R. & M. No. 3094
(17,263)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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S. NEUMARK, Techn.Sc.D., F.R.Ae.S., J. COLLINGBOURNE, B.Sc., and
H. H. B. M. THOMAS, B.Sc., A.F.R.Ae.S.

LONDON : HER MAJESTY'S STATIONERY OFFICE
1958
PRICE 15*s.* 0*d.* NET

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S. Neumark, Techn.Sc.D., F.R.Ae.S., J. Collingbourne, B.Sc., and H. H. B. M. Thomas, B.Sc., A.F.R.Ae.S.

Communicated by the Principal Director of Scientific Research (Air), Ministry of Supply

Reports and Memoranda No. 3094*

July, 1954

Summary.—Following flight experience of a particular aircraft, the effect of including a bob-weight and feel spring in the circuit of a power-operated longitudinal control on the dynamic stability of the aircraft is investigated.

The main findings of the investigation, which are given in the Introduction and fully set out in the discussion and conclusions at the end of the paper, can be summarised briefly as follows:

(*a*) With such a control system, instability of the aircraft short-period oscillatory mode can result. In these circumstances damping of this oscillatory mode deteriorates progressively with increase of speed.

(*b*) Friction in the control circuit is an important factor affecting the characteristics of the aircraft stability.

(*c*) It is considered that by care in design, particularly as regards positioning of the bob-weight, and choice of gearing, such instability can be avoided. Each case, however, requires examination on its own merits, on the lines of the analysis given here.

(*d*) For setting up the equations of motion, the transfer function of the power unit is required. In the present calculations a simple approximation is used, which raises the degree of the characteristic equation from a quartic to a sextic.

(*e*) Some consideration was given to the effect of changes in the more important design parameters, but it is found that apart from the gearing and position of the bob-weight mentioned above, they may have (within reasonable limits) only a mild palliative effect.

The main investigation was done by means of the usual mathematical analysis, with friction represented by an equivalent viscous damping. Additional results are obtained by the use of the Nyquist presentation and of the Philbrick Electronic Analog Computor.

1. *Introduction.*—On aircraft fitted with powered controls, it is necessary to provide some 'feel' for the pilot; the simplest way of doing this is to introduce some form of spring in the elevator circuit. A bob-weight is often added to the control circuit with the object of making the stick force per g less dependent on speed, and with the hope of restricting the amount of normal acceleration which can be imposed by the pilot (or by a gust). A control system containing these two features of spring and bob-weight is itself an oscillatory system and is sensitive to the symmetrical disturbances of the aircraft through normal accelerations because the aircraft,

* R.A.E. Report Aero. 2517, received 30th December, 1954.

elevator, power unit, and spring and bob-weight, form a closed loop. Thus, if the aircraft were subjected to any accidental disturbance involving normal acceleration, the bob-weight would be displaced and would move the control so that the oscillations might be augmented. In other words, there are the two oscillatory systems: (*a*) the control circuit and (*b*) the aircraft, which are coupled. This coupling may act in such a way as to reduce the damping of one of the oscillatory modes to a point where it becomes dangerous.

That this is so, is borne out by recent flight tests on an aircraft with these features. During the tests, the aircraft developed short-period oscillations in which the damping was inadequate and decreased as the aircraft speed increased.

This report discusses this kind of motion and shows that, in order to predict the behaviour of the aircraft with any accuracy, it is necessary to study its stick-free dynamic stability, taking into account the effect of friction in the control circuit and the effect of time constants of the power unit (which means treating the power unit as an additional oscillatory system). An attempt had been made originally to investigate the problem by considering the power unit as a simple proportional gear. This led, however, to the surprising result that, for zero friction, when the aircraft and bob-weight modes were linked by this gear, the aircraft damping improved but the bob-weight mode became unstable. This seemed to indicate that the oversimplified assumption as to the properties of the power unit may have led to serious errors although, with positive friction, reversed and more plausible results could be obtained. Accordingly, in this paper the characteristic equation of the power unit, as quoted by the manufacturers, has been included in the system of equations of motion, which have been derived in Section 2.

The equations in this form are difficult to discuss analytically, and they have been simplified for most of the analysis by substituting an equivalent viscous damping for the solid friction in the control circuit. The interpretation of this substitution is discussed in Section 3.

Three methods have been used to investigate these equations:

(*a*) Normal analytical method for linear differential equations with constant coefficients, discussed in Section 2

(*b*) Computation by an electronic analogue machine, discussed in Section 5

(*c*) The method of Nyquist diagrams, discussed in Section 6.

Various numerical examples have been examined using one or other of these methods. Details of the data are given in Tables 1 to 5.

When using the analogue computor it is possible to solve the equations with a true representation of solid friction. The accuracy of solution is poor but the results confirm qualitatively the deductions by the approximate method.

Analysis on these lines explains, generally, most of the features observed in the flight tests. In the tests, however, trailing-edge strips were fitted to the tail in an attempt to cure the oscillation. These did effect a little improvement by slightly increasing the speed at which the oscillations became dangerous. This improvement cannot be explained within the limitations of the above theory. In order to explain this, it is necessary to take into account the elasticity of the control circuit between the power unit and the tail surface. This is discussed in Section 4.

The main conclusions from this investigation are as follows:

(i) Inertia weight(s) included in the circuit of power-operated control, however irreversible, may lead to a dangerous dynamic instability in the form of steady or divergent oscillations, increasing with speed. No such systems should be included without a most careful numerical analysis for which the present paper gives a ready scheme, down to the computational stage.

(ii) Servo-units supplied for power-operated controls should be provided with their proper transfer functions and the numerical values of their respective time constants, to be used in assessing the aircraft dynamic characteristics, such as stability and response.

(iii) Friction plays an important part in the stick-free dynamic instability described here, and its increase, within usual limits, makes matters worse. The present method of estimating its effects is qualitatively correct, but requires further research to get more reliable numerical results. No doubts as to the reliability of the general results arise in the meantime.

(iv) The power-unit time constants (if in the range comparable with those used here) and the aircraft c.g. position do not affect appreciably the dynamic instability caused by weights and springs. The trailing-edge strips (or alternative devices increasing the hinge-moment coefficients of the elevator or all-moving tail) may help only a little, through the medium of elastic distortion of the rear part of the circuit, but this may only reduce the danger slightly and delay it to a somewhat higher range of speeds. The increase of the aircraft's own rotary damping, i.e., $(m_q + m_{\dot{w}})$ derivative, is somewhat more helpful, although again only in the delaying sense, unless the usual values of this derivative are strongly magnified by some special artificial auto-control.

(v) Decreasing the elevator-to-bob-weight gear ratio, G, may reduce or even eliminate the instability but, if accompanied (as would usually be the case) by an increase in the bob-weight-to-stick gear ratio, Γ, it would result in difficulties with the stick-force-per-g characteristics. A reasonable solution may sometimes be sought this way, but only if combined with a careful modification of all relevant design parameters.

(vi) The detrimental effect of the bob-weight and spring on dynamic stability decreases and may disappear when the bob-weight is moved forward. However, in view of the large number of parameters involved, the calculation is still required, even with a well forward position.

The standard nomenclature of Ref. 1 is used, together with the additional notation of Refs. 2 and 3.

2. *Derivation of the Equations.*—As the oscillations were of quite short period, the effect of change of forward speed has been neglected. The motion of the aircraft alone in level flight is then represented by:

$$\left.\begin{aligned}(D + \tfrac{1}{2}a)\hat{w} - \hat{q} &= 0\\ (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} + \delta\eta &= 0\end{aligned}\right\} \qquad (2.1)$$

If the displacement of the bob-weight is y (positive down), the equation of motion of the bob-weight is given by:

$$(M_1 + M_2)\ddot{y} \mp F + K_y y = M_2\{(n-1)g - r\dot{q}\}\,, \qquad (2.2a)$$

where F is the effective frictional force reduced to the bob-weight (the $\mp$ sign is to indicate that the friction always opposes the motion, *i.e.*, the sign is opposite to that of $\dot{y}$), and

M_1 = effective mass of the control system less bob-weight at the bob-weight (slugs)

M_2 = mass of bob-weight (slugs)

K_y = effective spring rate (force on the bob-weight per foot of static deflection)

r = distance of bob-weight aft of the aircraft c.g.

Now, if the frictional force is replaced, in the usual way (*see* Refs. 4 and 5 and Section 3 below), by an equivalent viscous friction proportional to the velocity of the bob-weight, the equation of motion of the bob-weight becomes:

$$(M_1 + M_2)\ddot{y} + K_{\dot{y}}\dot{y} + K_y y = M_2\{(n-1)g - r\dot{q}\}\,, \qquad (2.2b)$$

$K_{\dot{y}}$ being the appropriate derivative of equivalent viscous friction.

The incremental normal acceleration of the aircraft c.g. is:

$$(n-1)g = \frac{a\Delta\alpha\frac{1}{2}\rho V^2 S}{W} g = \frac{1}{2} a\hat{w}\frac{V}{\hat{t}} \; . \qquad \ldots \quad (2.3)$$

Substituting in equations (2.2*a*) and (2.2*b*), and reducing to non-dimensional units, we have

$$-k\hat{w} + sD\hat{q} + (D^2 + c)\hat{y} \mp \frac{F\hat{t}^2}{l(M_1 + M_2)} = 0 \qquad \ldots \quad (2.4a)$$

or

$$-k\hat{w} + sD\hat{q} + (D^2 + bD + c)\hat{y} = 0 \; ; \qquad \ldots \quad (2.4b)$$

where

$$\left.\begin{array}{ll} \hat{y} = y/l & k = \frac{1}{2} a \frac{\mu M_2}{M_1 + M_2} \\ l = \text{tail arm} & b = \frac{\hat{t}K_{\dot{y}}}{M_2 + M_2} \\ s = \frac{M_2}{M_1 + M_2}\frac{r}{l} & c = \frac{\hat{t}^2 K_y}{M_1 + M_2} \end{array}\right\} \qquad \ldots \quad (2.5)$$

The characteristic equation of the power unit, quoted by the manufacturers as representing its dynamic properties with sufficient accuracy, is:

$$T_1 T_v \ddot{\theta}_0 + T_v \dot{\theta}_0 + \theta_0 = \theta_i \, , \qquad \ldots \quad (2.6)$$

where T_1, T_v are time constants; θ_i and θ_0 are the input and output, being, in this case, proportional to the bob-weight displacement $\hat{y}$ and the control angle η, respectively, thus: $\theta_i = K_i\hat{y}$, $\theta_0 = K_0\eta$.

Equation (2.6) may be rewritten:

$$\frac{T_1 T_v}{\hat{t}^2} K_0 D^2\eta + \frac{T_v}{\hat{t}} K_0 D\eta + K_0\eta = K_i\hat{y} \, , \qquad \ldots \quad (2.7)$$

or

$$D^2\eta + MD\eta + N\eta = GN\hat{y} \; ; \qquad \ldots \quad (2.7a)$$

where

$$\left.\begin{array}{l} M = \frac{\hat{t}}{T_1} \\ N = \frac{\hat{t}^2}{T_1 T_v} \\ G = \frac{K_i}{K_0} \end{array}\right\} \qquad \ldots \quad (2.8)$$

G is the non-dimensional elevator-to-bob-weight gear ratio. This is seen from (2.7*a*) where, if the bob-weight is displaced and held, the first and second derivatives of η soon disappear and then $\eta = G\hat{y}$ which gives the gear ratio of $\eta : \hat{y}$ as G.

The complete set of equations of motion of the oscillating system is given by equations (2.1), (2.4*b*) and (2.7*a*) and may be written (by eliminating $\hat{q}$):

$$\left.\begin{array}{llll} \{D^2 + (\frac{1}{2}a + \nu + \chi)D + (\frac{1}{2}a\nu + \omega)\}\hat{w} & + \delta\eta & & = 0 \\ & (D^2 + MD + N)\eta & - GN\hat{y} & = 0 \\ -\{k - \frac{1}{2}asD - sD^2\}\hat{w} & & + (D^2 + bD + c)\hat{y} & = 0 \end{array}\right\} . \qquad (2.9)$$

The stability sextic in D is of the form:

$$\{D^2 + (\tfrac{1}{2}a + \nu + \chi)D + (\tfrac{1}{2}a\nu + \omega)\}(D^2 + MD + N)(D^2 + bD + c) + \delta GN(k - \tfrac{1}{2}asD - sD^2)$$
$$= D^6 + B_1D^5 + C_1D^4 + D_1D^3 + E_1D^2 + F_1D + G_1 = 0\,, \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

where:

$$\left.\begin{aligned}
B_1 &= (\tfrac{1}{2}a + \nu + \chi + M) + b \\
C_1 &= \{c + \tfrac{1}{2}a\nu + \omega + M(\tfrac{1}{2}a + \nu + \chi) + N\} + b(\tfrac{1}{2}a + \nu + \chi + M) \\
D_1 &= \{c(\tfrac{1}{2}a + \nu + \chi + M) + M(\tfrac{1}{2}a\nu + \omega) + N(\tfrac{1}{2}a + \nu + \chi)\} \\
&\quad + b\{\tfrac{1}{2}a\nu + \omega + M(\tfrac{1}{2}a + \nu + \chi) + N\} \\
E_1 &= [c\{\tfrac{1}{2}a\nu + \omega + M(\tfrac{1}{2}a + \nu + \chi) + N\} + N(\tfrac{1}{2}a\nu + \omega) - \delta GsN] \\
&\quad + b\{M(\tfrac{1}{2}a\nu + \omega) + N(\tfrac{1}{2}a + \nu + \chi)\} \\
F_1 &= [c\{M(\tfrac{1}{2}a\nu + \omega) + N(\tfrac{1}{2}a + \nu + \chi)\} - \tfrac{1}{2}a\delta GsN] + b(\tfrac{1}{2}a\nu + \omega)N \\
G_1 &= c(\tfrac{1}{2}a\nu + \omega)N + k\delta GN
\end{aligned}\right\} . \qquad (2.11)$$

3. *Interpretation of Results in the Basic Case.*—There are three oscillatory modes coupled together in the stability equation (2.10): the short-period aircraft mode, the mode of the spring and bob-weight system, and that of the power unit. Individually, all these three modes are well damped (the latter two may even become aperiodic under certain circumstances) but, when they are coupled together, the damping may become inadequate or even negative, the aircraft mode being particularly vulnerable. The resultant sextic may be factorized into three quadratic factors, each of which is recognizable as being a modified form of one of the original modes because the coupling does not alter drastically the constant terms of the quadratics.

The results of the calculations for various cases are given in Table 6 (1 to 5), and it is seen that the spring and bob-weight mode and the power-unit motion are always very well damped but that, for the aircraft mode, the damping is generally small and varies from positive to negative values, depending on the speed and the equivalent viscous damping coefficient b. This damping factor R (where the roots are represented as $-R \pm iJ$) is always positive for low speeds whatever the value of b; as the speed increases, however, the damping deteriorates. For a constant speed, as b is increased, R at first decreases and then increases again for larger values of b.

Let us now find the factors upon which the equivalent viscous damping coefficient b depends. The equivalent viscous friction is found by assuming that there is one oscillatory mode (that with low damping) and that the work done in each quarter of a cycle by the equivalent viscous friction is the same as that corresponding to the solid one, *i.e.*,

$$\int_0^{y_0} K_y \dot{y}\, dy = Fy_0\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where y_0 is the amplitude of the oscillations of the bob-weight.

It is sufficient to assume that the motion is simple harmonic of non-dimensional frequency J (which means that our formulae will apply well only for $R = 0$, but there should be only small discrepancies for small positive or negative R). We put therefore:

$$\left.\begin{aligned} y &= y_0 \sin J\tau \\ \dot{y} &= y_0 \frac{J}{\hat{t}} \cos J\tau \end{aligned}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

and then the work done in a quarter of a cycle by the viscous friction is:

$$\int_0^{\pi/2J} \frac{1}{\hat{t}} K_{\dot{y}} J^2 y_0^2 \cos^2 J\tau \, d\tau \, , \quad \ldots \qquad (3.3)$$

and hence:

$$F y_0 = \frac{1}{4} \frac{\pi}{\hat{t}} K_{\dot{y}} J y_0^2 \, , \quad \ldots \qquad (3.4)$$

and

$$K_{\dot{y}} = \frac{4F\hat{t}}{\pi J \hat{y}_0 l} \, ; \quad \ldots \qquad (3.5)$$

where

$$\hat{y}_0 = \frac{y_0}{l} \, .$$

The equation (3.5) is completely analogous to equation (36) of Ref. 5. From the expression for b in equation (2.5) we now obtain:

$$b = \frac{4F\hat{t}^2}{\pi(M_1 + M_2)l\hat{y}_0 J} \quad \ldots \qquad (3.6)$$

It is seen that b increases in proportion to the solid friction F but, in addition, it depends on the amplitude $\hat{y}_0$ and on the frequency J, decreasing in inverse proportion to either. It is impossible to determine the value of b, unless $\hat{y}_0$ and J are known in addition to F. The formula (3.6) can only be used in such a way that the stability sextic is factorised for varying b, the frequency J becoming determined as a function of b. This function, as seen from Table 6 (1 to 5), is always a decreasing one but the rate of decrease is low enough for the product bJ to increase with b. Writing (3.6) in the form:

$$\hat{y}_0 = \frac{4F\hat{t}}{\pi(M_1 + M_2)lbJ} \, , \quad \ldots \qquad (3.7)$$

we see that the amplitude $\hat{y}_0$ always decreases when b increases and *vice versa*.

The case of mid-weight c.g. position with trailing-edge strips on the tail has been taken as the basic case, and solutions of the stability sextic found for a range of values of b for different forward speeds. Table 1 gives the values of the derivatives assumed for the calculations for this case and Table 6 (1 to 5) full solutions of the sextics for a range of forward speed V.

Fig. 2 shows the variation of the damping factor of the aircraft mode, R, with the equivalent viscous damping coefficient b for varying speeds, still for the basic case. It will be seen that the damping factor falls as b increases from zero, reaches a minimum value and then rises. At 200 kt the damping is always positive but it deteriorates with increasing speed until, at a value just below 300 kt there will be a curve which touches the b axis and for any further increase of speed the damping is negative over a part of the range of b. For speeds greater than about 400 kt the damping is negative even when $b = 0$, but all curves must eventually, for large values of b, give positive damping factors (tending to the stick-fixed value) for, in the limiting case, there is no bob-weight displacement.

It follows, therefore, that for intermediate speeds there are two points at which there is zero damping. The point corresponding to the lower value of b gives the conditions for steady oscillations, as shown by the following reasoning. If the bob-weight originally oscillates with an amplitude corresponding to a certain b between the two zero damping values there will be negative damping, the amplitude will increase, *i.e.*, the value of b will decrease, until the zero damping point is reached when the amplitude will remain constant and steady oscillations will result. The reverse happens if the original amplitude is greater than that corresponding to the lower zero damping value of b, and the amplitude will settle down to the steady oscillations as before. These oscillations are therefore stable.

On the other hand, if the amplitude of the displacement is less than that corresponding to the second zero damping point (for the higher value of b) then the damping is positive and the motion dies out and so this point is associated with an unstable position. It gives the 'minimum conditions' for steady oscillations to develop, and any amplitude less than that represented by this point will die out.

For higher speeds, the damping factor is negative for $b = 0$ and we reach a state of divergent oscillations, the minimum conditions on which these depend being still defined by the intersection of the curve with the b-axis.

The amplitude of the control deflection at the two points of zero damping can now be evaluated. The two roots of the sextic corresponding to the aircraft mode are then purely imaginary ($D = \pm iJ$), and we find from (2.7):

$$\eta_0 = \frac{GN\hat{y}_0}{|D^2 + MD + N|} = \frac{GN\hat{y}_0}{|N - J^2 + iMJ|}$$

or

$$\eta_0 = \frac{GN\hat{y}_0}{\{(N - J^2)^2 + M^2J^2\}^{1/2}} \quad \ldots \quad (3.8)$$

The amplitude $\hat{y}_0$ is found from equation (3.7).

Similarly, the amplitude of the normal acceleration experienced at these two points may be found: we obtain from equation (2.1):

$$\hat{w}_0 = \frac{\delta\eta_0}{|\omega + \frac{1}{2}a\nu - J^2 + iJ(\nu + \chi + \frac{1}{2}a)|}, \quad \ldots \quad (3.9)$$

and from equation (2.3):

$$(n_0 - 1) = \frac{Va}{2g\hat{t}} \frac{\delta}{\{(\omega + \frac{1}{2}a\nu - J^2)^2 + (\nu + \chi + \frac{1}{2}a)^2J^2\}^{1/2}} \eta_0 . \quad \ldots \quad (3.10)$$

Figs. 8, 9 and 10 show the variation of the amplitudes of normal acceleration, control angle and bob-weight displacement, respectively, with speed, for the basic case and for different values of friction F. The basic value of F is taken as 10·15 lb and values of 50 per cent and 150 per cent of this value are also taken.

The validity of the approximation to solid friction by taking an equivalent viscous damping may be checked in the following way. In conditions of zero damping (i.e., at $R = 0$) every variable (such as $\hat{w}$, $\hat{q}$, η, y) is a simple harmonic function of time, and the motion of the bob-weight (equation (2.4b) may be regarded as a forced oscillation, the forcing function being:

$$-k\hat{w} + sD\hat{q} . \quad \ldots \quad (3.11)$$

Substituting for $\hat{w}$ and $D\hat{q}$ from equation (2.1), expression (3.11) becomes:

$$\frac{\{k - sD(D + \frac{1}{2}a)\}\,\delta\eta}{\omega + \frac{1}{2}a\nu - J^2 + iJ(\nu + \chi + \frac{1}{2}a)} . \quad \ldots \quad (3.12)$$

If we assume as before that this function is a simple harmonic one of time, its amplitude X_0 is given by:

$$X_0 = \delta\eta_0 \left[\frac{(k + sJ^2)^2 + \frac{1}{4}a^2s^2J^2}{(\omega + \frac{1}{2}a\nu - J^2)^2 + (\nu + \chi + \frac{1}{2}a)^2J^2}\right]^{1/2} \quad \ldots \quad (3.13)$$

or, if we substitute the value of η_0 given by equation (3.7):

$$X_0 = \frac{\delta GN}{\{(N - J^2)^2 + M^2J^2\}^{1/2}} \left[\frac{(k + sJ^2)^2 + \frac{1}{4}a^2s^2J^2}{(\omega + \frac{1}{2}a\nu - J^2)^2 + (\nu + \chi + \frac{1}{2}a)^2J^2}\right]^{1/2} \hat{y}_0 . \quad \ldots \quad (3.14)$$

Replacing the solid friction by an equivalent viscous friction is shown by Timoshenko in Ref. 6 to be valid if the ratio of the frictional force to the amplitude of the forcing function is less than $\pi/4$. Taking the non-dimensional forms of these quantities, the frictional force is found by equation (2.4a) and the amplitude of the forcing function is given by equation (3.14). Hence for the approximation to be valid the following inequality must hold:

$$\frac{F\hat{t}^2}{l(M_1 + M_2)} < \frac{\pi}{4} X_0 . \qquad \ldots \qquad (3.15)$$

This inequality has been checked in several cases and found to be well satisfied at the points corresponding to steady oscillations. At those for minimum conditions the inequality only just holds and this, of course, means a reduction in the accuracy. However, accuracy is not very important at these points where the amplitude is so small.

4. *Effect of the Elasticity of the Rear Part of the Control Circuit.*—Let k_e be the elastic constant for the part of the control circuit between the power unit and the control. Then, in the equilibrium state:

$$\Delta\eta = k_e\, \delta H , \qquad \ldots \qquad (4.1)$$

where H is the hinge moment on the tail.

If there were no distortion the hinge moment would be (taking $b_1 = b_2$ in this case of an all-moving tail):

$$H = \tfrac{1}{2}\rho V^2 S_t c_t b_2 (\Delta\alpha' + \eta) , \qquad \ldots \qquad (4.2)$$

where $\Delta\alpha'$ is the incremental incidence of the undeflected tailplane and η is the deflection (from equilibrium position).

If the distortion of the circuit takes place then η must be replaced by $(\eta + \Delta\eta)$, and if we now write:

$$Q = K_e \tfrac{1}{2}\rho V^2 S_t c_t b_2 , \qquad \ldots \qquad (4.3)$$

we have

$$\Delta\eta = \frac{Q}{1 - Q}\Delta\alpha' + \frac{Q}{1 - Q}\eta . \qquad \ldots \qquad (4.4)$$

From this equation it will be seen that, for an all-moving tail, we may interpret the elasticity of the control circuit as a change in α' and η in the ratio $\{1 + Q/(1 - Q)\}$, i.e., $1/(1 - Q)$. As both angles are effectively changed in the same ratio this may be further interpreted as a change in tailplane area by a factor $1/(1 - Q)$.

This means that we reduce all tailplane contributions to the derivatives by this ratio. The derivatives affected are m_η, m_w, $m_{\dot{w}}$ and m_q. The effect of the addition of the strips is shown by the change in b_2 which they bring about.

The values estimated for the derivatives for all cases considered are given in Tables 1 to 5. They include the effect of control-circuit elasticity.

5. *Numerical Solutions of Alternative Cases by Electronic Computation.*—As the solution of the stability sextics by manual computation was rather onerous, additional cases to the basic one were investigated with the aid of the Philbrick Electronic Analog Computor. This machine is described in some detail in Ref. 8. The resultant motion of the system is displayed on a cathode-ray oscilloscope and made it possible to obtain, with reasonable accuracy, the points of zero damping and also to measure the damping and frequency of the coupled aircraft mode for any value of b, as the other two modes are rapidly damped out.

The electronic-computer results are shown in Figs. 3, 4, 5, 6 and 7. The basic case was repeated and the Philbrick results are given in Fig. 3; it will be seen that the agreement between Figs. 2 and 3 is very good. Another four cases were investigated on the Philbrick, namely,

(*a*) Mid-weight c.g. position with trailing-edge strips on the tail but with a change of the time constant T_1 of the power-unit equation. This now becomes $T_1 = 0 \cdot 01$ sec.

(*b*) Mid-weight c.g. position without trailing-edge strips on the tail but with $T_1 = 0 \cdot 02$ sec as for the basic case.

(*c*) Forward c.g. position with trailing-edge strips on the tail, and $T_1 = 0 \cdot 02$ sec.

(*d*) As for the basic case but with increased aircraft damping obtained by doubling the values of ν and χ.

The results for these conditions are shown in Figs. 4 to 7, respectively, and it is clear from these that changing the time constant, T_1, has little effect on the damping of the aircraft mode. The presence of the trailing-edge strips does improve the stability as was found in the flight tests, but the effect is not very large; the movement of the aircraft c.g. to its forward position also improves the damping to some extent. The most marked improvement, however, comes with the doubling of the values of ν and χ, but even so the trouble is not eliminated, and there are still regions of negative damping at the higher speeds.

It is possible to represent solid friction on the Philbrick by the use of the so-called ' Bounding Component ' (*see* Ref. 8), and some calculations on these lines were done, although it was appreciated that the Philbrick would only give low numerical accuracy in the critical cases considered. The results confirmed qualitatively all those found using the concept of equivalent viscous friction, but appreciable numerical differences existed in amplitudes of the steady oscillations in particular. A discussion of the reasons underlying this is given in Section 7.

It may be interesting to compare the periods of the steady oscillations for various values of speed and for all the cases considered. These values are given in Table 8 for the stick-free case, and the period of the well damped stick-fixed aircraft mode is also given for comparison.

6. *The Method of Nyquist Diagrams.*—It was considered worth while to make a different approach to the problem in view of the time involved in manual computation and in even the electronic computation. This alternative was to employ the method of the Nyquist diagram or harmonic response locus[9, 10, 11], which does not involve solving the stability equation but indicates whether the resultant motion is positively or negatively damped. This method also has the advantage of rapidly demonstrating the effect of varying different parameters, especially of the gearing factor G.

We are going to use the simplest form of Nyquist criterion, applicable whenever all particular elements of the system are by themselves stable and it is required to find out whether the coupled system is also stable. The complete proof of the Nyquist criterion is rather long and complicated; it may be found in Ref. 13. In the given case, the elements of the system are the aircraft, the power unit, and the circuit with bob-weight and spring, and each of them is stable by itself.

We may write equation (2.10) in the form:

$$Y(D) = \frac{\delta GN(k - \frac{1}{2}asD - sD^2)}{(D^2 + AD + B)(D^2 + MD + N)(D^2 + bD + c)} = -1 \qquad (6.1)$$

where:

$$\left.\begin{aligned} A &= \tfrac{1}{2}a + \nu + \chi \\ B &= \tfrac{1}{2}a\nu + \omega \end{aligned}\right\} \qquad (6.2)$$

For the case of zero damping, $D = iJ$, and hence:

$$Y(iJ) = \frac{k\delta G}{Bc} \frac{\left(1 + \frac{s}{k}J^2\right) - i\frac{1}{2}a\frac{s}{k}J}{\left\{\left(1 - \frac{1}{B}J^2\right) + i\frac{A}{B}J\right\}\left\{\left(1 - \frac{1}{N}J^2\right) + i\frac{M}{N}J\right\}\left\{\left(1 - \frac{1}{c}J^2\right) + i\frac{b}{c}J\right\}} \quad \ldots \quad (6.3)$$

Thus

$$Y(iJ) = \frac{k\delta G}{Bc} \rho_c\rho_a\rho_p\rho_b \, e^{-i(\psi_c+\psi_a+\psi_p+\psi_b)}, \quad \ldots \quad (6.4)$$

where

$$\left.\begin{aligned} \rho_c^2 &= \frac{(k + sJ^2)^2 + (\frac{1}{2}asJ)^2}{k^2}, & \tan\psi_c &= \frac{\frac{1}{2}asJ}{k + sJ^2}, \\ \rho_a^2 &= \frac{B^2}{(B - J^2)^2 + A^2J^2}, & \tan\psi_a &= \frac{AJ}{B - J^2}, \\ \rho_p^2 &= \frac{N^2}{(N - J^2)^2 + M^2J^2}, & \tan\psi_p &= \frac{MJ}{N - J^2}, \\ \rho_b^2 &= \frac{c^2}{(c - J^2)^2 + b^2J^2}, & \tan\psi_b &= \frac{bJ}{c - J^2}. \end{aligned}\right\} \quad \ldots \quad (6.5)$$

If $Y(iJ)$ according to equation (6.4) is plotted on polar graph paper and if the curve passes through the point $(-1, 0)$ then, by virtue of equation (6.1), the damping is zero. Further, the Nyquist criterion states that if the curve as it approaches the origin leaves this point on the left, then the damping is positive; if on the right, it is negative.

It will be seen that it is quite a speedy procedure to change one of the parameters at a time as it only involves re-calculating the appropriate ρ and ψ, while leaving all other ρ's and ψ's unchanged. Figs. 11 and 12 are two Nyquist diagrams, both for 450 kt. Fig. 11 gives the curves for the assumed values of the derivatives in the previous calculations (basic case) and for three values of b; it is seen that the value of b giving zero damping ($b = 965$) agrees with the value already obtained from the calculations. The damping is negative for the other two values of b.

An enlargement of the part of Fig. 11 indicated by the circle is shown in Fig. 12 together with additional curves showing the effect of neglecting the power unit mode and of moving the bob-weight forward to the c.g. of the aircraft ($s = 0$), thus reducing the inertia term $r\dot{q}$ of equation (2.2*a*) to zero. This latter greatly improves the stability for the three values of b considered. The curve showing the effect of neglecting the power-unit mode bears out the results previously obtained, as it is clear from this diagram that there is an instability in the high-frequency mode.

The effect of varying the gear ratio G is quickly demonstrated in all cases, as it only means changing the scale of the curves, *i.e.*, it merely changes the position of the critical point on the original diagram. For instance, if G were halved in value the critical point would be at double its present distance from the origin, the curves remaining unaltered. Thus it is clear that the general stability can be vastly improved by this means if other conditions made it possible.

7. *Discussion of Results and Conclusions.*—While in normal conditions no case of stick-free instability arises for an aircraft with irreversible power operated controls, it is clear from the preceding analysis that the introduction of inertia weight(s) in the elevator circuit leads to a grave danger of large-amplitude steady oscillations, or even strongly divergent oscillations, in the vulnerable short-period mode*. It is a firm belief of the present writers that this sort of

* It can be shown that having a bob-weight in the circuit without a feel-spring leads to an even more violent dynamic instability, but this case has, of course, no practical importance. The presence of a spring without bob-weight can lead to no harm, as the aircraft motion may be coupled to the power-unit motion only through the bob-weight.

instability was a primary cause of the aircraft behaviour referred to in the Introduction to this paper. This particular matter is discussed in detail elsewhere, but whatever the final opinion may be, one point seems to be beyond doubt: the danger of inserting inertia weight(s) in the circuit of power-operated controls, without a proper investigation and careful analysis, seems to have been conclusively proved. Hence:

Conclusion 1.—Inertia weight(s) included in the circuit of power-operated controls, however irreversible, may lead to a dangerous dynamic instability in the form of steady or divergent oscillations, increasing with speed. No such systems should be included without a most careful numerical analysis, for which the present paper gives a ready scheme, down to the computational stage.

It is shown that the power unit must not be treated as a simple (proportional) gear but as a 'complex exponential delay' unit, with its own differential characteristics (transfer function) and its own time constants (*see* equation (2.6)). It is fortunate that the transfer function may be treated as linear with sufficient approximation and so fits, with no very serious difficulties, into the general framework of the usual theory of aircraft dynamic stability. It may be hoped that similar simple transfer functions will apply for other power units, and then the appropriate time constants are all that need to be estimated for each unit. It must be mentioned, however, that the practice in aircraft firms has been (and too often still is) to treat, in all problems of aircraft dynamics, the power unit as a proportional gear, and aircraft firms usually do not ask for or receive the information about the transfer function from the manufacturers of power units. This may be quite all right, if only static properties of the aircraft are concerned but, in all dynamic considerations, such as dynamic stability of the sort described in this paper, and especially response problems, the proper transfer function must be known. Hence:

Conclusion 2.—Servo-units supplied for power operated controls should be provided with their proper transfer functions and the numerical values of the respective time constants, to be used in assessing the aircraft dynamic characteristics, such as stability and response.

It should be repeated once more that, neglecting the delay characteristics of the power unit, and hence omitting the corresponding 'power-unit mode' in the dynamic stability calculation, may lead to erroneous results. A particularly striking example was found in the present case, when friction was assumed to be absent. The calculations actually done by the interested firm on these lines and repeated for a check by the Royal Aircraft Establishment (also by using Nyquist diagram (*see* Section 6 and Fig. 12)), have shown the surprising and unrealistic negative damping in the 'bob-weight mode' of very high frequency, instead of the opposite behaviour shown by our calculations described above. It is true that, with increasing friction, the results obtained by neglecting the delay characteristics of the power unit become qualitatively correct and comparable to the more exact ones, as shown in Appendix II and Fig. 14. Even then, however, the quantitative discrepancies are very considerable, and the simplification should never be considered as permissible.

The effect of friction has been investigated in detail and shown to play an important part in the phenomenon of dynamic instability under consideration. As a result of this detailed investigation we are in a position to determine not merely whether or not instability occurs. We gain in fact a much fuller knowledge of the aircraft behaviour, inasmuch as we can decide what is the minimum initial disturbance required for the instability to become operative and calculate the amplitude of the dangerous steady oscillations and finally, the limit for definitely divergent oscillations to occur.

The analytical treatment of friction was based consistently on the approximate theory of 'equivalent viscous friction', as commonly used hitherto, and expressed by the formula (3.6). Some doubts have been put forward as to the reliability of this approximation. Now, an attempt to improve it, and to work out an exact theory has been made by Den Hartog[12] (1931) in connection with the simple problem of forced oscillations of systems with one degree of freedom and with solid friction. A preliminary survey of this method made by one of the present writers has shown that:

(*a*) Den Hartog's theory is not complete and fails to provide information required in the case (most important in our particular problem) of the forced frequency being considerably lower than the natural one.

(*b*) Nevertheless, the structure of the formula (3.6) is valid in all cases including those not solved by Den Hartog, so that our qualitative results are confirmed and only some (not negligible) modifications in the value of the numerical constant involved may take place. This constant is $4/\pi \simeq 1 \cdot 27$ in (3.6), while its actual value may be considerably lower or higher. If the ratio of forced to natural frequency is greater than 1, the constant may be somewhat greater but the corrections are small. In the opposite case, relevant for the present application, the constant may be about 1 (hence only slightly less than $4/\pi$) when the F/X approaches its upper limit 1; hence no appreciable errors may be expected as to the amplitudes at minimum conditions. If, however, the ratio F/X is small (as relevant for the amplitudes of the steady oscillations), the constant may sometimes assume much higher values. This explains why there were practically no discrepancies between our computed values of minimum amplitudes by using both the equivalent viscous damping and the solid friction, but there were more serious differences as regards the steady oscillations.

The entire matter is irrelevant as to the validity of all our conclusions, and as to the entire qualitative contents of this report, but some caution is advisable as regards the final numerical results depending on friction. This is not important at this stage as, in view of the great number of derivatives involved (each only approximately known), the numerical values obtained should be treated with caution.

(*c*) Even adopting the most cautious and distrustful attitude towards the present treatment of frictional effects, the theory presented in this paper is valid, as the decrease of aircraft damping with rising speed and dangerous instability at high speed has been ascertained even in the absence of friction, and the detrimental effect of increasing friction (within reasonable limits) has been conclusively proved, irrespective of detailed numerical values.

(*d*) As the effects of friction are likely to be important in many analogous practical problems in the near future, a further enquiry into the dynamic problems with solid friction, with a view to obtaining exact solutions, is recommended.

All this leads to:

Conclusion 3.—Friction plays an important part in the stick-free dynamic instability described here, and its increase, within usual limits, makes matters worse. The present method of estimating its effects is qualitatively correct, but requires further research to get more reliable numerical results. No doubts as to the reliability of the general results arise in the meantime.

Furthermore, efforts have been made to show the influence of various derivatives and design data on the onset of the unstable phenomenon considered, and to find ways to prevent this instability while not absolutely precluding inertia weight(s) and spring(s) in the control circuit. It is not claimed that all the many relevant variables have been tested, but probably the most important ones have, and the results have been presented in Sections 5 and 6. Several variables have been shown to have little or no effect, as summarized in:

Conclusion 4.—The power-unit time constants (if in the range comparable with those used here) and the aircraft c.g. position do not affect appreciably the dynamic instability caused by weights and springs. The trailing-edge strips (or alternative devices increasing the hinge-moment coefficients of the elevator or all-moving tail) may help only a little, through the medium of elastic distortion of the rear part of the circuit, but this may only reduce the danger slightly and delay it to a somewhat higher range of speeds. The increase of the aircraft's own rotary damping, i.e., $(m_q + m_{\dot{w}})$ derivative, is somewhat more helpful, although again only in the delaying sense, unless the usual values of this derivative are strongly magnified by some special artificial auto-control.

Finally, two parameters have been found which affect the results very considerably, *viz.*

Firstly, the elevator-to-bob-weight gear ratio G plays a very great part in the stability characteristics. The instability becomes gradually stronger and more dangerous as G increases (*cf.* Section 6). It might appear, therefore, that if this ratio is kept low, the instability could be avoided altogether (in our case halving the original value of G prevents instability up to 450 kt speed). However, it should be kept in mind that the inertia weight has been introduced to serve a useful purpose as a g-restrictor, and to ensure advantageous or at least tolerable stick-force-per-g characteristics through the speed range. The latter effect is clearly seen in the following manoeuvrability formula (stick force per g) as derived in Appendix I:

$$\frac{P}{n-1} = \Gamma\left(M_2 g - \frac{C_L \bar{c} H_m K_y}{2Gm_\eta}\right), \quad \ldots \quad (7.1)$$

where P is the stick force, $(n - 1)$ the number of g's produced by an elevator deflection in the ensuing steady circle, H_m denotes the usual manoeuvre margin, $\bar{c}$ is the standard mean chord, and Γ the ' bob-weight-to-stick ' gear ratio. The meaning of the other symbols is as defined before. The second term in equation (7.1) shows the effect of the feel spring (through its constant K_y) and, owing mainly to the factor C_L, decreases uncomfortably with increasing speed, while the first term represents the effect of the bob-weight mass, which is constant, and not only increases the stick force per g but also keeps it ' more constant ' at varying speeds. The product ΓG being the full ' elevator-to-stick ' gear ratio, it must be kept within appropriate limits. Hence, decreasing G drastically, requires increasing Γ in the inverse ratio, which may result in impossible manoeuvrability characteristics. It is seen that the matter is not simple at all, and further modifications will be required, such as a suitable decrease of M_2 and K_y. The effect of such changes on the stability characteristics has not been examined, so we are not able to assess the effect of a combined change in K_y, M_2 and G so as to keep the stick-force-per-g characteristics more or less unaltered.

Conclusion 5.—Decreasing the gear ratio G may reduce or even eliminate the instability but, if accompanied (as would usually be the case) by an increase in Γ it would result in difficulties with the stick-force-per-g characteristics. A reasonable solution may sometimes be sought this way, but only if combined with a careful modification of all relevant design parameters involved.

The other factor which basically affects the problem, is the longitudinal position of the bob-weight. In the given case, the weight was placed well aft of the centre of gravity, and the aircraft was subject to dangerous instability as described. Moving this bob-weight right to the aircraft c.g. was investigated by means of the Nyquist diagram, as described in Section 6, and the aircraft became stable up to the highest speed considered (450 kt). When investigating other g-restrictors and, more recently, in a particular case examined by a designer in an aircraft factory, similar beneficial effects have been noted, so there are good grounds for believing that this behaviour is general. This does not mean, however, at this stage, that a designer may simply put a bob-weight near to the aircraft centre of gravity and dispense with a calculation indicated by *Conclusion* 1. Hence:

Conclusion 6.—The detrimental effect of bob-weight and spring on dynamic stability decreases and may disappear when the bob-weight is moved forward. However, in view of the large number of parameters involved, the calculation is still required, even with a well forward position.

LIST OF SYMBOLS

A, B		Coefficients of the short period mode of the aircraft (*see* (6.2))
$B_1, C_1, D_1, E_1, F_1, G_1$		Coefficients of the stability equation (*see* (2.10) and (2.11))
b, c		Coefficients of the equation of motion of the bob-weight (*see* (2.4*a*), (2.4*b*) and (2.5))
b_1	=	$\partial C_H/\partial \alpha'$ Rate of change of hinge moment coefficient with tail incidence
b_2	=	$\partial C_H/\partial \eta$ Rate of change of hinge moment coefficient with control angle
$\bar{c}$		Mean wing chord (ft)
c_t		Mean chord of tailplane (ft)
D	=	$d/d\tau$ Differential operator
F		Effective frictional force reduced to bob-weight (lb)
G		Non-dimensional elevator-to-bobweight gear ratio (*see* (2.8))
H		Hinge moment of tailplane (lb ft)
H_m		Manoeuvre margin
i_B		Inertia coefficient about y-axis
J		Non-dimensional frequency of aircraft mode
K_y		Effective spring rate (force in pounds per foot of static displacement)
$K_{\dot{y}}$		Derivative of equivalent viscous friction (replacing solid friction)
k		Non-dimensional parameter (*see* (2.4*a*), (2.4*b*) and (2.5))
k_i		Constant (*see* (2.6))
k_0		Constant (*see* 2.6))
k_e		Elastic constant for rear part of control circuit (*see* (4.1))
l		Tail arm (ft)
M, N		Coefficients of equation of motion of power unit (*see* (2.7*a*) and (2.8))
M_1		Effective mass of control system less bob-weight, reduced to bob-weight (slugs)
M_2		Mass of bob-weight (slugs)
m_q		Rotary damping derivative in pitch, dimensionless
m_w		Pitching-moment derivative due to w, dimensionless
$m_{\dot{w}}$		Pitching-moment derivative due to rate of change of w, dimensionless
m_η		Pitching-moment derivative due to tailplane deflection, dimensionless
n		Load factor as shown by the accelerometer (so that $(n - 1)g$ is the true normal acceleration in disturbed flight)
P		Stick force (lb)

Q		Portmanteau symbol (*see* (4.3))
$\hat{q}$	=	$q\hat{t}$ Non-dimensional rate of pitch
r		Distance of bob-weight aft of aircraft c.g. (ft)
R		Damping factor of aircraft mode (non-dimensional)
S		Wing area
S_t		Tailplane area
s		Non-dimensional parameter (*see* 2.5)
T_1, T_v		Time constants in power unit equation (*see* (2.6))
$\hat{t}$	=	$\frac{W}{g\rho SV}$ Unit of aerodynamic time (sec)
V		Forward speed of aircraft (ft/sec in text, knots in Tables)
W		Weight of aircraft (lb)
$\hat{w}$	=	w/V Increment of incidence in disturbed flight
X_0		Amplitude of forcing function (*see* (3.13))
y		Displacement of bob-weight, positive down (ft)
$\hat{y}$		Non-dimensional form of y (*see* (2.5))
α'		Incidence of the undeflected tailplane
Γ		Bob-weight-to-stick gear ratio, dimensionless
δ	=	$-\frac{\mu m_\eta}{i_B}$ Compound pitching moment derivative due to tailplane displacement, dimensionless
η		Deflection of tailplane
θ_i		Input to power unit (*see* (2.6))
θ_0		Output from power unit (*see* (2.6))
μ	=	$\frac{W}{g\rho Sl}$ Relative density of aircraft
ν	=	$-\frac{m_q}{i_B}$ Compound rotary damping derivative in pitch, dimensionless
$\rho_a, \rho_b, \rho_c, \rho_p$		Moduli of various factors in (6.3)
τ	=	$t/\hat{t}$ Non-dimensional time
χ	=	$-\frac{\mu m_{\dot{w}}}{i_B}$ Compound pitching moment derivative due to rate of change of w, dimensionless
$\psi_a, \psi_b, \psi_c, \psi_p$		Arguments of various factors in (6.3)
ω	=	$-\frac{\mu m_w}{i_B}$ Compound pitching-moment derivative due to w, dimensionless

The suffix ($_0$) indicates the amplitude of the respective quantity.

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APPENDIX I

Evaluation of the Stick Force per g

It is evident from the Nyquist diagrams of Figs. 11 and 12 that the value of the elevator-to-bob-weight gear ratio, G, will have a profound effect on the aircraft stability characteristics. Any reduction in the value of G will, however, in addition to affording a marked improvement of the stability, affect manoeuvrability, *viz.*, the stick-force-per-g characteristics of the control. These characteristics, for all aircraft with irreversible power-operated elevator control, do not depend on the hinge-moment derivatives (apart from small corrections connected with elasticity, *cf.* Section 4). They can be derived, however, in a similar manner to that used in orthodox manoeuvrability theory, that is, considering the asymptotic behaviour after a step deflection of the elevator, when the aircraft is supposed to describe a ' steady ' circle at constant speed, and neglecting variations of the weight component relative to the moving axes[14]. Following this procedure, we ignore the initial transient variations, thus disregarding any overshooting.

The equations of motion will then be similar to equation (2.9), except that now all terms involving derivatives with respect to time disappear as the aircraft is assumed to be flying in a 'steady' vertical turn. There will, of course, be an additional term in the bob-weight equation due to the applied stick force. The equations are thus:

$$\left.\begin{aligned}(\tfrac{1}{2}a\nu + \omega)\hat{w} + \delta\eta &= 0\\ \eta - G\hat{y} &= 0\\ -k\hat{w} + c\hat{y} &= -\frac{cP}{\Gamma l K_y}\end{aligned}\right\}, \quad \ldots \quad (\text{I.1})$$

where Γ is the bob-weight-to-stick gear ratio (strictly: ratio of bob-weight displacement to the corresponding stick travel), so that the overall elevator-to-stick non-dimensional gear ratio will be ΓG. Friction has not been included here, but its effect may be easily taken into account, as shown at the end of this appendix.

Solving equations (I.1) for $\hat{w}$, we obtain:

$$\hat{w} = \frac{P}{\Gamma l K_y\{k/c + (\omega + \frac{1}{2}a\nu)/G\delta\}}\,; \quad \ldots \quad (\text{I.2})$$

we have, however, from equation (2.5):

$$k/c = \frac{aM_2 g}{lC_L K_y}\,, \quad \ldots \quad (\text{I.3})$$

and hence

$$\hat{w} = \frac{P/\Gamma}{\dfrac{aM_2 g}{C_L} + lK_y\dfrac{\omega + \frac{1}{2}a\nu}{G\delta}}\,. \quad \ldots \quad (\text{I.4})$$

The normal acceleration will be expressed by

$$n - 1 = \frac{a}{C_L}\hat{w} = \frac{P/\Gamma}{M_2 g + \dfrac{C_L}{a}lK_y\dfrac{\omega + \frac{1}{2}a\nu}{G\delta}}\,, \quad \ldots \quad (\text{I.5})$$

and hence, using the relationships [2,14]:

$$\omega + \tfrac{1}{2}a\nu = \frac{\mu}{i_B}\frac{\bar{c}}{2l}aH_m\,; \qquad \delta = -\frac{\mu m_\eta}{i_B}\,, \quad \ldots \quad (\text{I.6})$$

the stick force per g becomes

$$\frac{P}{n-1} = \Gamma\left(M_2 g - \frac{C_L \bar{c} H_m K_y}{2Gm_\eta}\right). \quad \ldots \quad (\text{I.7})$$

The first term in equation (I.7) represents the effect of the bob-weight inertia, the second that of the feel spring*. The latter being proportional to C_L, varies considerably through the speed range, while the former is independent of speed and is a constant positive addition to the stick force per g throughout. The contribution of the bob-weight tends to result in a more uniform stick force per g.

The problem now is whether it is possible to improve the dynamic stability of the aircraft by reducing the value of G quite considerably, but at the same time to avoid any detrimental effects on the stick-force-per-g characteristics.

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* It may be mentioned that, m_η being essentially negative, this second term will be positive.

One important factor is that the overall elevator-to-stick gear ratio ΓG must remain within reasonable limits, and so any reduction in G must be accompanied by an appropriate increase of Γ. If this is done whilst still keeping the other components of equation (I.7) unchanged, the stick force per g will become very much larger and will also vary to a much greater extent through the speed range. It might perhaps be possible to offset these adverse results by suitably reducing M_2 and particularly K_y, but of course these must not become unreasonably small. Also such changes would necessarily affect a number of derivatives in the stability sextic (*see* equations 2.5, 2.9, 2.10), and the ultimate outcome as regards stability cannot be predicted without another very considerable computational effort. All this seems to offer scope to the designer's ingenuity, but it appears that the simple expedient of moving the bob-weight forward offers greater promise of a satisfactory solution.

The effect of friction has not been included in formula (I.7), but this merely means an additional term in it. Thus:

$$P = \Gamma\left[(n-1)\left(M_2 g - \frac{C_L \bar{c} H_m K_y}{2Gm_\eta}\right) \mp F\right], \quad \ldots \quad \ldots \quad \ldots \quad (I.8)$$

where F is still the equivalent frictional force at the bob-weight, and the $\mp$ sign depends on which way the control is moving.

APPENDIX II

The Effect of Neglecting the Power-Unit Transfer Function

If we had treated the power unit as a simple gear and had, therefore, neglected its transfer function, the characteristic stability equation (2.10) would reduce to:

$$(D^2 + AD + B)(D^2 + bD + c) + \delta G(k - \tfrac{1}{2}asD - sD^2) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (II.1)$$

or

$$D^4 + B_1 D^3 + C_1 D^2 + D_1 D + E_1 = 0, \quad \ldots \quad \ldots \quad \ldots \quad (II.2)$$

where

$$\left.\begin{aligned} B_1 &= A + b \\ C_1 &= B + Ab + c - \delta Gs \\ D_1 &= Ac + Bb - \tfrac{1}{2}\delta Gas \\ E_1 &= Bc + \delta Gk \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (II.3)$$

In order that all modes should be positively damped, all coefficients should be positive, and this leads to two conditions for complete stability (from the equations for C_1 and D_1 above):

$$\delta G < \frac{B + Ab + c}{s}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (II.4)$$

$$\delta G < \frac{Ac + Bb}{\tfrac{1}{2}as}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (II.5)$$

In addition to these two conditions, the Routh's descriminant must also be positive, that is:

$$C_1 D_1 - B_1 E_1 - \frac{D_1^2}{B_1} > 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (II.6)$$

This becomes, after substitution of the relationships (II.3):

$$m_0(\delta G)^2 + m_1(\delta G) + m_2 > 0\,, \qquad \ldots \quad \ldots \quad \ldots \text{(II.7)}$$

where:

$$\left.\begin{aligned} m_0 &= \tfrac{1}{2}as^2b + \tfrac{1}{4}as^2(2A - a) \\ m_1 &= -(k + sB + \tfrac{1}{2}aAs)b^2 + [\tfrac{1}{2}as(B - A^2 - c) - sA(B + c) \\ &\quad - 2kA]b - A^2(k + sc) - \tfrac{1}{2}aAs(B - c) \\ m_2 &= ABb^3 + A^2(B + c)b^2 + A[(B + c)^2 + A^2c]b\,. \end{aligned}\right\} \quad \ldots \text{(II.8)}$$

Let us now consider a particular case and substitute numerical values. For the basic case at a forward speed of 450 kt, the inequalities (II.4), (II.5) and (II.7) become

$$\left.\begin{aligned} &\delta G < 2210{\cdot}41 + 16{\cdot}142b \\ &\delta G < 3460{\cdot}96 + 17{\cdot}265b \\ &(0{\cdot}0433b + 0{\cdot}0418)(\delta G)^2 - (20{\cdot}838b^2 + 348{\cdot}595b + 280{\cdot}573)(\delta G) \\ &\quad + 11{\cdot}2837b^3 + 2448{\cdot}67b^2 + 341638b > 0 \end{aligned}\right\} \quad \ldots \text{(II.9)}$$

The first two of these conditions will always be satisfied, whatever the value of b, for all practical values of the gear ratio. It can be shown more generally, that the condition (II.7) always overrides the other two (II.4) and (II.5).

We can solve this third condition (II.7) for δG for a range of values of b:

b	δG	
0	0	6712
0·1	110	
0·5	394	
1·0	572	
5	753	9117
10	663	
15	583	
20	522	16647
30	441	
40	390	
50	356	Larger values
100	288	
150	277	
200	283	
400	360	
600	457	
800	560	

The inequality is satisfied if δG is greater than the larger values or less than the smaller ones. Obviously δG will never exceed the larger values given in the Table above, and so the question of whether the two modes represented by (II.2) are both positively damped reduces to determining whether δG is less than the lower values given in the Table.

The limiting curve for complete stability is given in Fig. 13. It will be seen that for $\delta G = 553{\cdot}4$ (as taken in the calculations), the damping is positive for a very narrow region of small values of b, and thus for all larger values there is always a negative damping in one of the modes, until b becomes very large when both modes again become stable.

To obtain a direct comparison with the results obtained by including the power-unit transfer function, the quartic has been solved for a range of value of b for this case. The results obtained are given in the following Table :

b	Coupled aircraft mode			Coupled bob-weight mode				
	Roots	Period (sec)	Time to halve amplitude (sec)	Roots		Period (sec)	Time to halve amplitude (sec)	
0	$-1{\cdot}833 \pm 6{\cdot}820i$	0·689	+ 0·283	$+0{\cdot}525 \pm 15{\cdot}037i$		0·313	−0·987	
10	$-0{\cdot}423 \pm 7{\cdot}273i$	0·646	+ 1·226	$-5{\cdot}885 \pm 14{\cdot}032i$		0·335	0·888	
35	$+0{\cdot}507 \pm 5{\cdot}776i$	0·814	− 1·022	−16·561	− 22·069	—	0·031	0·023
100	$+0{\cdot}617 \pm 4{\cdot}342i$	1·082	− 0·840	− 6·568	− 97·282	—	0·079	0·0053
200	$+0{\cdot}478 \pm 3{\cdot}521i$	1·335	− 1·085	− 4·899	−198·672	—	0·106	0·0026
400	$+0{\cdot}256 \pm 2{\cdot}839i$	1·655	− 2·025	− 3·787	−399·339	—	0·137	0·0013
600	$+0{\cdot}106 \pm 2{\cdot}502i$	1·878	− 4·891	− 3·267	−599·560	—	0·159	0·0009
800	$-0{\cdot}006 \pm 2{\cdot}288i$	2·054	+86·400	− 2·934	−799·670	—	0·177	0·0006

Negative time to halve amplitude means time to double amplitude.

These results are plotted in Fig. 14, where it is seen that the curve of R vs. b is of the same form as that found by including the power-unit transfer function but that the results found by solving the quartic are very much too optimistic. The peculiar result for $b = 0$ where the instability appears in the bob-weight mode is shown but is apparently only so for values of b very near to zero.

It seems clear from these results that neglecting the power-unit transfer function would give very misleading results and the power unit should not be treated as a simple gear in calculations of this kind.

TABLE 1

Constants and Derivatives for the Basic Case (Mid-Weight c.g. Position, with Trailing-Edge Strips on Tail; Time Constant $T_1 = 0 \cdot 02$ sec)

(*a*) *Constants Independent of Speed.*

Elevator-to-bob-weight gear ratio (non-dimensional), G	$= 33 \cdot 4$
Parameter s (*cf.* equation (2.5))	$= 0 \cdot 162$
Static value of frictional force of control circuit, F (measured at bob-weight)	$= 10 \cdot 15$ lb
Tail arm, l	$= 17 \cdot 3$ ft
Mass of bob-weight, M_2	$= 0 \cdot 451$ slugs
Equivalent mass of control circuit less bob-weight, M_1 (referred to bob-weight)	$= 0 \cdot 805$ slugs
Time constant T_v of power unit	$= 0 \cdot 05$ sec

(*b*) *Derivatives for Various Speeds.*

V (kt)	200	300	350	400	450
a	+ 3·93	+ 3·60	+ 3·45	+ 3·37	+ 3·30
ν	0·870	0·855	0·850	0·835	0·825
χ	0·195	0·155	0·150	0·145	0·140
ω	5·251	3·938	3·610	3·282	2·954
δ	17·231	17·066	16·902	16·902	16·574
M	84·16	56·10	48·10	42·10	37·40
N	2833·16	1258·88	925·444	708·964	559·50
c	1808·14	797·379	589·611	449·227	353·772
k	23·157	21·210	20·328	19·860	19·44
$\hat{t}$	+ 1·683	+ 1·122	+ 0·962	+ 0·842	+ 0·748
Q	− 0·0257	− 0·0578	− 0·0786	− 0·1027	− 0·1310

TABLE 2

Constants and Derivatives for Mid-Weight c.g. Position, with Trailing-Edge Strips on Tail. Time Constant $T_1 = 0 \cdot 01$ sec

(*a*) *Constants Independent of Speed.*
As in Table 1.

(*b*) *Derivatives for Various Speeds.*
As in Table 1 but with values of M and N doubled.

TABLE 3

Constants and Derivatives for Mid-Weight c.g. Position, without Trailing-Edge Strips on Tail. Time Constant $T_1 = 0 \cdot 02$ sec

(*a*) *Constants Independent of Speed.*

As in Table 1.

(*b*) *Derivatives for Various Speeds.*

As in Table 1 except as follows:

V (kt)	200	300	350	400	450
ν	+ 0·875	+ 0·865	—	+ 0·850	+ 0·840
χ	0·195	0·160	—	0·150	0·145
ω	5·415	4·103	—	3·610	3·282
δ	+17·395	+17·395	—	+17·231	+17·231
Q	− 0·0190	− 0·0428	—	− 0·0762	− 0·0964

TABLE 4

Constants and Derivatives for Forward c.g. Position, with Trailing-Edge Strips on Tail. Time Constant $T_1 = 0 \cdot 02$ sec

(*a*) *Constants Independent of Speed.*

As in Table 1 but with value of s changed:

$s = 0 \cdot 169.$

(*b*) *Derivatives for Various Speeds.*

As in Table 1 except as follows:

V (kt)	200	300	350	400	450
χ	0·20	0·16	—	0·145	0·14
ω	12·320	10·356	—	9·106	8·749
δ	18·748	18·569	—	18·391	18·034
M	91·56	61·045	—	45·78	40·69
N	3353·29	1490·60	—	838·32	662·28
c	2138·43	947·02	—	531·55	421·58
k	25·260	23·080	—	21·670	21·157
$\hat{t}$	1·832	1·221	—	0·916	0·814

TABLE 5

Constants and Derivatives for Mid-Weight c.g. Position, with Trailing-Edge Strips on Tail. Time Constant $T_1 = 0 \cdot 02$ sec. Increased Aircraft Damping

(*a*) *Constants Independent of Speed.*

As in Table 1.

(*b*) *Derivatives for Various Speeds.*

As in Table 1 but with ν and χ doubled.

TABLE 6

Solutions for the Basic Case

1. *Solutions for* 200 *knots.*

(*a*) *Coefficients of the Sextic.*

b	B_1	C_1	D_1	E_1	F_1	G_1
0	87·19	4903·27	166823	5352005	16062159	73417587
20	107·19	6647·07	228726	5535411	16454592	73417587
250	337·19	26700·8	940606	7644583	20992567	73417587
400	487·19	39779·3	1404875	9020129	23950811	73417587
600	687·19	57217·3	2023901	10854191	27895137	73417587
700	787·19	65936·3	2333414	11771222	29867300	73417587
800	887·19	74655·3	2642927	12688253	31839463	73417587
900	987·19	83374·3	2952440	13605284	33811626	73417587

(*b*) *Stability Roots.*

	Aircraft mode	Power-unit mode				
	Uncoupled (with fixed tail)	Uncoupled				
	−1·515 ± 2·160*i*	−42·08 ± 32·595*i*	Bob-weight mode			
b	Coupled ($-R \pm iJ$)	Coupled	Coupled		Uncoupled	
0	−1·419 ± 3·623*i*	−41·282 ± 31·785*i*	− 0·895 ± 42·257*i*		0 ± 42·522*i*	
20	−1·366 ± 3·636*i*	−40·881 ± 31·514*i*	−11·349 ± 41·204*i*		−10 ± 41·330*i*	
250	−0·836 ± 3·451*i*	−42·298 ± 32·946*i*	−8·352	−242·572	−7·455	−242·545
400	−0·689 ± 3·228*i*	−42·206 ± 32·787*i*	−5·967	−395·433	−4·573	−395·427
600	−0·621 ± 2·982*i*	−42·161 ± 32·714*i*	−4·655	−596·973	−3·029	−596·971
700	−0·613 ± 2·882*i*	−42·148 ± 32·695*i*	−4·261	−697·409	−2·593	−697·407
800	−0·613 ± 2·796*i*	−42·139 ± 32·682*i*	−3·951	−797·735	−2·267	−797·733
900	−0·620 ± 2·719*i*	−42·132 ± 32·671*i*	−3·699	−897·987	−2·014	−897·986

TABLE 6—*continued*

(*c*) *Periods and Times to Halve Amplitude* (200 kt).

Aircraft mode: Uncoupled (with tail fixed) — Period (sec)	Aircraft mode: Uncoupled (with tail fixed) — Time to halve amplitude (sec)	Power-unit mode: Uncoupled — Period (sec)	Power-unit mode: Uncoupled — Time to halve amplitude (sec)
4·896	0·770	0·324	0·028

	Aircraft mode		Power-unit mode		Bob-weight mode					
	Coupled		Coupled		Coupled			Uncoupled		
b	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)		Period (sec)	Time to halve amplitude (sec)	
0	2·919	0·822	0·333	0·028	0·250	1·303		0·250	∞	
20	2·909	0·854	0·336	0·029	0·257	0·103		0·256	0·117	
250	3·065	1·395	0·321	0·028	—	0·140	0·0048	—	0·156	0·0048
400	3·276	1·693	0·323	0·028	—	0·195	0·0029	—	0·255	0·0029
600	3·547	1·878	0·323	0·028	—	0·251	0·0020	—	0·385	0·0020
700	3·670	1·903	0·323	0·028	—	0·274	0·0017	—	0·450	0·0017
800	3·783	1·903	0·324	0·028	—	0·295	0·0015	—	0·515	0·0015
900	3·890	1·881	0·324	0·028	—	0·315	0·0013	—	0·579	0·0013

TABLE 6—*continued*

2. *Solutions for $V = 300$ knots.*

(*a*) *Coefficients of the Sextic.*

b	B_1	C_1	D_1	E_1	F_1	G_1
0	58·91	2219·38	50818·3	1024522	2856450	20717440
20	78·91	3397·58	82655·9	1101416	2994348	20717440
100	158·91	8110·38	193018	1408993	3545939	20717440
190 (st. osc.)	248·91	13412·3	320998	1755017	4166479	20717440
340 (min. cond.)	398·91	22248·8	534298	2331723	5200713	20717440
500	558·91	31674·4	761818	2946877	6303895	20717440
900	958·91	55238·4	1330618	4484761	9061851	20717440

(*b*) *Stability Roots.*

	Aircraft mode	Power-unit mode				
	Uncoupled (with tail fixed)	Uncoupled				
	$-1 \cdot 41 \pm 1 \cdot 87i$	$-28 \cdot 05 \pm 21 \cdot 73i$	Bob-weight mode			
b	Coupled ($-R \pm iJ$)	Coupled	Coupled		Uncoupled	
0	$-1 \cdot 016 \pm 4 \cdot 733i$	$-26 \cdot 896 \pm 20 \cdot 377i$	$-1 \cdot 539 \pm 27 \cdot 819i$		$0 \pm 28 \cdot 238i$	
20	$-0 \cdot 738 \pm 4 \cdot 686i$	$-25 \cdot 734 \pm 19 \cdot 249i$	$-12 \cdot 984 \pm 26 \cdot 889i$		$-10 \pm 26 \cdot 408i$	
100	$-0 \cdot 143 \pm 4 \cdot 121i$	$-28 \cdot 567 \pm 22 \cdot 915i$	$-9 \cdot 919$	$-91 \cdot 569$	$-8 \cdot 737$	$-91 \cdot 263$
190 (st. osc.)	$0 \pm 3 \cdot 626i$	$-28 \cdot 300 \pm 22 \cdot 175i$	$-6 \cdot 588$	$-185 \cdot 731$	$-4 \cdot 294$	$-185 \cdot 706$
340 (min. cond.)	$0 \pm 3 \cdot 131i$	$-28 \cdot 182 \pm 21 \cdot 944i$	$-4 \cdot 906$	$-337 \cdot 644$	$-2 \cdot 362$	$-337 \cdot 638$
500	$-0 \cdot 060 \pm 2 \cdot 820i$	$-28 \cdot 137 \pm 21 \cdot 871i$	$-4 \cdot 115$	$-498 \cdot 401$	$-1 \cdot 600$	$-498 \cdot 400$
900	$-0 \cdot 219 \pm 2 \cdot 388i$	$-28 \cdot 097 \pm 21 \cdot 802i$	$-3 \cdot 168$	$-899 \cdot 121$	$-0 \cdot 887$	$-899 \cdot 113$

TABLE 6—*continued*

(*c*) *Periods and Times to Halve Amplitude* (300 kt).

Aircraft mode		Power-unit mode	
Uncoupled (with tail fixed)		Uncoupled	
Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)
3·766	0·551	0·324	0·028

	Aircraft mode		Power-unit mode		Bob-weight mode					
	Coupled		Coupled		Coupled			Uncoupled		
b	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)		Period (sec)	Time to halve amplitude (sec)	
0	1·490	0·765	0·346	0·029	0·253	0·505		0·250	∞	
20	1·504	1·054	0·366	0·030	0·262	0·060		0·267	0·078	
100	1·711	5·438	0·308	0·027	—	0·078	0·0085	—	0·089	0·0085
190	1·944	∞	0·318	0·027	—	0·118	0·0042	—	0·181	0·0042
340	2·252	∞	0·321	0·028	—	0·158	0·0023	—	0·329	0·0023
500	2·500	12·960	0·322	0·028	—	0·189	0·0016	—	0·486	0·0016
900	2·952	3·551	0·323	0·028	—	0·245	0·0009	—	0·877	0·0009

TABLE 6—*continued*

3. *Solutions for* 350 *knots.*

(*a*) *Coefficients of the Sextic.*

b	B_1	C_1	D_1	E_1	F_1	G_1
0	50·825	1651·20	32733·0	545988	1484863	13389846
37·1 (st. osc.)	87·925	3536·81	72118·0	648606	1659142	13389846
100	150·825	6733·70	138892	822587	1954618	13389846
200	250·825	11816·2	245051	1099186	2424373	13389846
400	450·825	21981·2	457369	1652384	3363883	13389846
600	650·825	32146·2	669687	2205582	4303393	13389846
647 (min. cond.)	697·825	34535·0	719582	2335584	4524178	13389846
800	850·825	42311·2	882005	2758780	5242903	13389846

(*b*) *Stability Roots.*

	Aircraft mode	Power-unit mode	Bob-weight mode			
	Uncoupled (with tail fixed)	Uncoupled				
	−1·36 ± 1·79i	−24·05 ± 18·63i				
b	Coupled (− R ± iJ)	Coupled	Coupled		Uncoupled	
0	−0·692 ± 5·348i	−22·786 ± 16·944i	− 1·935 ± 23·819i		0 ± 24·282i	
37·1 (st. osc.)	0 ± 4·867i	−22·500 ± 6·301i	−21·463 ± 23·976i		−18·55 ± 15·669i	
100	+0·278 ± 4·122i	−24·520 ± 19·683i	−8·452	− 93·892	−6·292	− 93·708
200	+0·284 ± 3·487i	−24·266 ± 19·031i	−5·838	−197·022	−2·993	−197·007
400	+0·148 ± 2·885i	−24·152 ± 18·807i	−4·298	−398·522	−1·479	−398·521
600	+0·026 ± 2·570i	−24·117 ± 18·743i	−3·626	−599·016	−0·984	−599·016
647 (min. cond.)	0 ± 2·515i	−24·112 ± 18·734i	−3·514	−646·088	−0·913	−646·087
800	−0·076 ± 2·366i	−24·099 ± 18·714i	−3·211	−799·263	−0·738	−799·262

TABLE 6—*continued*

(*c*) *Periods and Times to Halve Amplitudes* (350 kt).

Aircraft mode		Power-unit mode	
Uncoupled (with tail fixed)		Uncoupled	
Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)
3·369	0·490	0·324	0·028

	Aircraft mode		Power-unit mode		Bob-weight mode			Bob-weight mode		
	Coupled		Coupled		Coupled			Uncoupled		
b	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)		Period (sec)	Time to halve amplitude (sec)	
0	1·130	+ 0·963	0·357	0·029	0·254	0·345		0·250	∞	
37·1	1·242	∞	0·959	0·030	0·252	0·031		0·386	0·036	
100	1·466	− 2·398	0·307	0·027	—	0·079	0·0071	—	0·106	0·0071
200	1·102	− 2·348	0·318	0·027	—	0·114	0·0034	—	0·223	0·0034
400	2·095	− 4·505	0·321	0·028	—	0·155	0·0017	—	0·451	0·0017
600	2·352	−25·642	0·322	0·028	—	0·184	0·0011	—	0·678	0·0011
647	2·403	∞	0·323	0·028	—	0·190	0·0010	—	0·730	0·0010
800	2·555	+ 8·772	0·323	0·028	—	0·208	0·0008	—	0·903	0·0008

Negative time to halve amplitude means time to double amplitude.

TABLE 6—*continued*

4. *Solutions for V = 400 knots.*

(*a*) *Coefficients of the Sextic.*

b	B_1	C_1	D_1	E_1	F_1	G_1
0	44·765	1275·08	22196·4	309482	828197	9441930
4·1 (st. osc.)	48·865	1458·62	25582·4	318038	841827	9441930
40	84·765	3065·68	55230·4	392954	961170	9441930
70	114·765	4408·63	80005·9	455558	1060900	9441930
100	144·765	5751·58	104781	518162	1160630	9441930
200	244·765	10228·1	187366	726842	1493063	9441930
400	444·765	19181·1	352536	1144202	2157929	9441930
834 (min. cond.)	878·765	38609·1	710955	2049873	3600688	9441930

(*b*) *Stability Roots.*

	Aircraft mode	Power-unit mode	Bob-weight mode			
	Uncoupled (with fixed tail)	Uncoupled				
	−1·33 ± 1·71*i*	−21·05 ± 16·30*i*				
b	Coupled (− R ± iJ)	Coupled	Coupled		Uncoupled	
0	−0·175 ± 6·051*i*	−19·722 ± 14·201*i*	−2·485 ± 20·737*i*		0 ± 21·195*i*	
4·1 (st. osc.)	0 ± 5·938*i*	−19·532 ± 13·864*i*	−4·899 ± 21·043*i*		−2·05 ± 21·096*i*	
40	+0·583 ± 4·942*i*	−20·367 ± 20·933*i*	−14·618	− 30·578	−20·0 ± 7·016*i*	
70	+0·644 ± 4·423*i*	−21·632 ± 18·053*i*	− 9·388	− 63·402	−7·147	− 62·853
100	+0·631 ± 4·070*i*	−21·474 ± 17·287*i*	− 7·675	− 95·408	−4·715	− 95·285
200	+0·505 ± 3·397*i*	−21·247 ± 16·701*i*	− 5·543	−197·739	−2·272	−197·728
400	+0·287 ± 2·795*i*	−21·144 ± 16·485*i*	− 4·170	−398·876	−1·126	−398·874
834 (min. cond.)	0 ± 2·257*i*	−21·094 ± 16·388*i*	− 3·116	−833·460	−0·539	−833·461

(73169)

TABLE 6—*continued*

(*c*) *Periods and Times to Halve Amplitude* (400 kt).

Aircraft mode		Power-unit mode	
Uncoupled (with tail fixed)		Uncoupled	
Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)
3·099	0·439	0·324	0·028

	Aircraft mode		Power-unit mode		Bob-weight mode			Bob-weight mode		
	Coupled		Coupled		Coupled			Uncoupled		
b	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)		Period (sec)	Time to halve amplitude (sec)	
0	0·874	0·334	0·373	0·030	0·255	0·235		0·250	∞	
4·1	0·891	∞	0·382	0·030	0·251	0·119		0·251	0·285	
40	1·070	—1·001	0·253	0·029	—	0·040	0·019	0·754	0·029	
70	1·196	—0·906	0·293	0·027	—	0·062	0·0092	—	0·082	0·0093
100	1·300	—0·925	0·306	0·027	—	0·076	0·0061	—	0·124	0·0061
200	1·557	—1·155	0·317	0·027	—	0·105	0·0030	—	0·257	0·0030
400	1·893	—2·033	0·321	0·028	—	0·140	0·0015	—	0·518	0·0015
834	2·344	∞	0·323	0·028	—	0·187	0·0007	—	1·083	0·0007

Negative time to halve amplitude means time to double amplitude.

TABLE 6—*continued*

5. *Solutions for* $V = 450$ *knots.*

(*a*) *Coefficients of the Sextic.*

b	B_1	C_1	D_1	E_1	F_1	G_1
0	40·015	1015·39	15780·7	186300	491904	6875112
20	60·015	1815·69	29013·0	218789	540189	6875112
100	140·015	5016·89	81942·3	348747	733328	6875112
200	240·015	9018·39	148104	511194	974752	6875112
400	440·015	17021·4	280427	836088	1457600	6785112
600	640·015	25024·4	412750	1160982	1940448	6875112
800	840·015	33027·4	545073	1485876	2423296	6875112
965 (min. cond.)	1005·015	39629·9	654240	1753914	2821646	6875112

(*b*) *Stability Roots.*

	Aircraft mode	Power-unit mode				
	Uncoupled (with fixed tail)	Uncoupled				
	$-1·308 \pm 1·61i$	$-18·7 \pm 14·48i$	Bob-weight mode			
b	Coupled ($-R \pm iJ$)	Coupled	Coupled		Uncoupled	
0	$+0·520 \pm 6·582i$	$-17·456 \pm 11·965i$	$-3·072 \pm 18·513i$		$0 \pm 18·809i$	
20	$+0·974 \pm 5·519i$	$-17·589 \pm 7·667i$	$-13·392 \pm 20·376i$		$-10 \pm 15·930i$	
100	$+0·874 \pm 3·964i$	$-19·074 \pm 15·407i$	−7·200	−96·412	−3·673	−96·327
200	$+0·653 \pm 3·290i$	$-18·877 \pm 14·871i$	−5·339	−198·219	−1·785	−198·215
400	$+0·378 \pm 2·699i$	$-18·785 \pm 14·663i$	−4·083	−399·119	−0·886	−399·114
600	$+0·205 \pm 2·397i$	$-18·756 \pm 14·600i$	−3·507	−599·413	−0·590	−599·410
800	$+0·082 \pm 2·202i$	$-18·741 \pm 14·571i$	−3·141	−799·559	−0·442	−799·558
965 (min. cond.)	$0 \pm 2·084i$	$-18·735 \pm 14·555i$	−2·915	−964·635	−0·367	−964·633

(73169)

TABLE 6—*continued*

(*c*) *Periods and Times to Halve Amplitude* (450 kt).

Aircraft mode, Uncoupled (with tail fixed): Period (sec)	Aircraft mode, Uncoupled (with tail fixed): Time to halve amplitude (sec)	Power-unit mode, Uncoupled: Period (sec)	Power-unit mode, Uncoupled: Time to halve amplitude (sec)
2·912	0·396	0·324	0·028

	Aircraft mode, Coupled		Power-unit mode, Coupled		Bob-weight mode, Coupled			Bob-weight mode, Uncoupled		
b	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)	Period (sec)	Time to halve amplitude (sec)		Period (sec)	Time to halve amplitude (sec)	
0	0·714	—0·997	0·393	0·030	0·254	0·169		0·250	∞	
20	0·852	—0·532	0·613	0·029	0·231	0·039		0·295	0·052	
100	1·186	—0·593	0·305	0·027	—	0·072	0·0054	—	0·141	0·0054
200	1·429	—0·794	0·316	0·027	—	0·097	0·0026	—	0·290	0·0026
400	1·741	—1·371	0·321	0·028	—	0·127	0·0013	—	0·585	0·0013
600	1·961	—2·529	0·322	0·028	—	0·148	0·0009	—	0·879	0·0009
800	2·134	—6·322	0·323	0·028	—	0·165	0·0006	—	1·173	0·0006
965	2·255	∞	0·323	0·028	—	0·178	0·0005	—	1·413	0·0005

Negative time to halve amplitude means time to double amplitude.

TABLE 7

Values of the Amplitudes of Oscillation of the Control Angle, Bob-Weight Displacement and Normal Acceleration at the Steady Oscillations and Minimum Conditions for Various Speeds

V (kt)	300	350	400	450
$\frac{(n_0 - 1)}{F}$ steady osc.	0·118	0·242	1·153	—
$\frac{(n_0 - 1)}{F}$ min. cond.	0·099	0·089	0·087	0·086
$\frac{\eta_0^\circ}{F}$ steady osc.	0·205	0·570	3·22	—
$\frac{\eta_0^\circ}{F}$ min. cond.	0·132	0·064	0·042	0·031
$\frac{y_0(\text{in.})}{F}$ steady osc.	0·023	0·062	0·355	—
$\frac{y_0(\text{in.})}{F}$ min. cond.	0·0143	0·0069	0·0046	0·0033

TABLE 8

(a) Values of the Periods of Steady Oscillations for All Cases Considered

V (kt)	Period (sec) of the steady oscillations					
	Desk calculations	Philbrick results				
	Basic case	Basic case	Change of time constant $T_1 = 0 \cdot 01$ (sec)	Without trailing-edge strips	Forward position of c.g.	Increased aircraft damping (ν and χ doubled)
200	None	None	None	None	None	None
300	1·944	1·890	1·905	1·849	None	None
350	1·242	—	—	—	—	—
400	0·891	0·925	0·883	0·872	0·863	0·976
450	div.	div.	div.	div.	div.	0·696

(b) Values of Periods of Aircraft Mode, Stick Fixed, for All Cases Considered

V (kt)	Period (sec) of stick-fixed aircraft mode				
	Basic case	Change of time constant $T_1 = 0 \cdot 01$ sec	Without trailing-edge strips	Forward c.g. position	Increased airdraft damping (ν and χ doubled)
200	4·896	4·896	4·814	3·359	4·998
300	3·766	3·766	3·683	2·437	3·842
350	3·369	3·369	—	—	3·448
400	3·099	3·099	2·941	1·942	3·177
450	2·912	2·912	2·745	1·763	2·990

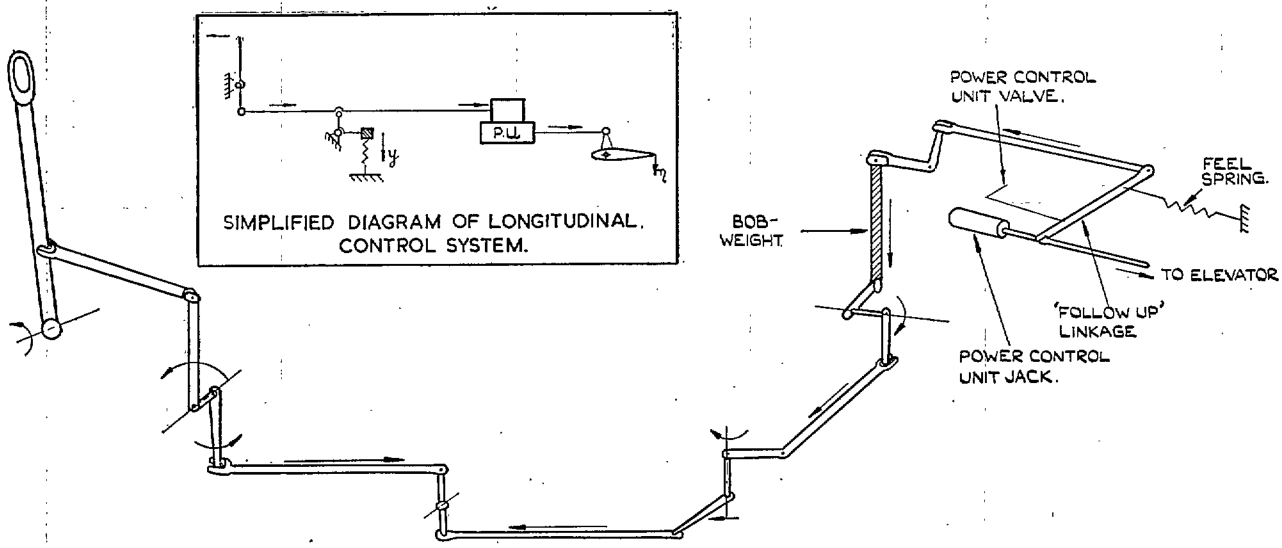

FIG. 1. Diagram of longitudinal control system.

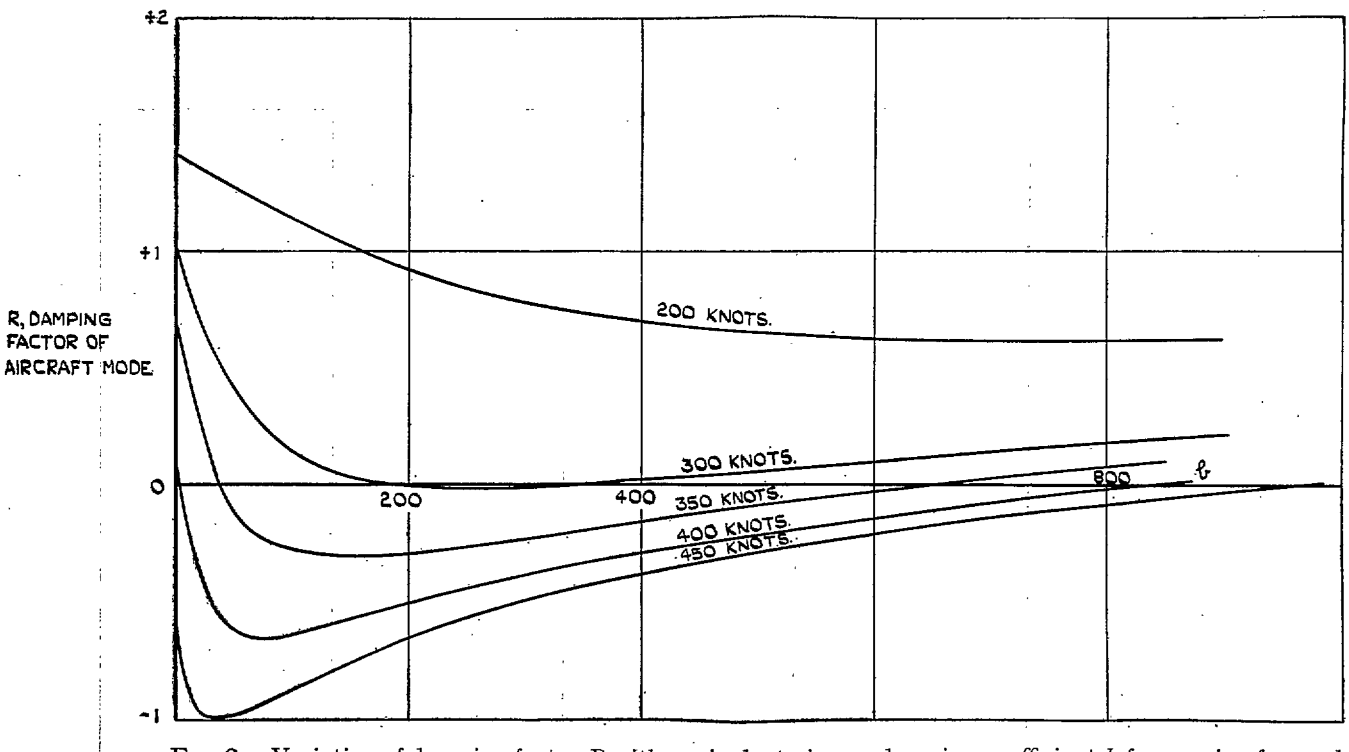

FIG. 2. Variation of damping factor R with equivalent viscous damping coefficient b for varying forward speed V (Mid-weight c.g. position, with strips. $T_1 = 0 \cdot 02$ sec).

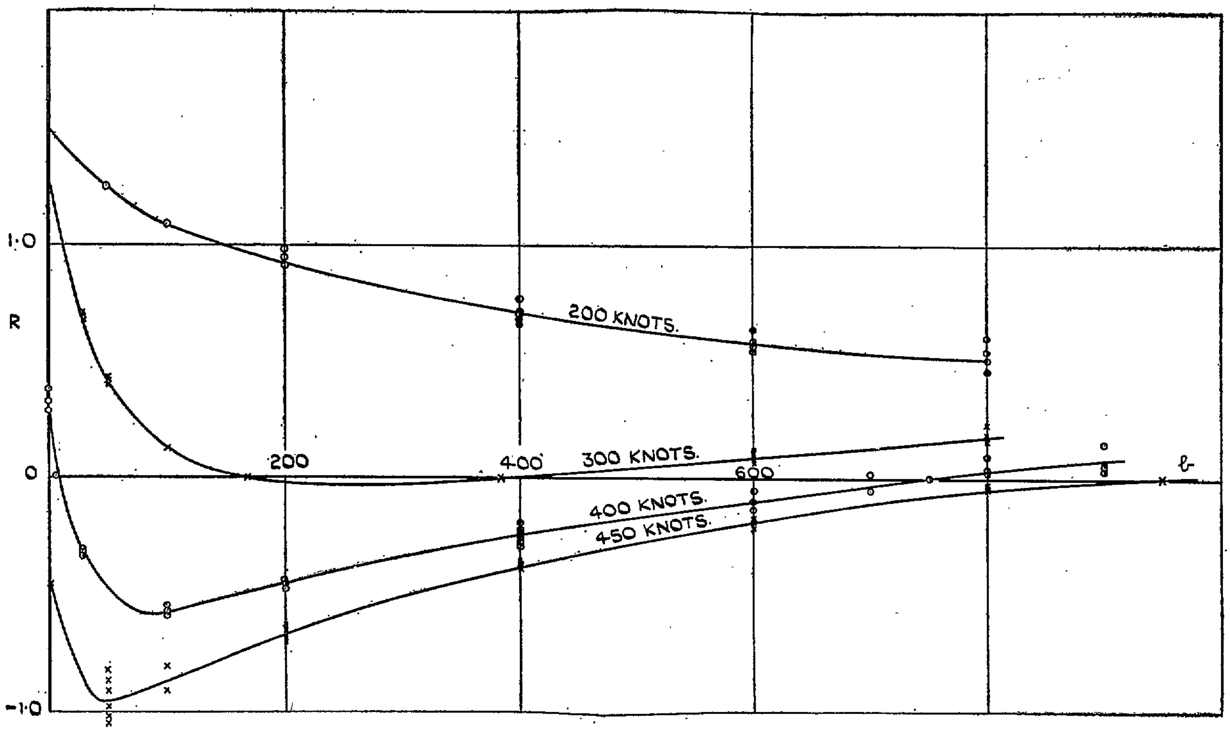

FIG 3. Variation of damping factor R with equivalent viscous damping coefficient b for varying forward speed V (Philbrick results. Mid-weight c.g. position, with strips. $T_1 = 0 \cdot 02$ sec).

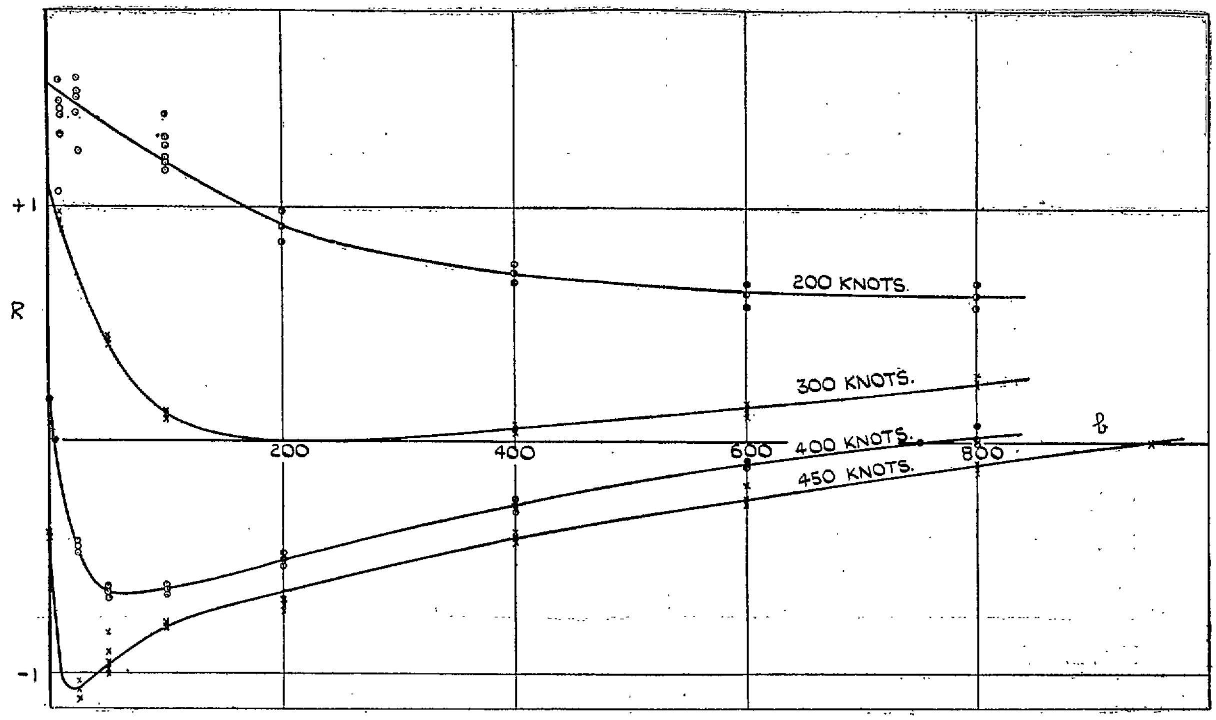

FIG. 4. Variation of damping factor R with equivalent viscous damping coefficient b for varying forward speed V (Philbrick results. Mid-weight c.g. position, with strips. $T_1 = 0 \cdot 01$ sec).

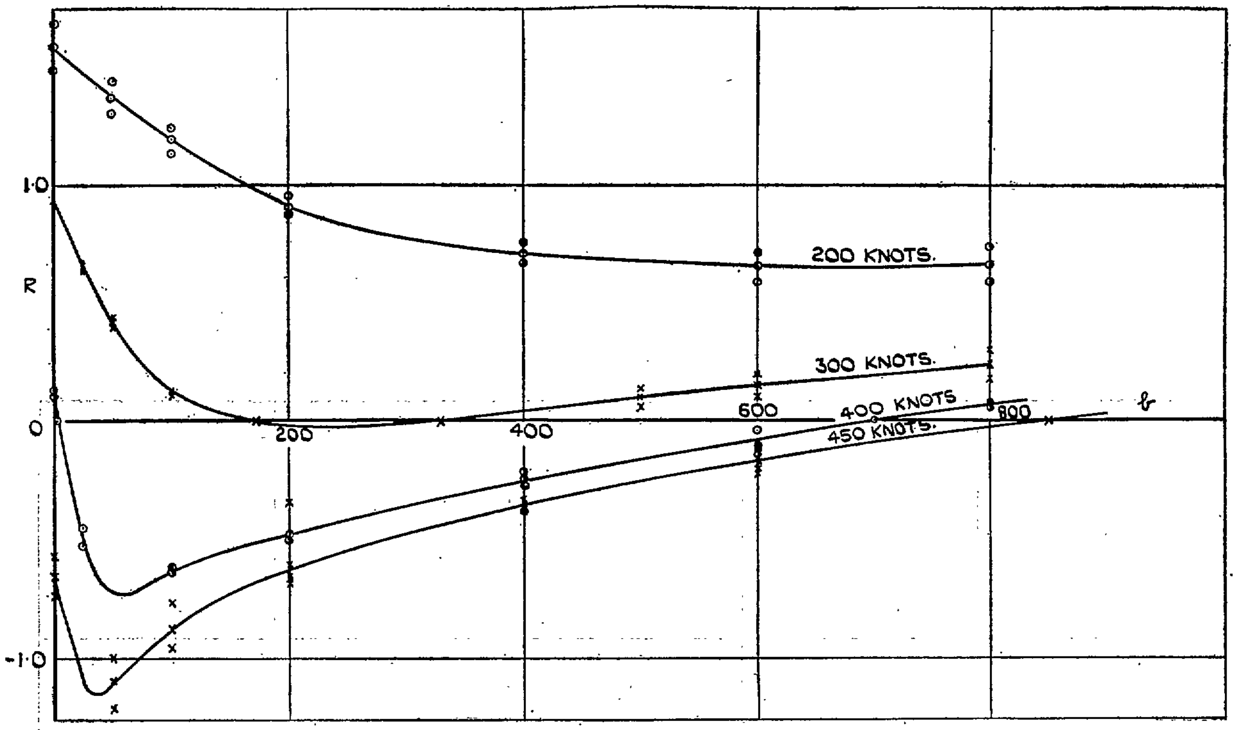

FIG. 5. Variation of damping factor R with equivalent viscous damping coefficient b for varying forward speed V (Philbrick results. Mid-weight c.g. position, without strips. $T_1 = 0 \cdot 02$ sec).

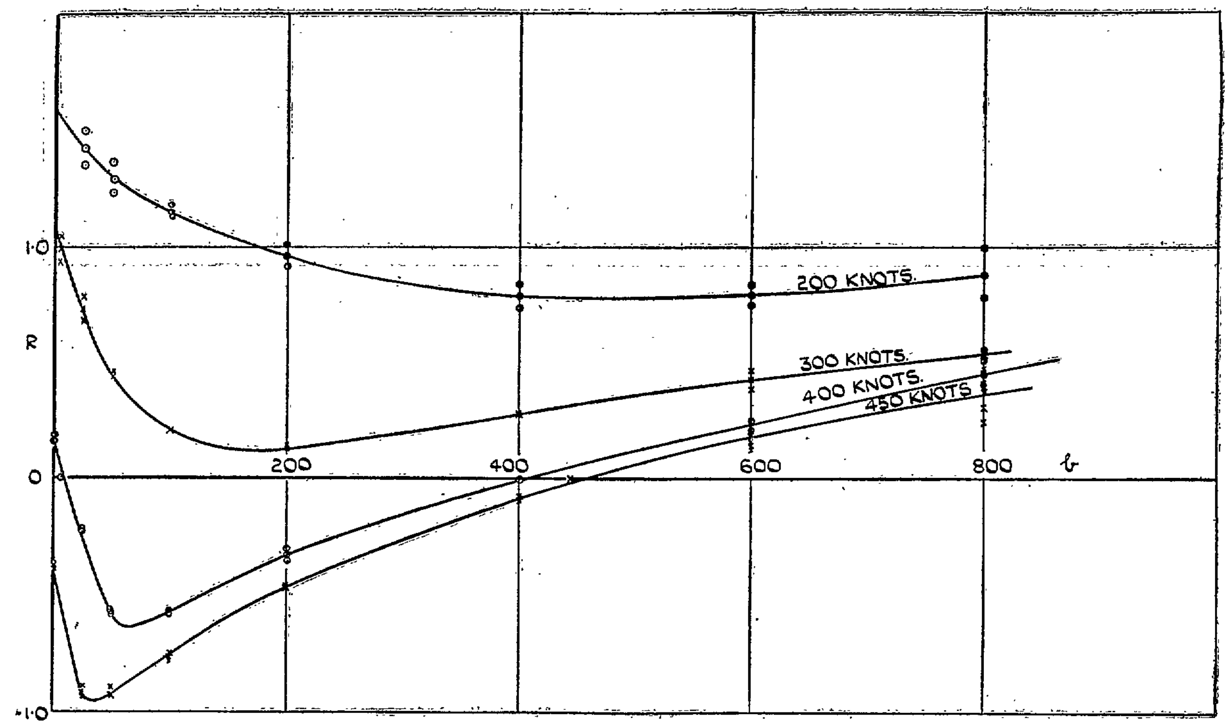

FIG. 6. Variation of damping factor R with equivalent viscous damping coefficient b for varying forward speed V (Philbrick results. Forward c.g. position, with strips. $T_1 = 0 \cdot 02$ sec).

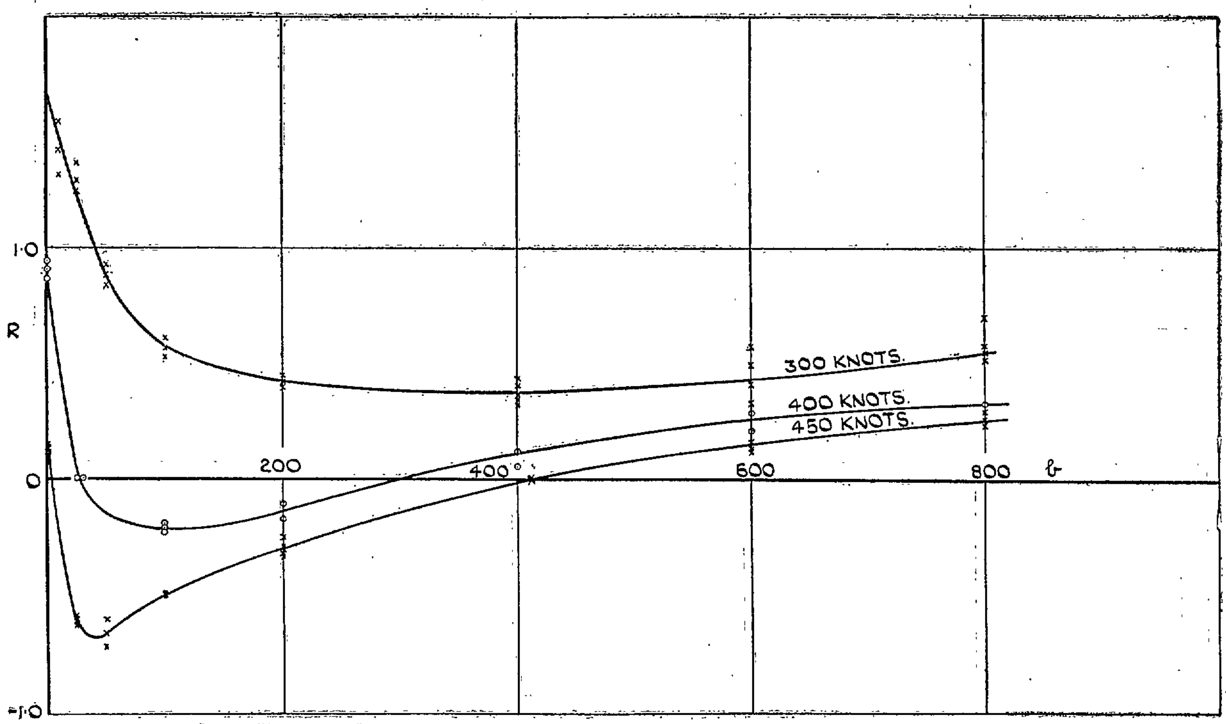

FIG. 7. Variation of damping factor R with equivalent viscous damping coefficient b for varying forward speed V (Philbrick results. Mid-weight c.g. position, with strips. $T_1 = 0 \cdot 02$ sec; ν and χ doubled).

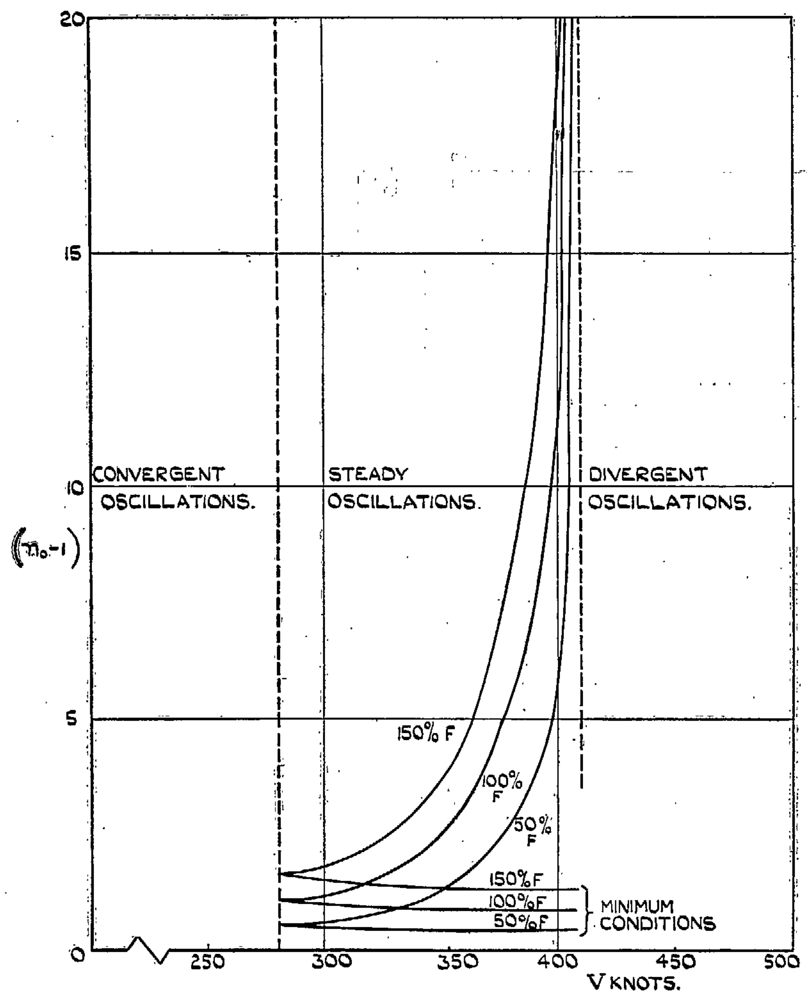

FIG. 8. Variation of the amplitude of the normal acceleration $(n_0 - 1)$ for the steady oscillations and at minimum conditions with speed, for varying friction F (Basic case).

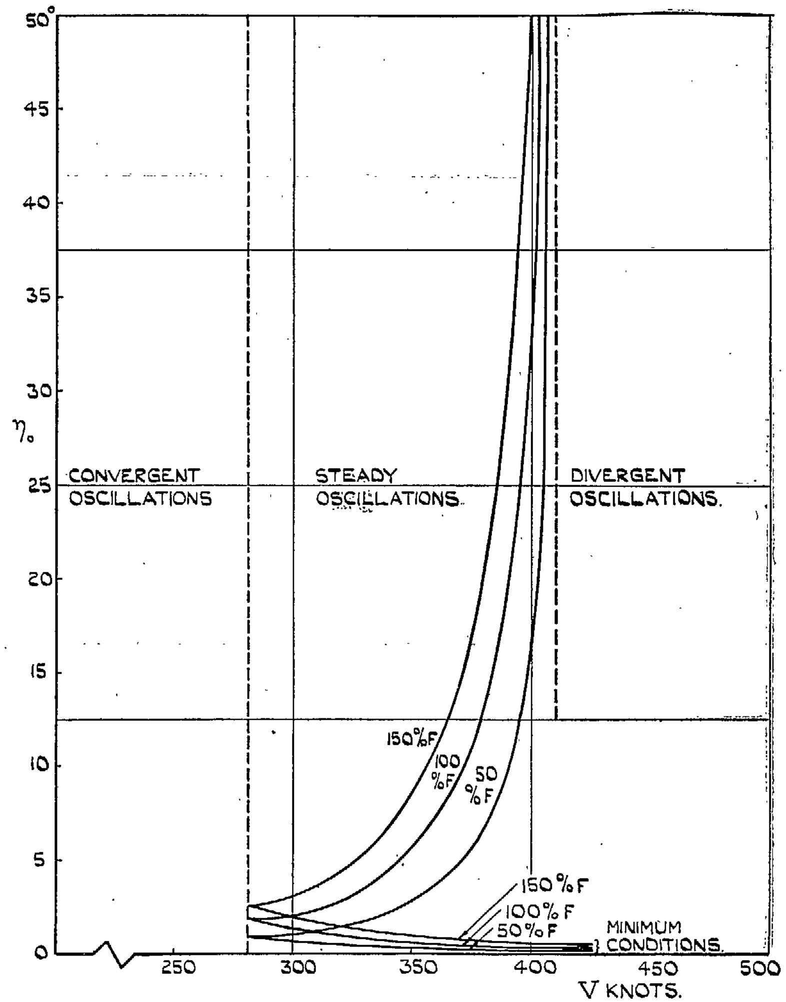

FIG. 9. Variation of the amplitude of the control angle η_0 with speed V, for different values of friction F (Basic case).

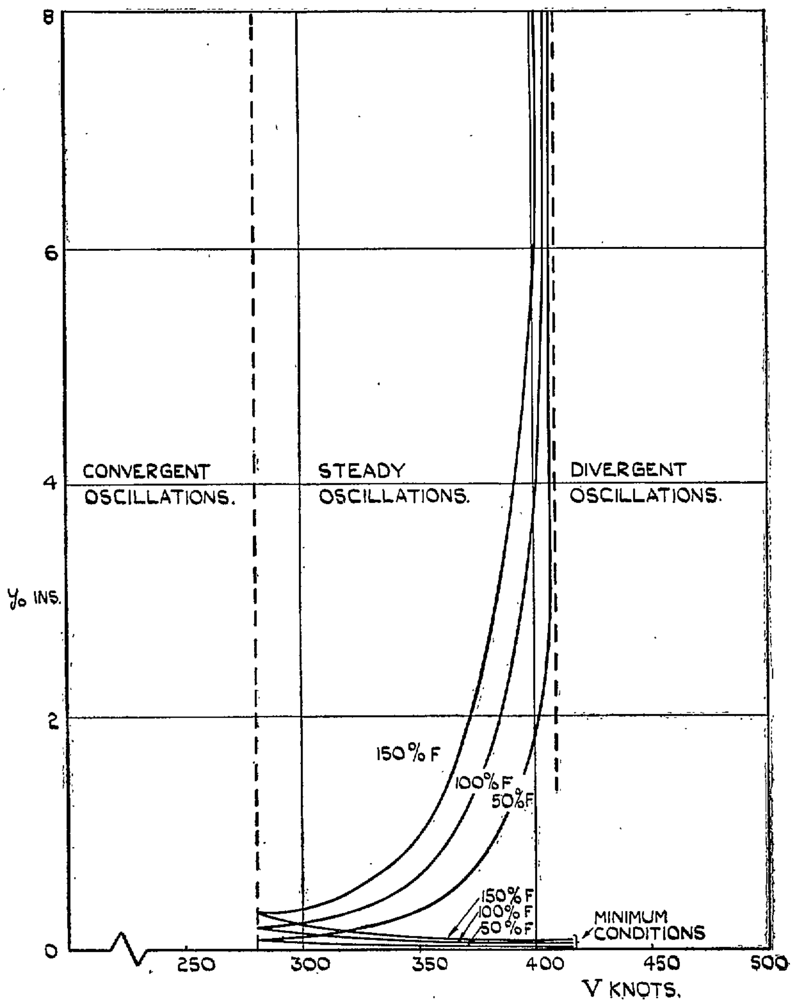

FIG. 10. Variation of the amplitude of the bob-weight displacement y_0 (in.) with speed V, for different values of friction F (Basic case).

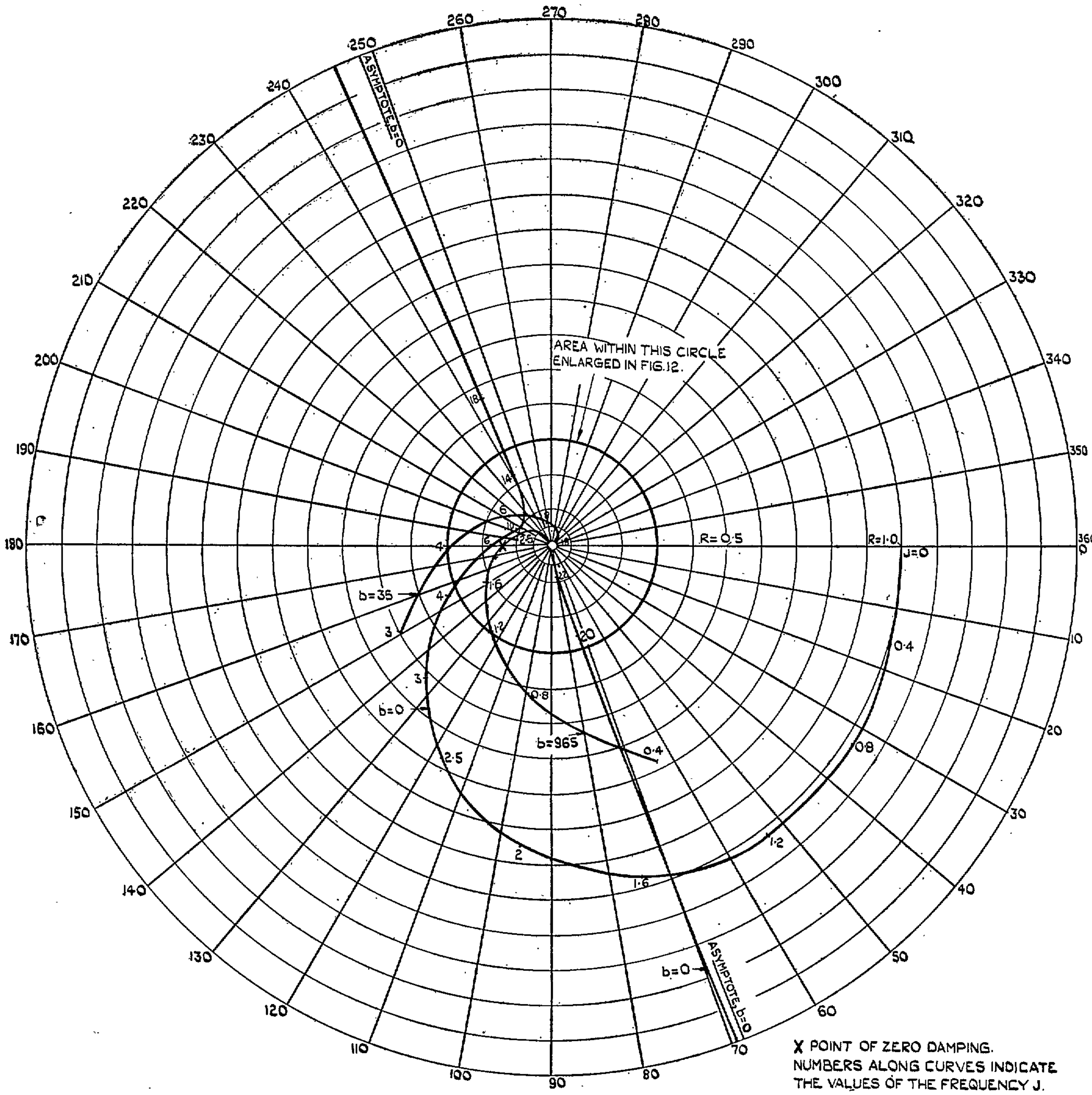

FIG. 11. Nyquist diagram for $V = 450$ kt for basic case of mid-weight c.g. position, with trailing-edge strips on tail ($T_1 = 0 \cdot 02$ sec).

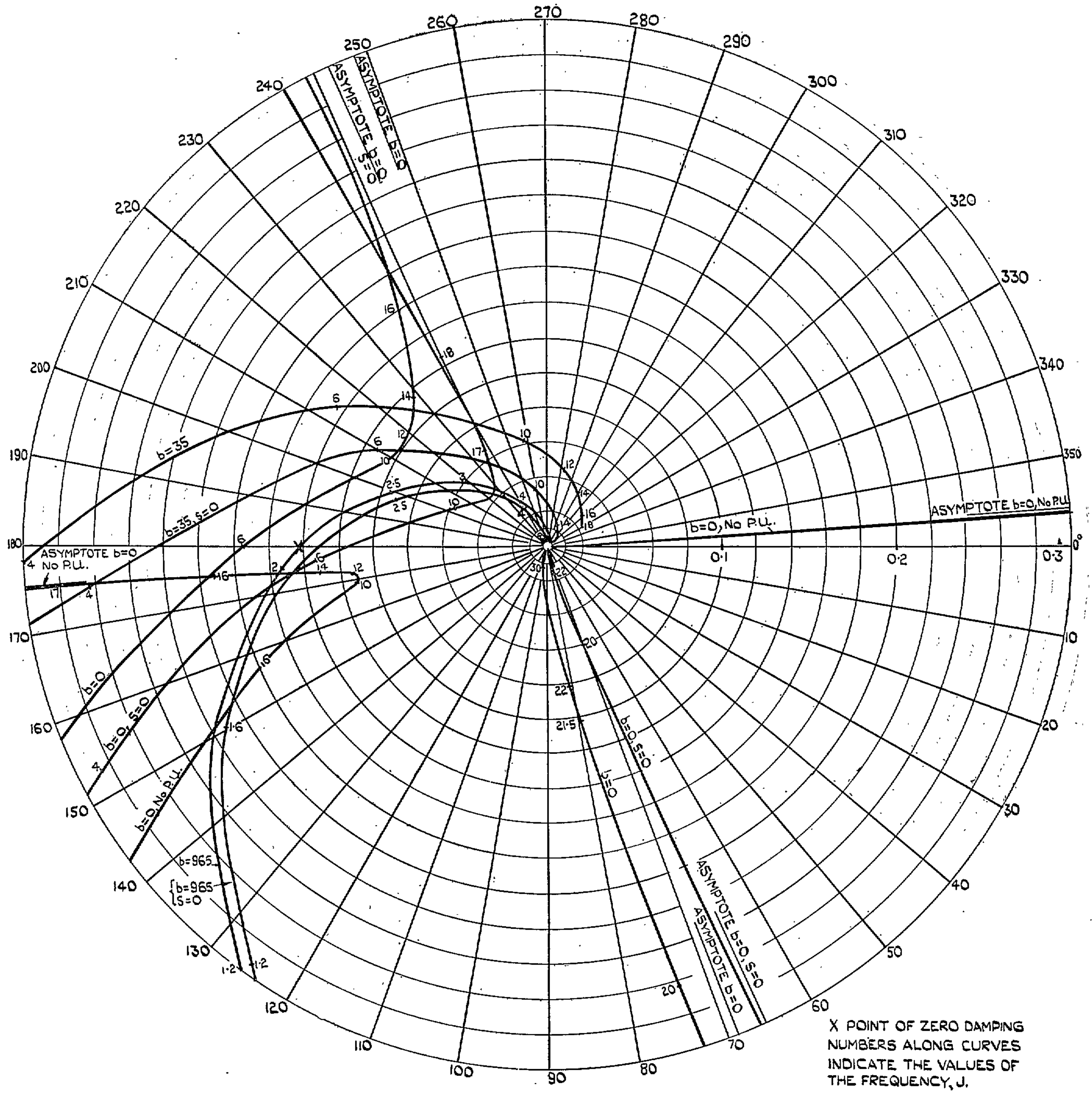

FIG. 12. Enlargement from Nyquist diagram for 450 kt, basic case, with additional curves showing effect of neglecting power unit at $b = 0$ and the effect of putting bob-weight at aircraft c.g. for the three values of b.

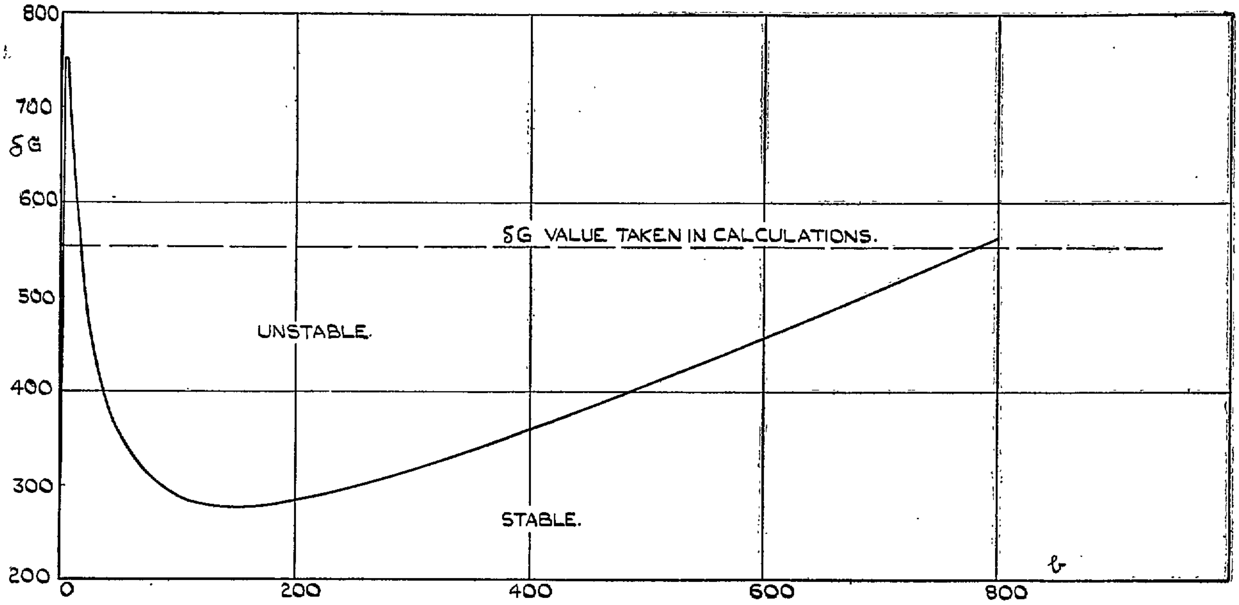

FIG. 13. Stability boundary for δG for varying values of equivalent viscous damping coefficient b, transfer function of power unit neglected (Basic case; $V = 450$ kt).

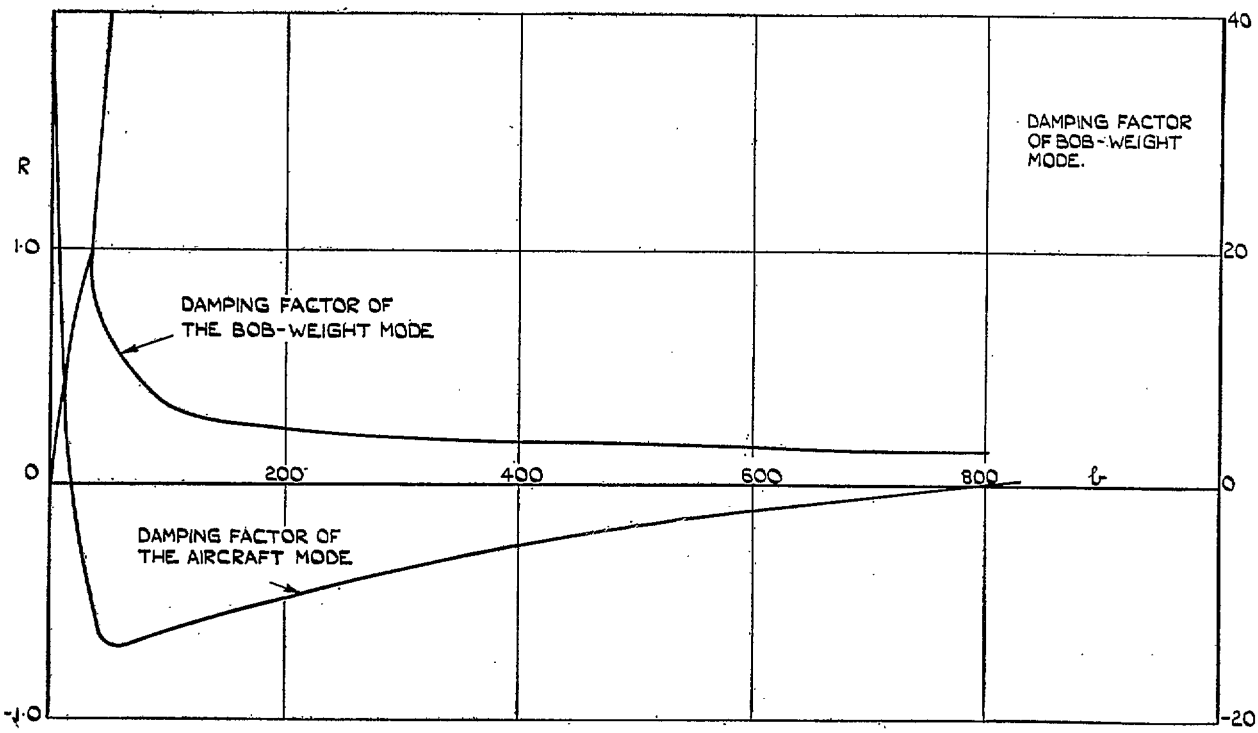

FIG. 14. Variation of aircraft damping R and bob-weight damping with the equivalent viscous damping coefficient b, for the basic case, at a forward speed $V = 450$ kt, transfer function of power unit neglected.

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