p) = p/2 \, ,$$

the two results being clearly equivalent.

It is convenient to take out as a factor the same quantity as appears for the uniform shell, namely, $\pi Ehl/4a(1 - \nu^2)$ and this leads naturally to the definition of the dimensionless constants L_0, L_1, L_2, depending on the section constants. As longerons are normally inside the skin, the sign of L_1 has been arranged to be positive then : as z is positive outwards, this corresponds to a negative first moment of area. The final expression for strain energy of the longerons is given in equation (III.5).

An exactly parallel procedure for the frames leads to equation (III.10).

It will be noted that, as $z = 0$ is not the neutral surface for bending we do not attempt to distinguish between extensional and bending strain energies. The procedure used here automatically introduces all the required section constants and actually defines them explicitly, this being particularly convenient for the frames.

The strain energy of the skin is given by equation (II.16*a*), as for the uniform shell, and we can now write the total strain energy of the shell. It is found that the bending strain energy of the skin is negligible and it has been omitted in equation (III.11). This is equivalent to omitting a very small term when calculating the combined second moment of area of the skin and stiffener.

4.4. *Kinetic Energy*.—Assuming that the mass of one longeron is concentrated on a plane of constant ϕ, its kinetic energy can be written, using equations (II.10), in terms of z and the components of velocity $\dot{u}$, $\dot{v}$, $\dot{w}$, of the reference surface, at that value of ϕ (equation (III.14)). Following the procedure used for the strain energy this can be integrated and summed to give the total kinetic energy of the longerons in terms of the time-dependent displacement coefficients q_1, q_2, q_3 (equation (III.18)).

It is convenient to take out the factor $\pi\rho hla/4$, which appeared in the expression for the kinetic energy of the uniform shell, and this leads naturally to the dimensionless inertial constants H_0, H_1, H_2. One unfortunate result of using a reference surface which does not coincide with the mass centroids of the longeron-skin combinations is that we cannot neglect terms in H_1 and H_2, for they are not merely rotary inertia terms in the usual sense of the word. The terms in $H_1\dot{q}_1\dot{q}_3$, $H_1\dot{q}_2\dot{q}_3$ and $H_2\dot{q}_3{}^2$ are due principally to this offset and would still arise if the inertia due to translation of the mass centroids were calculated. A change of variables to displacements of mass centroids would not be entirely satisfactory since the mass centroids of the frames and stringers are at different radii.

The only term in equation (III.18) which could reasonably be dropped is that in $H_2\dot{q}_2{}^2$, since $H_2 \ll H_0$ generally. Neglect of rotary inertia would merely change H_2 by subtracting an amount proportional to the moment of inertia about the centroid, with no real simplification.

An identical procedure leads to equation (III.24) for the kinetic energy of the frames. The form of the approximate expressions (III.23) and (III.25) is another result of the difference between the radii to the reference surface and to the mass centroids. The kinetic energy of the skin is still given by equation (II.16*b*), as for the uniform shell, the effect of rotary inertia of the skin about its own middle surface being completely negligible for fairly heavy stiffening, and so the total kinetic energy of the shell can be written as in equation (III.26), a crude approximation being given by equation (III.27). However, $(F_2 + L_2)$ can easily be of the order $0 \cdot 01$ to $0 \cdot 05$ so that equation (III.27) can only be expected to be useful for the very lowest modes.

4.5. *The Dynamic Equations*.—We now have the strain energy as a function of the q_r and the kinetic energy as a function of the $\dot{q}_r$, and since the q_r are independent we can use them as generalised co-ordinates in a set of Lagrange equations (III.29), giving three homogeneous linear equations (III.30), in the maxima of the displacement components, U, V and W. The eliminant of these equations can be expanded to give a cubic in the frequency parameter Δ :

$$K_3\Delta^3 - K_2\Delta^2 + K_1\Delta - K_0 = 0$$

where, for instance, K_3 is the determinant of the coefficients of the inertial matrix $[B]$ and K_1 is the determinant of the coefficients of the strain energy matrix $[C]$.

This cubic can be solved for three values of Δ and the corresponding frequencies obtained from

$$f = \frac{1}{2\pi a}\sqrt{\left(\frac{E\Delta}{\rho(1-\nu^2)}\right)}.$$

There seems to be little advantage in actually obtaining expressions for the K from equations (III.30) as the results would be extremely involved. When obtaining numerical solutions for a particular cylinder the best course seems to be to first substitute numerical values of the F_r, G_r, H_r, L_r and ν into equations (III.30), obtaining the elements of $[B]$ and $[C]$ as functions of λ, n only. Particular values of λ, n are chosen, the K_r evaluated and the roots of the frequency equation obtained. Alternatively the K_r could be expanded as functions of the λ, n only, after the first numerical substitution.

The lowest root of the frequency equation can be obtained rapidly by iteration from the form

$$\Delta = \frac{K_0}{K_1} + \frac{K_2}{K_1}\Delta^2 - \frac{K_3}{K_1}\Delta^3$$

since, for that root, Δ is generally appreciably less than 1.

4.6. *The Generalised Mass.*—We are concerned with radial excitations and so refer the generalised mass M to the co-ordinate q_3, it being defined by

$$T_t = \tfrac{1}{2} M \dot{q}_3^2 .$$

The ratios $(\dot{q}_1/\dot{q}_3)$, $(\dot{q}_2/\dot{q}_3)$ are the same as (U/W), (V/W), these being obtained from equations (III.30) after evaluating the roots of the frequency equation, so that by substituting for T_t from equation (III.26), we find M as in equation (III.33).

In general, for the lowest, radial, mode, (U/W) and (V/W) will be rather larger for the stiffened cylinder than for the uniform cylinder. This is due partly to the relatively lower extensional stiffness of the stiffened cylinder, and partly to the difference between the radius of the reference surface and the radii to the centroids of the shell wall.

4.7. *Note on the Assumed Mode Shape.*—Strictly the deflections should be expanded completely in the forms

$$u = \sum_n \sum_m \{q_{1nm} \cos n\phi \ \cos (\lambda x/a) + q'_{1nm} \sin n\phi \ \cos (\lambda x/a)\}$$

$$v = \sum_n \sum_m \{q_{2nm} \sin n\phi \ \sin (\lambda x/a) + q'_{2nm} \cos n\phi \ \sin (\lambda x/a)\}$$

$$w = \sum_n \sum_m \{q_{3nm} \cos n\phi \ \sin (\lambda x/a) + q'_{3nm} \sin n\phi \ \sin (\lambda x/a)\} .$$

These are complete expressions, capable of describing any arbitrary mode shape and would yield $6nm$ independent Lagrange equations. For the uniform shell or for the shell with closely spaced stiffeners these $6nm$ equations would be grouped in $2nm$ sets of 3 coupled equations with no coupling between the groups. Also the nm sets in the q_r would be identical with the nm sets in the q_r' so that the q_r' may be ignored completely. This is equivalent to saying that the choice of origin for co-ordinate ϕ is immaterial.

The substitution of equations (II.14), with no summation over n and m is therefore justified for the uniform or closely stiffened cylinders.

When considering higher modes, however, these coupling terms will not all disappear and we must retain the full expressions as above.

4.8. *The Special Cases* $n = 0$, $\lambda = 0$.

(i) $\underline{n = 0,\ \lambda \neq 0.}$

This amounts to replacing deflection forms (II.14) by

$$u = q_1 \cos (\lambda x/a) ,$$

$$v = 0 ,$$

$$w = q_3 \sin (\lambda x/a) .$$

It is easily verified that the strain and kinetic energies are obtained correctly by putting $n = 0$ and $q_2 = 0$ in equations (III.11) and (III.26) and doubling the remaining terms. The frequency equation is then obtained from equations (III.30) in the same way except that the doubling cancels out. Putting $q_2 = 0$ has eliminated the trivial solution $\Delta = 0$ and we are left with a quadratic in Δ. The generalised mass is obtained from equation (III.33) by putting $n = 0$ and $V = 0$ and doubling the remaining terms.

(ii) $\underline{n \neq 0,\ \lambda = 0.}$

This is a limiting case obtained by letting $l \to \infty$ with $m = 1$. The corresponding deflection forms are

$$u = 0 ,$$

$$v = q_2 \sin n\phi ,$$

$$w = q_3 \cos n\phi ,$$

where, paradoxically, sin $(\lambda x/a)$ has been replaced by 1 and cos $(\lambda x/a)$ by 0. This arises since as $l \to \infty$ we must let $x \to \infty$ as well. The expressions for the strain and kinetic energies must be taken over a finite length, so we will consider a finite length l out of an infinitely long cylinder. We then find that the strain and kinetic energies are given from equations (III.11) and (III.26) by putting $\lambda = 0$ and $q_1 = 0$ and doubling. The dynamic equations (III.30) are treated similarly, the doubling cancelling, and the frequency equation is reduced to a quadratic in Δ, a root $\Delta = 0$ having been eliminated. The generalised mass is obtained from equation (III.33) by putting $\lambda = 0$ and $U = 0$ and doubling.

(iii) $\underline{n = 0 \text{ and } \lambda = 0.}$

This corresponds to displacements

$$u = 0\,, \qquad v = 0\,, \qquad w = q_3$$

and we simply apply (i) above together with (ii), multiplying by four where appropriate.

5. *Extensions of the Theory.*—5.1. *The Coupling Effects due to Stiffener Spacing.*—We consider what happens when we use the full expansion for the deflections as given in Section 4.7. All coupling terms vanish during the integrations, and couplings between the q and q' terms vanish during summations as seen from equation (IV.11), but equations (IV.9), (IV.10), (IV.12) and (IV.13) show that there will be coupling between sinusoidal modes with

$$n_1 \pm n_2 = 0\,, \quad \pm p\,, \quad \pm 2p \ldots.$$

and/or

$$m_1 \pm m_2 = 0\,, \quad \pm 2q\,, \quad \pm 4q \ldots.$$

If we start from a mode (n, m) we find that it is coupled to the following modes, assuming $n < p/2$, $m < q$:

(n, m)	$(n, 2q - m)$	$(n, 2q + m) \ldots.$
$(p - n, m)$	$(p - n, 2q - m)$	$(p - n, 2q + m) \ldots.$
$(p + n, m)$	$(p + n, 2q - m)$	$(p + n), 2q + m)$
$\ldots.$		

This is a doubly infinite set of modes, every one of which is coupled to every other one, but to no others, *i.e.*, it is a closed set. There are in fact about $p/2 \times q$ such sets. Although the magnitude of the coupling problem is reduced in that large proportion it is still considerable. Assuming n, m to be small, as assumed in Section 4, the other eight modes in the above array represent only the next order of magnitude of mode number, and yet there are already 27 coupled equations. It appears that any complete investigation will require a digital computer.

The magnitude of the coefficients of the uncoupled equations is dependent on the order of the mode numbers. The direct coupling terms between (n, m) and the other modes will be of the general order of the (n, m) coefficients, and consequently much less than the coefficients of the other modes. To a first approximation then, if n, m are sufficiently small, we can split the array up into four groups as shown by the dashed lines. This is the justification for the procedure of Section 4. It should be possible to obtain some idea of how small n, m are required to be, by some approximate method and this is being followed up. The other three groups could also be investigated since they are of amenable order.

A group which will be interesting when investigating these couplings is that starting from (0, 0), *viz.*,

$$\begin{array}{lll} (0, 0) & (0.\ 2q) & (0, 4q) \ldots \\ (p, 0) & (p, 2q) & (p, 4q) \ \ldots \\ (2p, 0) & (2p, 2q) & (2p, 4q) \ldots. \end{array}$$

This set should give something like the so-called ' panel ' and ' stringer ' modes.

Similar coupling will also arise from irregularities in the stiffening. For instance, some extra heavy longerons could be added as extra stiffening with large spacing, so giving coupling between quite low modes.

It is proposed in the first instance to get some idea of the importance of these effects in a few numerical calculations, starting from the uncoupled modes and adding first order corrections for the coupling.

5.2. *Relaxation of Thin-Shell Assumptions.*–The most important step in this will be the allowance for shear deflections. This can be done, in a similar manner to that already used on beams, by separating the shear and bending deflections. The present analysis applied to bending deflections will allow the resultant bending moments to be calculated and the equations of equilibrium will then give the corresponding shear forces so that shear deflections can be calculated. The details of this analysis have not yet been worked out. For instance, it is not yet clear whether it will be better to have the final equations in terms of bending deflections or total deflections. The principle is, however, reasonably clear and should lead to a straightforward development, the first object of which will be to find when the effects become important.

6. *Remarks and Conclusions.*—A theory has been developed which gives the lower resonant frequencies and corresponding mode shapes of a right circular cylindrical shell with uniform, closely spaced stiffeners in the form of circular frames and straight longerons. Further numerical work is required to establish the influence of the various parameters, but, within the limits of the assumptions, the general character of the results appears to be similar to that of the uniform cylinder results. Some indication is given of the possible extensions of the theory to cover higher-order modes where stiffener spacing becomes important, and to investigate the importance of various assumptions, including some which are common to usual shell theory. This has involved an investigation of the general theory of thin shells, leading, it is believed, to some degree of rationalisation of the method and assumptions.

No numerical results are presented here, but some preliminary calculations on a typical fuselage indicated a very large number of modes with frequencies in the range present in jet noise ; a number which would probably be further increased by the inclusion of higher modes. If thorough calculations are carried out for some typical fuselages and plotted in the form of curves of frequency parameter, Δ, against longitudinal wavelength parameter, λ, for a series of values of circumferential mode number, n, then it is possible that on any fresh fuselage it would only be necessary to calculate a few key modes, the rest being obtained by interpolation. The complete investigations would require the use of a digital computer, for which the problem is well suited, involving systematic variation of parameters.

The results would need to be incorporated in the investigations, which are proceeding, of the effects of the acoustic medium, the final presentation possibly being of the modified modes.

Some experimental checks would also be desirable but the overall theory is not yet advanced enough for a model to be designed, there being a large number of structural and acoustic parameters to be considered. For instance, it is not yet certain whether we will wish to have the stiffeners very close together, as in full-scale practice, or whether to have relatively few stiffeners to accentuate the coupling effects due to stiffener spacing. The overall scale of the model will depend on the nature of the interaction of acoustic and mechanical effects which is not yet fully understood. The answer may be to have two models ; one small with few stiffeners for studies of the mechanical features ; the other larger and more complicated for studies of overall response, including acoustic effects and damping, both acoustic and mechanical.

NOTATION

General Notation and Co-ordinate Systems

(x, ϕ, r)		Right-handed cylindrical polar co-ordinates, origin at one end of cylinder
z	$=$	$r - a$ (Perpendicular distance outwards from middle surface of skin)
$\delta_1, \delta_2, \delta_3$		Linear displacements in the longitudinal, tangential and radial directions at any point in the shell
u, v, w		Linear displacements in the longitudinal, tangential and radial directions at any point on the middle surface of the skin.
q_1, q_2, q_3		Coefficients in the Fourier expansions, in space, of u, v, w. Functions of time only : used as generalised co-ordinates
U, V, W		Maximum values of q_1, q_2, q_3 = maximum values of u, v, w
t		Time
ω	$=$	$2\pi f$ (Circular frequency : time dependence taken in form $\cos \omega t$)
Δ	$=$	$\rho a^2(1 - \nu^2)\omega^2/E$ (Non-dimensional frequency parameter)
n		Number of full waves round circumference
m		Number of half waves in length l
λ	$=$	$\dfrac{m\pi a}{l} = \dfrac{\text{circumference at skin median}}{\text{longitudinal wavelength}}$
S_l, S_f, S_t		Strain energy of longerons, frames, and total shell
T_l, T_f, T_t		Kinetic energy of longerons, frames, and total shell
e_1, e_2, e_3		Direct } strain components at any point in shell, referred to (x, ϕ, r) co-ordinates
e_{23}, e_{31}, e_{12}		Shear }
$\sigma_1, \sigma_2, \sigma_3$		Direct } stress components at any point in shell, referred to (x, ϕ, r) co-ordinates
$\sigma_{23}, \sigma_{31}, \sigma_{12}$		Shear }
Suffixes $_{1, 2, 3}$		Refer to co-ordinates (x, ϕ, r) or (x, ϕ, z)
Suffixes $_{x, \phi}$		Denote partial differentiation $\partial/\partial x$
Suffixes $_{l, f, t}$		Refer to longerons (or stringers), frames, and total shell
Dots (e.g., $\dot{u}$)		Denote partial differentiation $\partial/\partial t$

Also, in Appendix II, para. 1 only, we use

(ξ_1, ξ_2, ξ_3)	General curvilinear co-ordinates
η_1, η_2, η_3	Quantities converting elements $d\xi$ to lengths
(ξ_1, ξ_2, z)	Co-ordinates for general shell
R_1, R_2	Principal radii of curvature of surface $z = 0$
A, B	Values of η_1, η_2 at $z = 0$
Suffixes $_{1, 2, 3}$	Refer to (ξ_1, ξ_2, ξ_3) or (ξ_1, ξ_2, z) co-ordinates

Physical Constants

E		Youngs Modulus of shell material, assumed constant (differences could be incorporated in areas)
G	$=$	$E/2(1+\nu)$ (Modulus of Rigidity)
ν		Poissons ratio for skin material
ρ		Density of skin material
ρ_l, ρ_f		Densities of longerons, frames, variagle with r, but averaged over width
$\bar{\rho}_{,l}$ $\bar{\rho}_f$		Densities of longerons, frames, averaged over complete section

Dimensions of Cylinder

a		Radius to skin median
h		Skin thickness
l		Length of cylinder
p		Number of longerons ; longeron spacing $= 2\pi a/p$
q		Number of frame lags in length l spacing $= l/q$
b_l, b_f		Net breadth of longerons, frames : functions of r
A_l	$=$	$\int_{\text{depth}} b_l\, dz$ (Area of longeron section)
$A_l^{(r)}$	$=$	$\int_{\text{depth}} b_l z^r\, dz$ (rth moment of area about skin median of longeron section)
A_f	$=$	$\int_{\text{depth}} b_f\, dz$ (Area of frame section)
$A_f^{(r)}$	$=$	$\int_{\text{depth}} b_f z^r\, dz$ (rth moment of area about skin median of frame section)
m_l	$=$	$\int_{\text{depth}} \rho_l b_l\, dz$ (Mass of one longeron per unit length)
$I_l^{(r)}$	$=$	$\int_{\text{depth}} \rho_l b_l z^r\, dz$ (rth moment of mass of longeron per unit length about skin median)
m_f	$=$	$\int_{\text{depth}} p_f b_f \frac{r}{a}\, dz$ (Mass of one frame per unit circumference at skin median)
$I_f^{(r)}$	$=$	$\int_{\text{depth}} \rho_f b_f \frac{r}{a} z^r\, dz$ (rth moment of mass of frame per unit circumference at skin median, about skin median)

Dimensionless Constants

$$L_r = (-1)^r \frac{pA_e^{(r)}(1-\nu^2)}{2\pi h a^{r+1}}$$

$$F_r = (-1)^r \frac{qA_f^{(r)}(1-\nu^2)}{hla^r}$$

$$H_0 = \frac{pm_l}{2\pi\rho ha} \simeq \frac{\bar{\rho}_l}{\rho(1-\nu^2)} L_0$$

$$H_r = (-1)^r \frac{pI_l^{(r)}}{2\pi\rho h a^{r+1}} \simeq \frac{\bar{\rho}_l}{\rho(1-\nu^2)} L_r$$

$$G_0 = \frac{qm_f}{\rho hl} \simeq \frac{\rho_f}{\rho(1-\nu^2)} (F_0 - F_1)$$

$$G_r = (-1)^r \frac{qI_f^{(r)}}{\rho hla^r} \simeq \frac{\rho_f}{\rho(1-\nu^2)} (F_r - F_{r+1})$$

Note that for r odd, L_r, F_r, H_r, G_r are negative when the longerons/frames are outside the skin.

$\beta = h^2/12a^2$

Δ, n, λ Defined above

Miscellaneous

K_0, K_1, K_2, K_3 Coefficients of frequency equation expressed as cubic in Δ

$[C]$ Matrix of coefficients arising from strain energy

$[B]$ Matrix of coefficients arising from inertia

REFERENCES

No.	Author	Title, etc.
1	M. C. Junger	Vibration of elastic shells in a fluid medium and the associated radiation of sound. *J. App. Mech.* Vol. 19. 1952.
2	M. C. Junger	The dynamic behaviour of reinforced cylindrical shells in a vacuum and in a fluid. *J. App. Mech.* Vol. 21. 1954.
3	H. H. Bleich and M. L. Baron ..	Free and forced vibrations of an infinitely long cylindrical shell in an infinite acoustic medium. *J. App. Mech.* Vol. 21. 1954.
4	M. L. Baron and H. H. Bleich ..	Table for frequencies and modes of free vibration of infinitely long thin cylindrical shells. *J. App. Mech.* Vol. 21. 1954.
5	R. N. Arnold and G. B. Warburton	Flexural vibrations of the walls of thin cylindrical shells having freely supported ends. *Proc. Roy. Soc.* (A). Vol. 197. 1949.
6	R. N. Arnold and G. B. Warburton	The flexural vibrations of thin cylinders. *Proc. Inst. Mech. Eng.* (A). Vol. 167. 1953.
7	E. H. Kennard	The new approach to shell theory.—Circular cylinders. *J. App. Mech.* Vol. 20. 1953. Discussion, same Vol.
8	P. S. Epstein	On the theory of elastic vibrations in plates and shells. *J. Math. Phys.* Vol. 21. 1942.
9	P. M. Naghdi and J. G. Berry ..	On the equation of motion of cylindrical shells. *J. App. Mech.* Vol. 21. 1954.
10	Yi-Yuan Yu	Free vibrations of thin cylindrical shells having finite lengths with freely supported and clamped edges. *J. App. Mech.* Vol. 22. 1955.
11	H. H. Bleich and F. Di Maggio ..	A strain energy expression for thin cylindrical shells. *J. App. Mech.* Vol. 20. 1953.
12	H. L. Langhaar	A strain energy expression for thin elastic shells. *J. App. Mech.* Vol. 16. 1949. Discussion. Vol. 17. 1950.
13	W. R. Osgood and J. A. Joseph ..	On the general theory of thin shells. *J. App. Mech.* Vol. 17. 1950. Discussion. Vol. 18. 1951.
14	A. E. H. Love	*The Mathematical Theory of Elasticity.* 4th Ed. Cambridge University Press. 1952.
15	H. L. Langhaar and D. R. Carver ..	On the strain energy of shells. *J. App. Mech.* Vol. 21. 1954.
16	E. H. Kennard	Cylindrical shells: energy, equilibrium, addenda and erratum. *J. App. Mech.* Vol. 22. 1955.
17	R. Byrne	Theory of small deformations of a thin elastic shell. Univ. of California. Pubs. in Math. N.S. Vol. 2. 1944.
18	C. T. Wang	*Applied Elasticity.* McGraw Hill. 1953.
19	W. Flugge	*Statik und Dynamik der Schalen.* Edwards Bros. Mich. 1943.
20	A. Powell	The problem of structural failure due to jet noise. C.P. 17, 514. March, 1955.
21	A. Powell	On the fatigue failure of structures due to vibrations excited by random pressure fields. C.P. 17,925. October, 1955.

APPENDIX I

Summary of Assumptions

1. *The Cylinder Model.*

(i) It is a right circular cylinder with constant skin thickness.

(ii) The stiffening consists of uniform equally spaced circular frames and uniform equally spaced straight stringers.

(iii) Frames and stringers are both attached directly to the skin.

(iv) The cylinder is closed at the end by thin diaphragms rigid in their own plane but free to warp, *i.e.*,

$$\text{at ends} \quad v = w = 0 \; ; \; u \text{ unspecified}$$

$$u_x = 0 = w_{xx} \; ; \; v_x, \, u_\phi \text{ unspecified.}$$

(v) The distances between successive nodal lines is large compared with the stiffener spacing.

2. *Analysis of Stress and Strain.*

(vi) Strains and displacements are small. This is essential in order to use a linearised theory of elasticity.

(vii) Normals to the middle surface of the skin remain normal to it and straight after straining, *i.e.*,

$$e_{23} = e_{31} = 0 \, .$$

(viii) The effect of the changes in length of the normal are negligible, *i.e.*,

$$\delta_3 \simeq w \, .$$

(ix) The normal stresses are negligible, *i.e.*,

$$\sigma_3 = 0.$$

For the stiffeners we also neglect strain energy arising from

(x) . . . shear in the plane of the skin

(xi) . . . torsion

(xii) . . . transverse stresses in the plane of the skin.

Assumptions (i), (iv), (vi), (vii), (viii) and (ix) are common to the usual theory for the uniform cylinder.

3. *Comments.—Future Developments.*—Work is in hand or contemplated to relax some of these assumptions, or, at least, to investigate their importance, as follows :

(i) The effects of taper on the cylinder will be considered. This is one reason for presenting Appendix II, para. 1 in terms of the general shell.

(ii) Irregularities of stiffening will give effects similar to those discussed in Section 5.1 and under (v) below.

(iii) When frames are attached to the inside of longerons a separate degree of freedom is introduced, the circumferential frame displacements not being the same as those of the skin. A first approximation is given by dropping bending strain energy of the frames, but the problem can be isolated and the frames solved separately under arbitrary radial displacements.

(iv) This is covered by Section 2.4.3.

(v) First thoughts on this problem are given in Section 5.1.

(vi) Strains are certainly small but radial displacements may be appreciable compared with the shell thickness. If displacements are not assumed small, then the strain expressions become non-linear in displacements. This was followed through for the uniform cylinder giving rise to a very large number (over 300) non-linear terms in the strain-energy expression. It is hoped to apply a simplified procedure to the stiffened shell, retaining the most important non-linear terms, so as to get some estimate of the magnitude of displacement at which the effects become important. This knowledge will be important in any experimental work undertaken.

(vii) The relaxation of this is mentioned in Section 5.2.

(viii) to (xii) These effects are almost certainly small, except possibly (xi), but their importance can be estimated, at least for a given numerical example.

It will be appreciated that a number of these points require numerical investigation to establish orders of magnitude and importance of various factors. These would, in the first instance, be carried out by hand to obtain a general picture, but any detailed investigation, particularly of coupling effects, or covering a wide range of mode numbers, could be programmed for a digital computer.

APPENDIX II

The Unstiffened Shell

1. *General Theory of Thin Shells.—Strain Components.*—Let (ξ_1, ξ_2, ξ_3) be a set of right-handed orthogonal curvilinear co-ordinates.

Let η_1, η_2, η_3 be factors converting distances along co-ordinates to lengths, thus an element of length is given by

$$(ds)^2 = (\eta_1\, d\xi_1)^2 + (\eta_2\, d\xi_2)^2 + (\eta_3\, d\xi_3)^2 . \tag{II.1}$$

In general η_1, η_2, η_3 are functions of ξ_1. ξ_2, ξ_3.

Let δ_1, δ_2, δ_3 be the three components of displacement at any point referred to the tangents at that point to the ξ_r co-ordinates.

Then the six components of strain are given by

$$e_1 = \frac{1}{\eta_1}\frac{\partial\delta_1}{\partial\xi_1} + \frac{\delta_2}{\eta_1\eta_2}\frac{\partial\eta_1}{\partial\xi_2} + \frac{\delta_3}{\eta_3\eta_1}\frac{\partial\eta_1}{\partial\xi_3}, \tag{II.2a}$$

$$e_2 = \frac{1}{\eta_2}\frac{\partial\delta_2}{\partial\xi_2} + \frac{\delta_3}{\eta_2\eta_3}\frac{\partial\eta_2}{\partial\xi_3} + \frac{\delta_1}{\eta_1\eta_2}\frac{\partial\eta_2}{\partial\xi_1}, \tag{II.2b}$$

$$e_3 = \frac{1}{\eta_3}\frac{\partial\delta_3}{\partial\xi_3} + \frac{\delta_1}{\eta_3\eta_1}\frac{\partial\eta_3}{\partial\xi_1} + \frac{\delta_2}{\eta_2\eta_3}\frac{\partial\eta_3}{\partial\xi_2}, \tag{II.2c}$$

$$e_{12} = \frac{\eta_2}{\eta_1}\frac{\partial}{\partial\xi_1}\left(\frac{\delta_2}{\eta_2}\right) + \frac{\eta_1}{\eta_2}\frac{\partial}{\partial\xi_2}\left(\frac{\partial_1}{\eta_1}\right), \tag{II.2d}$$

$$e_{23} = \frac{\eta_3}{\eta_2}\frac{\partial}{\partial\xi_2}\left(\frac{\delta_3}{\eta_3}\right) + \frac{\eta_2}{\eta_3}\frac{\partial}{\partial\xi_3}\left(\frac{\delta_2}{\eta_2}\right), \tag{II.2e}$$

$$e_{31} = \frac{\eta_1}{\eta_3}\frac{\partial}{\partial\xi_3}\left(\frac{\delta_1}{\eta_1}\right) + \frac{\eta_3}{\eta_1}\frac{\partial}{\partial\xi_1}\left(\frac{\delta_3}{\eta_3}\right) \tag{II.2f}$$

See, for example, Wang[18], page 336, equations (12.45). These equations assume only small strains and small displacements, as always assumed in the linearised theory of elasticity and apply to any three dimensional body.

We now change the co-ordinate system to one which is suitable for describing a thin shell.

Let the surface $\xi_3 = 0$ be the mid-surface of the shell.

Replace ξ_3 by z, the normal distance from the mid-surface.

Thus

$$\eta_3 \equiv 1, \text{ for all } \xi_1, \xi_2, z . \tag{II.3a}$$

Since (ξ_1, ξ_2, z) are orthogonal co-ordinates then $\xi_2 =$ constant, $\xi_1 =$ constant must be lines of curvature of $z = 0$ (theorem due to Dupin ; *see* Love (Ref. 14), page 51). The z co-ordinates are straight lines and the z co-ordinates through two adjacent points on the line ($\xi_2 =$ constant, $z = 0$) must meet at the centre of curvature of that line. It follows that

$$\eta_1 = A(1 - z/R_1) \tag{II.3b}$$

and similarly

$$\eta_2 = B(1 - z/R_2) , \tag{II.3c}$$

where A, B are the values of η_1, η_2 at $z = 0$, and R_1 R_2 are the principal radii of curvature of $z = 0$, taken positive when the centre of curvature has a positive z co-ordinate.

Changing the co-ordinates (ξ_1, ξ_2, ξ_3) to (ξ_1, ξ_2, z) and using equations (II.3), we can write equations (II.2) as :

$$e_2 = \frac{1}{\eta_1}\frac{\partial \delta_1}{\partial \xi_1} + \frac{\delta_2}{\eta_1\eta_2}\frac{\partial \eta_1}{\partial \xi_2} - \frac{A}{\eta_1 R_1}\delta_3 , \qquad \text{(II.4}a\text{)}$$

$$e_2 = \frac{1}{\eta_2}\frac{\partial \delta_2}{\partial \xi_2} + \frac{\delta_1}{\eta_2\eta_1}\frac{\partial \eta_2}{\partial \xi_1} - \frac{B}{\eta_2 R_2}\delta_3 , \qquad \text{(II.4}b\text{)}$$

$$e_3 = \frac{\partial \delta_3}{\partial Z} , \qquad \text{(II.4}c\text{)}$$

$$e_{12} = \frac{\eta_2}{\eta_1}\frac{\partial}{\partial \xi_1}\left(\frac{\delta_2}{\eta_2}\right) + \frac{\eta_1}{\eta_2}\frac{\partial}{\partial \xi_2}\left(\frac{\delta_1}{\eta_1}\right) , \qquad \text{(II.4}d\text{)}$$

$$e_{23} = \frac{1}{\eta_2}\frac{\partial \delta_3}{\partial \xi_2} + \eta_2\frac{\partial}{\partial z}\left(\frac{\delta_2}{\eta_2}\right) , \qquad \text{(II.4}e\text{)}$$

$$e_{31} = \frac{1}{\eta_1}\frac{\partial \delta_3}{\partial \xi_1} + \eta_1\frac{\partial}{\partial z}\left(\frac{\delta_1}{\eta_1}\right) . \qquad \text{(II.4}f\text{)}$$

These equations are just as general as equations (II.2).

Now δ_1, δ_2, δ_3 are functions of ξ_1 ξ_2 and z, but, provided the shell is thin, it is possible to make assumptions which enable us to describe the displacements, and hence the strains, at any point in the shell in terms of the displacements at the middle surface, which are functions of ξ_1, ξ_2 only. We assume that normals to the middle surface remain normal to the strained middle surface, *i.e.*,

$$\text{shear strains } e_{23} = e_{21} = 0 , \qquad \text{(II.5}a\text{)}$$

and that changes in length of the normal are small compared with the radial displacements, *i.e.*,

$$\text{normal strain } e_3 \ll w/h . \qquad \text{(II.5}b\text{)}$$

Equations (II.4*c*), (II.4*e*) and (II.4*f*) then reduce to

$$\frac{\partial \delta_3}{\partial Z} \ll \frac{\delta_3}{h} , \qquad \text{(II.6}a\text{)}$$

$$\frac{\partial}{\partial z}\left(\frac{\delta_2}{\eta_2}\right) = -\left(\frac{1}{\eta_2}\right)^2 \frac{\partial \xi_3}{\partial \xi_2} , \qquad \text{(II.6}b\text{)}$$

$$\frac{\partial}{\partial z}\left(\frac{\delta_1}{\eta_1}\right) = -\left(\frac{1}{\eta_1}\right)^2 \frac{\partial \delta_3}{\partial \xi_1} . \qquad \text{(II.6}c\text{)}$$

Equations (II.6) can be integrated, substituting for η_1, η_2 from equations (II.3*b*) and (II.3*c*), to give

$$\delta_1 = \frac{\eta_1}{A}u - \frac{Z}{A}\frac{\partial w}{\partial \xi_1} , \qquad \text{(II.7}a\text{)}$$

$$\delta_2 = \frac{\eta_2}{B}v - \frac{Z}{B}\frac{\partial w}{\partial \xi_2} , \qquad \text{(II.7}b\text{)}$$

$$\delta_3 = w , \qquad \text{(II.7}c\text{)}$$

where u, v, w are the displacements at the middle surface.

Substituting from equations (II.7) into equations (II.4a), (II.4b) and (II.4d) we find

$$e_1 = \frac{1}{\eta_1}\frac{\partial}{\partial \xi_1}\left\{\frac{\eta_1}{A}u - \frac{Z}{A}\frac{\partial w}{\partial \xi_1}\right\} + \frac{\partial \eta_1}{\partial \xi_2}\left\{\frac{v}{\eta_1 B} - \frac{Z}{\eta_1\eta_2 B}\frac{\partial w}{\partial \xi_2}\right\} - \frac{Aw}{\eta_1 R_1}, \quad \text{(II.8}a\text{)}$$

$$e_2 = \frac{1}{\eta_2}\frac{\partial}{\partial \xi_2}\left\{\frac{\eta_2}{B}v - \frac{Z}{B}\frac{\partial w}{\partial \xi_2}\right\} + \frac{\partial \eta_2}{\partial \xi_1}\left\{\frac{u}{\eta_2 A} - \frac{Z}{\eta_1\eta_2 A}\frac{\partial w}{\partial \xi_1}\right\} - \frac{Bw}{\eta_2 R_2}, \quad \text{(II.8}b\text{)}$$

$$e_{12} = \frac{\eta_2}{\eta_1}\frac{\partial}{\partial \xi_1}\left\{\frac{v}{B} - \frac{Z}{\eta_2 B}\frac{dw}{\partial \xi_2}\right\} + \frac{\eta_1}{\eta_2}\frac{\partial}{\partial \xi_2}\left\{\frac{u}{A} - \frac{Z}{\eta_1 A}\frac{\partial w}{\partial \xi_1}\right\}, \quad \text{(II.8}c\text{)}$$

which are the generally accepted expressions for strains in terms of displacements at the middle surface of the shell.

The thin shell assumptions only enable us to define these three components of strain. The subsequent assumption that $\sigma_3 = 0$ (equations (II.11)) effectively defines e_3 in terms of e_1 and e_2 but it is not useful to write it out unless assumption (II.5a) is being relaxed.

2. *Uniform Cylindrical Shell—Strain Components.*—We now apply equations (II.8) to the special case of the uniform circular cylinder.

The co-ordinates (ξ_1, ξ_2, z) become (x, ϕ, z)

and

$$\eta_1, \eta_2 \text{ become } 1, r(= a + z),$$
$$A, B \text{ become } 1, a,$$
$$R_1, R_2 \text{ become } \infty, -a.$$

Equations (II.8) now become :

$$e_1 = u_x - zw_{xx}, \quad \text{(II.9}a\text{)}$$

$$e_2 = \frac{1}{a}r + \frac{w}{r} - \frac{z}{ar}w_{\phi\phi}, \quad \text{(II.9}b\text{)}$$

$$e_{12} = \frac{1}{r}u_\phi + \frac{r}{a}v_x - z\left(\frac{1}{a} + \frac{1}{r}\right)w_{x\phi}. \quad \text{(II.9}c\text{)}$$

It is, of course, possible to simplify the preceding work by restricting ourselves to cylindrical polar co-ordinates from the start, but the steps in the argument are unaltered.

It is useful to write the particular form of equations (II.7) for the cylinder. *viz.*,

$$\delta_1 = u - zw_x, \quad \text{(II.10}a\text{)}$$

$$\delta_2 = \frac{r}{a}v - \frac{z}{a}w_\phi, \quad \text{(II.10}b\text{)}$$

$$\delta_3 = w. \quad \text{(II.10}c\text{)}$$

Equations (II.9) are identical with those derived by Flugge[19].

3. *Uniform Cylinder.—Dynamic Equations.*—Assuming plane stress in the shell we find

$$\sigma_1 = \frac{E}{1-\nu^2}\{e_1 + \nu e_2\}, \quad \text{(II.11}a\text{)}$$

$$\sigma_2 = \frac{E}{1-\nu^2}\{e_2 + \nu e_1\}, \quad \text{(II.11}b\text{)}$$

$$\sigma_{12} = Ge_{12} = \frac{E}{2(1+\nu)}e_{12}, \quad \text{(II.11}c\text{)}$$

$$\sigma_3 = 0 = \sigma_{23} = \sigma_{31}. \quad \text{(II.11}d\text{)}$$

The strain energy of the shell can therefore be written

$$S = \frac{E}{2(1-\nu^2)} \int_0^l \int_0^{2\pi} \int_{-h/2}^{h/2} \left\{ e_1^2 + e_2^2 + 2\nu e_1 e_2 + \frac{1-\nu}{2} e_{12}^2 \right\} r \, dx \, d\phi \, dz \,. \quad \ldots \quad \text{(II.12}a\text{)}$$

The kinetic energy is given by

$$T = \frac{\rho}{2} \int_0^l \int_0^{2\pi} \int_{-h/2}^{h/2} \{\dot{\delta}_1^2 + \dot{\delta}_2^2 + \dot{\delta}_3^2\} r \, dx \, d\phi \, dz \,. \quad \ldots \quad \text{(II.12}b\text{)}$$

After substituting from equations (II.9) and (II.10) we can carry out the z-integration to find

$$S = \frac{Eh}{2a(1-\nu^2)} \int_0^l \int_0^{2\pi} \left[\begin{array}{l} a^2 u_{xx}^2 + (v_\phi + w)^2 + 2\nu a u_x (v_\phi + w) + \dfrac{1-\nu}{2} (u_\phi + a v_x)^2 + \\ + \beta \left\{ \begin{array}{l} a^2 w_{xx} (a^2 w_{xx} - 2a u_x) + (w + w_{\phi\phi})^2 + 2\nu a^2 w_{xx} (w_{\phi\phi} - v_\phi) + \\ + \dfrac{1-\nu}{2} (a w_{x\phi} + u_\phi)^2 + \dfrac{3}{2} (1-\nu)(a w_{x\phi} - a v_x)^2 \end{array} \right\} \end{array} \right] d\phi \, dx \,, \quad \ldots \quad \text{(II.13}a\text{)}$$

$$T = \frac{\rho h a}{2} \int_0^l \int_0^{2\pi} \left[\begin{array}{l} \dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \\ + \beta \{ a \dot{w}_x (a \dot{w}_x - 2\dot{u}) + (\dot{w}_\phi - \dot{v})(\dot{w}_\phi - 3\dot{v}) \} \end{array} \right] d\phi \, dx \,, \quad \ldots \quad \text{(II.13}b\text{)}$$

where $\beta = h^2/12a^2$.

Logarithmic terms arising from integration of $1/r$ have been expanded as far as $(h/a)^2$. The next terms in the expansion are in $(h/a)^4$, i.e., β^2, so that equations (II.13) can be considered as mathematically exact within the limitations of the physical assumptions in equations (II.5) and (II.11d).

Equation (II.13a) is identical with one derived by Bleich and Di-Maggio[11].

We now assume the displacements to have the form

$$u = U \cos n\phi \, \cos \lambda \frac{x}{a} \, \cos \omega t = q_1 \cos n\phi \, \cos \lambda \frac{x}{a} \,, \quad \ldots \quad \text{(II.14}a\text{)}$$

$$v = V \sin n\phi \, \sin \lambda \frac{x}{a} \, \cos \omega t = q_2 \sin n\phi \, \sin \lambda \frac{x}{a} \,, \quad \ldots \quad \text{(II.14}b\text{)}$$

$$w = W \cos n\phi \, \sin \lambda \frac{x}{a} \, \cos \omega t = q_3 \cos n\phi \, \sin \lambda \frac{x}{a} \,, \quad \ldots \quad \text{(II.14}c\text{)}$$

where $\lambda = m\pi/a$; $m = 1, 2, 3, \ldots$; $n = 1, 2, \ldots$.

Equations (II.14) satisfy the boundary conditions of thin diaphragms, *i.e.*,

$$v = 0, \; w = 0 \text{ at } x = 0, l \,. \quad \ldots \quad \text{(II.15)}$$

They are also self-consistent and on substituting into equations (II.13) every term in the integrands involves squares of circular functions only. As a corollary, if displacement forms involving different circular functions (e.g., $u = q_1' \sin n\phi \cos \lambda x/a$) were added to equations (II.14), then, on substitution into equations (II.13) terms in, for instance $q_1 q_1'$ or $q_2 q_1'$ would also involve a function such as $\sin n\phi \cos n\phi$, and would therefore vanish on integration. Similarly, if we added terms with different values of n and m, all the resulting cross product terms would vanish on integration. We are therefore justified in selecting equations (II.14) from the complete Fourier type expansions for u, v, w.

Substituting from equations (II.14) into (II.13), integrating and re-arranging the terms we have

$$S = \frac{\pi Ehl}{4a(1-\nu^2)}\left[\begin{array}{l}\left\{\lambda^2 + \frac{1-\nu}{2}n^2\right\}q_1^2 - (1+\nu)\lambda n q_1 q_2 - 2\nu\lambda q_1 q_3 + \\ + \left\{n^2 + \frac{1-\nu}{2}\lambda^2\right\}q_2^2 + 2nq_2q_3 + q_3^2 + \\ + \beta\left[\begin{array}{l}\frac{1-\nu}{2}n^2q_1^2 + \{(1-\nu)\lambda n^2 - 2\lambda^3\}q_1q_3 + \frac{3}{2}(1-\nu)\lambda^2q_2^2 + \\ + (3-\nu)\lambda^2 n q_2 q_3 + \{(\lambda^2+n^2)^2 - 2n^2 + 1\}\,q_3^2\end{array}\right]\end{array}\right], \qquad \text{(II.16}a\text{)}$$

$$T = \frac{\pi\rho hla}{4}\left[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \beta\left[\begin{array}{l}-2\lambda\dot{q}_1\dot{q}_3 + 3\dot{q}_2^2 + \\ + 4n\dot{q}_2\dot{q}_3 + (\lambda^2+n^2)\dot{q}_3^2\end{array}\right]\right]. \quad \ldots \quad \text{(II.16}b\text{)}$$

We take q_1, q_2, q_3 as the generalised co-ordinates of Lagranges equations for free vibrations :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial S}{\partial q_r} = 0\,, \qquad r = 1, 3, 3\,. \quad \ldots \quad \text{(II.17)}$$

Substituting from equations (II.15) and (II.16) and multiplying through by $2a(1-\nu^2)/\pi Ehl \cos \omega t$ we find, since $\ddot{q}_r = -\omega^2 q_r$,

$$[-\Delta + \lambda^2 + \tfrac{1}{2}(1-\nu)n^2 + \tfrac{1}{2}(1-\nu)n^2\beta]\,U + [-\tfrac{1}{2}(1+\nu)\lambda n]V + $$
$$+ [\lambda\beta\Delta - \nu\lambda + \beta\{\tfrac{1}{2}(1-\nu)\lambda n^2 - \lambda^3\}]W = 0\,, \quad \ldots \quad \text{(II.18}a\text{)}$$

$$[-\tfrac{1}{2}(1+\nu)\lambda n]U + [-\Delta\,(1+\tfrac{3}{2}\beta) + n^2 + \tfrac{1}{2}(1-\nu)\lambda^2 + \tfrac{3}{2}(1-\nu)\lambda^2\beta]V + $$
$$+ [-2n\beta\Delta + n + \tfrac{1}{2}(3-\nu)\lambda^2 n\beta]W = 0\,, \quad \ldots \quad \text{(II.18}b\text{)}$$

$$[\lambda\beta\Delta - \nu\lambda + \beta\{\tfrac{1}{2}(1-\nu)\lambda n^2 - \lambda_3\}]U + [-2n\beta\Delta + n + \tfrac{1}{2}(3-\nu)\lambda^2 n\beta]V + $$
$$+ [-\Delta\{1 + (\lambda^2+n^2)\beta\} + 1 + \beta\{(\lambda^2+n^2)^2 - 2n^2 + 1\}]W = 0\,, \quad \ldots \quad \text{(II.18}c\text{)}$$

where $\Delta = \{\rho a^2(1-\nu^2)\omega^2\}/E$.

Note that it is dangerous to start neglecting too many of the β terms at this stage since many of the extensional terms cancel out in the frequency equation. Equations (II.18) yield the frequency equation

$$K_3\Delta^3 - K_2\Delta^2 + K_1\Delta - K_0 = 0\,, \quad \ldots \quad \text{(II.19)}$$

where

$$K_0 = \tfrac{1}{2}(1-\nu)(1-\nu^2)\lambda^4 + \tfrac{1}{2}(1-\nu)\beta\{(\lambda^2+n^2)^4 - 2\nu\lambda^6 - 6\lambda^4n^2 - (8-2\nu)\lambda^2n^4 + $$
$$+ 2n^6 + (4-3\nu^2)\lambda^4 + \tfrac{1}{2}(7-3\nu)\lambda^2n^2 + n^4\}\,, \quad \ldots \quad \text{(II.20}a\text{)}$$

$$K_1 = \tfrac{1}{2}(1-\nu)(\lambda^2+n^2)^2 + \tfrac{1}{2}(1-\nu)(3+2\nu)\lambda^2 + \tfrac{1}{2}(1-\nu)n^2 + \beta\{(2-\nu)(\lambda^2+n^2)^3 + $$
$$+ \tfrac{1}{2}(3 - 9\nu + 2\nu^2)\lambda^4 - (6-\nu-\nu^2)\lambda^2n^2 - \tfrac{1}{2}(7+3\nu)n^4 + $$
$$+ \tfrac{1}{2}(9 - 4\nu - 3\nu^2)\lambda^2 + \tfrac{1}{4}(11-7\nu)n^2\}\,, \quad \ldots \quad \text{(II.20}b\text{)}$$

$$K_2 = 1 + \tfrac{1}{2}(3-\nu)(\lambda^2+n^2) + \tfrac{1}{2}\beta\{(5-\nu)(\lambda^2+n^2)^2 + (6-7\nu)\lambda^2 - $$
$$- \tfrac{1}{2}(3+5\nu)n^2 + 5\}\,, \quad \ldots \quad \text{(II.20}c\text{)}$$

$$K_3 = 1 + \beta\{(\lambda^2+n^2) + \tfrac{3}{2}\}\,. \quad \ldots \quad \text{(II.20}d\text{)}$$

Owing to the neglect of shear deflections many of the β terms in equations (II.20) are of doubtful value. It is to be expected that shear deflections would introduce terms of the same order as those arising from rotary inertia, *i.e.*, of order $\lambda^2\beta$ or $n^2\beta$ times the extensional terms. It seems then that the present theory should be restricted to cases where shear deflections and rotary inertia are small. *i.e.*, where

$$(\lambda^2 + n^2)\beta \ll 1 . \qquad \text{(II.21)}$$

Equations (II.20) can then be simplified to

$$K_0 = \tfrac{1}{2}(1 - \nu)(1 - \nu^2)\lambda^4 + \tfrac{1}{2}(1 - \nu)\beta\{\lambda^2 + n^2)^4 - 2n^6 + n^4\} , \qquad \text{(II.22}a\text{)}$$

$$K_1 = \tfrac{1}{2}(1 - \nu)(\lambda^2 + n^2)^2 + \tfrac{1}{2}(1 - \nu)(3 + 2\nu)\lambda^2 + \tfrac{1}{2}(1 - \nu)n^2 , \qquad \text{(II.22}b\text{)}$$

$$K_2 = 1 + \tfrac{1}{2}(3 - \nu)(\lambda^2 + n^2) , \qquad \text{(II.22}c\text{)}$$

$$K_3 = 1 . \qquad \text{(II.22}d\text{)}$$

Had the assumption (II.21) been made in equations (II.18), then the n^6 and n^4 terms in (II.22a) would have been omitted : these are essential when λ and n are small (*i.e.*, for a long cylinder with n and m fairly small). However, all the terms in equations (II.22), together with some additional small β terms, are obtained from the following simplified forms of equations (II.18) :

$$[- \Delta + \lambda^2 + \tfrac{1}{2}(1 - \nu)n^2]U + [- \tfrac{1}{2}(1 + \nu)\lambda n]V + [- \nu\lambda]W = 0 , \qquad \text{(II.23}a\text{)}$$

$$[- \tfrac{1}{2}(1 + \nu)\lambda n]U + [- \Delta + n^2 + \tfrac{1}{2}(1 - \nu)\lambda^2]V + [n]W = 0 , \qquad \text{(II.23}b\text{]}$$

$$[- \nu\lambda]U + [n]V + [- \Delta + 1 + \beta\{(\lambda^2 + n^2)^2 - 2n^2 + 1\}]W = 0 . \qquad \text{(II.23}c\text{)}$$

Thus we have effectively neglected rotary inertia and all the bending terms in the strain energy except those associated with W^2.

As has been noted before (*e.g.*, Arnold and Warburton[5], equation (II.19) yields three solutions for Δ, corresponding to three normal modes with the same nodal pattern but with different values of U/W and V/W. In each mode one of the three components of displacement predominates and the three values of Δ are of different orders of magnitude. We are principally interested in the lowest root corresponding to predominantly radial displacements.

4. *Approximation to Lowest Root of Frequency Equation.*—A first approximation to this root is obtained by neglecting the tangential components of inertia, *i.e.*, omitting the Δ terms in equations (II.23a) and (II.23b). The frequency equation then becomes

$$\Delta = \frac{(1 - \nu^2)\lambda^4 + \beta\{(\lambda^2 + n^2)^4 - 2n^6 + n^4\}}{(\lambda^2 + n^2)^2} . \qquad \text{(II.24)}$$

A better numerical approximation is actually obtained by putting

$$\Delta = \frac{K_0}{K_1} , \qquad \text{(II.25)}$$

and it can be improved by iteration from equation (II.19), a single iteration giving quite good accuracy as pointed out by Arnold and Warburton[6]. If the convergence is slow it follows that the assumptions of this theory are no longer valid.

5. *Note on the Anomalous Variation of Frequency with Mode Numbers.*—Equation (II.24) (or (II.25)) is adequate to demonstrate the variation of frequency with λ and n : for instance, with λ constant it indicates a minimum frequency at about

$$n = \left[\lambda\left(\frac{\beta}{1 - \nu^2}\right)^{1/4} - \lambda^2\right]^{1/2} , \qquad \text{(II.26)}$$

which agrees very well with the curves given by Arnold and Warburton. This curious effect, namely, that an increase in mode number does not necessarily increase the frequency, was pointed out by them, but they only demonstrated it in theory by carrying out a large number of calculations. Equation (II.24) (or (II.25)) has the merit of summing up this behaviour in a compact form. The explanation (given by Arnold and Warburton) is also readily apparent from equation (II.24) : for when n is small the first term in the numerator predominates (provided λ is such that the right-hand side of equation (II.26) is $\geqslant 1$) and we have

$$\Delta \sim \frac{\lambda^4}{(\lambda^2 + n^2)^2}, \text{ decreasing as } n \text{ increases.}$$

Eventually the second term of the numerator takes over and we have $\Delta \sim \beta(\lambda^2 + n^2)^2$, increasing with n.

The minimum of n is reached when the two terms are equal giving equation (II.26). It is clear from equation (II.24) that when n is less than the value given by equation (II.26) stretching energy predominates and when it it is greater bending energy predominates.

Remembering that n is a positive integer, i.e., $n \geqslant 1$, equation (II.26) indicates the range of λ over which this reversal can take place.

APPENDIX III

Details of Analysis of Stiffened Cylinder

(The analysis is described in Section 4)

1. *Strain Energy of Longerons.*—It is reasonable to assume that σ_2 and σ_3 are small so that longitudinal stress and strain are related by

$$\sigma_1 = Ee_1 \qquad \text{(III.1)}$$

and the total strain energy of one longeron is then

$$dS_l = \tfrac{1}{2}E \int_0^l \int_{\text{depth}} e_1^2 b_l dz\, dx\,, \qquad \text{(III.2)}$$

where b_l = breadth of longeron at any height z, and strain energy of shear is neglected (this includes torsion).

Substituting for e_1 from equation (II.9*a*)

$$dS_l = \tfrac{1}{2}E \int_0^l dx \int_{\text{depth}} (u_x^2 - 2zu_x w_{xx} + w_{xx}^2) b_l\, dz$$

$$= \tfrac{1}{2}E \int_0^l (A_l u_x^2 - 2A_l^{(1)} u_x w_{xx} + A_l^{(2)} w_{xx}^2)\, dx\,, \qquad \text{(III.3)}$$

where $A_l = \int_{\text{depth}} b_l\, dz$ = area of longeron section,

$A_l^{(1)} = \int_{\text{depth}} zb_l\, dz$ = first moment of area of longeron section about skin median,

$A_l^{(2)} = \int_{\text{depth}} z^2 b_l\, dz$ = second moment of area of longeron section about skin median.

Note that $A_l^{(1)}$ is only positive when the longeron is outside the skin.

Assuming sinusoidal deflection forms we can substitute for u and w from equations (II.14) obtaining, for the rth longeron, provided $\lambda \neq 0$,

$$dS_l = \frac{El}{4}\left\{\frac{\lambda^2}{a^2} A_l q_1^2 - 2\frac{\lambda^3}{a^3} A_l^{(1)} q_1 q_3 + \frac{\lambda^4}{a^4} A_l^{(2)} q_3^2\right\} \cos^2 (2\pi nr/p)\,, \qquad \text{(III.4)}$$

where p = number of longerons, $r = 0, 1 \ldots p - 1$, the longerons being assumed to lie at $\phi = 0, 2\pi n/p, 4\pi n/p, \ldots$, their finite widths being ignored.

From equation (IV.3) we see that

$$\sum_{r=0}^{p-1} \cos^2 (2\pi n/p) = p/2\,;\ n \neq 0, p/2, p \ldots.$$

$n = 0$ is dealt with separately. The other singular values are excluded by the assumption of closely spaced stiffeners, implying $p/2 \gg n$.

Hence equation (III.4) can be summed over the p longerons to give

$$S_l = \frac{\pi Ehl}{4a(1 - \nu^2)} \left\{\lambda^2 L_0 q_1^2 + 2\lambda^3 L_1 q_1 q_3 + \lambda^4 L_2 q_3^2\right\}, \qquad \text{(III.5)}$$

where $L_0 = \dfrac{pA_l(1 - \nu^2)}{2\pi ah}$; $L_1 = -\dfrac{pA_l^{(1)}(1 - \nu^2)}{2\pi a^2 h}$; $L_2 = \dfrac{pA_l^{(2)}(1 - \nu^2)}{2\pi a^3 h}$.

Note now that L_1 is (positive/negative) when stringers are (inside/outside) the skin.

2. *Strain Energy of Frames.*—Neglecting σ_1 and σ_3 we have

$$\sigma_2 = Ee_2\,, \qquad \text{(III.6)}$$

$$dS_l = \tfrac{1}{2}E\int_0^{2\pi}\int_{\text{depth}} b_f e_2{}^2 r\,d\phi\,dz\,, \qquad \text{(III.7)}$$

where b_f = breadth of frame at any radius $r = a + z$.

Substituting for e_2 from equation (II.9*b*)

$$dS_f = \tfrac{1}{2}E\int_0^{2\pi} d\phi \int_{\text{depth}} b_f\left\{\frac{1}{a}v_\phi + \frac{w}{r} - \frac{z}{ar}w_{\phi\phi}\right\}^2 r dz$$

$$= \frac{E}{2a}\int_0^{2\pi}\left[\begin{aligned} & v_\phi{}^2\left(A_f + \frac{1}{a}A_f{}^{(1)}\right) + 2v_\phi w A_f - \frac{2}{a}v_\phi w_{\phi\phi}A_f{}^{(1)} + \\ & + w^2\left(A_f - \frac{1}{a}A_f{}^{(1)} + \frac{1}{a^2}A_f{}^{(2)} - \ldots\right) - 2ww_{\phi\phi}\left(\frac{1}{a}A_f{}^{(1)} - \frac{1}{a^2}A_f{}^{(2)} + \ldots\right) \\ & + w_{\phi\phi}{}^2\left(\frac{1}{a^2}A_f{}^{(2)} - \frac{1}{a^3}A_f{}^{(3)}\right) \end{aligned}\right] \qquad \text{(III.8)}$$

where

$$A_f = \int_{\text{depth}} b_f\,dz = \text{area of frame section}$$

$$A_f{}^{(r)} = \int_{\text{depth}} z^r b_f\,dz = r\text{th moment of area of frame section about skin median.}$$

Note : for r odd, $A_f{}^{(r)}$ is negative when frame is inside skin.

Assuming deflection forms (II.14), for the rth frame, provided $n \neq 0$,

$$dS_f = \frac{\pi E}{2a}\left[\begin{aligned} & n^2\left(A_f + \frac{1}{a}A_f{}^{(1)}\right)q_2{}^2 + 2n\left(A_f + n^2\frac{1}{a}A_f{}^{(1)}\right)q_2 q_3 + \\ & + \left\{\left(A_f - \frac{1}{a}A_f{}^{(1)} + \ldots\right) + 2n^2\left(A_f{}^{(1)}\frac{1}{a} - \frac{1}{a^2}A_f{}^{(2)} + \ldots\right) + \right. \\ & \left. + n^4\left(\frac{1}{a^2}A_f{}^{(2)} - \ldots\right)\right\}q_3{}^2 \end{aligned}\right]\sin^2\left(\frac{r\lambda l}{qa}\right). \qquad \text{(III.9)}$$

Where frame spacing $= l/q$,

$$r = (0), 1, 2 \ldots q - 1, (q).$$

From equation (IV.7)

$$\sum_{r=1}^{q-1}\sin^2\left(\frac{r\lambda l}{qa}\right) \equiv \sum_{r=0}^{q-1}\sin^2\left(m\pi\frac{r}{q}\right) = \tfrac{1}{2}q\,, \quad m \neq q, 2q \ldots, \quad \left(\lambda = \frac{m\pi a}{l}\right),$$

where again the singular values are excluded by the assumption of close spacing which implies $q \gg m$.

The limiting case $\lambda = 0$ is dealt with separately.

Thus equation (III.14) can be summed over the frames to give

$$S_f = \frac{\pi Ehl}{4a(1-\nu^2)} \begin{bmatrix} n^2(F_0 - F_1)q_2^2 + (nF_0 - n^3F_1)q_2q_3 \\ +\{(F_0 + F_1 + \ldots) - 2n^2(F_1 + F_2 + \ldots) + n^4(F_2 + F_3 + \ldots)\}q_3^2 \end{bmatrix}, \quad \text{(III.10)}$$

where

$$F_0 = \frac{qA_f(1-\nu^2)}{hl}; \qquad F_r = (-1)^r \frac{qA_f^{(r)}(1-\nu^2)}{a^r hl}.$$

Note now that for r odd, F_r is (positive/negative) when frames are (inside/outside) the skin. The series of F_r in the q_3^2 term of equation (III.10) will generally converge rapidly, probably only two terms being required.

3. *Total Strain Energy of Shell.*—The strain energy of the skin is given by equation (II.16*a*). Adding equations (II.16*a*), (III.5) and (II.10) we have

$$S_t = \frac{\pi Ehl}{4a(1-\nu^2)} \begin{bmatrix} \{\lambda^2(1 + L_0) + \frac{1}{2}(1-\nu)n^2\}q_1^2 - (1+\nu)\lambda n q_1 q_2 - \\ - 2\{\nu\lambda - \lambda^3 L_1\}q_1q_3 + \{n^2(1 + F_0 + F_1) + \frac{1}{2}(1-\nu)\lambda^2\}q_2^2 + \\ + 2\{n(1+F_0) - n^3F_1\}q_2q_3 + \{(1 + F_0 + F_1 + \ldots) - \\ - 2n^2(F_1 + F_2 + \ldots) + n^4(F_2 + F_3 + \ldots) + \lambda^4 L_2\}q_3^2 \end{bmatrix}. \quad \text{(III.11)}$$

The bending energy of the skin has been omitted since we are concerned with relatively heavy stiffening and can assume

$$\beta \ll F_2,\ L_2 < F_2,\ L_1 \quad \text{(III.12)}$$

Most of the skin bending terms appear in direct company with these quantities : the others are of little importance even for the uniform shell.

4. *Kinetic Energy of Longerons.*—The kinetic energy of one longeron is given by

$$dT_l = \tfrac{1}{2}\int_0^l \int_{\text{depth}} \rho_l b_l \{\dot\delta_1^2 + \dot\delta_2^2 + \dot\delta_3^2\}\, dz\, dx, \quad \text{(III.13)}$$

where $\rho_l b_l$ is the mass of the longeron per unit depth and length.

Substituting for the δ from equations (II.10),

$$dT_l = \tfrac{1}{2}\int_0^l dx \int_{\text{depth}} \rho_l b_l \left[(\dot u - z\dot w_x)^2 + \left(\frac{r}{a}\dot v - \frac{z}{a}\dot w_\phi\right)^2 + \dot w^2\right] dz \quad \text{(III.14)}$$

$$= \tfrac{1}{2}\int_0^l \begin{bmatrix} m_l \dot u^2 - 2I_l^{(1)}\dot u \dot w_x + \left(m_l + \frac{2}{a}I_l^{(1)} + \frac{1}{a^2}I_l^{(2)}\right)\dot v^2 + \\ - 2\left(\frac{1}{a}I_l^{(1)} + \frac{1}{a^2}I_l^{(2)}\right)\dot v \dot w_\phi + I_l^{(2)}\dot w_x^2 + \frac{1}{a^2}I_l^{(2)}\dot w_\phi^2 + m_l \dot w^2 \end{bmatrix} dx, \quad \text{(III.15)}$$

where

$m_l = \int_{\text{depth}} \rho_l b_l\, dz$ = mass of longeron per unit length,

$I_l^{(1)} = \int_{\text{depth}} z\rho_l b_l\, dz$

$I_l^{(2)} = \int_{\text{depth}} z^2 \rho_l b_l\, dz$ = moment of inertia of longeron per unit length about skin median.

If we average out the rivets, etc., give to a mean density $\bar{\rho}_l$, then

$$m_l \simeq \bar{\rho}_l A_l \; ; \; I_l^{(1)} \simeq \bar{\rho}_l A_l^{(1)} \; ; \; I_l^{(2)} \simeq \bar{\rho}_l A_l^{(2)} \, . \qquad \text{(III.16)}$$

Substituting from deflection forms (II.14) into equation (III.15),

$$dT_l = \frac{l}{4}\begin{bmatrix} \left\{ m_l \dot{q}_1^2 - 2\frac{\lambda}{a} I_l^{(1)} \dot{q}_1 \dot{q}_3 + \frac{\lambda^2}{a^2} I_l^{(2)} \dot{q}_3^2 + m_l \dot{q}_3^2 \right\} \cos^2\left(2\pi n \frac{r}{p}\right) + \\ + \left\{ \left(m_l + \frac{2}{a} I_l^{(1)} + \frac{1}{a^2} I_l^{(2)}\right) \dot{q}_2^2 + \right. \\ \left. - 2n\left(\frac{1}{a} I_l^{(1)} + \frac{1}{a^2} I_l^{(2)}\right) \dot{q}_2 \dot{q}_3 + \frac{n^2}{a^2} I_l^{(2)} \dot{q}_3^2 \right\} \sin^2\left(2\pi n \frac{r}{p}\right) \end{bmatrix} \qquad \text{(III.17)}$$

As with the strain energy we can sum over the longerons to find, finally, provided $n \neq 0$, $\lambda \neq 0$,

$$T_l = \frac{\pi \rho h l a}{4}\begin{bmatrix} H_0 \dot{q}_1^2 + 2\lambda H_1 \dot{q}_1 \dot{q}_3 + (H_0 - 2H_1 + H_2)\dot{q}_2^2 + \\ + 2n(H_1 - H_2)\dot{q}_2 \dot{q}_3 + (H_0 + \lambda^2 H_2 + n^2 H_2)\dot{q}_3^2 \end{bmatrix}, \qquad \text{(III.18)}$$

where

$$H_0 = \frac{p m_l}{2\pi a h \rho} \; ; \quad H_1 = -\frac{p I_l^{(1)}}{2\pi h \rho a^2} \; ; \quad H_2 = \frac{p I_l^{(2)}}{2\pi h \rho a^3}$$

when density is averaged out to $\bar{\rho}_l$,

$$H_0 \simeq \frac{\bar{\rho}_l}{\rho(1-\nu^2)} L_0 \; ; \quad H_1 \simeq \frac{\bar{\rho}_l}{\rho(1-\nu^2)} L_1 \; ; \quad H_2 \simeq \frac{\bar{\rho}_l}{\rho(1-\nu^2)} L_2 \, . \qquad \text{(III.19)}$$

Note : signs of $I_l^{(1)}$ and H_1 are as for $A_l^{(1)}$ and L_1.

5. *Kinetic Energy of Frames.*—The kinetic energy of one frame is given by

$$dT_f = \tfrac{1}{2}\int_0^{2\pi} \int_{\text{depth}} \rho_f b_f \{\dot{\delta}_1^2 + \dot{\delta}_2^2 + \dot{\delta}_3^2\} r \, d\phi \, dz \, , \qquad \text{(III.20)}$$

where $\rho_f b_f$ is the mass of the frame per unit depth and per unit circumference at radius $r = a + z$.

Substituting for the δ from equations (II.10),

$$dT_f = \tfrac{1}{2}\int_0^{2\pi} d\phi \int_{\text{depth}} r \rho_f b_f \left\{ (\dot{u} - z\dot{w}_x)^2 + \left(\frac{r}{a}\dot{v} - \frac{z}{a}\dot{w}_\phi\right)^2 + \dot{w}^2 \right\} dz \qquad \text{(III.21)}$$

$$= \tfrac{1}{2}\int_0^{2\pi} \begin{bmatrix} m_f \dot{u}^2 - 2I_f^{(1)} \dot{u}\dot{w}_x + I_f^{(2)} \dot{w}_x^2 + \left(m_f + \frac{2}{a} I_f^{(1)} + \frac{1}{a^2} I_f^{(2)}\right) \dot{v}^2 + \\ - 2\left(\frac{1}{a} I_f^{(1)} + \frac{1}{a^2} I_f^{(2)}\right) \dot{v}\dot{w}_\phi + \frac{1}{a^2} I_f^{(2)} \dot{w}_\phi^2 + m_f \dot{w}^2 \end{bmatrix} d\phi \, , \qquad \text{(III.22)}$$

where

$$m_f = \int_{\text{depth}} \rho_f b_f \frac{r}{a} dz$$ = mass of frame per unit circumference at the reference surface.

= (total mass of frame)$/2\pi a$,

$$\left.\begin{aligned} I_f^{(1)} &= \int_{\text{depth}} \rho_f b_f \frac{r}{a} z \, dz \\ I_f^{(2)} &= \int_{\text{depth}} \rho_f b_f \frac{r}{a} z^2 \, dz \end{aligned}\right\} \text{Both per unit circumference at the reference surface.}$$

If the density is averaged out to ρ_f then

$$m_f \simeq \bar{\rho}_f \left\{ A_f + \frac{1}{a} A_f^{(1)} \right\},$$

$$I_f^{(1)} \simeq \bar{\rho}_f \left\{ A_f^{(1)} + \frac{1}{a} A_f^{(2)} \right\}, \quad \ldots \qquad \text{(III.23)}$$

$$I_f^{(2)} \simeq \bar{\rho}_f \left\{ A_f^{(2)} + \frac{1}{a} A_f^{(3)} \right\}.$$

Substituting from deflection forms (II.14) we obtain an equation similar to equation (III.17), which can then be summed over the frames to give, finally, provided $n \neq 0$, $\lambda \neq 0$,

$$T_f = \frac{\pi \rho h l a}{4} \begin{bmatrix} G_0 \dot{q}_1^2 + 2\lambda G_1 \dot{q}_1 \dot{q}_3 + (G_0 - 2G_1 + G_2)\dot{q}_2^2 + \\ + 2n(G_1 - G_2)\dot{q}_2\dot{q}_3 + (G_0 + \lambda^2 G_2 + n^2 G_2)\dot{q}_3^2 \end{bmatrix}, \quad \ldots \qquad \text{(III.24)}$$

where

$$G_0 = \frac{q m_f}{h l \rho}; \quad G_1 = -\frac{q I_f^{(1)}}{h l a \rho}; \quad G_2 = \frac{q I_f^{(2)}}{h l a^2 \rho}.$$

When density is averaged to $\bar{\rho}_f$,

$$G_0 \simeq \frac{\bar{\rho}_f}{\rho(1-\nu^2)}(F_0 - F_1); \quad G_1 \simeq \frac{\bar{\rho}_f}{\rho(1-\nu^2)}(F_1 - F_2); \quad G_2 \simeq \frac{\bar{\rho}_f}{\rho(1-\nu^2)}(F_2 - F_3). \qquad \text{(III.25)}$$

6. *Total Kinetic Energy of Shell.*—The kinetic energy of the skin is given with sufficient accuracy by equation (II.16*b*). Adding equations (II.16*b*), (III.18) and (III.24) we have

$$T_t = \frac{\pi \rho h l a}{4} \begin{bmatrix} (1 + G_0 + H_0)\dot{q}_1^2 + 2\lambda(G_1 + H_1)\dot{q}_1\dot{q}_3 + \\ + \{(1 + G_0 + H_0) - 2(G_1 + H_1) + G_2 + H_2)\}\dot{q}_2^2 + \\ + 2n\{(G_1 + H_1) - (G_2 + H_2)\}\dot{q}_2\dot{q}_3 + \\ + \{(1 + G_0 + H_0) + (\lambda^2 + n^2)(G_2 + H_2)\}\dot{q}_3^2 \end{bmatrix}. \quad \ldots \qquad \text{(III.26)}$$

A crude approximation is given by

$$T_t = \frac{\pi \rho h l a}{4}(1 + G_0 + H_0)(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2). \quad \ldots \qquad \text{(III.27)}$$

This assumes, for instance, that

$$(\lambda^2 + n^2)(G_2 + H_2) \ll 1 + G_0 + H_0, \quad \ldots \qquad \text{(III.28}a\text{)}$$

which is equivalent to

$$(\lambda^2 + n^2)(F_2 + L_2) \ll 1. \quad \ldots \qquad \text{(III.28}b\text{)}$$

7. *The Dynamic Equations.*—We write Lagrange equations in q_1, q_2, q_3 :

$$\frac{d}{dt}\left(\frac{\partial T_t}{\partial \dot{q}_r}\right) - \frac{\partial T_t}{\partial q_r} + \frac{\partial S_t}{\partial q_r} = 0, \qquad r = 1, 2, 3. \quad \ldots \qquad \text{(III.29)}$$

Now $$\frac{d}{dt}\dot{q}_r = \ddot{q}_r = -\omega^2 q_r = -\omega^2 \cos \omega t \, (u, v, \text{or } w),$$

so if we substitute for T_t and S_t from equations (III.11) and (III.24) and multiply through by $2a(1 - \nu^2)/\pi E h l \cos \omega t$ we can write the three resulting equations in the form :

$$[C - \Delta B] \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0, \quad \ldots \qquad \text{(III.30)}$$

where $\Delta = \rho a^2(1 - \nu^2)\omega^2/E$, and where $[C]$ and $[B]$ are square matrices defined by

$$[C] \equiv \begin{bmatrix} \lambda^2(1 + L_0) + \frac{1}{2}(1 - \nu)n^2 \, , & -\frac{1}{2}(1 + \nu)\lambda n \, , & -\nu\lambda + \lambda_3 L_1 \\ -\frac{1}{2}(1 + \nu)\lambda n \, , & n^2(1 + F_0 - F_1) + \frac{1}{2}(1 - \nu)\lambda^2 \, , & n(1 + F_0) - n^3 F_1 \\ -\nu\lambda + \lambda^3 F_1 \, , & n(1 + F_0) - n^3 F_1 \, , & (1 + F_0 + \ldots) - 2n^2(F_1 + \ldots) + n^4(F_2 + \ldots) + \lambda^4 L_2 \end{bmatrix}$$

$$[B] \equiv \begin{bmatrix} 1 + G_0 + H_0 \, , & 0 \, , & \lambda(G_1 + H_1) \\ 0 \, , & (1 + G_0 + H_0) - 2(G_1 + H_1) + (G_2 + H_2) \, , & n(G_1 + H_1 - G_2 - H_2) \\ \lambda(G_1 + H_1) \, , & n(G_1 + H_1 - G_2 - H_2) \, , & (1 + G_0 + H_0) + (\lambda^2 + n^2)(G_2 + H_2) \end{bmatrix} .$$

The eliminant of equations (III.30), *i.e.*,

$$[C - \Delta B] = 0 \quad \ldots \quad \text{(III.31}a\text{)}$$

can be expanded to give a cubic in Δ,

$$K_3\Delta^3 - K_2\Delta^2 + K_1\Delta - K_0 = 0 \, , \quad \ldots \quad \text{(III.31}b\text{)}$$

giving three real positive roots, in general.

After substituting these roots back in equations (III.30) the mode shapes can be determined, as defined by the ratios U/W, V/W.

8. *The Generalised Mass.*—The generalised mass M is defined by

$$T_t = \tfrac{1}{2}M\dot{q}_3^{\,2} \, . \quad \ldots \quad \text{(III.32)}$$

Comparing equations (III.32) and (III.26), we find

$$M = \frac{\pi\rho hla}{2}\left[\begin{array}{l} (1 + G_0 + H_0)\left(\dfrac{U}{W}\right)^2 + 2\lambda(G_1 + H_1)\left(\dfrac{U}{W}\right) + \\ + \{(1 + G_0 + H_0) - 2(G_1 + H_1) + (G_1 + H_2)\}\left(\dfrac{V}{W}\right)^2 + \\ + 2n\{(G_1 + H_1) - (G_2 + H_2)\}\left(\dfrac{V}{W}\right) + \\ + \{(1 + G_0 + H_0) + (\lambda^2 + n^2)(G_2 + H_2)\} \end{array}\right] , \quad \ldots \quad \text{(III.33)}$$

since, *e.g.*,

$$\frac{\dot{q}_1}{\dot{q}_3} = \frac{-\omega U \sin \omega t}{-\omega W \sin \omega t} = \frac{U}{W} \, .$$

APPENDIX IV

Evaluation of Summations over Frames and Longerons

1. *Two Preliminary Summations.*—First we evaluate $\sum_{r=0}^{s-1} \cos\left(k\pi \frac{r}{s}\right)$,

where $\qquad k =$ integer or zero,

$\qquad r, s =$ integers.

Now $\quad \sum_{r=0}^{s-1} \cos\left(k\pi \frac{r}{s}\right) = \mathscr{R} \sum_{r=0}^{s-1} \exp\,(ik\pi\, r/s)\,, \qquad i = \sqrt{-1}\,.$

But this is a geometric progression with s terms, common ratio of $\exp\,(ik\pi/s)$ and first term $= 1$

$$\text{Therefore } \sum_{r=0}^{s-1} \cos\left(k\pi \frac{r}{s}\right) = \mathscr{R}\,\frac{1 - \exp\,(ik\pi)}{1 - \exp\,(ik\pi/s)}$$

$$= \mathscr{R}\,\frac{\exp\,(-\,ik\pi/2s) - \exp\,(i\overline{2s-1}\;k\pi/2a)}{\exp\,(-\,ik\pi/2s) - \exp\,(ik\pi/2s)}$$

$$= \frac{-\sin\,(k\pi/2s) - \sin\,(\overline{2s-1}\;k\pi/2s)}{-\,2\sin\,(k\pi/2s)}$$

$$= +\,\tfrac{1}{2} + \frac{\sin\,(k\pi)\;\cos\,(k\pi/2s) - \cos\,(k\pi)\;\sin\,(k\pi/2s)}{2\sin\,(k\pi/2s)}$$

$$= +\,\tfrac{1}{2} - \tfrac{1}{2}(-\,1)^k\,,$$

provided $\sin\,(k\pi/2s) \neq 0$, i.e., $k \neq 0, 2s, 4s \ldots$.

But if $\sin\,(k\pi/2s) = 0$, then $\cos\,(k\pi/s) = \cos\,(k\pi r/s) = 1$ and hence

$$\sum_{r=0}^{s-1} \cos\left(k\pi \frac{r}{s}\right) = \left\{\begin{matrix} 0, & k \text{ even but} \neq 0, \pm\,2s, \pm\,4s\ldots \\ s, & k = 0, \pm\,2s, \pm\,4s\ldots \\ 1, & k \text{ odd} \end{matrix}\right\}. \qquad \text{(IV.1)}$$

$$\text{Similarly } \sum_{r=0}^{s-1} \sin\left(k\pi \frac{r}{s}\right) = \frac{\cos\,(k\pi/2s) - \cos\,(\overline{2s-1}\;k\pi/2s)}{2\sin\,(k\pi/2s)}$$

$$= \frac{\cos\,(k\pi/2s) - \cos\,(k\pi)\;\cos\,(k\pi/2s) + \sin\,(k\pi)\;\sin\,(k\pi/2s)}{2\sin\,(k\pi/2s)}$$

$$= \{\tfrac{1}{2} - \tfrac{1}{2}(-\,1)^k\}\,\frac{\cos\,(k\pi/2s)}{\sin\,(k\pi/2s)}\,.$$

But when $\sin\,(k\pi/2s) = 0$, $\sin\,(k\pi r/s) = 0$: hence

$$\sum_{r=0}^{s-1} \sin\left(k\pi \frac{r}{s}\right) = \left\{\begin{matrix} 0 & , & k \text{ even} \\ \dfrac{\cos\,(k\pi/2s)}{\sin\,(k\pi/2s)}, & & k \text{ odd} \end{matrix}\right\}. \qquad \text{(IV.2)}$$

2. *Summations Arising from Single Sinusoidal Modes.*

Longerons

We have p longerons spaced at $\phi = 2\pi r/p$, $r = 0, 1, 2 \ldots p - 1$.

$$\sum_{r=0}^{p-1} \cos^2\,(2\pi nr/p) = \tfrac{1}{2}\sum_{r=0}^{p-1}\{1 + \cos\,(4\pi nr/p)\}$$

$$= \tfrac{1}{2}\,p + \sum_{r=0}^{p-1} \cos\,(4\pi nr/p)\,.$$

Comparing this last summation with equation (IV.1), $k = 4n$, $s = p$, we see that

$$\sum_{r=0}^{p-1} \cos^2 (2\pi nr/p) = \begin{Bmatrix} \frac{1}{2}p \,, & n \neq 0 \,, & \frac{1}{2}p \,, & p \ldots \\ p \quad, & n = 0 \,, & \frac{1}{2}p \,, & p \ldots \end{Bmatrix} . \qquad \ldots \quad \ldots \quad \ldots \quad \text{(IV.3)}$$

Similarly,

$$\sum_{r=0}^{p-1} \sin^2 (2\pi nr/p) = \begin{Bmatrix} \frac{1}{2}p \,, & n \neq 0 \,, & \frac{1}{2}p \,, & p \ldots \\ 0 \quad, & n = 0 \,, & \frac{1}{2}p \,, & p \ldots \end{Bmatrix} . \qquad \ldots \quad \ldots \quad \ldots \quad \text{(IV.4)}$$

Also

$$\sin (2\pi nr/p) \quad \cos (2\pi nr/p) = \tfrac{1}{2} \sin (4\pi nr/p) \,.$$

Comparing with equation (IV.2), with $k = 4n$, we see that

$$\sum_{r=0}^{p-1} \sin (2\pi nr/p) \quad \cos (2\pi nr/p) = 0 \text{ for all } n \,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(IV.5)}$$

Frames

We have frames spaced at intervals of l/q, *i.e.*,

at
$$x = l\frac{r}{q}\,, \quad r = 1, 2, \ldots q - 1 \,.$$

The addition of $r = 0$ will not affect the summation.

$$\cos^2 \left(m\pi \frac{r}{q}\right) = \tfrac{1}{2} + \tfrac{1}{2} \cos \left(2m\pi \frac{r}{q}\right) .$$

Comparing with equation (IV.1), with $k = 2m$, $s = q$, we see that

$$\sum_{r=0}^{q-1} \cos^2 \left(m\pi \frac{r}{q}\right) = \begin{Bmatrix} \frac{1}{2}q \,, & m \neq 0 \,, & q \,, & 2q \ldots \\ q \quad, & m = 0 \,, & q \,, & 2q \ldots \end{Bmatrix} . \qquad \ldots \quad \ldots \quad \ldots \quad \text{(IV.6)}$$

Similarly

$$\sum_{r=0}^{q-1} \sin^2 \left(m\pi \frac{r}{q}\right) = \begin{Bmatrix} \frac{1}{2}q \,, & m \neq 0 \,, & q \,, & 29 \ldots \\ 0 \quad, & m = 0 \,, & q \,, & 2q \ldots \end{Bmatrix} . \qquad \ldots \quad \ldots \quad \ldots \quad \text{(IV.7)}$$

Also

$$\sin \left(m\pi \frac{r}{q}\right) \quad \cos \left(m\pi \frac{r}{q}\right) = \frac{1}{2} \sin \left(2m\pi \frac{r}{q}\right) .$$

Comparing with equation (IV.2), with $k = 2m$, we see that

$$\sum_{r=0}^{q-1} \sin \left(m\pi \frac{r}{q}\right) \quad \cos \left(m\pi \frac{r}{q}\right) = 0 \text{ for all } m \,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(IV.8)}$$

3. *Coupling Terms between Different Sinusoidal Modes.*

Longerons

e.g., terms which would arise from

$$\sum_{r=0}^{p-1} \left\{ \sum_n \cos \left(2\pi n \frac{r}{p}\right) + \sum_n \sin \left(2\pi n \frac{r}{p}\right) \right\}^2 .$$

Now

$$\cos \left(2\pi n_1 \frac{r}{p}\right) \quad \cos \left(2\pi n_2 \frac{r}{p}\right) = \tfrac{1}{2} \cos \left(2\pi \overline{n_1 + n_2} \frac{r}{p}\right) + \tfrac{1}{2} \cos \left(2\pi \overline{n_1 - n_2} \frac{r}{p}\right) .$$

Therefore $\sum_{r=0}^{p-1} \cos\left(2\pi n_1 \frac{r}{p}\right) \quad \cos\left(2\pi n_2 \frac{r}{p}\right)$

$$= \begin{cases} 0 \quad , \quad n_1 \pm n_2 \neq 0 & , \quad \pm p\,, \quad \pm 2p \ldots \\ \frac{1}{2}p\,, \quad (n_1 + n_2) \text{ or } (n_1 - n_2) = 0 & , \quad \pm p\,, \quad \pm 2p \ldots \\ p \quad , \quad (n_1 + n_2) \text{ and } (n_1 - n_2) = 0\,, & \quad \pm p\,, \quad \pm 2p \ldots \end{cases} \ldots \quad \text{(IV.9)}$$

Also $\qquad \sin\left(2\pi n_1 \frac{r}{p}\right) \quad \sin\left(2\pi n_2 \frac{r}{p}\right)$

$$= -\tfrac{1}{2} \cos\left(2\pi \overline{n_1 + n_2} \frac{r}{p}\right) + \tfrac{1}{2} \cos\left(2\pi \overline{n_1 - n_2} \frac{r}{p}\right).$$

Therefore $\sum_{r=0}^{p-1} \sin\left(2\pi n_1 \frac{r}{p}\right) \quad \sin\left(2\pi n_2 \frac{r}{p}\right)$

$$= \begin{cases} 0 \quad , \quad n_1 \pm n_2 \neq 0 & , \quad \pm p\,, \quad \pm 2p \ldots \\ \frac{1}{2}p\,, \quad (n_1 + n_2) \text{ or } (n_1 - n_2) = 0 & , \quad \pm p\,, \quad \pm 2p \ldots \\ 0 \quad , \quad (n_1 + n_2) \text{ and } (n_1 - n_2) = 0\,, & \quad \pm p\,, \quad \pm 2p \ldots \end{cases} \ldots \quad \text{(IV.10)}$$

Also $\qquad \sin\left(2\pi n_1 \frac{r}{p}\right) \quad \cos\left(2\pi n_2 \frac{r}{p}\right)$

$$= \tfrac{1}{2} \sin\left(2\pi \overline{n_1 + n_2} \frac{r}{p}\right) + \tfrac{1}{2} \sin\left(2\pi \overline{n_1 - n_2} \frac{r}{p}\right).$$

Therefore $\sum_{r=0}^{p-1} \sin\left(2\pi n_1 \frac{r}{p}\right) \quad \cos\left(2\pi n_2 \frac{r}{p}\right)$

$$= 0 \text{ for all } n_1, n_2\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(IV.11)}$$

Equations (IV.3), (IV.4), (IV.5) are special cases of (IV.9), (IV.10), (IV.11).

Frames

e.g., terms which would arise from

$$\sum_{r=0}^{q-1} \left\{ \sum_m \cos\left(m\pi \frac{r}{q}\right) + \sum_m \sin\left(m\pi \frac{r}{q}\right) \right\}^2.$$

Now $\qquad \cos\left(m_1 \pi \frac{r}{q}\right) \quad \cos\left(m_2 \pi \frac{r}{q}\right)$

$$= \tfrac{1}{2} \cos\left(\overline{m_1 + m_2}\, \pi \frac{r}{q}\right) + \tfrac{1}{2} \cos\left(\overline{m_1 - m_2}\, \pi \frac{r}{q}\right).$$

Therefore $\sum_{r=0}^{q-1} \cos\left(m_1 \pi \frac{r}{q}\right) \quad \cos\left(m_2 \pi \frac{r}{q}\right)$

$$= \begin{cases} 0 \quad , \quad m_1 \pm m_2 \text{ even but } \neq 0 & , \quad \pm 2q\,, \quad \pm 4q \ldots \\ \frac{1}{2}q\,, \quad (m_1 + m_2) \text{ or } (m_1 - m_2) = 0 & , \quad \pm 2q\,, \quad \pm 4q \ldots \\ q \quad , \quad (m_1 + m_2) \text{ and } (m_1 - m_2) = 0\,, & \quad \pm 2q\,, \quad \pm 4q \ldots \\ 1 \quad , \quad m_1 \pm m_2 \text{ odd} & \end{cases} \quad \text{(IV.12)}$$

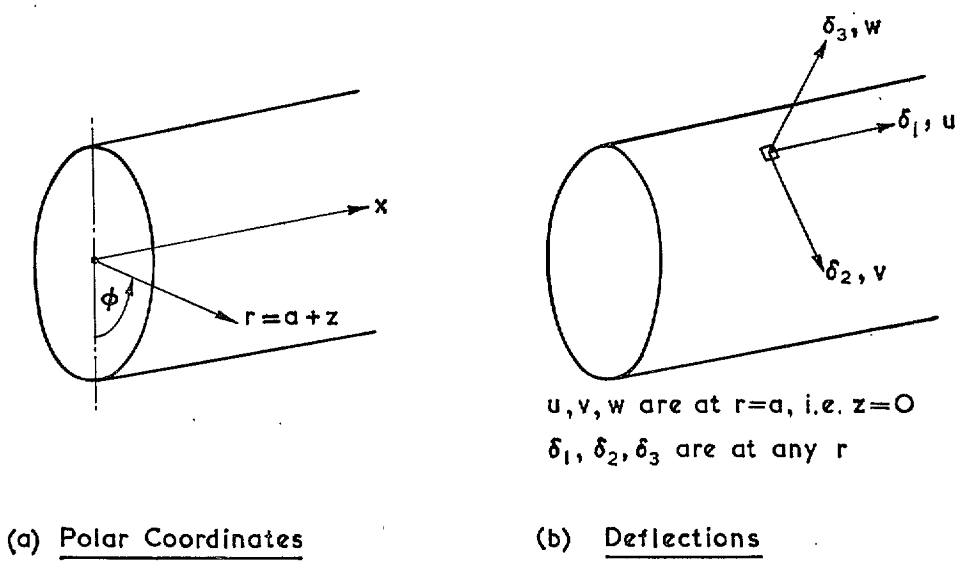
Similarly $\sum\limits_{r=0}^{q-1} \sin\left(m_1\pi \frac{r}{q}\right) \sin\left(m_2\pi \frac{r}{q}\right)$

$$= \begin{cases} 0\ , & m_1 \pm m_2 \neq 0 \qquad\qquad\qquad\quad ,\ \pm 2q\ ,\ \pm 4q \ldots \\ \frac{1}{2}q\ , & (m_1 + m_2) \text{ or } (m_1 - m_2) = 0 \quad ,\ \pm 2q\ ,\ \pm 4q \ldots \\ 0\ , & (m_1 + m_2) \text{ and } (m_1 - m_2) = 0\ ;\ \pm 2q\ ,\ \pm 4q \ldots \end{cases} . \tag{IV.13}$$

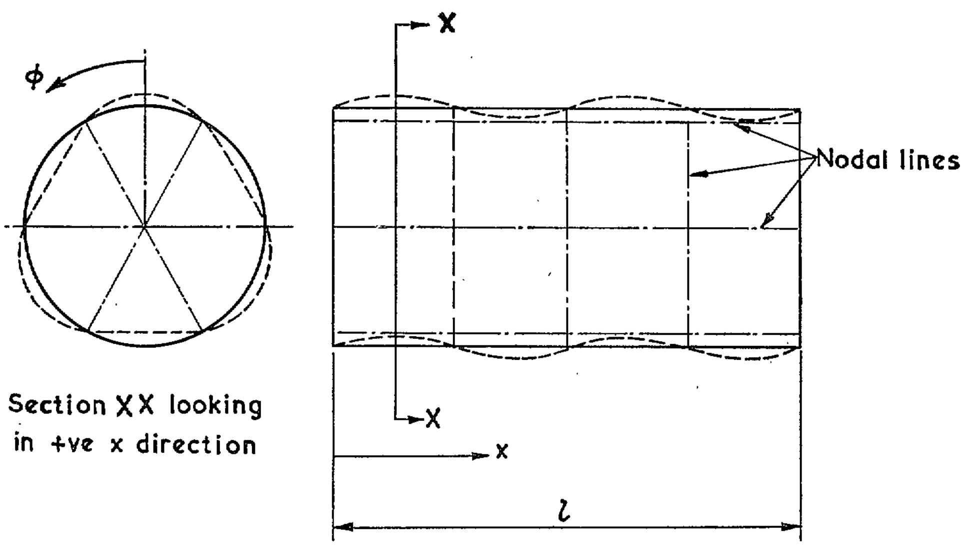
and $\quad \sum\limits_{r=0}^{q-1} \sin\left(m_1\pi \frac{r}{q}\right) \cos\left(m_2\pi \frac{r}{q}\right)$

$$= \begin{cases} 0\ , & m_1 \pm m_2 \text{ even} \\ \dfrac{\cos\,(\overline{m_1 + m_2}\,\pi/2q)}{\sin\,(\overline{m_1 + m_2}\,\pi/2q)} + \dfrac{\cos\,(\overline{m_1 - m_2}\,\pi/2q)}{\sin\,(\overline{m_1 - m_2}\,\pi/2q)}\ , & m_1 \pm m_2 \text{ odd} \end{cases} . \tag{IV.14}$$

Note: This last summation can only occur in a very long cylinder where the end conditions are virtually indeterminate so that we can have displacements varying as either $\sin\,(m\pi x/l)$ or $\cos\,(m\pi x/l)$.



FIGS. 1a and 1b. Co-ordinate system.



$$w = q_3 \cos n\phi \sin \frac{m\pi x}{l} \; ; \quad n = 3, \quad m = 4$$

FIG. 2. Typical radial deflection form.

(78425) Wt. 3523/8210 K7 8/60 Hw. PRINTED IN ENGLAND

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50s. (52s.).
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63s. (65s.)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50s. (52s.)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control, Structures. 63s. (65s.)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75s. (77s.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47s. 6d. (49s. 6d.)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80s. (82s.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90s. (92s. 9d.)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84s. (86s. 6d.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84s. (86s. 6d.)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130s. (132s. 9d.)
Vol. II. Aircraft, Airscrews, Controls. 130s. (132s. 9d.)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130s. (132s. 6d.)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130s. (132s. 6d.)

Annual Reports of the Aeronautical Research Council—

1937 2s. (2s. 2d.) 1938 1s. 6d. (1s. 8d.) 1939–48 3s. (3s. 5d.)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - R. & M. 2600 2s. 6d. (2s. 10d.)

Author Index to all Reports and Memoranda of the Aeronautical Research Council—

1909—January, 1954 R. & M. No. 2570 15s. (15s. 8d.)

Indexes to the Technical Reports of the Aeronautical Research Council—

December 1, 1936—June 30, 1939	R. & M. No. 1850	1s. 3d. (1s. 5d.)
July 1, 1939—June 30, 1945	R. & M. No. 1950	1s. (1s. 2d.)
July 1, 1945—June 30, 1946	R. & M. No. 2050	1s. (1s. 2d.)
July 1, 1946—December 31, 1946	R. & M. No. 2150	1s. 3d. (1s. 5d.)
January 1, 1947—June 30, 1947	R. & M. No. 2250	1s. 3d. (1s. 5d.)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2251–2349	R. & M. No. 2350	1s. 9d. (1s. 11d.)
Between Nos. 2351–2449	R. & M. No. 2450	2s. (2s. 2d.)
Between Nos. 2451–2549	R. & M. No. 2550	2s. 6d. (2s. 10d.)
Between Nos. 2551–2649	R. & M. No. 2650	2s. 6d. (2s. 10d.)
Between Nos. 2651–2749	R. & M. No. 2750	2s. 6d. (2s. 10d.)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1; 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol 1; 80 Chichester Street, Belfast, *or through any bookseller.*

S.O. Code No. 23-3154

R. & M. No. 315