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Operational Formulae for Response Calculations

By

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Summary.—The paper presents systematic tables of formulae whose purpose is to facilitate the operational solution of response problems reducible to linear differential equations with constant coefficients and with simple forcing functions. The formulae enable the user to find operational equivalents of a wide class of simple functions and, inversely, to find functional equivalents of a great number of operational expressions, in the most rapid and direct manner. In such a way, it is possible to reduce to a minimum the usual heavy algebraical work involved in response calculations. The tables include only such functions whose operational equivalents are algebraic fractions, but these cover a wide field of practical applications. Operational fractions of the 1st, 2nd, 3rd and 4th order are treated in a comprehensive way, so that all possible particular cases are included. Additional tables make it possible to reduce every fraction of 5th or 6th order to a combination of fractions of lower order.

The introductory text describes the method of deriving the formulae and explains how to use them in solving response problems. A number of examples are appended which show the advantages of the tables and give solutions of several typical problems.

1. *Introduction.*—Response calculations for mechanical, electrical and processing systems, with various sorts of controls, have become indispensable in many branches of engineering. They appear prominently in the modern theory of control and servo-mechanisms. In aeronautics, response problems made their appearance almost immediately after the fundamentals of aircraft dynamic stability had been established. Now, such problems become increasingly important, and more and more complicated cases must be dealt with, to enable us to understand and assess handling properties and structural loadings of aircraft.

The complexity of problems, often involving large numbers of elements and degrees of freedom and also various kinds of non-linearities, has led to a rapid development of differential analysers and simulators. The old analytical methods may seem inadequate or obsolete. There remains, however, a multitude of problems of great practical importance, which are definitely suitable for analytical treatment, and for which the use of costly analogue computors is neither justified nor indeed the most appropriate. This applies especially to linear problems (strictly: problems leading to linear differential equations with constant coefficients, of not very high order). Such problems often admit of comparatively concise and elegant analytical solutions which, if properly handled, may yield simple practical rules and criteria.

The usual analytical method for linear problems is the operational calculus. The advantages of operational treatment have been best described by H. Jeffreys[2]: ' The operational method will give the answer in a page when ordinary methods take five pages; also, it gives the correct answer when the ordinary methods, through human fallibility, are liable to give a wrong one '. Nevertheless, even the operational method, when applied to anything above the level of simple textbook

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examples, usually involves serious algebraical work, consisting mainly of the resolution of operational fractions of higher order into partial fractions. The procedure is quite elementary, of course, but tedious and often exasperatingly long. Many workers in this field try to compile their own sets of auxiliary formulae, so as to avoid repeating identical or similar drudgery over and over again; but such efforts are usually of limited scope and seldom become available to anybody but the originator. There is no lack of publications* containing more or less spacious tables of operational formulae, but they invariably tend to develop more in width than in depth, *i.e.*, there is a tendency to include a great number of higher transcendental functions, while the simple algebraic operational fractions (most often needed in practice) are never treated in a really comprehensive way. It is here that extensive tables of formulae may serve a useful purpose. The present paper is an effort in this direction.

The principle of the operational calculus (in its simplest form as required here) is as follows. A function $F(\tau)$ is replaced by its operational equivalent $\varphi(D)$, defined by Carson's formula:

$$\varphi(D) = D\int_0^\infty e^{-D\tau}\, F(\tau)\, d\tau \; . \qquad \text{(I)}$$

If $F(\tau)$ is an 'elementary' function (*i.e.*, either an integer power of τ, or an exponential, or sine or cosine, or any linear combination of such functions and their products), then $\varphi(D)$ is a rational algebraic fraction in D, the order of the numerator never exceeding that of the denominator. Inversely, any such operational fraction corresponds to a certain 'elementary function' of the type described. Suppose that a sufficient number of operational equivalents of various functions are known. A linear differential equation (relating, *e.g.*, an unknown function x and an independent variable τ), with constant coefficients and with arbitrary initial conditions, can then be solved in an extremely simple way. The original equation is replaced by an operational 'subsidiary equation', algebraic and rational in D (in the manner described in section 3). The latter is then solved for x, whose operational expression is thus found, and the problem is reduced to finding the function of τ, equivalent to this operational expression. The method can also be applied to an equation with a forcing function, in which case this function must be replaced by its operational equivalent, in the subsidiary equation. Finally, a system of n linear differential equations with n unknown functions $x, y, z, \ldots$ can also be solved in a similar way. We have then n subsidiary equations, linear in $x, y, z, \ldots$, algebraic and rational in D, which must be solved for $x, y, z, \ldots$ in the usual way, and then the operational solutions interpreted as functions of τ.

The interpretation of the operational solutions usually requires the tedious procedure of resolving complex fractions into simple ones for which the functional equivalents are available. Here, a great simplification may be obtained by use of tables as presented in this paper. They enable the user to find the functional equivalents of a great number of operational expressions (or inversely) in the most direct and rapid way, thus avoiding the complicated algebraic work. Our tables include only operational expressions in the form of rational algebraic fractions, but these cover an enormous field of the most common applications. The essential feature of the tables is that, up to a certain order, all possible forms of fractions are dealt with, including all the various combinations of real, imaginary, complex and zero roots (also multiple roots) in the denominators, and the most general form of the numerators.

The derivation of the tables is explained in section 2. Section 3 recapitulates the procedure of producing subsidiary equations, which is essential for solving the response problems correctly. The procedure is well known, but it was thought advisable to include this section, so as to avoid misunderstandings and to enable the user to solve his problems without referring to textbooks, in which it is often difficult to find simple working instructions, usually hidden in the maze of theory. This is particularly important because of the unfortunate existence of two very similar, but not identical, operational methods, commonly referred to as the method of Heaviside and

* *See*, for example, Refs. 5, 6, 9.

of Laplace transform, respectively. The fundamental formula (I) corresponds to the original Heaviside's treatment, while the Laplace transform is usually defined[4] by:

$$\psi(p) = \int_0^\infty e^{-p\tau} F(\tau)\, d\tau \,, \quad \ldots \qquad \text{(II)}$$

the only difference being that the factor p is missing in (II). The confusion is made worse by the fact that some authors[8] do include this factor but still use the term 'Laplace transform'. The definition (I) has been adopted here because it has the advantage that $\varphi(D)$ and $F(\tau)$ have the same dimensions; also, if $F(\tau) = 1$ (unit step function), we have $\varphi(D) = 1$, while $\psi(p) = 1/p$, which seems unnatural. It is hoped that the use of the letter D instead of p (following Ref. 1) will help to avoid misunderstandings. It may be mentioned that all formulae of this paper may be employed by readers used to the nomenclature of the Laplace transform in the form (II); it will suffice to divide every operational expression $\varphi(D)$ by D and then replace D by p, while the function $F(\tau)$ remains unchanged.

The Appendix contains a number of examples with complete solutions and discussion. It is hoped that they will be useful, not only as a help to beginners, but also because several examples treat response of simple typical control units as often encountered in practice.

A grateful acknowledgment is due to Mrs. J. Collingbourne who has checked all formulae and helped in working out the examples.

2. *Derivation of Tables*.—The general formula (I) is not convenient for deriving operational equivalents of any but the simplest functions. There exist, however, several simple rules which facilitate the procedure by indicating how to build up operational equivalents gradually, starting from the simplest cases. The only rules needed within the scope of the present paper are as follows:

If the operational equivalent of $F(\tau)$ is $\varphi(D)$, then the operational equivalents of several related functions may be obtained as shown below:

Function	*Operational equivalent*	
$\tau F(\tau)$	$-D \dfrac{d}{dD}\left\{\dfrac{\varphi(D)}{D}\right\}$, ...	(*a*)
$e^{-R\tau} F(\tau)$	$\dfrac{D}{D+R}\varphi(D+R)$, ...	(*b*)
$(\cos J\tau + i \sin J\tau) F(\tau)$	$\dfrac{D}{D-iJ}\varphi(D-iJ)$, ...	(*c*)
$F'(\tau)$	$D\{\varphi(D) - F(o)\}$, ...	(*d*)
$\int_0^\tau F(\tau)\, d\tau$	$\varphi(D)/D$.	(*e*)

Also, if the operational equivalents of $F_1(\tau)$, $F_2(\tau)$, $F_3(\tau)$, . . . are $\varphi_1(D)$, $\varphi_2(D)$, $\varphi_3(D)$, . . ., respectively, and a_1, a_2, a_3, . . . are arbitrary constants, then:

$$\left.\begin{array}{ll}\text{the operational equivalent of} & a_1F_1(\tau) + a_2F_2(\tau) + a_3F_3(\tau) \ldots \\ \text{is} & a_1\varphi_1(D) + a_2\varphi_2(D) + a_3\varphi_3(\mathrm{D}) \ldots\end{array}\right\} \quad \ldots \qquad (f)$$

The proofs of the above rules, based on the definition (I), may be found in textbooks on operational calculus, and are omitted here.

It is easily seen that the above rules lead immediately to all formulae of the fundamental Table 1. Thus, for instance, each of the formulae (2 to 7) is obtained by applying the rule (*e*) to the preceding formula. Formulae (8 to 14) are derived from (1 to 7) by applying the rule (*b*) in each case but, alternatively, the rule (*a*) may be used to derive (9) from (8), (10) from (9), etc.

Formulae (15 to 22) have been obtained from (1 to 4) by applying the rule (*c*), and in each case a single operation yields two formulae at one stroke. For example, applying the rule (*c*) to (2), we find that the operational equivalent of $\tau(\cos J\tau + i \sin J\tau)$ is:

$$\frac{D}{D-iJ}\frac{1}{D-iJ} = \frac{D(D+iJ)^2}{(D^2+J^2)^2} = \frac{D(D^2-J^2)+2iJD^2}{(D^2+J^2)^2}$$

and, separating the real and imaginary parts, we get (16) and (20) at once. Alternatively, (16) may be obtained from (15) by applying the rule (*a*), and so on.

Finally, the formulae (23 to 30) have been derived from (15 to 22) respectively, by applying the rule (*b*), but alternative derivations by means of the rules (*a*) and (*c*) are available for checking.

Each group of formulae in Table 1 may, of course, be extended indefinitely, introducing higher powers of τ. The formulae given are, however, more than sufficient for all ordinary purposes and for deriving the subsequent tables.

By inspecting Table 1 it is seen that the operational equivalents of all simple functions of the type considered are algebraical fractions in D, the denominator being in each case an appropriate power of one of the factors D, $D+R$, D^2+J^2, or $D^2+2RD+(R^2+J^2)$, and the numerator being a polynomial in D, of the order never exceeding that of the denominator. By combining various formulae according to the rule (*f*), it is possible to obtain every algebraical fraction (the order of the numerator never exceeding that of the denominator), and hence the way is open to find the functional equivalent of an arbitrary operational fraction, and to attempt a systematic tabulation. The procedure adopted here consisted of two steps : firstly, Table 2 was compiled, giving functional equivalents of simplest operational fractions, *i.e.*, those whose denominators were powers of only one of the factors D, $D+R$, D^2+J^2 or $D^2+2RD+\mathrm{H}$ (where $\mathrm{H}=R^2+J^2$), and the numerators were either constant or single powers of D; and secondly, we have derived Table 3, giving functional equivalents of the most general operational fractions of the 1st, 2nd, 3rd and 4th order.

The derivation of Table 2 was comparatively easy. Formulae (31 to 37) are simply inversions of (1 to 7). In each of the eight subsequent groups, it was always possible to find one formula directly by inspection of Table 1, *e.g.*, the formula (48) is simply an inversion of (11). All remaining formulae of each particular group could then be obtained by simple differentiation or integration, *i.e.*, by applying the rules (*d*) and (*e*). Only the last four groups (formulae 73 to 94) required some little more effort. For example, (75) was derived by combining (17) and (20) so as to obtain a fraction with the numerator D^2; then the remaining formulae of the group (73 to 79) could be found immediately by differentiation or integration. The tabulation has included all simple operational fractions up to the 6th order.

Table 3 required much more effort. This table consists of four groups, including operational fractions of the first four orders. In each group there are a number of operational fractions with various types of factorized denominators, including all possible combinations of real, imaginary, complex and zero roots, also all cases involving multiple roots. It was found that there were two different types of fractions of the 1st order, six of the 2nd order, eleven of the 3rd order, and twenty-five of the 4th order, and all these had to be tackled in turn. The procedure was simple whenever the denominator was a power of one of the factors D, $D+R$, D^2+J^2 or $D^2+2RD+\mathrm{H}$ because, in such cases, the functional equivalents could be determined by combining appropriate formulae of Table 2. For example, the formula (100) was obtained directly by combining (80, 81, 82); similarly, (118) is a simple combination of (47 to 51). If, however, the denominator contained different linear or quadratic factors, the fraction had to be resolved into simple fractions, whose functional equivalents were then found in Table 2 and combined into the final formula. The algebraical work was complicated in many cases, and extensive checks were made to ensure reliability of all results. In many cases simple checking formulae are given in the table, as they may be helpful for the user.

Table 3 could be continued to include operational fractions of 5th, 6th and higher orders. This would require, however, an immense amount of work. For instance, there would be forty-three different forms of fractions of 5th order, and eighty-four of those of 6th order, and the corresponding formulae would be increasingly complicated. It was found impossible to perform such an enormous task. Instead, the reader will find Tables 4 and 5, which contain a comparatively small (but sufficient) number of formulae for reducing every fraction of 5th or 6th order respectively, to a combination of fractions of lower order. The latter can be dealt with by using Table 3 or 2.

3. *Use of Tables for Response Calculations.*—We consider only problems which reduce to the solution of a linear differential equation with constant coefficients, or of a system of simultaneous differential equations of the same type. The forcing function (if any) in each equation must be one of those appearing in Table 1 (or any linear combination of such functions).

Let us consider, for simplicity, one equation of the 3rd order :

$$\frac{d^3x}{d\tau^3} + K_2\frac{d^2x}{d\tau^2} + K_1\frac{dx}{d\tau} + K_0x = F_1(\tau)\,, \quad \ldots \quad \text{(III)}$$

where x is an unknown function of τ, and $F_1(\tau)$ the forcing function; and let the initial conditions be $x = x_0$, $dx/d\tau = \dot{x}_0$, $d^2x/d\tau^2 = \ddot{x}_0$, at $\tau = 0$. The equation (III) can be transformed into operational form by applying the operation (I) to each term. Suppose that $\varphi_1(D)$ is the operational equivalent of $F_1(\tau)$, and $X = \varphi(D)$ that of $x(\tau)$. Applying the rule (*d*) of section 2 three times, we find that :

the operational equivalent of $dx/d\tau$ is $DX - Dx_0$,
the operational equivalent of $d^2x/d\tau^2$ is $D^2X - D^2x_0 - D\dot{x}_0$,
the operational equivalent of $d^3x/d\tau^3$ is $D^3X - D^3x_0 - D^2\dot{x}_0 - D\ddot{x}_0$,

and substituting these expressions into (III), we obtain the subsidiary equation in the form :

$$(D^3 + K_2D^2 + K_1D + K_0)X = \varphi_1(D) + x_0(D^3 + K_2D^2 + K_1D) + \dot{x}_0(D^2 + K_2D) + \ddot{x}_0D\,. \quad \text{(IV)}$$

The procedure of obtaining the subsidiary equation is thus:

(i) To replace x by X, $dx/d\tau$ by DX, $d^2x/d\tau^2$ by D^2X, etc., and write the left-hand part of the equation as a product of X by an appropriate polynomial in D

(ii) To replace the forcing function by its operational equivalent

(iii) To add on the right terms accounting for initial conditions, formed in such a way; x_0 is multiplied by the left-hand polynomial in D less the constant term, and $\dot{x}_0$, $\ddot{x}_0$, etc., multiplied by shorter polynomials obtained each time by dropping the next term and dividing by D.

It remains then to solve the subsidiary equation (IV) for X, whereupon we obtain:

$$X = \varphi(D) = \frac{\varphi_1(D) + x_0(D^3 + K_2D^2 + K_1D) + \dot{x}_0(D^2 + K_2D) + \ddot{x}_0D}{D^3 + K_2D^2 + K_1D + K_0}\,, \quad \ldots \quad \text{(V)}$$

which is the operational equivalent of x, in the form of an algebraic fraction. This must be interpreted by using the appropriate tables. In our case Table 3 is sufficient if the order of the fraction does not exceed 4 but, if the order is 5 or 6, Table 4 or 5 must also be used.

A similar method may be applied to solve a system of n simultaneous differential equations with n unknown functions $x, y, z, \ldots$ of a single independent variable τ. All equations are then transformed into operational form by applying the operation (I) term by term. Let the symbols $X, Y, Z, \ldots$ denote operational equivalents of $x, y, z, \ldots$, respectively. The left-hand part of each equation becomes a linear combination of $X, Y, Z, \ldots$, multiplied respectively by appropriate polynomials in D; on the right, we write the operational equivalent of the respective forcing function, and add products of initial values $x_0, \dot{x}_0, \ddot{x}_0, \ldots, y_0, \dot{y}_0, \ddot{y}_0, \ldots$, etc., by polynomials formed in the same way as described above. We obtain n subsidiary equations. linear in $X, Y, Z, \ldots$ and, by solving them algebraically for $X, Y, Z, \ldots$ (preferably by using determinants), we find the operational equivalents of $x, y, z, \ldots$, which must then be interpreted by making use of the tables.

Several examples are given in the Appendix, and the reader will find that the solution of problems by this method is extremely simple and rapid. It may be mentioned that there is really no need to write X instead of x, etc., in subsidiary equations, as there is never a risk of confusing the operational equivalent with the function itself.

4. *Additional Remarks.*—The following remarks may not be superfluous:

(*a*) In our tables, all exponentials are written in the form $e^{-R\tau}$, $e^{-r\tau}$, etc., the exponents being assumed to be negative. The reason is that systems normally encountered in engineering are (or should be) stable, so that the real parts of the stability roots are usually negative. The formulae can, of course, be used when one or more exponents are positive, which simply means writing R instead of $(-R)$ etc., both in the function and in its operational equivalent.

(*b*) If a power series in τ is required for any function $F(\tau)$ whose operational equivalent $\varphi(D)$ is known, then the easiest way is usually to expand $\varphi(D)$ as a power series in $1/D$, simply by dividing the numerator by the denominator, and then to interpret by means of the formulae (31 to 37) of Table 2.

(*c*) The initial value of $F(\tau)$, for $\tau = 0$, can be found by letting D tend to infinity in the expansion of $\varphi(D)$ mentioned above or, which comes to the same, in the fractional expression of $\varphi(D)$. Thus, we have:

$$F(o) = \lim_{D \to \infty} \varphi(D) \qquad \text{(or simpler } F(o) = \varphi(\infty)) \,. \quad \ldots \quad \ldots \quad \text{(VI)}$$

In view of the notation used in Tables 3, 4, 5, the initial value of every function appearing in these tables is equal to a.

(*d*) A relation analogous to (VI) also exists for the limit of $F(\tau)$ when τ tends to infinity, if such a limit exists. We have then:

$$\lim_{\tau \to \infty} F(\tau) = \varphi(o) \qquad \text{(or simpler } F(\infty) = \varphi(o)) \,, \quad \ldots \quad \ldots \text{(VII)}$$

and it is seen that this limit is always equal to the ratio of the constant terms of the numerator and denominator of the operational fraction. This formula is very useful in practical applications.

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TABLE 1

Operational Equivalents of Simple Functions

Function $F(\tau)$	Operational equivalent $\varphi(D)$	Number
1	1	1
τ	$1/D$	2
τ^2	$2/D^2$	3
τ^3	$6/D^3$	4
τ^4	$24/D^4$	5
τ^5	$120/D^5$	6
τ^6	$720/D^6$	7
$e^{-R\tau}$	$D/(D+R)$	8
$\tau\, e^{-R\tau}$	$D/(D+R)^2$	9
$\tau^2\, e^{-R\tau}$	$2D/(D+R)^3$	10
$\tau^3\, e^{-R\tau}$	$6D/(D+R)^4$	11
$\tau^4\, e^{-R\tau}$	$24D/(D+R)^5$	12
$\tau^5\, e^{-R\tau}$	$120D/(D+R)^6$	13
$\tau^6\, e^{-R\tau}$	$720D/(D+R)^7$	14
$\cos J\tau$	$\dfrac{D^2}{D^2+J^2}$	15
$\tau \cos J\tau$	$\dfrac{D(D^2-J^2)}{(D^2+J^2)^2}$	16
$\tau^2 \cos J\tau$	$\dfrac{2D^2(D^2-3J^2)}{(D^2+J^2)^3}$	17
$\tau^3 \cos J\tau$	$\dfrac{6D(D^4-6J^2D^2+J^4)}{(D^2+J^2)^4}$	18
$\sin J\tau$	$\dfrac{JD}{D^2+J^2}$	19
$\tau \sin J\tau$	$\dfrac{2JD^2}{(D^2+J^2)^2}$	20
$\tau^2 \sin J\tau$	$\dfrac{2JD(3D^2-J^2)}{(D^2+J^2)^3}$	21
$\tau^3 \sin J\tau$	$\dfrac{24JD^2(D^2-J^2)}{(D^2+J^2)^4}$	22

TABLE 1—*continued*

Function $F(\tau)$	Operational equivalent	Number
$e^{-R\tau} \cos J\tau$	$\dfrac{D(D+R)}{D^2+2RD+(R^2+J^2)}$	23
$\tau\, e^{-R\tau} \cos J\tau$	$\dfrac{D\{D^2+2RD+(R^2-J^2)\}}{\{D^2+2RD+(R^2+J^2)\}^2}$	24
$\tau^2\, e^{-R\tau} \cos J\tau$	$\dfrac{2D(D+R)\{D^2+2RD+(R^2-3J^2)\}}{\{D^2+2RD+(R^2+J^2)\}^3}$	25
$\tau^3\, e^{-R\tau} \cos J\tau$	$\dfrac{6D\{D^4+4RD^3+6(R^2-J^2)D^2+4R(R^2-3J^2)D+(R^4-6R^2J^2+J^4)\}}{\{D^2+2RD+(R^2+J^2)\}^4}$	26
$e^{-R\tau} \sin J\tau$	$\dfrac{JD}{D^2+2RD+(R^2+J^2)}$	27
$\tau\, e^{-R\tau} \sin J\tau$	$\dfrac{2JD(D+R)}{\{D^2+2RD+(R^2+J^2)\}^2}$	28
$\tau^2\, e^{-R\tau} \sin J\tau$	$\dfrac{2JD\{3D^2+6RD+(3R^2-J^2)\}}{\{D^2+2RD+(R^2+J^2)\}^3}$	29
$\tau^3\, e^{-R\tau} \sin J\tau$	$\dfrac{24JD(D+R)\{D^2+2RD+(R^2-J^2)\}}{\{D^2+2RD+(R^2+J^2)\}^4}$	30

TABLE 2

Functions Equivalent to Simple Operational Expressions

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
1	1	31
$1/D$	τ	32
$1/D^2$	$\tau^2/2$	33
$1/D^3$	$\tau^3/6$	34
$1/D^4$	$\tau^4/24$	35
$1/D^5$	$\tau^5/120$	36
$1/D^6$	$\tau^6/720$	37
$R/(D+R)$	$1 - e^{-R\tau}$	38
$D/(D+R)$	$e^{-R\tau}$	39
$R^2/(D+R)^2$	$1 - (1 + R\tau)\, e^{-R\tau}$	40
$D/(D+R)^2$	$\tau\, e^{-R\tau}$	41
$D^2/(D+R)^2$	$(1 - R\tau)\, e^{-R\tau}$	42
$R^3/(D+R)^3$	$1 - (1 + R\tau + \frac{1}{2}R^2\tau^2)\, e^{-R\tau}$	43
$D/(D+R)^3$	$\frac{1}{2}\tau^2\, e^{-R\tau}$	44
$D^2/(D+R)^3$	$(\tau - \frac{1}{2}R\tau^2)\, e^{-R\tau}$	45
$D^3/(D+R)^3$	$(1 - 2R\tau + \frac{1}{2}R^2\tau^2)\, e^{-R\tau}$	46
$R^4/(D+R)^4$	$1 - (1 + R\tau + \frac{1}{2}R^2\tau^2 + \frac{1}{6}R^3\tau^3)\, e^{-R\tau}$	47
$D/(D+R)^4$	$\frac{1}{6}\tau^3\, e^{-R\tau}$	48
$D^2/(D+R)^4$	$(\frac{1}{2}\tau^2 - \frac{1}{6}R\tau^3)\, e^{-R\tau}$	49
$D^3/(D+R)^4$	$(\tau - R\tau^2 + \frac{1}{6}R^2\tau^3)\, e^{-R\tau}$	50
$D^4/(D+R)^4$	$(1 - 3R\tau + \frac{3}{2}R^2\tau^2 - \frac{1}{6}R^3\tau^3)\, e^{-R\tau}$	51
$R^5/(D+R)^5$	$1 - (1 + R\tau + \frac{1}{2}R^2\tau^2 + \frac{1}{6}R^3\tau^3 + \frac{1}{24}R^4\tau^4)\, e^{-R\tau}$	52
$D/(D+R)^5$	$\frac{1}{24}\tau^4\, e^{-R\tau}$	53
$D^2/(D+R)^5$	$(\frac{1}{6}\tau^3 - \frac{1}{24}R\tau^4)\, e^{-R\tau}$	54
$D^3/(D+R)^5$	$(\frac{1}{2}\tau^2 - \frac{1}{3}R\tau^3 + \frac{1}{24}R^2\tau^4)\, e^{-R\tau}$	55
$D^4/(D+R)^5$	$(\tau - \frac{3}{2}R\tau^2 + \frac{1}{2}R^2\tau^3 - \frac{1}{24}R^3\tau^4)\, e^{-R\tau}$	56
$D^5/(D+R)^5$	$(1 - 4R\tau + 3R^2\tau^2 - \frac{2}{3}R^3\tau^3 + \frac{1}{24}R^4\tau^4)\, e^{-R\tau}$	57

TABLE 2—*continued*

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$R^6/(D+R)^6$	$1-(1+R\tau+\frac{1}{2}R^2\tau^2+\frac{1}{6}R^3\tau^3+\frac{1}{24}R^4\tau^4+\frac{1}{120}R^5\tau^5)\,e^{-R\tau}$	58
$D/(D+R)^6$	$\frac{1}{120}\tau^5\,e^{-R\tau}$	59
$D^2/(D+R)^6$	$(\frac{1}{24}\tau^4-\frac{1}{120}R\tau^5)\,e^{-R\tau}$	60
$D^3/(D+R)^6$	$(\frac{1}{6}\tau^3-\frac{1}{12}R\tau^4+\frac{1}{120}R^2\tau^5)\,e^{-R\tau}$	61
$D^4/(D+R)^6$	$(\frac{1}{2}\tau^2-\frac{1}{2}R\tau^3+\frac{1}{8}R^2\tau^4-\frac{1}{120}R^3\tau^5)\,e^{-R\tau}$	62
$D^5/(D+R)^6$	$(\tau-2R\tau^2+R^2\tau^3-\frac{1}{6}R^3\tau^4+\frac{1}{120}R^4\tau^5)\,e^{-R\tau}$	63
$D^6/(D+R)^6$	$(1-5R\tau+5R^2\tau^2-\frac{5}{3}R^3\tau^3+\frac{5}{24}R^4\tau^4-\frac{1}{120}R^5\tau^5)\,e^{-R\tau}$	64
$J^2/(D^2+J^2)$	$1-\cos J\tau$	65
$D/(D^2+J^2)$	$\frac{\sin J\tau}{J}$	66
$D^2/(D^2+J^2)$	$\cos J\tau$	67
$J^4/(D^2+J^2)^2$	$1-\cos J\tau-\frac{1}{2}J\tau\sin J\tau$	68
$D/(D^2+J^2)^2$	$\frac{1}{2J^2}\left(\frac{\sin J\tau}{J}-\tau\cos J\tau\right)$	69
$D^2/(D^2+J^2)^2$	$\frac{1}{2}\tau\frac{\sin J\tau}{J}$	70
$D^3/(D^2+J^2)^2$	$\frac{1}{2}\left(\tau\cos J\tau+\frac{\sin J\tau}{J}\right)$	71
$D^4/(D^2+J^2)^2$	$\cos J\tau-\frac{1}{2}J\tau\sin J\tau$	72
$J^6/(D^2+J^2)^3$	$1-(1-\frac{1}{8}J^2\tau^2)\cos J\tau-\frac{5}{8}J\tau\sin J\tau$	73
$D/(D^2+J^2)^3$	$\frac{1}{8J^4}\left\{\left(3-J^2\tau^2\right)\frac{\sin J\tau}{J}-3\tau\cos J\tau\right\}$	74
$D^2/(D^2+J^2)^3$	$\frac{1}{8J^2}\left(\tau\frac{\sin J\tau}{J}-\tau^2\cos J\tau\right)$	75
$D^3/(D^2+J^2)^3$	$\frac{1}{8J^2}\left\{(1+J^2\tau^2)\frac{\sin J\tau}{J}-\tau\cos J\tau\right\}$	76
$D^4/(D^2+J^2)^3$	$\frac{1}{8}\left(\tau^2\cos J\tau+3\tau\frac{\sin J\tau}{J}\right)$	77
$D^5/(D^2+J^2)^3$	$\frac{1}{8}\left\{5\tau\cos J\tau+(3-J^2\tau^2)\frac{\sin J\tau}{J}\right\}$	78
$D^6/(D^2+J^2)^3$	$(1-\frac{1}{8}J^2\tau^2)\cos J\tau-\frac{7}{8}J\tau\sin J\tau$	79

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{H}{D^2+2RD+H}$	$1-\left(\cos J\tau+\dfrac{R}{J}\sin J\tau\right)\mathrm{e}^{-R\tau}$	80
$\dfrac{D}{D^2+2RD+H}$	$\mathrm{e}^{-R\tau}\dfrac{\sin J\tau}{J}$	81
$\dfrac{D^2}{D^2+2RD+H}$	$\left(\cos J\tau-\dfrac{R}{J}\sin J\tau\right)\mathrm{e}^{-R\tau}$	82
$\dfrac{H^2}{(D^2+2RD+H)^2}$	$1-\left[\left(1-\dfrac{RH}{2J^2}\tau\right)\cos J\tau+\tfrac{1}{2}\left\{\dfrac{R(3H-2R^2)}{J^2}+H\tau\right\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	83
$\dfrac{D}{(D^2+2RD+H)^2}$	$\dfrac{1}{2J^2}\left(\dfrac{\sin J\tau}{J}-\tau\cos J\tau\right)\mathrm{e}^{-R\tau}$	84
$\dfrac{D^2}{(D^2+2RD+H)^2}$	$\dfrac{1}{2J^2}\left\{R\tau\cos J\tau+(J^2\tau-R)\dfrac{\sin J\tau}{J}\right\}\mathrm{e}^{-R\tau}$	85
$\dfrac{D^3}{(D^2+2RD+H)^2}$	$\dfrac{1}{2J^2}\left\{(H-2R^2)\tau\cos J\tau+(H-2RJ^2\tau)\dfrac{\sin J\tau}{J}\right\}\mathrm{e}^{-R\tau}$	86
$\dfrac{D^4}{(D^2+2RD+H)^2}$	$\dfrac{1}{2J^2}\left[\left\{2J^2-R(3H-4R^2)\tau\right\}\cos J\tau+\{J^2(4R^2-H)\tau-R(3H-2R^2)\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	87
$\dfrac{H^3}{(D^2+2RD+H)^3}$	$1-\left[\left\{1-\dfrac{RH}{2J^2}\left(1+\dfrac{3H}{4J^2}\right)\tau-\dfrac{H^2}{8J^2}\tau^2\right\}\cos J\tau+\dfrac{1}{8J^2}\left\{\dfrac{R}{J^2}(15H^2-20R^2H+8R^4)+\right.\right.$ $\left.\left.+H(5H-4R^2)\tau-H^2R\tau^2\right\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	88
$\dfrac{D}{(D^2+2RD+H)^3}$	$\dfrac{1}{8J^4}\left\{(3-J^2\tau^2)\dfrac{\sin J\tau}{J}-3\tau\cos J\tau\right\}\mathrm{e}^{-R\tau}$	89
$\dfrac{D^2}{(D^2+2RD+H)^3}$	$\dfrac{1}{8J^4}\left\{(3R\tau-J^2\tau^2)\cos J\tau+(RJ^2\tau^2+J^2\tau-3R)\dfrac{\sin J\tau}{J}\right\}\mathrm{e}^{-R\tau}$	90
$\dfrac{D^3}{(D^2+2RD+H)^3}$	$\dfrac{1}{8J^4}\left[\{2RJ^2\tau^2-(H+2R^2)\tau\}\cos J\tau+\right.$ $\left.+\{(H+2R^2)-2RJ^2\tau+J^2(H-2R^2)\tau^2\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	91
$\dfrac{D^4}{(D^2+2RD+H)^3}$	$\dfrac{1}{8J^4}\left[\{3RH\tau+J^2(H-4R^2)\tau^2\}\cos J\tau-\right.$ $\left.-\{3RH-3J^2H\tau+J^2R(3H-4R^2)\tau^2\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	92
$\dfrac{D^5}{(D^2+2RD+H)^3}$	$\dfrac{1}{8J^4}\left[\{4RJ^2(2R^2-H)\tau^2+(5H^2-16R^2H+8R^4)\tau\}\cos J\tau+\right.$ $\left.+\{3H^2-4RJ^2(3H-2R^2)\tau-J^2(H^2-8R^2H+8R^4)\tau^2\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	93
$\dfrac{D^6}{(D^2+2RD+H)^3}$	$\dfrac{1}{8J^4}\left[\{8J^4-R(25H^2-60R^2H+32R^4)\tau-J^2(H^2-12R^2H+16R^4)\tau^2\}\cos J\tau\right.$ $-\{R(15H^2-20R^2H+8R^4)+J^2(7H^2-44R^2H+32R^4)\tau-$ $\left.-J^2R(5H^2-20R^2H+16R^4)\tau^2\}\dfrac{\sin J\tau}{J}\right]\mathrm{e}^{-R\tau}$	94

TABLE 3

Functions Equivalent to General Operational Expressions in Form of Fractions with Denominators of 1st to 4th Order

(1) *Fractions with denominators linear in D*

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD+b}{D+R}$	$\dfrac{b}{R}+\left(a-\dfrac{b}{R}\right)e^{-R\tau}$	95
$\dfrac{aD+b}{D}$	$a+b\tau$	96

(2) *Fractions with denominators quadratic in D*

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^2+bD+c}{(D+R)(D+r)}$	$\dfrac{c}{Rr}+\dfrac{aR-b+c/R}{R-r}e^{-R\tau}-\dfrac{ar-b+c/r}{R-r}e^{-r\tau}$	97
$\dfrac{aD^2+bD+c}{(D+r)^2}$	$\dfrac{c}{r^2}+\left\{\left(a-\dfrac{c}{r^2}\right)-\left(ar-b+\dfrac{c}{r}\right)\tau\right\}e^{-r\tau}$	98
$\dfrac{aD^2+bD+c}{D^2+J^2}$	$\dfrac{c}{J^2}+\left(a-\dfrac{c}{J^2}\right)\cos J\tau+\dfrac{b}{J}\sin J\tau$	99
$\dfrac{aD^2+bD+c}{D^2+2RD+H}$	$\dfrac{c}{H}+\left\{\left(a-\dfrac{c}{H}\right)\cos J\tau-\left(aR-b+\dfrac{cR}{H}\right)\dfrac{\sin J\tau}{J}\right\}e^{-R\tau}$	100
$\dfrac{aD^2+bD+c}{D(D+R)}$	$\left(\dfrac{b}{R}-\dfrac{c}{R^2}\right)+\dfrac{c}{R}\tau+\left(a-\dfrac{b}{R}+\dfrac{c}{R^2}\right)e^{-R\tau}$	101
$\dfrac{aD^2+bD+c}{D^2}$	$a+b\tau+\frac{1}{2}c\tau^2$	102

(3) *Fractions with denominators cubic in D*

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^3+bD^2+cD+d}{(D+R_1)(D+R_2)(D+R_3)}$	$A+B_1e^{-R_1\tau}+B_2e^{-R_2\tau}+B_3e^{-R_3\tau}$, where: $A=\dfrac{d}{R_1R_2R_3}$, $B_1=\dfrac{aR_1^2-bR_1+c-d/R_1}{(R_1-R_2)(R_1-R_3)}$, $B_2=\dfrac{aR_2^2-bR_2+c-d/R^2}{(R_2-R_1)(R_2-R_3)}$, $B_3=\dfrac{aR_3^2-bR_3+c-d/R_3}{(R_3-R_1)(R_3-R_2)}$. Check: $A+B_1+B_2+B_3=a$.	103

TABLE 3—*continued*

(3) *Fractions with denominators cubic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^3+bD^2+cD+d}{(D+R)(D+r)^2}$	$A+B\,\mathrm{e}^{-R\tau}+(C+E\tau)\,\mathrm{e}^{-r\tau}$, where: $A=\dfrac{d}{Rr^2}$, $B=\dfrac{aR^2-bR+c-d/R}{(R-r)^2}$, $C=\dfrac{a(r^2-2Rr)+bR-c+d\left(\dfrac{2}{r}-\dfrac{R}{r^2}\right)}{(R-r)^2}$, $E=\dfrac{ar^2-br+c-d/r}{R-r}$. Check: $A+B+C=a$, $BR+Cr-E=a(R+2r)-b$.	104
$\dfrac{aD^3+bD^2+cD+d}{(D+r)^3}$	$\dfrac{d}{r^3}+\left\{\left(a-\dfrac{d}{r^3}\right)-\left(2ar-b+\dfrac{d}{r^2}\right)\tau+\tfrac{1}{2}\left(ar^2-br+c-\dfrac{d}{r}\right)\tau^2\right\}\mathrm{e}^{-r\tau}$	105
$\dfrac{aD^3+bD^2+cD+d}{(D^2+J^2)(D+r)}$	$A+B\cos J\tau+C\dfrac{\sin J\tau}{J}+E\,\mathrm{e}^{-r\tau}$, where: $A=\dfrac{d}{rJ^2}$, $B=\dfrac{aJ^2+br-c-rd/J^2}{J^2+r^2}$, $C=\dfrac{(b-ar)J^2+cr-d}{J^2+r^2}$, $E=\dfrac{ar^2-br+c-d/r}{J^2+r^2}$. Check: $A+B+E=a$, $C-Er=b-ar$.	106
$\dfrac{aD^3+bD^2+cD+d}{(D^2+2RD+H)(D+r)}$	$A+\left(B\cos J\tau+C\dfrac{\sin J\tau}{J}\right)\mathrm{e}^{-R\tau}+E\,\mathrm{e}^{-r\tau}$, where: $A=\dfrac{d}{rH}$, $B=\dfrac{a(H-2Rr)+br-c+d(2R-r)/H}{H-2Rr+r^2}$, $C=\dfrac{a(2R^2r-RH-rH)+b(H-Rr)-c(R-r)+d(2R^2-Rr-H)/H}{H-2Rr+r^2}$, $E=\dfrac{ar^2-br+c-d/r}{H-2Rr+r^2}$. Check: $A+B+E=a$, $C-BR-Er=b-a(2R+r)$.	107

(3) *Fractions with denominators cubic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^3 + bD^2 + cD + d}{D(D+R)(D+r)}$	$\dfrac{d}{Rr}\tau + \left(\dfrac{c}{Rr} - d\,\dfrac{R+r}{R^2r^2}\right) + \dfrac{aR - b + c/R - d/R^2}{R-r}\,\mathrm{e}^{-R\tau} - \dfrac{ar - b + c/r - d/r^2}{R-r}\,\mathrm{e}^{-r\tau}$	108
$\dfrac{aD^3 + bD^2 + cD + d}{D(D+r)^2}$	$\dfrac{d}{r^2}\tau + \left(\dfrac{c}{r^2} - \dfrac{2d}{r^3}\right) + \left\{\left(a - \dfrac{c}{r^2} + \dfrac{2d}{r^3}\right) - \left(ar - b + \dfrac{c}{r} - \dfrac{d}{r^2}\right)\tau\right\}\mathrm{e}^{-r\tau}$	109
$\dfrac{aD^3 + bD^2 + cD + d}{D(D^2+J^2)}$	$\dfrac{d}{J^2}\tau + \dfrac{c}{J^2} + \left(a - \dfrac{c}{J^2}\right)\cos J\tau + \left(b - \dfrac{d}{J^2}\right)\dfrac{\sin J\tau}{J}$	110
$\dfrac{aD^3 + bD^2 + cD + d}{D(D^2+2RD+H)}$	$\dfrac{d}{H}\tau + \left(\dfrac{c}{H} - 2d\,\dfrac{R}{H^2}\right) + \left\{\left(a - \dfrac{c}{H} + 2d\,\dfrac{R}{H^2}\right)\cos J\tau - \left(aR - b + \dfrac{cR}{H} - d\,\dfrac{2R^2 - H}{H^2}\right)\dfrac{\sin J\tau}{J}\right\}\mathrm{e}^{-R\tau}$	111
$\dfrac{aD^3 + bD^2 + cD + d}{D^2(D+R)}$	$\dfrac{d}{2R}\tau^2 + \left(\dfrac{c}{R} - \dfrac{d}{R^2}\right)\tau + \left(\dfrac{b}{R} - \dfrac{c}{R^2} + \dfrac{d}{R^3}\right) + \left(a - \dfrac{b}{R} + \dfrac{c}{R^2} - \dfrac{d}{R^3}\right)\mathrm{e}^{-R\tau}$	112
$\dfrac{aD^3 + bD^2 + cD + d}{D^3}$	$a + b\tau + \frac{1}{2}c\tau^2 + \frac{1}{6}d\tau^3$	113

(4) *Fractions with denominators quartic in D*

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{(D+R_1)(D+R_2)(D+R_3)(D+R_4)}$	$A + B_1\,\mathrm{e}^{-R_1\tau} + B_2\,\mathrm{e}^{-R_2\tau} + B_3\,\mathrm{e}^{-R_3\tau} + B_4\,\mathrm{e}^{-R_4\tau}$, where: $A = \dfrac{f}{R_1R_2R_3R_4}, \quad B_1 = \dfrac{aR_1^3 - bR_1^2 + cR_1 - d + f/R_1}{(R_1-R_2)(R_1-R_3)(R_1-R_4)}$, and analogous formulae for B_2, B_3, B_4. Check: $A + B_1 + B_2 + B_3 + B_4 = a$.	114

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D+R_1)(D+R_2)(D+r)^2}$	$A + B_1\,e^{-R_1\tau} + B_2\,e^{-R_2\tau} + (C - E\tau)\,e^{-r\tau}$, where: $A = \dfrac{f}{R_1R_2r^2}$, $B_1 = \dfrac{aR_1^3 - bR_1^2 + cR_1 - d + f/R_1}{(R_1-R_2)(R_1-r)^2}$ (analogous for B_2), $E = \dfrac{ar^3 - br^2 + cr - d + f/r}{(R_1-r)(R_2-r)}$, $C(R_1-r)^2(R_2-r)^2 = ar^2\{r^2 - 2r(R_1+R_2) + 3R_1R_2\} +$ $+ br\{r(R_1+R_2) - 2R_1R_2\} + c(R_1R_2 - r^2) - d(R_1+R_2-2r) -$ $- f\left(3 - 2\dfrac{R_1+R_2}{r} + \dfrac{R_1R_2}{r^2}\right)$. Check: $A + B_1 + B_2 + C = a$.	115
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D+R)^2(D+r)^2}$	$A + (K - L\tau)\,e^{-R\tau} + (k - l\tau)\,e^{-r\tau}$, where: $A = \dfrac{f}{R^2r^2}$, $K = \dfrac{aR^2(R-3r) + 2bRr - c(R+r) + 2d - \dfrac{f}{R^2}(3R-r)}{(R-r)^3}$, $L = \dfrac{aR^3 - bR^2 + cR - d + f/R}{(R-r)^2}$, and analogous formulae for k, l. Check: $A + K + k = a$, $L + l + KR + kr = 2a(R+r) - b$.	116
$\dfrac{aD^4+bD^3+cD^2-dD+f}{(D+R)(D+r)^3}$	$A + B\,e^{-R\tau} + (C + E\tau - \frac{1}{2}G\tau^2)\,e^{-r\tau}$, where: $A = \dfrac{f}{Rr^3}$, $\quad B = \dfrac{aR^3 - bR^2 + cR - d + f/R}{(R-r)^3}$, $C = \dfrac{-ar(3R^2-3Rr+r^2)+bR^2-cR+d-\dfrac{f}{r^3}(R^2-3Rr+3r^2)}{(R-r)^3}$, $E = \dfrac{ar^2(3R-2r) - br(2R-r) + cR - d + \dfrac{f}{r^2}(2r-R)}{(R-r)^2}$, $G = \dfrac{ar^3 - br^2 + cr - d + \dfrac{f}{r}}{R-r}$. Check: $A + B + C = a$, $ARr + E(R-r) - G = c - 2br + 3ar^2$.	117

TABLE 3—*continued*

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{(D + r)^4}$	$\dfrac{f}{r^4} + \left\{\left(a - \dfrac{f}{r^4}\right) - \left(3ar - b + \dfrac{f}{r^3}\right)\tau + \tfrac{1}{2}\left(3ar^2 - 2br + c - \dfrac{f}{r^2}\right)\tau^2 - \tfrac{1}{6}\left(ar^3 - br^2 + cr - d + \dfrac{f}{r}\right)\tau^3\right\} \mathrm{e}^{-r\tau}$	118
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{(D + R_1)(D + R_2)(D^2 + j^2)}$	$A + B_1\,\mathrm{e}^{-R_1\tau} - B_2\,\mathrm{e}^{-R_2\tau} + C\cos j\tau - E\dfrac{\sin j\tau}{j}$, where: $A = \dfrac{f}{R_1R_2j^2}$, $B_1 = \dfrac{aR_1^3 - bR_1^2 + cR_1 - d + f/R_1}{(R_1 - R_2)(R_1^2 + j^2)}$, $B_2 = \dfrac{aR_2^3 - bR_2^2 + cR_2 - d + f/R_2}{(R_1 - R_2)(R_2^2 + j^2)}$, $C = \dfrac{(j^2 - R_1R_2)(aj^2 - c + f/j^2) + (R_1 + R_2)(bj^2 - d)}{(R_1^2 + j^2)(R_2^2 + j^2)}$, $E = \dfrac{j^2(R_1 + R_2)(aj^2 - c + f/j^2) - (j^2 - R_1R_2)(bj^2 - d)}{(R_1^2 + j^2)(R_2^2 + j^2)}$. Check: $A + B_1 - B_2 + C = a$, $AR_1 - B_2(R_1 - R_2) + CR_1 - E = b - aR_2$.	119
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{(D + R)^2(D^2 + j^2)}$	$A + (B - B'\tau)\,\mathrm{e}^{-R\tau} + C\cos j\tau - E\dfrac{\sin j\tau}{j}$, where: $A = \dfrac{f}{R^2j^2}$, $B' = \dfrac{aR^3 - bR^2 + cR - d + f/R}{R^2 + j^2}$, $B = \dfrac{aR^2(R^2 + 3j^2) - 2bRj^2 + c(j^2 - R^2) + 2dR - f(3 + j^2/R^2)}{(R^2 + j^2)^2}$, $C = \dfrac{(j^2 - R^2)(aj^2 - c + f/j^2) + 2R(bj^2 - d)}{(R^2 + j^2)^2}$, $E = \dfrac{2Rj^2(aj^2 - c + f/j^2) - (j^2 - R^2)(bj^2 - d)}{(R^2 + j^2)^2}$. Check: $A + B + C = a$, $BR + B' + E = 2aR - b$.	120

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D+R_1)(D+R_2)(D^2+2rD+h)}$	$A+B_1\,\mathrm{e}^{-R_1\tau}-B_2\,\mathrm{e}^{-R_2\tau}+\left(l\cos j\tau+n\dfrac{\sin j\tau}{j}\right)\mathrm{e}^{-r\tau}$, where: $A=\dfrac{f}{R_1R_2h}$, $B_1=\dfrac{aR_1^3-bR_1^2+cR_1-d+f/R_1}{(R_1-R_2)M_1}$, $B_2=\dfrac{aR_2^3-bR_2^2+cR_2-d+f/R_2}{(R_1-R_2)M_2}$, $M_1M_2l=a\{h^2-hR_1R_2-2rh(R_1+R_2)+4r^2R_1R_2\}+$ $+b\{h(R_1+R_2)-2rR_1R_2\}+c(R_1R_2-h)+$ $+d(2r-R_1-R_2)+f\left\{1-\dfrac{(R_1-2r)(R_2-2r)}{h}\right\}$, $M_1M_2n=a\{-h^2(R_1+R_2+r)+3hR_1R_2r+2r^2h(R_1+R_2)-$ $-4R_1R_2r^3\}+b\{h^2-R_1R_2h-rh(R_1+R_2)+$ $+2R_1R_2r^2\}+c\{h(R_1+R_2-r)-R_1R_2r\}+$ $+d\{(R_1-r)(R_2-r)+r^2-h\}+$ $+f\left\{3r-R_1-R_2-\dfrac{r}{h}(R_1-2r)(R_2-2r)\right\}$, and $M_1=R_1^2-2rR_1+h$, $M_2=R_2^2-2rR_2+h$. Check: $A+B_1-B_2+l=a$, $B_1R_1-B_2R_2+rl-n=a(2r+R_1+R_2)-b$.	121
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D+R)^2(D^2+2rD+h)}$	$A+(B-B'\tau)\,\mathrm{e}^{-R\tau}+\left(l\cos j\tau+n\dfrac{\sin j\tau}{j}\right)\mathrm{e}^{-r\tau}$, where: $A=\dfrac{f}{R^2h}$, $B'=\dfrac{aR^3-bR^2+cR-d+f/R}{M}$, $M^2B=aR^2(R^2-4Rr+3h)+2bR(Rr-h)+c(h-R^2)+$ $+2d(R-r)-f(3-4r/R+h/R^2)$, $M^2l=a\{h^2-hR(R+4r)+4R^2r^2\}+2bR(h-Rr)+$ $+c(R^2-h)-2d(R-r)+f\{1-(R-2r)^2/h\}$; $M^2n=a\{-h^2(2R+r)+hRr(3R+4r)-4R^2r^3\}+$ $+b\{h^2-hR(R+2r)+2R^2r^2\}+c\{h(2R-r)-R^2r\}+$ $+d\{(R-r)^2+r^2-h\}+f\{3r-2R-r(R-2r)^2/h\}$, and $M=R^2-2Rr+h$. Check: $A+B+l=a$, $BR+B'+rl-n=2a(R+r)-b$.	122

TABLE 3—*continued*

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D^2+J^2)(D^2+j^2)}$	$\dfrac{f}{J^2j^2}+\dfrac{1}{J^2-j^2}\left\{\left(aJ^2-c+\dfrac{f}{J^2}\right)\cos J\tau+(bJ^2-d)\dfrac{\sin J\tau}{J}\right.$ $\left.-\left(aj^2-c+\dfrac{f}{j^2}\right)\cos j\tau-(bj^2-d)\dfrac{\sin j\tau}{j}\right\}$	123
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D^2+j^2)^2}$	$\dfrac{f}{j^4}+\left\{\left(a-\dfrac{f}{j^4}\right)+\tfrac{1}{2}\left(b-\dfrac{d}{j^2}\right)\tau\right\}\cos j\tau+$ $+\tfrac{1}{2}\left\{\left(b+\dfrac{d}{j^2}\right)-\left(aj^2-c+\dfrac{f}{j^2}\right)\tau\right\}\dfrac{\sin j\tau}{j}$	124
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D^2+2RD+H)(D^2+j^2)}$	$A+\left(L\cos J\tau+N\dfrac{\sin J\tau}{J}\right)e^{-R\tau}+l\cos j\tau+n\dfrac{\sin j\tau}{j}$, where: $A=\dfrac{f}{Hj^2}$, $QL=a\{H(H-j^2)+4R^2j^2\}-2bRj^2-c(H-j^2)+$ $+2dR+f\left(1-\dfrac{4R^2+j^2}{H}\right)$, $QN=aR\{j^2(3H-4R^2)-H^2\}+b\{H(H-j^2)+2R^2j^2\}-$ $-cR(H+j^2)+d(2R^2-H+j^2)+fR\left(3-\dfrac{4R^2+j^2}{H}\right)$, $Ql=(j^2-H)\left(aj^2-c+\dfrac{f}{j^2}\right)+2R(bj^2-d)$, $Qn=-2Rj^2\left(aj^2-c+\dfrac{f}{j^2}\right)+(j^2-H)(bj^2-d)$, and $Q=(j^2-H)^2+4R^2j^2$. Check: $A+L+l=a$, $N+n-RL=b-2aR$.	125
$\dfrac{aD^4+bD^3+cD^2+dD+f}{(D^2+2RD+H)(D^2+2rD+h)}$	$A+\left(L\cos J\tau+N\dfrac{\sin J\tau}{J}\right)e^{-R\tau}+\left(l\cos j\tau+n\dfrac{\sin j\tau}{j}\right)e^{-r\tau}$, where: $QL=a\{H(H-h)+4R(Rh-rH)\}-2b(Rh-rH)-$ $-c(H-h)+2d(R-r)+\dfrac{f}{H}\{H-h-4R(R-r)\}$, $QN=a\{(3H-4R^2)(Rh-rH)-H^2(R-r)\}+$ $+b\{H(H-h)+2R(Rh-rH)\}-$ $-c\{(H+h)R-2rH\}-d\{H-h-2R(R-r)\}+$ $+\dfrac{f}{H}\{R(H-h)+2(R-r)(H-2R^2)\}$, (analogous expressions for l and n) $A=\dfrac{f}{Hh}$, and $Q=(H-h)^2+4(R-r)(Rh-rH)$. Check: $A+L+l=a$, $N+n-RL-rl=b-2a(R+r)$.	126

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{(D^2 + 2rD + h)^2}$	$A + \left\{(B + B'\tau)\cos j\tau + (C + C'\tau)\dfrac{\sin j\tau}{j}\right\} e^{-r\tau}$, where: $A = \dfrac{f}{h^2}$, $\quad B = a - \dfrac{f}{h^2}$, $2j^2B' = ar(4r^2 - 3h) + b(h - 2r^2) + cr - d + fr/h$, $2j^2C = ar(2r^2 - 3h) + bh - cr + d + fr\left(\dfrac{2r^2}{h^2} - \dfrac{3}{h}\right)$, $2C' = a(4r^2 - h) - 2br + c - f/h$. Check: $Br - B' - C = 4ar - b$; $2C'r - B'h - Ch = (2ar - b)(2r^2 + h) + cr$.	127
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{D(D + R_1)(D + R_2)(D + R_3)}$	$A\tau + B + C_1 e^{-R_1\tau} + C_2 e^{-R_2\tau} + C_3 e^{-R_3\tau}$, where: $A = \dfrac{f}{R_1R_2R_3}$, $\quad B = \dfrac{d}{R_1R_2R_3} - \dfrac{f}{R_1R_2R_3}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right)$ $C_1 = \dfrac{aR_1^2 - bR_1 + c - d/R_1 + f/R_1^2}{(R_1 - R_2)(R_1 - R_3)}$, and similar formulae for C_2, C_3. Check: $B + C_1 + C_2 + C_3 = a$.	128
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{D(D + R)(D + r)^2}$	$A\tau + B + C\, e^{-R\tau} + (E + G\tau)\, e^{-r\tau}$, where: $A = \dfrac{f}{Rr^2}$, $\quad B = \dfrac{d}{Rr^2} - \dfrac{f}{Rr^2}\left(\dfrac{1}{R} + \dfrac{2}{r}\right)$. $C = \dfrac{aR^2 - bR + c - d/R + f/R^2}{(R - r)^2}$, $G = \dfrac{ar^2 - br + c - d/r + f/r^2}{R - r}$, $E = \dfrac{ar(r - 2R) + bR - c + d(2r - R)/r^2 + f(2R - 3r)/r^3}{(R - r)^2}$. Check: $B + C + E = a$, $\quad A + BR + E(R - r) + G = b - 2ar$.	129
$\dfrac{aD^4 + bD^3 + cD^2 + dD + f}{D(D + r)^3}$	$\dfrac{f}{r^3}\tau + \left(\dfrac{d}{r^3} - \dfrac{3f}{r^4}\right) + \left\{\left(a - \dfrac{d}{r^3} + \dfrac{3f}{r^4}\right) - \left(2ar - b + \dfrac{d}{r^2} - \dfrac{2f}{r^3}\right)\tau + \right.$ $\left. + \tfrac{1}{2}\left(ar^2 - br + c - \dfrac{d}{r} + \dfrac{f}{r^2}\right)\tau^2\right\} e^{-r\tau}$	130

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D(D+R)(D^2+j^2)}$	$A\tau+B+C\,\mathrm{e}^{-R\tau}+E\cos j\tau-G\dfrac{\sin j\tau}{j}$, where: $A=\dfrac{f}{Rj^2}$, $B=\dfrac{d}{Rj^2}-\dfrac{f}{R^2j^2}$, $C=\dfrac{aR^2-bR+c-d/R+f/R^2}{R^2+j^2}$, $E=\dfrac{aj^2+bR-c-dR/j^2+f/j^2}{R^2+j^2}$, $G=\dfrac{aRj^2-bj^2-cR+d+fR/j^2}{R^2+j^2}$. Check: $B+C+E=a$, $A-CR-G=b-aR$.	131
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D(D+R)(D^2+2rD+h)}$	$A\tau+B+C\,\mathrm{e}^{-R\tau}+\left(l\cos j\tau+n\dfrac{\sin j\tau}{j}\right)\mathrm{e}^{-r\tau}$, where: $A=\dfrac{f}{Rh}$, $B=\dfrac{d}{Rh}-\dfrac{f}{Rh}\left(\dfrac{1}{R}+\dfrac{2r}{h}\right)$, $C=\dfrac{aR^2-bR+c-d/R+f/R^2}{M}$, $Ml=a(h-2Rr)+bR-c+\dfrac{d}{h}(2r-R)+\dfrac{f}{h^2}(h+2Rr-4r^2)$, $Mn=a(2Rr^2-hR-hr)+b(h-Rr)+c(R-r)+$ $+\dfrac{d}{h}(2r^2-Rr-h)+\dfrac{f}{h^2}\{2r^2(R-2r)+h(3r-R)\}$, and $M=R^2-2Rr+h$. Check: $B+C+l=a$, $A+BR+l(R-r)+n=b-2ar$.	132
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D^2(D+R)(D+r)}$	$\tfrac{1}{2}A\tau^2+B\tau+C+K\,\mathrm{e}^{-R\tau}-k\,\mathrm{e}^{-r\tau}$, where: $A=\dfrac{f}{Rr}$ $B=\dfrac{d}{Rr}-\dfrac{f}{Rr}\left(\dfrac{1}{R}+\dfrac{1}{r}\right)$, $RrC=c-d\left(\dfrac{1}{R}+\dfrac{1}{r}\right)+\dfrac{f}{Rr}\left(\dfrac{r}{R}+1+\dfrac{R}{r}\right)$. $K=\dfrac{aR-b+c/R-d/R^2+f/R^3}{R-r}$, $k=\dfrac{ar-b+c/r-d/r^2+f/r^3}{R-r}$. Check: $C+K-k=a$.	133

TABLE 3—*continued*

(4) *Fractions with denominators quartic in D*—continued

Operational expression $\varphi(D)$	Equivalent function $F(\tau)$	Number
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D^2(D+r)^2}$	$\dfrac{f}{2r^2}\tau^2+\left(\dfrac{d}{r^2}-\dfrac{2f}{r^3}\right)\tau+\left(\dfrac{c}{r^2}-\dfrac{2d}{r^3}+\dfrac{3f}{r^4}\right)+$ $+\left\{\left(a-\dfrac{c}{r^2}+\dfrac{2d}{r^3}-\dfrac{3f}{r^4}\right)-\left(ar-b+\dfrac{c}{r}-\dfrac{d}{r^2}+\dfrac{f}{r^3}\right)\tau\right\}e^{-r\tau}$	134
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D^2(D^2+j^2)}$	$\dfrac{f}{2j^2}\tau^2+\dfrac{d}{j^2}\tau+\left(\dfrac{c}{j^2}-\dfrac{f}{j^4}\right)+\left(a-\dfrac{c}{j^2}+\dfrac{f}{j^4}\right)\cos j\tau+$ $+\left(b-\dfrac{d}{j^2}\right)\dfrac{\sin j\tau}{j}$	135
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D^2(D^2+2rD+h)}$	$\frac{1}{2}A\tau^2+B\tau+C+\left(l\cos j\tau+n\dfrac{\sin j\tau}{j}\right)e^{-r\tau}$, where: $A=\dfrac{f}{h}$, $B=\dfrac{d}{h}-\dfrac{2fr}{h^2}$, $C=\dfrac{c}{h}-\dfrac{2dr}{h^2}+\dfrac{f}{h^3}(4r^2-h)$, $l=a-\dfrac{c}{h}+\dfrac{2dr}{h^2}-\dfrac{f}{h^3}(4r^2-h)$, $n=-ar+b-\dfrac{cr}{h}+\dfrac{d}{h^2}(2r^2-h)+\dfrac{fr}{h^3}(3h-4r^2)$. Check: $C+l=a$, $B-lr+n=b-2ar$.	136
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D^3(D+R)}$	$\dfrac{f}{6R}\tau^3+\frac{1}{2}\left(\dfrac{d}{R}-\dfrac{f}{R^2}\right)\tau^2+\left(\dfrac{c}{R}-\dfrac{d}{R^2}+\dfrac{f}{R^3}\right)\tau+$ $+\left(\dfrac{b}{R}-\dfrac{c}{R^2}+\dfrac{d}{R^3}-\dfrac{f}{R^4}\right)+\left(a-\dfrac{b}{R}+\dfrac{c}{R^2}-\dfrac{d}{R^3}+\dfrac{f}{R^4}\right)e^{-R\tau}$	137
$\dfrac{aD^4+bD^3+cD^2+dD+f}{D^4}$	$a+b\tau+\frac{1}{2}c\tau^2+\frac{1}{6}d\tau^3+\frac{1}{24}f\tau^4$	138

TABLE 4

Operational Fractions, with Denominators Quintic in D, Reduced to Simpler Fractions

(A) No apparent zero roots in denominators	Number
Denominator with at least one single real root: $$\frac{aD^5 + bD^4 + cD^3 + dD^2 + fD + g}{(D + R)(D^4 + BD^3 + CD^2 + ED + G)} = \frac{kD}{D + R} + \frac{a_1D^4 + b_1D^3 + c_1D^2 + d_1D + f_1}{D^4 + BD^3 + CD^2 + ED + G},$$ where: $$k = \frac{aR^2 - bR + c - d/R + f/R^2 - g/R^3}{R^2 - BR + C - E/R + G/R^2}, \qquad f_1 = g/R,$$ $$d_1 = \frac{f - f_1 - kG}{R}, \qquad c_1 = \frac{d - d_1 - kE}{R}, \qquad b_1 = \frac{c - c_1 - kC}{R}, \qquad a_1 = \frac{b - b_1 - kB}{R}.$$ Check: $k + a_1 = a$.	139
Denominator with a double real and a triple real root: $$\frac{aD^5 + bD^4 + cD^3 + dD^2 + fD + g}{(D + R)^2(D + r)^3} = \frac{a_1D^2 + b_1D + c_1}{(D + R)^2} + \frac{a_2D}{D + r} + \frac{b_2D}{(D + r)^2} + \frac{c_2D}{(D + r)^3},$$ where: $$c_1 = g/r^3, \qquad c_2 = \frac{ar^4 - br^3 + cr^2 - dr + f - g/r}{(R - r)^2},$$ $$b_2(R - r)^3 = 2ar^3(r - 2R) + br^2(3R - r) - 2cRr + d(R + r) - 2f + \frac{g}{r^2}(3r - R),$$ $$a_2(R - r)^4 = ar^2(6R^2 - 4Rr + r^2) - 3bR^2r + cR(R + 2r) - d(2R + r) + 3f - \frac{g}{r^3}(R^2 - 4Rr + 6r^2),$$ $$a_1 = a - a_2, \qquad b_1 = b - 3ar - a_2(2R - r) - b_2.$$	140
Denominator with a pair of imaginary roots and a triple real root: $$\frac{aD^5 + bD^4 + cD^3 + dD^2 + fD + g}{(D^2 + J^2)(D + r)^3} = \frac{a_1D^2 + b_1D + c_1}{D^2 + J^2} + \frac{a_2D}{D + r} + \frac{b_2D}{(D + r)^2} + \frac{c_2D}{(D + r)^3},$$ where: $$c_1 = g/r^3, \qquad c_2 = \frac{ar^4 - br^3 + cr^2 - dr + f - g/r}{J^2 + r^2},$$ $$b_2(J^2 + r^2)^2 = -2ar^3(2J^2 + r^2) + br^2(3J^2 + r^2) - 2crJ^2 + d(J^2 - r^2) + 2fr - \frac{g}{r^2}(J^2 + 3r^2),$$ $$a_2(J^2 + r^2)^3 = ar^2(6J^4 + 3J^2r^2 + r^4) - brJ^2(3J^2 - r^2) + cJ^2(J^2 - 3r^2) + dr(3J^2 - r^2) - f(J^2 - 3r^2) - \frac{g}{r^3}(J^4 + 3J^2r^2 + 6r^4),$$ $$a_1 = a - a_2, \qquad b_1 = b - 3ar + a_2r - b_2.$$	141

(A) No apparent zero roots—*continued*	Number

Denominator with a triple real root and any two remaining roots:

$$\frac{aD^5 + bD^4 + cD^3 + dD^2 + fD + g}{(D^2 + 2RD + H)(D + r)^3} = \frac{a_1D^2 + b_1D + c_1}{D^2 + 2RD + H} + \frac{a_2D}{D + r} + \frac{b_2D}{(D + r)^2} + \frac{c_2D}{(D + r)^3},$$

where:

$$c_1 = g/r^3, \qquad c_2 = \frac{ar^4 - br^3 + cr^2 - dr + f - g/r}{H - 2Rr + r^2},$$

$$b_2(H - 2Rr + r^2)^2 = -2ar^3(2H - 3Rr + r^2) + 6r^2(3H - 4Rr + r^2) + 2cr(Rr - H) + d(H - r^2) - 2f(R - r) - \frac{g}{r^2}(H - 4Rr + 3r^2),$$

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$$a_2(H - 2Rr + r^2)^3 = ar^2\{6H^2 - Hr(16R - 3r) + r^2(12R^2 - 6Rr + r^2)\} - br\{3H^2 - Hr(6R + r) + 4R^2r^2\} + c(H^2 - 3r^2H + 2Rr^3) - d\{H(2R - 3r) + r^3\} + f(4R^2 - 6Rr + 3r^2 - H) - \frac{g}{r^3}\{H^2 - 3Hr(2R - r) + 2r^2(6R^2 - 8Rr + 3r^2)\},$$

$$a_1 = a - a_2, \qquad b_1 = b - 3ar - a_2(2R - r) - b_2.$$

(B) Single or multiple zero roots in denominators

Denominator with a single zero root and any non-zero remaining roots:

$$\frac{aD^5 + bD^4 + cD^3 + dD^2 + fD + g}{D(D^4 + BD^3 + CD^2 + ED + G)} = \frac{g/G}{D} + \frac{aD^4 + \left(b - \frac{g}{G}\right)D^3 + \left(c - \frac{gB}{G}\right)D^2 + \left(d - \frac{gC}{G}\right)D + \left(f - \frac{gE}{G}\right)}{D^4 + BD^3 + CD^2 + ED + G}$$

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Denominator with a double zero root and a triple real root:

$$\frac{aD^5 + bD^4 + cD^3 + dD^2 + fD + g}{D^2(D + r)^3} = \frac{g/r^3}{D^2} + \left(\frac{f}{r^3} - \frac{3g}{r^4}\right)\frac{1}{D} + \left(\frac{d}{r^3} - \frac{3f}{r^4} + \frac{6g}{r^5}\right) + \left(a - \frac{d}{r^3} + \frac{3f}{r^4} - \frac{6g}{r^5}\right)\frac{D}{D + r} - \left(2ar - b + \frac{d}{r^2} - \frac{2f}{r^3} + \frac{3g}{r^4}\right)\frac{D}{(D + r)^2} + \left(ar^2 - br + c - \frac{d}{r} + \frac{f}{r^2} - \frac{g}{r^3}\right)\frac{D}{(D + r)^3}$$

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TABLE 4—*continued*

(B) Multiple zero roots—*continued*	Number
Denominator with a triple zero root and a double real root: $$\frac{aD^5+bD^4+cD^3+dD^2+fD+g}{D^3(D+r)^2}=\frac{g/r^2}{D^3}+\left(\frac{f}{r^2}-\frac{2g}{r^3}\right)\frac{1}{D^2}+\left(\frac{d}{r^2}-\frac{2f}{r^3}+\frac{3g}{r^4}\right)\frac{1}{D}+$$ $$+\frac{aD^2+\left(b-\frac{d}{r^2}+\frac{2f}{r^3}-\frac{3g}{r^4}\right)D+\left(c-\frac{2d}{r}+\frac{3f}{r^2}-\frac{4g}{r^3}\right)}{(D+r)^2}$$	145
Denominator with a triple zero root and a pair of imaginary roots: $$\frac{aD^5+bD^4+cD^3+dD^2+fD+g}{D^3(D^2+J^2)}=\frac{g/J^2}{D^3}+\frac{f/J^2}{D^2}+\left(\frac{d}{J^2}-\frac{g}{J^4}\right)\frac{1}{D}+$$ $$+\frac{aD^2+\left(b-\frac{d}{J^2}+\frac{g}{J^4}\right)D+\left(c-\frac{f}{J^2}\right)}{D^2+J^2}$$	146
Denominator with a triple zero root and any two remaining roots: $$\frac{aD^5+bD^4+cD^3+dD^2+fD+g}{D^3(D^2+2RD+H)}=\frac{g/H}{D^3}+\left(\frac{f}{H}-\frac{2gR}{H^2}\right)\frac{1}{D^2}+\left(\frac{d}{H}-\frac{2fR}{H^2}+g\,\frac{4R^2-H}{H^3}\right)\frac{1}{D}+$$ $$+\frac{aD^2+\left(b-\frac{d}{H}+\frac{2fR}{H^2}-g\,\frac{4R^2-H}{H^3}\right)D+\left(c-\frac{2dR}{H}+f\,\frac{4R^2-H}{H^2}-4gR\,\frac{2R^2-H}{H^3}\right)}{D^2+2RD+H}$$	147
Denominator with a quadruple zero root and a single real root: $$\frac{aD^5+bD^4+cD^3+dD^2+fD+g}{D^4(D+R)}=\frac{g/R}{D^4}+\left(\frac{f}{R}-\frac{g}{R^2}\right)\frac{1}{D^3}+\left(\frac{d}{R}-\frac{f}{R^2}+\frac{g}{R^3}\right)\frac{1}{D^2}+$$ $$+\left(\frac{c}{R}-\frac{d}{R^2}+\frac{f}{R^3}-\frac{g}{R^4}\right)\frac{1}{D}+\left(\frac{b}{R}-\frac{c}{R^2}+\frac{d}{R^3}-\frac{f}{R^4}+\frac{g}{R^5}\right)+$$ $$+\left(a-\frac{b}{R}+\frac{c}{R^2}-\frac{d}{R^3}+\frac{f}{R^4}-\frac{g}{R^5}\right)\frac{D}{D+R}$$	148

TABLE 5

Operational Fractions, with Denominators Sextic in D, Reduced to Simpler Fractions

(A) No apparent zero roots in denominators	Number

Denominator with at least one single pair of imaginary roots:

$$\frac{aD^6 + bD^5 + cD^4 + dD^3 + fD^2 + gD + k}{(D^2 + J^2)(D^4 + BD^3 + CD^2 + ED + G)} = \frac{a_1D^2 + b_1D}{D^2 + J^2} + \frac{a_2D^4 + b_2D^3 + c_2D^2 + d_2D + f_2}{D^4 + BD^3 + CD^2 + ED + G},$$

where:

$$a_1 = \frac{(aJ^4 - cJ^2 + f - k/J^2)(J^4 - CJ^2 + G) + (bJ^4 - dJ^2 + g)(BJ^2 - E)}{(J^4 - CJ^2 + G)^2 + J^2(BJ^2 - E)^2},$$

$$b_1 = \frac{(bJ^4 - dJ^2 + g)(J^4 - CJ^2 + G) - J^2(aJ^4 - cJ^2 + f - k/J^2)(BJ^2 - E)}{(J^4 - CJ^2 + G)^2 + J^2(BJ^2 - E)^2},$$

$$a_2 = a - a_1, \qquad b_2 = b - b_1 - Ba_1, \qquad f_2 = k/J^2,$$

$$c_2 = c - aJ^2 - Bb_1 + (J^2 - C)a_1, \qquad d_2 = d - bJ^2 + (J^2 - C)b_1 + (BJ^2 - E)a_1.$$

Check: $Gb_1 + J^2d_2 = g$.

Number: 149

Denominator with at least one double real root:

$$\frac{aD^6 + bD^5 + cD^4 + dD^3 + fD^2 + GD + k}{(D + R)^2(D^4 + BD^3 + CD^2 + ED + G)} = \frac{a_1D^2 + b_1D}{(D + R)^2} + \frac{a_2D^4 + b_2D^3 + c_2D^2 + d_2D + f_2}{D^4 + BD^3 + CD^2 + ED + G},$$

where:

$$a_1 = \frac{a\Lambda + b\alpha - c\beta + d\gamma - f\delta + g\theta - k\Gamma}{(R^4 - BR^3 + CR^2 - ER + G)^2},$$

$$b_1 = R^2\frac{-a\alpha + b\beta - c\gamma + d\delta - f\theta + g\Gamma - kK}{(R^4 - BR^3 + CR^2 - ER + G)^2},$$

$$\Lambda = R^4(R^4 - 2BR^3 + 3CR^2 - 4ER + 5G), \qquad \alpha = R^3(BR^3 - 2CR^2 + 3ER - 4G),$$

$$\beta = R^2(R^4 - CR^2 + 2ER - 3G), \qquad \gamma = R(2R^4 - BR^3 + ER - 2G),$$

$$\delta = 3R^4 - 2BR^3 + CR^2 - G, \qquad \theta = 4R^3 - 3BR^2 + 2CR - E,$$

$$\Gamma = 5R^2 - 4BR + 3C - 2E/R + G/R^2 \qquad K = 6R - 5B + 4C/R - 3E/R^2 + 2G/R^3,$$

$$a_2 = a - a_1, \qquad b_2 = (b - 2aR) + (2R - B)a_1 - b_1,$$

$$c_2 = (c - 2bR + 3aR^2) + (2R - B)b_1 - (3R^2 - 2BR + C)a_1,$$

$$d_2 = (d - 2cR + 3bR^2 - 4aR^3) - (3R^2 - 2BR + C)b_1 + (4R^3 - 3BR^2 + 2CR - E)a_1,$$

$$f_2 = k/R^2.$$

Check: $b_1G + d_2R^2 + 2f_2R = g$.

Number: 150

(A) No apparent zero roots—*continued*	Number
Denominator factorized into a quadratic and a quartic with no common roots: $$\frac{aD^6+bD^5+cD^4+dD^3+fD^2+gD+k}{(D^2+2RD+H)(D^4+BD^3+CD^2+ED+G)}=\frac{a_1D^2+b_1D}{D^2+2RD+H}+\frac{a_2D^4+b_2D^3+c_2D^2+d_2D+f_2}{D^4+BD^3+CD^2+ED+G},$$ where: $$a_1=\frac{aA+b\alpha-c\beta+d\gamma-f\delta+g\theta-k\Gamma}{H(\theta^2-\delta\Gamma)},\qquad b_1=\frac{-a\alpha+b\beta-c\gamma+d\delta-f\theta+g\Gamma-kK}{\theta^2-\delta\Gamma},$$ $$A=H^4-2BRH^3+CH^2(4R^2-H)-4ERH(2R^2-H)+G(H^2-12R^2H+16R^4),$$ $$\alpha=BH^3-2CRH^2+EH(4R^2-H)-4GR(2R^2-H),\quad \beta=H^3-CH^2+2ERH-G(4R^2-H),$$ $$\gamma=2RH^2-BH^2+EH-2GR,\qquad \delta=H(4R^2-H)-2BRH+CH-G,$$ $$\theta=4R(2R^2-H)-B(4R^2-H)+2CR-E,$$ $$H\Gamma=(H^2-12R^2H+16R^4)-4BR(2R^2-H)+C(4R^2-H)-2ER+G,$$ $$H^2K=2R(3H^2-16R^2H+16R^4)-B(H^2-12R^2H+16R^4)+4CR(2R^2-H)-E(4R^2-H)+2GR.$$ $$a_2=a-a_1,\qquad b_2=(b-2aR)+(2R-B)a_1-b_1,\qquad f_2=k/H,$$ $$c_2=\{c-2Rb+(4R^2-H)a\}+(2R-B)b_1+(H-4R^2+2BR-C)a_1,$$ $$d_2=\{d-2Rc+(4R^2-H)b-4R(2R^2-H)a\}+(H-4R^2+2BR-C)b_1+$$ $$+\{4R(2R^2-H)-B(4R^2-H)+2CR-E\}a_1.$$ Check: $b_1G+d_2H+2f_2R=g$.	151
Denominator with two triple real roots: $$\frac{aD^6+bD^5+cD^4+dD^3+fD^2+gD+k}{(D+R)^3(D+r)^3}=A+\frac{a_1D}{D+R}+\frac{b_1D}{(D+R)^2}+\frac{c_1D}{(D+R)^3}+$$ $$+\frac{a_2D}{D+r}+\frac{b_2D}{(D+r)^2}+\frac{c_2D}{(D+r)^3},$$ where: $$(R-r)^5a_1=aR^3(R^2-5Rr+10r^2)-6bR^2r^2+3cRr(R+r)-d(R^2+4Rr+r^2)+3f(R+r)-$$ $$-6g+\frac{k}{R^3}(10R^2-5Rr+r^2),$$ $$(R-r)^4b_1=aR^4(5r-2R)+bR^3(R-4r)+3cR^2r-dR(R+2r)+f(2R+r)-3g+\frac{k}{R^2}(4R-r),$$ $$(R-r)^3c_1=aR^5-bR^4+cR^3-dR^2+fR-g+k/R,\qquad A=k/R^3r^3,$$ and analogous formulae for a_2, b_2, c_2. Check: $A+a_1+a_2=a$, $\quad A(R+r)+b_1+b_2+a_1r+a_2R=b-2a(R+r)$.	152

TABLE 5—*continued*

(A) No apparent zero roots—*continued*	Number
Denominator with a single and a quintuple root: $\dfrac{aD^6 + bD^5 + cD^3 + dD^2 + fD^2 + gD + k}{(D+R)(D+r)^5} = \dfrac{AD}{D+R} + \dfrac{a_1D^5 + b_1D^4 + c_1D^3 + d_1D^2 + f_1D + g_1}{(D+r)^5},$ where: $A = \dfrac{aR^5 - bR^4 + cR^3 - dR^2 + fR - g + k/R}{(R-r)^5}, \quad g_1 = \dfrac{k}{R}, \quad f_1 = \dfrac{g - g_1 - Ar^5}{R},$ $d_1 = \dfrac{f - f_1 - 5Ar^4}{R}, \quad c_1 = \dfrac{d - d_1 - 10Ar^3}{R}, \quad b_1 = \dfrac{c - c_1 - 10Ar^2}{R}, \quad a_1 = \dfrac{b - b_1 - 5Ar}{R},$ Check: $A + a_1 = a$.	153
(B) Single or multiple zero roots in denominators	
Denominator with a single zero and a quintuple root: $\dfrac{aD^6 + bD^5 + cD^4 + dD^3 + fD^2 + gD + k}{D(D+r)^5} =$ $\dfrac{k/r^5}{D} + \dfrac{aD^5 + (b - k/r^5)D^4 + (c - 5k/r^4)D^3 + (d - 10k/r^3)D^2 + (f - 10k/r^2)D + (g - 5k/r)}{(D+r)^5}$	154
Denominator with a single zero, single real and a double pair of any other roots: $\dfrac{aD^6 + bD^5 + cD^4 + dD^3 + fD^2 + gD + k}{D(D+R)(D^2 + 2rD + h)^2} = \dfrac{A}{D} + \dfrac{BD}{D+R} + \dfrac{a_1D^4 + b_1D^3 + c_1D^2 + d_1D + f_1}{(D^2 + 2rD + h)^2},$ where: $A = \dfrac{k}{Rh^2}, \quad B = \dfrac{aR^4 - bR^3 + cR^2 - dR + f - g/R + k/R^2}{(R^2 - 2Rr + h)^2}, \quad f_1 = \dfrac{g - Ah(h + 4Rr)}{R},$ $d_1 = \dfrac{f - f_1 - 2A(Rh + 2rh + 2Rr^2) - Bh^2}{R}, \quad c_1 = \dfrac{d - d_1 - 2A(h + 2Rr + 2r^2) - 4Brh}{R},$ $b_1 = \dfrac{c - c_1 - A(R + 4r) - 2B(h + 2r^2)}{R}, \quad a_1 = \dfrac{b - b_1 - A - 4Br}{R}.$ Check: $B + a_1 = a$.	155

TABLE 5—*continued*

(B) Multiple zero roots —*continued*	Number
Denominator with a double zero, single real and triple real root: $$\frac{aD^6+bD^5+cD^4+dD^3+fD^2+gD+k}{D^2(D+R)(D+r)^3}=\frac{A}{D^2}+\frac{B}{D}+\frac{CD}{D+R}+\frac{a_1D^3+b_1D^2+c_1D+d_1}{(D+r)^3},$$ where: $$A=\frac{k}{Rr^3},\qquad B=\frac{g}{Rr^3}-k\frac{3R+r}{R^2r^4},\qquad C=\frac{aR^3-bR^2+cR-d+f/R-g/R^2+k/R^3}{(R-r)^3},$$ $$d_1=\frac{f-3Ar(R+r)-Br^2(3R+r)}{R},\qquad c_1=\frac{d-d_1-A(R+3r)-3Br(R+r)-Cr^3}{R},$$ $$b_1=\frac{c-c_1-A-B(R+3r)-3Cr^2}{R},\qquad a_1=\frac{b-b_1-B-3Cr}{R}.$$ Check: $a_1+C=a$.	156
Denominator with a double zero and a double pair of any other roots: $$\frac{aD^6+bD^5+cD^4+dD^3+fD^2+gD+k}{D^2(D^2+2rD+h)^2}=\frac{A}{D^2}+\frac{B}{D}+$$ $$+\frac{aD^4+(b-B)D^3+(c-A-4Br)D^2+\{d-4Ar-2B(h+2r^2)\}D+\{f-2A(h+2r^2)-4Brh\}}{(D^2+2rD+h)^2},$$ $$A=\frac{k}{h^2},\qquad B=\frac{g}{h^2}-\frac{4kr}{h^3}.$$	157
Denominator with a triple zero and triple real root: $$\frac{aD^6+bD^5+cD^4+dD^3+fD^2+gD+k}{D^3(D+r)^3}=$$ $$\frac{A}{D^3}+\frac{B}{D^2}+\frac{C}{D}+\frac{aD^3+(b-C)D^2+(c-B-3Cr)D+(d-A-3Br-3Cr^2)}{(D+r)^3},$$ $$A=\frac{k}{r^3},\qquad B=\frac{g}{r^3}-\frac{3k}{r^4},\qquad C=\frac{f}{r^3}-\frac{3g}{r^4}+\frac{6k}{r^5}.$$	158

APPENDIX

Examples of Solutions of Differential Equations and Response Problems by Using the Formulae of Tables 1 to 5

The purpose of the following examples is to show how to apply the formulae correctly, to exhibit advantages in avoiding all the usual drudgery of resolving operational expressions into partial fractions, and finally to give a few interesting analytical discussions of response problems thus solved. Very simple, trivial or well-known problems have been excluded, but greatly complicated ones have also been avoided.

The introductory examples 1 and 2 treat simple differentiation and integration; examples 3 to 5 ordinary differential equations; examples 6 to 11 some response problems which may find immediate application in aeronautics or other branches of engineering. In all these examples the order of operational fractions never exceeds 4, but example 12 shows some applications of the more complex formulae of Tables 4 and 5 (fifth and sixth order fractions).

Example 1.—Find the third derivative of the function

$$\underline{F(\tau) = \tau\, e^{-R\tau}\left(\cos J\tau + \frac{R}{J}\sin J\tau\right).} \qquad \text{(A.1)}$$

Let us denote by $\varphi(D)$, $\varphi_1(D)$, $\varphi_2(D)$ and $\varphi_3(D)$ the operational equivalents of $F(\tau)$, $F'(\tau)$, $F''(\tau)$ and $F'''(\tau)$, respectively. Combining the formulae (24) and (28) of Table 1, we find

$$\varphi(D) = \frac{D^3 + 4RD^2 + (4R^2 - H)D}{(D^2 + 2RD + H)^2}. \qquad \text{(A.2)}$$

We now use the formula (*d*) of section 2 and, as $F(0) = 0$, we obtain

$$\varphi_1(D) = \frac{D^4 + 4RD^3 + (4R^2 - H)D^2}{(D^2 + 2RD + H)^2}, \qquad \text{(A.3)}$$

and it is seen that $F'(0) = \varphi_1(\infty) = 1$. Applying (*d*) again, we get

$$\varphi_2(D) = -H\frac{3D^3 + 4RD^2 + HD}{(D^2 + 2RD + H)^2}. \qquad \text{(A.4)}$$

We have $F''(0) = \varphi_2(\infty) = 0$ and, applying (*d*) once more, we obtain

$$\varphi_3(D) = -H\frac{3D^4 + 4RD^3 + HD^2}{(D^2 + 2RD + H)^2}. \qquad \text{(A.5)}$$

To determine $F'''(\tau)$, we look for an appropriate formula in Table 3, which is (127), and putting $a = 3$, $b = 4R$, $c = H$, $d = f = 0$, we find $A = 0$, $B = 3$, $B' = -2R$, $C = -3R$, $C' = R^2 - J^2$, and hence

$$\underline{F'''(\tau) = H\left\{\left(2R\tau - 3\right)\cos J\tau + \left(\frac{3R}{J} + \frac{J^2 - R^2}{J}\tau\right)\sin J\tau\right\} e^{-R\tau}.} \qquad \text{(A.6)}$$

If the first and second derivatives were also required, we could apply (127) to (A.3) and (A.4), which would yield

$$\left.\begin{aligned} F'(\tau) &= \left(\cos J\tau + \frac{R - H\tau}{J}\sin J\tau\right) e^{-R\tau} \\ F''(\tau) &= -H\left(\tau\cos J\tau + \frac{2 - R\tau}{J}\sin J\tau\right) e^{-R\tau} \end{aligned}\right\}, \qquad \text{(A.7)}$$

and all results can be obviously checked by direct differentiation. The advantage of the operational method is that the 3rd derivative could be found without determining the 1st and 2nd ones. Simple differentiation by use of operators is seldom shorter than the usual process, and cannot be generally recommended. The reader may notice, however, that each group of formulae in Table 2 contains a series of successive derivatives, and hence this table may often be useful for rapidly finding derivatives (or integrals) of such functions as appear therein.

Example 2.—Find the integral $I = \int_0^\tau F(\tau)\, d\tau$, the integrand $F(\tau)$ being the function (A.1) of the previous example.

Applying the formula (*e*) of section 2, we find the operational equivalent of I:

$$\varphi_1(D) = \frac{\varphi(D)}{D} = \frac{D^2 + 4RD + (4R^2 - H)}{(D^2 + 2RD + H)^2}, \qquad \text{(A.8)}$$

and then, putting in (127) $a = b = 0$, $c = 1$, $d = 4R$, $f = 4R^2 - H$, we obtain at once

$$I = \frac{4R^2 - H}{H^2} + \left\{\left(\frac{H - 4R^2}{H^2} - \frac{2R\tau}{H}\right)\cos J\tau + \left(R\,\frac{3H - 4R^2}{H^2} + \frac{H - 2R^2}{H}\,\tau\right)\frac{\sin J\tau}{J}\right\} e^{-R\tau}. \qquad \text{(A.9)}$$

This can be checked immediately by direct differentiation but, to determine the integral I by usual methods, without using operational formulae, would be a rather laborious procedure.

If the upper limit of integration were assumed to be (∞) right from the start, only the constant term A of I would have to be determined, and we would obtain

$$\int_0^\infty \tau\, e^{-R\tau}\left(\cos J\tau + \frac{R}{J}\sin J\tau\right) d\tau = \frac{3R^2 - J^2}{(R^2 + J^2)^2}. \qquad \text{(A.10)}$$

Example 3.—Solve the system of simultaneous differential equations

$$\left.\begin{aligned} \frac{dx}{d\tau} + \frac{dy}{d\tau} + J(y - x) &= 2aJ\cos J\tau \\ \frac{dy}{d\tau} - \frac{dx}{d\tau} - J(y + x) &= 2aJ\sin J\tau \end{aligned}\right\}, \qquad \text{(A.11)}$$

assuming initial conditions $x = y = 0$ at $\tau = 0$.

We follow the instruction of section 3, find the operational equivalents of forcing functions from formulae (15) and (19) of Table 1, and obtain the following subsidiary equations:

$$\left.\begin{aligned} (D - J)x + (D + J)y &= 2aJ\,\frac{D^2}{D^2 + J^2} \\ -(D + J)x + (D - J)y &= 2aJ^2\,\frac{D}{D^2 + J^2} \end{aligned}\right\}, \qquad \text{(A.12)}$$

The main determinant of these equations is $2(D^2 + J^2)$ and hence, if there were no forcing functions, we would have to deal with a simple undamped oscillating system, of natural frequency J. As the forcing functions are simple harmonic of the same frequency, we have a case of resonance, which it would not be too easy to solve by usual methods. Solving the subsidiary equations, however, and interpreting by means of the formula (124), we obtain at once

$$\left.\begin{aligned} \frac{x}{a}\left[= J\,\frac{D^3 - 2JD^2 - J^2D}{(D^2 + J^2)^2}\right] &= J\tau(\cos J\tau - \sin J\tau) = \theta(\cos\theta - \sin\theta) \\ \frac{y}{a}\left[= J\,\frac{D^3 + 2JD^2 - J^2D}{(D^2 + J^2)^2}\right] &= J\tau(\cos J\tau + \sin J\tau) = \theta(\cos\theta + \sin\theta) \end{aligned}\right\}, \qquad \text{(A.13)}$$

where $\theta = J\tau$, for abbreviation.

The solution is illustrated in Fig. 1. It is seen at once that $x = 0$ for $\theta = \pi(\frac{1}{4} + n)$, and $y = 0$ for $\theta = \pi(\frac{3}{4} + n)$, when n is an arbitrary positive integer. It is also found easily that the curves of x/a and y/a are both contained in the angle between two straight lines from the origin, of the slope $\pm\sqrt{2}$ against θ. The curve x/a touches the upper limiting line at an infinite number of points corresponding to $\theta = \pi(7/4 + 2n)$, the lower one similarly at $\theta = \pi(\frac{3}{4} + 2n)$. The same applies to the curve y/a, for which the points of contact occur at $\theta = \pi(\frac{1}{4} + 2n)$ and $\theta = \pi(5/4 + 2n)$ respectively.

The exact turning values of x and y can be found by determining the first derivatives (directly, or multiplying the operational equivalents by D and re-interpreting):

$$\left.\begin{aligned} \frac{1}{Ja}\frac{dx}{d\tau} &= (1 - \theta)\cos\theta - (1 + \theta)\sin\theta \\ \frac{1}{Ja}\frac{dy}{d\tau} &= (1 + \theta)\cos\theta + (1 - \theta)\sin\theta \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.14)$$

The variable x becomes maximum or minimum for an infinity of values θ_m making $dx/d\tau = 0$, so that

$$\tan\theta_m = \frac{1 - \theta_m}{1 + \theta_m}, \quad \text{or} \quad \theta_m = \tan\left(\frac{\pi}{4} - \theta_m\right), \quad \ldots \quad \ldots \quad \ldots \quad (A.15)$$

whence the turning values become

$$\frac{x_m}{a} = \frac{1 - \sin 2\theta_m}{\cos\theta_m + \sin\theta_m} = \pm\,\theta_m\sqrt{\left(\frac{2\theta_m^{\,2}}{1 + \theta_m^{\,2}}\right)}, \quad \ldots \quad \ldots \quad \ldots \quad (A.16)$$

the first three being $x_m/a \simeq 0{\cdot}213$, $-\,3{\cdot}595$, $7{\cdot}895$, at $\theta_m \simeq 0{\cdot}128\pi$, $\pi(\frac{3}{4} + 0{\cdot}113)$, $\pi(7/4 + 0{\cdot}056)$ respectively. Further turning values occur for values of φ_m exceeding only slightly $11\pi/4$, $15\pi/4$, $19\pi/4$, etc., the differences becoming rapidly very small, and the turning values themselves being almost equal to $\pm\,\theta_m\sqrt{2}$. The turning points get gradually nearer and nearer to the consecutive points of contact with the limiting lines, as seen in Fig. 1.

Similarly, y becomes maximum or minimum for all values θ_n making $dy/d\tau = 0$, *i.e.*, satisfying

$$\tan\theta_n = \frac{\theta_n + 1}{\theta_n - 1}, \quad \text{or} \quad \theta_n = \cot\left(\theta_n - \frac{\pi}{4}\right), \quad \ldots \quad \ldots \quad \ldots \quad (A.17)$$

and the turning values are

$$\frac{y_n}{a} = \frac{1 + \sin 2\theta_n}{\sin\theta_n - \cos\theta_n} = \pm\,\theta_n\sqrt{\left(\frac{2\theta_n^{\,2}}{1 + \theta_n^{\,2}}\right)}, \quad \ldots \quad \ldots \quad \ldots \quad (A.18)$$

the first three being $y_n/a \simeq 1{\cdot}618$, $-\,5{\cdot}728$, $10{\cdot}11$ at $\theta_n \simeq \pi(\frac{1}{4} + 0{\cdot}197)$, $\pi(5/4 + 0{\cdot}075)$, $\pi(9/4 + 0{\cdot}044)$ respectively. Further turning values are again very near to the consecutive points of contact with the limiting lines.

Example 4.—Solve the differential equation

$$\left.\begin{aligned} &\frac{d^4x}{d\tau^4} + 6\frac{d^3x}{d\tau^3} + 14\frac{d^2x}{d\tau^2} + 14\frac{dx}{d\tau} + 5x = 0 \\ \text{with} \quad & \\ &x_0 = \dot{x}_0 = 0,\ \ddot{x}_0 = 2,\ \dddot{x}_0 = -\,8 \quad (\text{at } \tau = 0) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (A.19)$$

and the equation:

$$\left.\begin{aligned} &\frac{d^4y}{d\tau^4} + 6\frac{d^3y}{d\tau^3} + 14\frac{d^2y}{d\tau^2} + 16\frac{dy}{d\tau} + 8y = 0 \\ \text{with} \quad & \\ &y_0 = \dot{y}_0 = 0,\ \ddot{y}_0 = -\,2,\ \dddot{y}_0 = 10 \quad (\text{at } \tau = 0) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (A.20)$$

The subsidiary equations are respectively:

$$(D^4 + 6D^3 + 14D^2 + 14D + 5)x = 2(D^2 + 6D) - 8D \,, \qquad \text{(A.21)}$$

$$(D^4 + 6D^3 + 14D^2 + 16D + 8)y = -2(D^2 + 6D) + 10D \,, \qquad \text{(A.22)}$$

and the solutions become (using formula 122 after factorizing quartics) :

$$x \left[= \frac{2D^2 + 4D}{(D+1)^2(D^2 + 4D + 5)}\right] = \tau\, e^{-\tau} - e^{-2\tau} \sin \tau \,, \qquad \text{(A.23)}$$

$$y \left[= \frac{-2D^2 - 2D}{(D+2)^2(D^2 + 2D + 2)}\right] = \tau\, e^{-2\tau} - e^{-\tau} \sin \tau \,. \qquad \text{(A.24)}$$

The two solutions are very similar, the two exponentials being simply interchanged. This difference accounts for a different behaviour of two functions, the former reaching only one maximum $x_{max} \simeq 0{\cdot}2855$ at $\tau \simeq 1{\cdot}444$ and then subsiding to 0 without oscillation, while the latter reaches first $y_{min} \simeq -0{\cdot}1750$ at $\tau \simeq 1{\cdot}064$, then $y_{max} \simeq 0{\cdot}0156$ at $\tau \simeq 3{\cdot}835$, and later performs a decaying oscillation. The functions and their derivatives are illustrated in Fig. 2.

Example 5.—Solve the following system of simultaneous differential equations:

$$\left.\begin{array}{ll} \left.\begin{array}{l} \dfrac{d^2x}{d\tau^2} + 4\dfrac{dx}{d\tau} + 2x - 4\dfrac{dy}{d\tau} + 2y = 0 \\ 4\dfrac{dx}{d\tau} + x + \dfrac{d^2y}{d\tau^2} - 4\dfrac{dy}{d\tau} + 3y = 0 \end{array}\right\} & \begin{array}{r} \text{with } x_0 = 3 \,,\ \dot{x}_0 = 0 \,, \\ y_0 = 0 \,,\ \dot{y}_0 = 6 \\ (\text{at } \tau = 0) \end{array} \end{array}\right\} . \qquad \text{(A.25)}$$

The subsidiary equations are

$$\left.\begin{array}{l} (D^2 + 4D + 2)x - (4D - 2)y = 3(D^2 + 4D) \\ (4D + 1)x + (D^2 - 4D + 3)y = 18D \end{array}\right\} , \qquad \text{(A.26)}$$

the main determinant being $(D^2 + 4)(D^2 + 1)$, so that the solution consists of two undamped simple harmonic oscillations, of frequencies 1 and 2. The appropriate formula is (123), and the solutions become:

$$\left.\begin{array}{l} x \left[= \dfrac{3D^4 + 33D^2}{(D^2 + 4)(D^2 + 1)}\right] = 10 \cos \tau - 7 \cos 2\tau \\ y \left[= \dfrac{6D^3 + 21D^2 + 24D}{(D^2 + 4)(D^2 + 1)}\right] = 7 \cos \tau + 6 \sin \tau - 7 \cos 2\tau \end{array}\right\} . \qquad \text{(A.27)}$$

Example 6.—Determine the response of a simple ‘ exponential delay unit ’ (of gain factor G and time constant t) to a finite parabolic input of duration $2\tau_1$ and maximum value A (parabolic impulse). Find the position and magnitude of the maximum output, for varying time constants t and τ_1.

The parabolic input I, illustrated in Fig. 3 (small inset diagram) seems to be a good first approximation to a ‘ smooth to-and-fro impulse ’ as often used in practice (*e.g.*, an ordinary push-pull stick movement applied in flight tests of aircraft). Such an input is represented by the simple formula

$$I = A\left(2\frac{\tau}{\tau_1} - \frac{\tau^2}{\tau_1^2}\right) , \qquad \text{for } 0 < \tau < 2\tau_1 \,, \qquad \text{(A.28)}$$

so that $I_{max} = A$ at $\tau = \tau_1$, and for $\tau > 2\tau_1$ the input remains equal to 0. The law governing the output x of our unit is then:

$$G\left(x + t\frac{dx}{d\tau}\right) = I = A\left(2\frac{\tau}{\tau_1} - \frac{\tau^2}{\tau_1^2}\right) , \qquad \text{for } 0 < \tau < 2\tau_1 \,, \qquad \text{(A.29)}$$

$$G\left(x + t\frac{dx}{d\tau}\right) = 0 \,, \qquad \text{for } \tau > 2\tau_1 \,. \qquad \text{(A.30)}$$

Let us consider one simple particular case first, *viz.*, $t = 0$. In this case the unit is a simple proportional gear, and we have, for $0 < \tau < 2\tau_1$

$$x = \frac{I}{G}, \text{ the maximum being } \bar{x} = \frac{I_{\max}}{G} = \frac{A}{G}, \text{ at } \tau = \tau_1 . \qquad \text{(A.31)}$$

In the general case, it will be convenient to introduce the auxiliary notation

$$R = 1/t = G/k , \qquad \text{(A.32)}$$

where R may be termed ' damping factor ' of the e.d. unit. The equation (A.29) can then be written in the following operational form (replacing the input function in the right-hand part by its operational equivalent by means of formulae 2 and 3 of Table 1):

$$k(D + R)x = \frac{2A}{\tau_1^2}\frac{\tau_1 D - 1}{D^2}, \qquad \text{(A.33)}$$

and the solution (valid for $0 < \tau < 2\tau_1$) will be found immediately by using formula (112) of Table 3:

$$x \left[= \frac{2A}{k\tau_1^2}\frac{\tau_1 D - 1}{D^2(D + R)}\right] = \frac{2A}{kRz_1^2}\left\{(1 + z_1)(z - 1 + e^{-z}) - \tfrac{1}{2}z^2\right\}, \qquad \text{(A.34)}$$

where, for abbreviation, we have introduced auxiliary symbols:

$$z = R\tau , \qquad z_1 = R\tau_1 = \tau_1/t . \qquad \text{(A.35)}$$

Similarly, the first derivative of x will be (using formula 101)

$$\frac{dx}{d\tau}\left[= \frac{2A}{k\tau_1^2}\frac{\tau_1 D - 1}{D(D + R)}\right] = \frac{2A}{kz_1^2}\left\{(1 + z_1)(1 - e^{-z}) - z\right\}. \qquad \text{(A.36)}$$

Initially, both x and $dx/d\tau$ increase from 0 to positive values. However, at $\tau = 2\tau_1$ (end of input), we have

$$x_e = \frac{2A}{kRz_1^2}(1 + e^{-2z_1})(z_1 - \tanh z_1) > 0 , \quad \left(\frac{dx}{d\tau}\right)_e = - Rx_e < 0 , \qquad \text{(A.37)}$$

so that $dx/d\tau$ has changed sign, and x must have reached a maximum before the input has ceased. If the maximum occurs at $z_m = R\tau_m$, then z_m must make (A.36) equal to 0, hence

$$1 + z_1 = \frac{z_m}{1 - e^{-z_m}}. \qquad \text{(A.38)}$$

If z_1 is known, this equation can only be solved for z_m by trial and error, or by any other approximate method. It is easily seen, however, that there is only one solution for z_m (whatever the value of z_1), and that

$$z_1 < z_m < 2z_1 , \qquad \text{(A.39)}$$

so that the maximum occurs during the second half of the input duration. A few sample response curves (for $z_1 = 1$, 2 and 3) are shown in Fig. 3a, where the ordinate is conveniently chosen as $x/\bar{x}$. The thick parts of the curves correspond to (A.34), while the continuing thin parts have been obtained by solving (A.30), the solution being obviously

$$x = x_e\, e^{-R(\tau - 2\tau_1)} . \qquad \text{(A.40)}$$

To obtain general characteristics of the response for variable R and τ_1, it is convenient to tabulate z_1 and z_m from (A.38), whereupon the ratio $z_m/z_1 = \tau_m/\tau_1$ can be plotted against z_1 (Fig. 3b). A simple formula for $x_{\max}$ may be obtained by combining (A.34) and (A.38), as follows:

$$x_{\max} = \frac{A}{kR}\frac{z_m}{z_1}\left(2 - \frac{z_m}{z_1}\right). \qquad \text{(A.41)}$$

This can be presented in two alternative forms, comparing the maximum output obtained here to either of the corresponding maxima in two particular limiting cases :

(*a*) *Limiting case* $R \to \infty$.—This is equivalent to $t = 0$, if G is supposed to be a given constant (*cf.* A.32). The case has been considered already (proportional gear with no delay), and the respective maximum has been given by (A.31). We have thus

$$\underline{\frac{x_{\max}}{\bar{x}} = \frac{z_m}{z_1}\left(2 - \frac{z_m}{z_1}\right)} \quad (G \text{ assumed constant}) \quad \ldots \quad (A.42)$$

(*b*) Limiting case $R = 0$, *i.e.*, $t = \infty$ and $G = 0$, the assumption being now that k is a given constant. The equation (A.29) then becomes

$$k\frac{kx}{d\tau} = I = A\left(\frac{2\tau}{\tau_1} - \frac{\tau^2}{\tau_1^2}\right); \quad (0 < \tau < 2\tau_1) \quad \ldots \quad (A.43)$$

our unit becomes a simple integrator, and we obtain

$$x = \frac{A\tau_1}{k}\left(\frac{\tau^2}{\tau_1^2} - \frac{\tau^3}{3\tau_1^3}\right) \quad \ldots \quad (A.44)$$

In this case the maximum output x^*, reached at $\tau = 2\tau_1$, becomes

$$x^* = \frac{4}{3}\frac{A\tau_1}{k}, \quad \ldots \quad (A.45)$$

and we get from (A.41) and (A.45)

$$\underline{\frac{x_{\max}}{x^*} = \frac{3}{4z_1}\frac{z_m}{z_1}\left(2 - \frac{z_m}{z_1}\right)} \quad (k \text{ assumed constant}) \quad \ldots \quad (A.46)$$

The formulae (A.42) and (A.46) are illustrated by curves in Fig. 3b, where an additional curve represents the ratio $x_e/\bar{x}$, obtained by combining (A.37) and (A.31), and all quantities are plotted against $z_1 = R\tau_1 = \tau_1/t$. This ratio of two time constants is seen to be the only parameter on which the form of the response depends.

Example 7.—Consider the previous problem (example 6), replacing the parabolic input by one proportional to $\tau e^{-r\tau}$.

We may write the formula for input:

$$I = v\tau\, e^{-r\tau}, \qquad dI/d\tau = v(1 - r\tau)\, e^{-r\tau}, \quad \ldots \quad (A.47)$$

so that v is the initial rate of increase of I. The input is illustrated by a small inset diagram in Fig. 4b. The maximum value A of the input occurs at

$$\tau_1 = 1/r \quad \ldots \quad (A.48)$$

and is obtained as

$$A = \frac{v\tau_1}{e} = \frac{v}{er}. \quad \ldots \quad (A.49)$$

The input increases originally from 0 to A, in a way rather similar to that of the parabolic input (Fig. 3b) but, instead of falling back to 0 at $\tau = 2\tau_1$, it decreases slowly, reaching 0 at infinity. The differential equation for the output is now, for all positive values of τ

$$G\left(x + t\frac{dx}{d\tau}\right) = I = v\tau\, e^{-r\tau}. \quad \ldots \quad (A.50)$$

Let us consider again the simple particular case $t = 0$ first. The unit is a simple proportional gear, and we have

$$x = \frac{I}{G}, \text{ the maximum being } \bar{x} = \frac{I_{\max}}{G} = \frac{A}{G}, \text{ at } \tau = \tau_1 \quad \ldots \quad (A.51)$$

In the general case, we introduce again the auxiliary notation (*cf.* A.32)

$$R = 1/t = G/k\,, \qquad \text{(A.52)}$$

and the operational form of equation (A.50), using formula (9), will be

$$k(D + R)x = \frac{vD}{(D + r)^2}\,. \qquad \text{(A.53)}$$

The solution will be found immediately, by applying formula (104):

$$x \left[= \frac{v}{k} \frac{D}{(D + R)(D + r)^2}\right] = \frac{v}{k(R - r)^2} \left[\mathrm{e}^{-R\tau} - \{1 - (R - r)\tau\}\,\mathrm{e}^{-r\tau}\right] \qquad \text{(A.54)}$$

and, similarly,

$$\frac{dx}{d\tau} \left[= \frac{v}{k} \frac{D^2}{(D + R)(D + r)^2}\right] = \frac{v}{k(R - r)^2} \left[\{R - r(R - r)\tau\}\,\mathrm{e}^{-r\tau} - R\,\mathrm{e}^{-R\tau}\right]. \qquad \text{(A.55)}$$

A few sample response curves are shown in Fig. 4a, the ordinates being $x/\bar{x}$, and the abscissae $z = R\tau$, by analogy with example 6 and Fig. 3a. The sample curves correspond again to the values 1, 2 and 3 of the parameter $z_1 = R\tau_1$ (*cf.* A.35), and may be compared with those of Fig. 3a. The ascending parts do not differ much, but the following decrease of the output is, of course, very much slower in Fig. 4a. The exact maxima and their location can be determined as follows. Supposing that the maximum occurs at $z_m = R\tau_m$, and introducing for abbreviation

$$\gamma = (R - r)\tau_m = z_m\left(1 - \frac{1}{z_1}\right), \qquad \text{(A.56)}$$

we equate (A.55) to 0, and the condition for maximum output is obtained in the form

$$1 - \frac{\gamma}{z_1} = \mathrm{e}^{-\gamma}\,, \quad \text{or} \quad z_1 = \frac{\gamma}{1 - \mathrm{e}^{-\gamma}}\,. \qquad \text{(A.57)}$$

If z_1 is known, this equation can only be solved for γ (and thus z_m) by approximate methods, there being only one solution (whatever the value of z_1). It is easily proved that $z_m > z_1$, *i.e.*, the output reaches its maximum later than the input. To obtain general characteristics of the response, it is convenient to tabulate z_1 and z_m against γ, whereupon $z_m/z_1 = \tau_m/\tau_1$ can be plotted against z_1 (Fig. 4b). A simple formula for $x_{\max}$ may be found by combining (A.54) with (A.56 and A.57), as follows :

$$x_{\max} = \frac{v\gamma}{kR(R - r)}\,\mathrm{e}^{-z_m/z_1} = \frac{Ar\gamma}{kR(R - r)}\,\mathrm{e}^{1-z_m/z_1} \ldots \qquad \text{(A.58)}$$

This can be presented again in two alternative forms, comparing the maximum output obtained here to either of the corresponding maxima in two limiting cases:

(*a*) *Limiting case* $R \to \infty$ (Thus $t = 0$, if G is supposed to be a given constant (*cf.* A.52)). The case has been considered already, the respective maximum output being (A.51), so that we have

$$\underline{\frac{x_{\max}}{\bar{x}} = \frac{\gamma r}{R - r}\,\mathrm{e}^{1-z_m/z_1} = \frac{z_m}{z_1}\,\mathrm{e}^{1-z_m/z_1}} \quad (G \text{ assumed constant}) \ldots \qquad \text{(A.59)}$$

(*b*) *Limiting case* $R = 0$ (Thus $t = \infty$ and $G = 0$, the parameter k being now assumed to be a given constant). The equation (A.50) becomes

$$k\frac{dx}{d\tau} = I = v\tau\,\mathrm{e}^{-r\tau}\,; \qquad \text{(A.60)}$$

our unit is a simple integrator, so that we obtain (*cf.* formulae 40 or 98)

$$x\left[=\frac{v}{k(D+r)^2}\right]=\frac{v}{kr^2}\left\{1-(1+r\tau)\,e^{-r\tau}\right\}, \qquad (A.61)$$

and the maximum output, reached when $\tau\to\infty$, becomes

$$x^*=\frac{v}{kr^2}=\frac{e^2A^2}{kv}. \qquad (A.62)$$

We then get from (A.58) and (A.62)

$$\frac{x_{\max}}{x^*}=\frac{\gamma r^2}{R(R-r)}\,e^{-z_m/z_1}=\frac{z_m}{z_1^2}\,e^{-z_m/z_1}. \qquad (A.63)$$

The formulae (A.59) and (A.63) are illustrated by curves in Fig. 4b, the abscissa being still $z_1=R\tau_1=\tau_1/t$, the only parameter on which the form of response depends.

Example 8.—Find the response of a simple exponential delay unit (of gain factor G and time constant t) to a simple harmonic (sinusoidal) input of amplitude A and periodic time T. In particular, find whether there is any transient overshoot of the output maxima over its ultimate steady amplitude.

The differential equation for the output x,

$$G\left(x+t\frac{dx}{d\tau}\right)=A\sin 2\pi\tau/T, \qquad (A.64)$$

can also be written, putting

$$R=1/t=G/k, \quad \text{and} \quad j=2\pi/T, \qquad (A.65)$$

in the following form, more convenient for using our tables:

$$k\left(\frac{dx}{d\tau}+Rx\right)=A\sin j\tau, \qquad (A.66)$$

or in the operational form:

$$k(D+R)x=Aj\,\frac{D}{D^2+j^2}. \qquad (A.67)$$

The solution is found by making use of (106):

$$x\left[=\frac{Aj}{k}\,\frac{D}{(D^2+j^2)(D+R)}\right]=\frac{Aj}{k(R^2+j^2)}\left(e^{-R\tau}-\cos j\tau+\frac{R}{j}\sin j\tau\right) \qquad (A.68)$$

or, putting for abbreviation

$$\theta=j\tau, \qquad R/j=T/2\pi t=s, \qquad x^*=A/G, \qquad (A.69)$$

in the simpler form

$$x=x^*\,\frac{s}{1+s^2}\,(e^{-s\theta}-\cos\theta+s\sin\theta). \qquad (A.70)$$

Similarly, the first derivative becomes

$$\frac{dx}{d\tau}=jx^*\,\frac{s}{1+s^2}\,(s\cos\theta+\sin\theta-s\,e^{-s\theta}). \qquad (A.71)$$

If the time constant t were $=0$, i.e., $s=\infty$, we would have to deal with an ordinary proportional gear, and the output x would be simply $x^*\sin j\tau$, with the amplitude x^*. In the general case, the output is not simple harmonic (because of the exponential term in the solution) but after a sufficiently long time (with $\tau \to \infty$ and $\theta \to \infty$) the transient exponential term becomes vanishingly small and we have, with gradually improving accuracy, steady-state oscillations expressed by

$$x \simeq x^* \frac{s}{1+s^2} (s \sin \theta - \cos \theta), \frac{dx}{d\tau} \simeq jx^* \frac{s}{1+s^2} (s \cos \theta + \sin \theta) \,. \quad \cdots \quad \cdots \quad (A.72)$$

The maxima and minima then occur alternately for an infinite series of values of θ which may be denoted by $\bar{\theta}$ and for which

$$\tan \bar{\theta} = -s \,, \quad \sin \bar{\theta} = \pm s/\sqrt{(1+s^2)}, \quad \cos \bar{\theta} = \pm 1/\sqrt{(1+s^2)} \,. \quad \cdots \quad \cdots \quad (A.73)$$

The consecutive values of $\bar{\theta}$ form an arithmetical progression, and we may write $\bar{\theta} = n\pi - \tan^{-1} s$, where $0 < \tan^{-1} s < \frac{1}{2}\pi$, and maxima correspond to odd values of n. The turning values of the input occur when $\theta = n\pi - \frac{1}{2}\pi$, the maxima again corresponding to odd values of n. The phase delay of the output is therefore $\frac{1}{2}\pi - \tan^{-1} s = \tan^{-1} 2\pi t/T$. The steady-state amplitude is

$$\bar{x} = x^* \frac{s}{\sqrt{(1+s^2)}} = \frac{A}{G\sqrt{(1+j^2t^2)}} = \frac{A}{k\sqrt{(R^2+j^2)}} \,. \quad \cdots \quad \cdots \quad (A.74)$$

A few illustrative curves of the output, represented conveniently by

$$\frac{x}{\bar{x}} = \frac{1}{\sqrt{(1+s^2)}} (e^{-s\theta} - \cos \theta + s \sin \theta) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (A.75)$$

and including the early transient stage are given in Fig. 5, for several values of $2\pi s$, or T/t. It is seen that, during the transient stage, the minima of x have smaller values than $\bar{x}$, but the maxima are bigger, the overshoots increasing as s falls. The exact transient turning values x_m occur for the values θ_m satisfying the equation (from A.71)

$$s \cos \theta_m + \sin \theta_m = s\, e^{-s\theta_m} \,, \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (A.76)$$

which can only be solved by trial and error, or other approximate methods, whereupon the turning values themselves, and the correspondong overshoots, may be found from

$$\frac{x_m}{\bar{x}} = \frac{\sqrt{(1+s^2)}}{s} \sin \theta_m \,. \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (A.77)$$

The greatest overshoots occur when $s = 0$, or $R = 0$, in which case the solution is

$$x = \frac{A}{kj} (1 - \cos \theta) = \bar{x}(1 - \cos \theta) \,, \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (A.78)$$

and the overshoot is obviously 100 per cent. In this particular case our unit becomes a simple integrator; there is a steady oscillation right from the start, but the mean value is $\bar{x}$, instead of 0.

The inset diagram in Fig. 5 gives the ratio $\bar{x}/x^*$ against $2\pi s$ or T/t (*cf.* A.74). This graph seems to imply that there is no output when $s = 0$, but in this case, with k finite, we have $x^* = \infty$, $\bar{x} = A/kj$.

The reader may consider the analogous case when the input is proportional to $\cos j\tau$, instead of $\sin j\tau$. He will find similar results, except that the overshoots are practically negligible.

Example 9.—Determine the response of an undercritically damped oscillator (complex delay unit) to a step input, and find the maximum output overshoot over its final steady value. Also determine the response of a system consisting of two such oscillators in series to a step input, and find whether the overshoot in such a system may be greater than in the case of a single oscillator.

For a single oscillator we have

$$\frac{d^2x}{d\tau^2} + 2R\frac{dx}{d\tau} + Hx = AH \,, \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (A.79)$$

where $H - R^2 = J^2 > 0$, J is the natural frequency of the oscillator, R its damping factor, and A denotes the final steady output. By using formula (80) or (100), the response is obtained in the form

$$\frac{x}{A}\left[= \frac{H}{D^2 + 2RD + H}\right] = 1 - \left(\cos J\tau + \frac{R}{J}\sin J\tau\right) e^{-R\tau}, \quad \ldots \quad \ldots \quad \ldots \quad (A.80)$$

and the derivative

$$\frac{1}{A}\frac{dx}{d\tau} = \frac{H}{J} e^{-R\tau} \sin J\tau. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.81)$$

The turning values occur at $J\tau_n = n\pi$ (where $n = 1, 2, 3, \ldots$) and are

$$\frac{x_n}{A} = 1 + (-1)^{n-1} e^{-n\pi R/J}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.82)$$

The maximum overshoot (for $n = 1$) is

$$\frac{x_{\max}}{A} - 1 = e^{-\pi R/J}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.82a)$$

A specimen response curve (for $R/J = 0{\cdot}2$, $H/J^2 = 1{\cdot}04$) is traced in Fig. 6a, the overshoot being 53·4 per cent. Fig. 6b gives the graph of the percentage overshoot against the ratio R/J, the greatest value being 100 per cent for $R = 0$. The overshoot decreases rapidly when R/J rises, and becomes only 4·3 per cent for $R/J = 1$. Beyond that value, the overshoot becomes practically negligible and it disappears completely for $R/J = \infty$, i.e., $J = 0$, $H = R^2$ (the case of critical damping). In this particular case, we obtain, using formula (40),

$$\frac{x}{A}\left[= \frac{R^2}{(D + R)^2}\right] = 1 - (1 + R\tau) e^{-R\tau}; \quad \frac{1}{A}\frac{dx}{d\tau} = R^2\tau\, e^{-R\tau}, \quad \ldots \quad \ldots \quad (A.83)$$

and it is seen that the output increases monotonically from 0 to its ultimate steady value. The alternative case of overcritical damping (when $H < R^2$ and the transfer function $D^2 + 2RD + H$ has two real roots, say $-r_1$ and $-r_2$) is also easily solved by means of formula (97) :

$$\frac{x}{a}\left[= \frac{r_1 r_2}{(D + r_1)(D + r_2)}\right] = 1 + \frac{r_2 e^{-r_1\tau} - r_1 e^{-r_2\tau}}{r_1 - r_2}; \quad \frac{1}{A}\frac{dx}{d\tau} = \frac{r_1 r_2}{r_1 - r_2}\left(e^{-r_2\tau} - e^{-r_1\tau}\right), \quad (A.84)$$

and the output is seen again to increase monotonically, with no overshoot.

The above simple problem is well known and has various applications. In particular, its solution represents the rise of incidence (or normal acceleration) of an aircraft, following a sudden elevator displacement.

Let us suppose now that the output of the oscillator is fed as an input into another similar oscillator, with an analogous transfer function $(D^2 + 2rD + h)$, the constants r, h differing, in general, from R, H. We have then two oscillators in series, and the final output (obviously independent of the order in which the two oscillators follow each other) will be given by the following operational formula:

$$\frac{x}{A}\left[= \frac{Hh}{(D^2 + 2RD + H)(D^2 + 2rD + h)}\right] \quad (A \text{ is final steady output}) \ldots \quad \ldots \quad (A.85)$$

The solution is obtained directly from formula (126):

$$\left.\begin{aligned} &\frac{x}{A} = 1 + \frac{h}{Q}\left(a \cos J\tau + \frac{b}{J}\sin J\tau\right) e^{-R\tau} + \frac{H}{Q}\left(a' \cos j\tau + \frac{b'}{j}\sin j\tau\right) e^{-r\tau}, \\ &\text{where:} \\ &a = H - h - 4\mathrm{R}(\mathrm{R} - \mathrm{r}), \quad b = R(H - h) + 2(R - r)(H - 2R^2), \\ &a' = h - H + 4r(R - r), \quad b' = r(h - H) - 2(R - r)(h - 2r^2), \\ &Q = (H - h)^2 + 4(R - r)(Rh - rH) \end{aligned}\right\}, \quad \ldots \quad (A.86)$$

but this solution applies only if both oscillators are undercritically damped, i.e., $H > R^2$, $h > r^2$, and not in resonance, i.e., $R \neq r$, or $H \neq h$. An analogous solution for the derivative $dx/d\tau$ can be readily obtained but, in view of its complexity, it would not be of much use for determining the overshoot. Of several alternative cases, let us consider only one, *viz.*, when $r = R$, $h = H$, i.e., the oscillators are in resonance, both as to frequency and damping (still assuming $H > R^2$). We obtain, using (83) or (127),

$$\left.\begin{aligned} \frac{x}{a}\left[= \frac{H^2}{(D^2 + 2RD + H)^2}\right] &= 1 - \left\{\left(1 - \frac{RH}{2J^2}\tau\right)\cos J\tau \right.\\ &\quad \left. + \left(R\frac{3H - 2R^2}{2J^2} + \frac{H\tau}{2}\right)\frac{\sin J\tau}{J}\right\} e^{-R\tau}, \\ \text{and similarly} \qquad & \\ \frac{1}{A}\frac{dx}{d\tau}\left[= \frac{H^2 D}{(D^2 + 2RD + H)^2}\right] &= \frac{H^2}{2J^3}\left(\sin J\tau - J\tau\cos J\tau\right) e^{-R\tau} \end{aligned}\right\} \quad \text{(A.87)}$$

In Fig. 6a, two additional curves have been plotted, one of which shows the response according to (A.86), with assumed ratios $R/J = r/j = 0 \cdot 2$, $j/J = 0 \cdot 5$, the other one a similar response in the case of resonance, the ratio R/J being still $0 \cdot 2$. It is seen that in both cases the first overshoot is considerably greater than with a single oscillator, the former curve showing ~ 80 per cent, the latter ~ 110 per cent overshoot, as against $53 \cdot 4$ per cent for a single oscillator. The higher overshoots occur, of course, with considerable delays relative to the lower one especially, when the second frequency j is lower than the first one J. (It should be noticed that all curves are plotted against the same abscissa $J\tau$.) The matter is of considerable practical interest, *e.g.*, for an aircraft with power-operated controls, where the power unit may often behave as a complex delay unit. A complete analysis of overshoots for all possible combinations of frequency and damping ratios (3 independent parameters) would be rather involved. In the case or resonance, however, the analysis is simple. The turning values occur at $J\tau = \theta$, the angle θ being one of the consecutive roots of the equation

$$\tan\theta = \theta \, , \qquad \text{(A.88)}$$

i.e., $\theta_1 = 4 \cdot 493$, $\theta_2 = 7 \cdot 725$, $\theta_3 = 10 \cdot 904$, etc., and the maximum overshoot, corresponding to θ_1, is found to be

$$\frac{x_{\max}}{A} - 1 = \frac{e^{-R\theta_1/J}}{\sqrt{(1 + \theta_1^2)}}\left(1 + \frac{R}{J}\theta_1 + \frac{H}{2J^2}\theta_1^2\right) \ldots \qquad \text{(A.89)}$$

The corresponding graph is given in Fig. 6b where it may be compared with that referring to a single oscillator.

Example 10.—Determine the response of an undamped oscillator, of frequency $J = 2\pi/T$, to a finite parabolic impulse of duration $2\tau_1$ and maximum value A. In particular, find the turning values during the transient stage and the final steady amplitudes.

The differential equation for the output x, while the input lasts, is

$$\frac{d^2x}{d\tau^2} + J^2 x = A\left(2\frac{\tau}{\tau_1} - \frac{\tau^2}{\tau_1^2}\right) \qquad (0 < \tau < 2\tau_1) \qquad \text{(A.90)}$$

or, in operational form,

$$(D^2 + J^2)x = \frac{2A}{\tau_1^2}\frac{\tau_1 D - 1}{D^2} \, . \qquad \text{(A.90a)}$$

Before solving this equation, let us remark that, if the oscillator was subject to a step input of magnitude A, the response would oscillate about its mean value A/J^2, with $\pm$ 100 per cent overshoot or undershoot, so that the maximum would be

$$x^* = 2A/J^2 \, , \qquad \text{(A.91)}$$

and this is a convenient value for comparison in the following analysis.

The solution of (A.90) is obtained immediately by using formulae (135) and (110):

$$x\left[=\frac{2A}{\tau_1^2}\frac{\tau_1 D-1}{D^2(D^2+J^2)}\right]=\frac{x^*}{\theta_1^2}\left[(1-\cos\theta)+\theta_1(\theta-\sin\theta)-\tfrac{1}{2}\theta^2\right], \quad \ldots \quad \ldots \quad \text{(A.92)}$$

$$v=\frac{dx}{d\tau}\left[=\frac{2A}{\tau_1^2}\frac{\tau_1 D-1}{D(D^2+J^2)}\right]=\frac{Jx^*}{\theta_1^2}\left[\theta_1(1-\cos\theta)-(\theta-\sin\theta)\right], \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.92a)}$$

where, for abbreviation,

$$\theta=J\tau, \qquad \theta_1=J\tau_1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.93)}$$

It may be noticed that, for $\tau=0$ (hence $\theta=0$) both x and v are 0. The next interesting point is to find the values of x and v at the end of the input, *i.e.*, for $\tau_e=2\tau_1$, $\theta_e=2\theta_1$:

$$x_e=2x^*\frac{\sin\theta_1(\sin\theta_1-\theta_1\cos\theta_1)}{\theta_1^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.94)}$$

$$v_e=2Jx^*\frac{\cos\theta_1(\sin\theta_1-\theta_1\cos\theta_1)}{\theta_1^2}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.94a)}$$

Once the input has ceased, i.e., for $\theta>2\theta_1$, the differential equation for x is simply

$$\frac{d^2x}{d\tau^2}+J^2x=0 \qquad (\tau>2\tau_1)\text{ with }\left.\begin{matrix}x=x_e\\ v=v_e\end{matrix}\right\}\text{ at }\tau=2\tau_1 \quad \ldots \quad \ldots \quad \text{(A.95)}$$

or, in operational form,

$$(D^2+J^2)x=D^2x_e+Dv_e \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.95a)}$$

and the solution becomes (*cf.* formulae 15, 19, 99)

$$x=x_e\cos(\theta-\theta_e)+\frac{v_e}{J}\sin(\theta-\theta_e) \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.96)}$$

or substituting (A.94, A.94a) and simplifying,

$$x=2x^*\frac{\sin\theta_1-\theta_1\cos\theta_1}{\theta_1^2}\sin(\theta-\theta_1) \qquad (\theta\geqslant 2\theta_1). \quad \ldots \quad \text{(A.96a)}$$

The motion is obviously harmonic, with the amplitude

$$\bar{x}=2x^*\left|\frac{\sin\theta_1-\theta_1\cos\theta_1}{\theta_1^2}\right|. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.97)}$$

We may observe now that the form of the response (apart from scale), both for the duration of the input and after, depends on one parameter only,

$$\theta_1=J\tau_1=2\pi\,\tau_1/T, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.98)}$$

proportional to the ratio of time constants τ_1 and T.

A remarkable feature of the present problem is that the steady amplitude (A.97) may sometimes become nil, *i.e.*, that the output may stop simultaneously with the input. This will happen, in fact, for an infinite number of values of θ_1, viz., those satisfying the equation

$$\sin\theta_1-\theta_1\cos\theta_1=0\,,\text{ or }\tan\theta_1=\theta_1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.99)}$$

It is so because the expression ($\sin \theta_1 - \theta_1 \cos \theta_1$) happens to be a common factor of formulae for x_e and v_e (*see* A.94 and A.94a)†. The well-known equation (A.99), which we have encountered already in example 9 formula (A.88) has an infinite number of solutions:

$$\theta_1 = \frac{3}{2}\pi - 0\cdot219 = 4\cdot493\,,$$

or

$$\frac{5}{2}\pi - 0\cdot129 = 7\cdot725\,,$$

or

$$\frac{7}{2}\pi - 0\cdot092 = 10\cdot904, \text{ etc.},$$

the subsequent solutions being slightly smaller than, but differing less and less from, $9\pi/2$, $11\pi/2$, etc.

In Fig. 7a, several sample response curves (for some chosen values of θ_1) have been traced, the ordinate x/x^* being plotted against θ. The thick parts of all curves correspond to (A.92) and the thin ones to (A.96), the points of junction marking the end of the input in each case. Small values of θ_1 (e.g., $\pi/4$, $\pi/2$) apply when the duration of input $2\tau_1$ is only a fraction of the natural period T of the oscillator. With large values of θ_1 (*e.g.*, 10 or 15), the input lasts over several periods, and this results in several oscillations of the thick parts of the curves. The curve corresponding to $\theta_1 = \pi/2$ is an example of a case in which $v_e = 0$ but $x_e \simeq 0$; so that $\bar{x} = x_e$, and the point of junction gives a turning value. Finally, the curve corresponding to $\theta_1 = 4\cdot493$ is the simplest example of the output stopping simultaneously with the input. It may be noticed that all maxima of x/x^* within the duration of input are less than 1 (although some are near enough to 1), and both maxima and minima are positive; x_e may, however, be either positive or negative and, of course, the subsequent turning values are alternatively $(+\,\bar{x})$ and $(-\,\bar{x})$.

It remains to find a general way to determine the positions and values of the maxima and minima during the transient stage. Denoting by $\theta_m = J\tau_m$ the relevant values of θ, the equation for θ_m is obtained by making (A.92a) equal to zero, *i.e.*,

$$\theta_1(1 - \cos \theta_m) - (\theta_m - \sin \theta_m) = 0 \quad \text{(A.100)}$$

For a given θ_1, this equation can only be solved by approximate methods. The best way, however, is to write it in the form

$$\theta_1 = \frac{\theta_m - \sin \theta_m}{1 - \cos \theta_m}\,, \quad \text{(A.100a)}$$

and plot θ_1 against θ_m, as shown in Fig. 7b. It is seen that only one θ_1 corresponds to each θ_m, but the inverse is not true. For a given θ_1 there may be one, or three, five, seven values of θ_m, etc., their number increasing as θ_1 increases. The minima of the curves in Fig. 7b are easily found to lie on the straight line $\theta_m = 2\theta_1$ and satisfy (A.99)‡. As obviously θ_m must be $\leqslant 2\theta_1$, it follows that the minute arcs of the curve below that straight line do not give any real turning values and should be considered as meaningless.

Substituting (A.100a) into (A.92), we obtain, after some considerable simplification, the following formula for turning values of x:

$$x_m = 2x^* \left(\frac{1 - \cos \theta_m - \frac{1}{2}\theta_m \sin \theta_m}{\theta_m - \sin \theta_m}\right)^2, \quad \text{(A.101)}$$

and this confirms, in a general way, that all transient maxima and minima must be positive.

† It may be noticed that x_e (but not v_e) becomes 0 also when $\sin \theta_1 = 0$, and v_e (but not x_e) becomes 0 when $\cos \theta_1 = 0$; in both cases the response continues indefinitely after the input has ceased.

‡ Of two other straight lines marked in Fig. 7b, the one $\theta_m = 3\theta_1$ is tangent at the origin, and that $\theta_m = \theta_1$ (corresponding to $\tan \theta_m/2 = \theta_m/2$) intersects the curve at points for which the output attains a turning value when the input is maximum.

To complete the illustration of this example (rather more complicated than the preceding ones), Fig. 8a presents graphs of x_e/x^* and $\bar{x}/x^*$ according to (A.94) and (A.97) respectively, and Fig. 8b graphs of the few transient turning values x_m/x^* calculated from (A.100a) and (A.101). The abscissa is θ_1 in all cases. As regards Fig. 8b, it should be mentioned that, for $\theta_1 < \pi/2$ the output has no maximum in the transient region (*cf.* Fig. 7a). As θ_1 increases, more and more turning values make their appearance, but the illustration does not go beyond the third maximum.

Example 11.—Determine the response of a simple oscillator, with any amount of damping, to an instantaneous pulse (Dirac's pulse, or delta-function input). Find the maximum output.

It may not be superfluous to recollect the definition of Dirac's pulse. Suppose any dynamic system is subject to a step input of magnitude a at $\tau = 0$, and the response is represented by $a\,f(\tau)$. If a subsequent negative step input of magnitude $(-\,a)$ is applied at $\tau = \tau_1$, then the final response, from τ_1 onwards, will be

$$a\,f(\tau) - a\,f(\tau - \tau_1) = a\tau_1 \frac{f(\tau) - f(\tau - \tau_1)}{\tau_1} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.102)$$

Let us now suppose that τ_1 decreases while a increases, in such a way that $a\tau_1 = \text{const} = A$, say, so that the total impulse is constant, while its duration becomes smaller and ultimately tends to 0. In the limiting case the response becomes

$$F(\tau) = Af'(\tau) \qquad (\tau > 0) . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.103)$$

Hence the operational equivalent of such an input is AD. A finite impulse of strictly zero duration is, of course, physically impossible. Nevertheless, $F(\tau)$ represents, with a fair approximation, the response to a ' rectangular ' impulse of magnitude A, of very short duration.

In the case of an oscillator, the subsidiary equation will be

$$k(D^2 + 2RD + H)x = AD , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.104)$$

k being an arbitrary constant factor. We have to consider five cases :

(*a*) Undercritical damping: $H - R^2 = J^2 > 0$, the solution being from (100) or (81, 82):

$$x = \frac{A}{kJ}\,e^{-R\tau} \sin J\tau\,; \quad \frac{dx}{d\tau} = \frac{A}{k}\left(\cos J\tau - \frac{R}{J}\sin J\tau\right) e^{-R\tau} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.105)$$

There are an infinite number of turning values, and the first (and highest) maximum occurs at τ_m, where $J\tau_m = \tan^{-1} J/R$, $\sin J\tau_m = J/\sqrt{H}$, so that

$$x_{\max} = \frac{A}{k\sqrt{H}}\,e^{-(R/J)\tan^{-1}(J/R)} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.106)$$

(*b*) Supercritical damping: $H - R^2 < 0$. We have then

$$\left.\begin{aligned} D^2 + 2RD + H &= (D + r_1)(D + r_2) , \\ \text{where} \qquad r_1 &= R + \sqrt{(R^2 - H)} , \qquad r_2 = R - \sqrt{(R^2 - H)} , \\ r_1 + r_2 &= 2R , \qquad\qquad\qquad r_1 r_2 = H \end{aligned}\right\} , \quad \ldots \quad (A.107)$$

and the solution is, from (97),

$$\left.\begin{aligned} x \left[= \frac{A}{k}\,\frac{D}{(D + r_1)(D + r_2)}\right] &= \frac{A}{k(r_1 - r_2)}\,(e^{-r_2\tau} - e^{-r_1\tau}) , \\ \frac{dx}{d\tau} &= \frac{A}{k(r_1 - r_2)}\,(r_1\,e^{-r_1\tau} - r_2\,e^{-r_2\tau}) \end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots \quad (A.108)$$

The only maximum occurs at τ_m, where $e^{(r_1-r_2)\tau_m} = r_1/r_2$, and

$$x_{\max} = \frac{A}{kr_1}\left(\frac{r_1}{r_2}\right)^{-r_1/(r_1-r_2)} = \frac{A}{k\sqrt{H}}\left(\frac{R+\sqrt{(R^2-H)}}{\sqrt{H}}\right)^{-R/\sqrt{(R^2-H)}} \quad \ldots \quad \ldots \quad \ldots \quad (A.109)$$

(*c*) Critical damping: $H = R^2$, in which case (formulae 98 or 41, 42)

$$x\left[=\frac{A}{k}\frac{D}{(D+R)^2}\right] = \frac{A}{k}\tau\, e^{-R\tau}; \quad \frac{dx}{d\tau} = \frac{A}{k}(1-R\tau)\, e^{-R\tau} \quad \ldots \quad \ldots \quad \ldots \quad (A.110)$$

The only maximum occurs at $\tau_m = 1/R$, and

$$x_{\max} = \frac{A}{ekR}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.111)$$

(*d*) Zero spring constant: $H = 0$, the solution being, from (101),

$$x\left[=\frac{A}{k}\frac{D}{D^2+2RD}\right] = \frac{A}{2kR}(1-e^{-2R\tau}), \text{ and the maximum value of } x \text{ is}$$

$$\bar{x} = \frac{A}{2kR}(\text{at } \tau = \infty). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.112)$$

(*e*) Zero damping: $R = 0$, $H = J^2$, and then the solution, from (99) or (66, 67) becomes

$$x\left[=\frac{A}{k}\frac{D}{D^2+J^2}\right] = \frac{A}{kJ}\sin J\tau, \quad \frac{dx}{d\tau} = \frac{A}{k}\cos J\tau; \quad x_{\max} = \frac{A}{kJ}. \quad \ldots \quad \ldots \quad (A.113)$$

The solutions (*a*) to (*d*) are illustrated by five response curves in Fig. 9a, corresponding to the following values of the convenient parameter $\sqrt{H}/R$:

$$0, \quad 0{\cdot}6, \quad 1, \quad 2{\cdot}125, \quad 4{\cdot}0625.$$

In all these cases $R > 0$ and $R\tau$ is a convenient abscissa. The figure shows how the response varies at constant R with varying H. In case (*e*) the response would be represented by a simple sine curve, but the convenient abscissa would be $J\tau$.

Fig. 9b gives the graph of the maximum output ratio $x_{\max}/\bar{x}$ against $\sqrt{H}/R$.

Example 12.—Find functions $F_1(\tau)$, $F_2(\tau)$, $F'(\tau)$, equivalent to the following operational expressions

$$\varphi_1(D) = \frac{2D^5+13D^4+39D^3+59D^2+37D+2}{(D^2+4D+5)(D+1)^3}, \quad \ldots \quad \ldots \quad \ldots \quad (A.114(i))$$

$$\varphi_2(D) = \frac{3D^6+7D^5+12D^4+17D^3+14D^2+9D+2}{(D^2+2D+2)(D^4+D^3+2D^2+3D+1)}, \quad \ldots \quad \ldots \quad (A.114(ii))$$

$$\varphi_3(D) = \frac{5D^6+60D^5+285D^4+675D^3+836D^2+477D+54}{(D+3)^3(D+1)^3} \quad \ldots \quad (A.114(iii))$$

respectively.

(i) In the first case we apply formula (142) of Table 4, with $R = 2$, $H = 5$, $r = 1$, $a = 2$, $b = 13$, $c = 39$, $d = 59$, $f = 37$, $g = 2$, and we obtain $c_1 = 2$, $c_2 = 2$, $b_2 = 3$, $a_2 = 1$, $a_1 = 1$, $b_1 = 1$, so that

$$\varphi_1(D) = \frac{D^2 + D + 2}{D^2 + 4D + 5} + \frac{D}{D + 1} + \frac{3D}{(D + 1)^2} + \frac{2D}{(D + 1)^3}, \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.115)}$$

a direct check being easy. Using now formulae (100) and (8, 9, 10), we get

$$F_1(\tau) = \frac{2}{5} + \left(\frac{3}{5}\cos\tau - \frac{9}{5}\sin\tau\right) e^{-2\tau} + (1 + 3\tau + \tau^2)\, e^{-\tau}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.116)}$$

(ii) In the second case we apply formula (151) of Table 5, with $a = 3$, $b = 7$, $c = 12$, $d = 17$, $f = 14$, $g = 9$, $k = 2$, $R = 1$, $H = 2$, $B = 1$, $C = 2$, $E = 3$, $F = 1$, and we obtain $A = 12$, $\alpha = 4$, $\beta = 10$, $\gamma = 8$, $\delta = 3$, $\theta = -1$, $\Gamma = -5/2$, $K = -2$, and hence $a_1 = 2$, $b_1 = 1$, $a_2 = 1$, $b_2 = 2$, $f_2 = 1$, $c_2 = 1$, $d_2 = 3$, so that

$$\varphi_2(D) = \frac{2D^2 + D}{D^2 + 2D + 2} + \frac{D^4 + 2D^3 + D^2 + 3D + 1}{D^4 + D^3 + 2D^2 + 3D + 1}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.117)}$$

a direct check being again simple. It remains to factorise the quartic denominator,

$$D^4 + D^3 + 2D^2 + 3D + 1 = (D + 1)(D + 0{\cdot}4534)(D^2 - 0{\cdot}4534\, D + 2{\cdot}066) \quad \ldots \quad \text{(A.118)}$$

and to use (100) and (121), and the solution is

$$\begin{aligned} F_2(\tau) &= (2\cos\tau - \sin\tau)\, e^{-\tau} + 1 - e^{-\tau} \\ &\quad + 0{\cdot}4606 e^{-0{\cdot}4534\tau} + (0{\cdot}5396\cos 2{\cdot}1949\tau + 0{\cdot}0395\sin 2{\cdot}1949\tau) e^{0{\cdot}2267\tau}. \end{aligned} \quad \ldots \quad \text{(A.119)}$$

(iii) In the third case we apply formula (152) of Table 5, with $R = 3$, $r = 1$, $a = 5$, $b = 60$, $c = 285$, $d = 675$, $f = 836$, $g = 477$, $k = 54$, and we obtain $A = 2$, $a_1 = 1$, $b_1 = 2$, $c_1 = 3$, $a_2 = 2$, $b_2 = 3$, $c_2 = 4$, so that

$$\varphi_3(D) = 2 + \frac{D}{D + 3} + \frac{2D}{(D + 3)^2} + \frac{3D}{(D + 3)^3} + \frac{2D}{D + 1} + \frac{3D}{(D + 1)^2} + \frac{4D}{(D + 1)^3} \quad \text{(A.120)}$$

and, using (8, 9, 10) twice, we get the final answer:

$$F_3(\tau) = 2 + \left(1 + 2\tau + \frac{3}{2}\tau^2\right) e^{-3\tau} + (2 + 3\tau + 2\tau^2)\, e^{-\tau}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(A.121)}$$

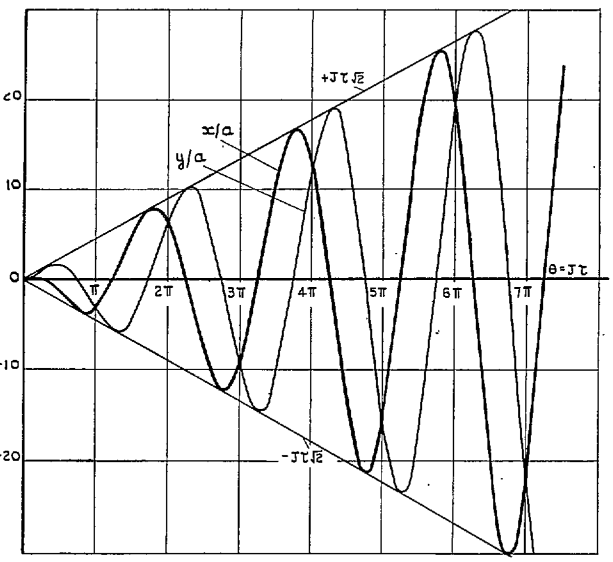


FIG. 1. Solution of example 3.

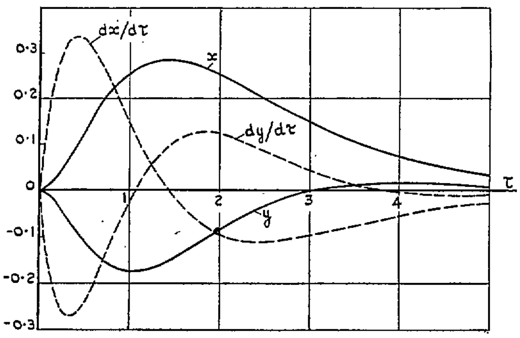
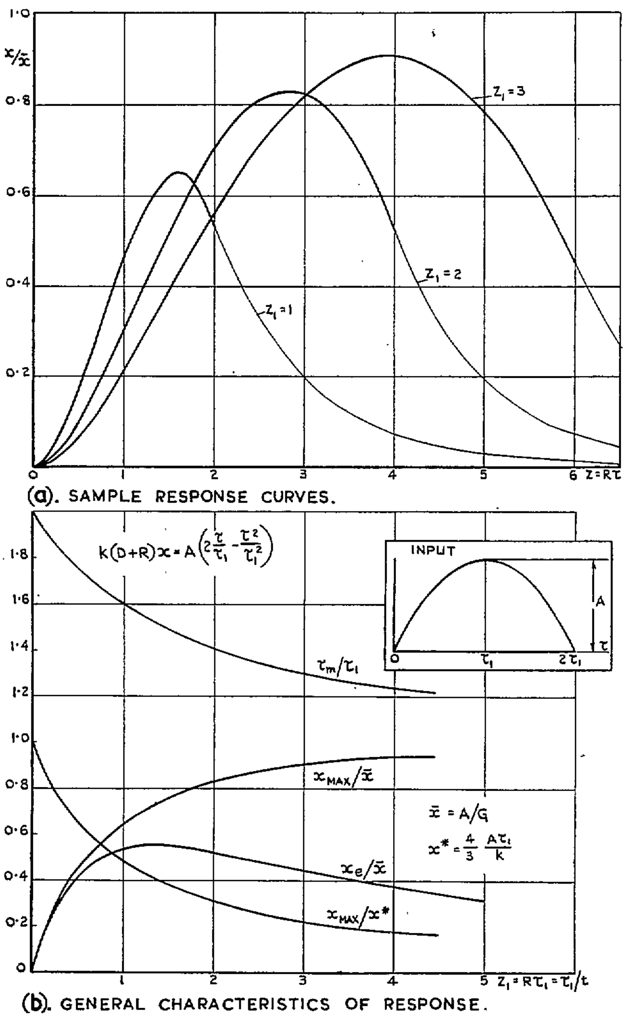
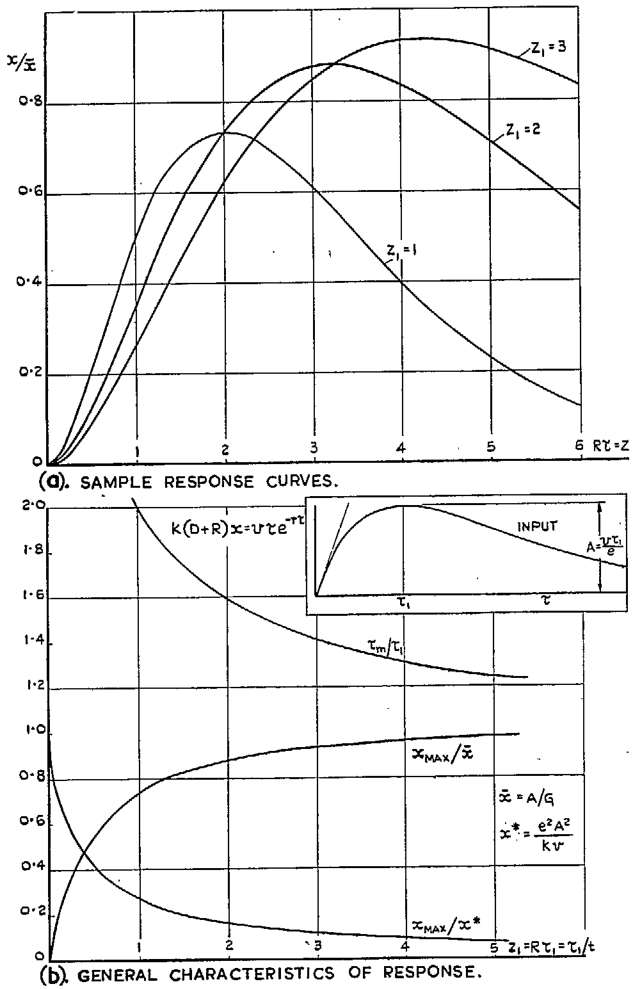


FIG. 2. Solution of example 4.



FIGS. 3a and 3b. Response of simple delay unit to a parabolic input of finite duration (example 6).



FIGS. 4a and 4b. Response of a simple delay unit to an input proportional to $\tau\, e^{-r\tau}$ (example 7).

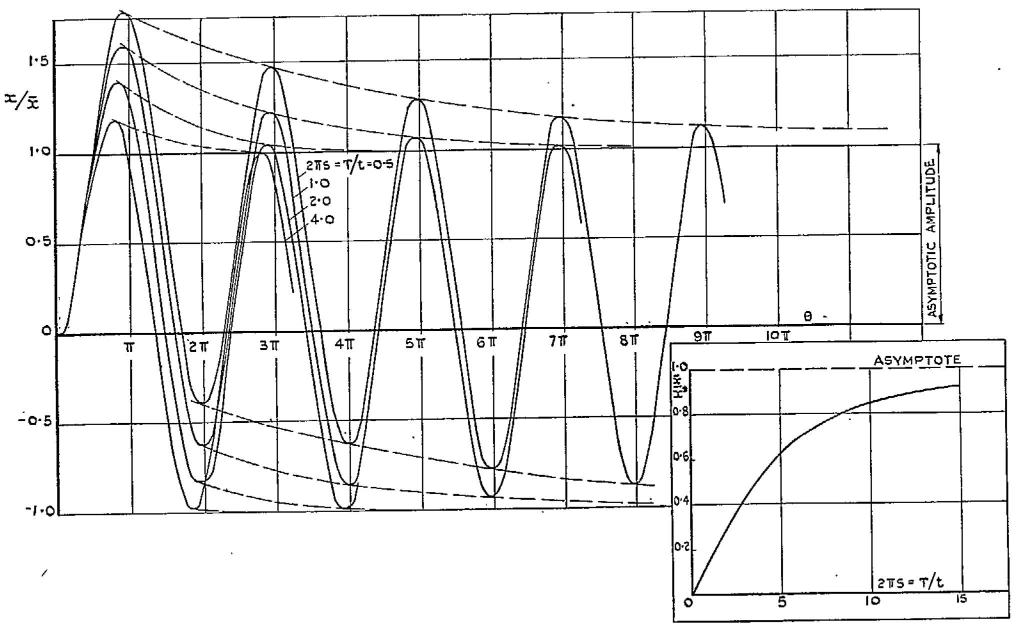


FIG. 5. Transient response of a simple delay unit to a sinusoidal input—Sample curves (example 8). (Inset: Variation of the steady amplitude with the ratio of time constants.)

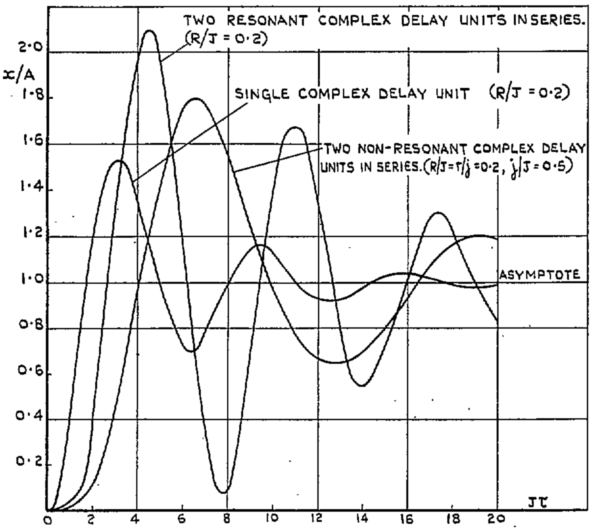


FIG. 6a. Response to a step input into damped oscillatory systems (complex delay units) in series, compared to the case of a single unit (example 9).

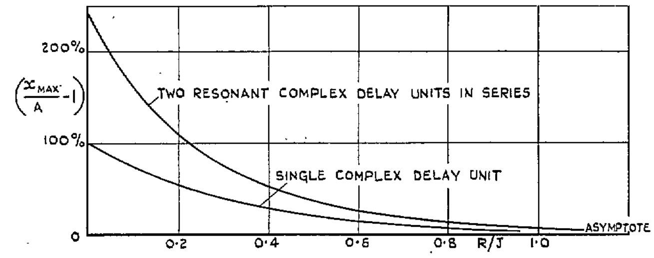
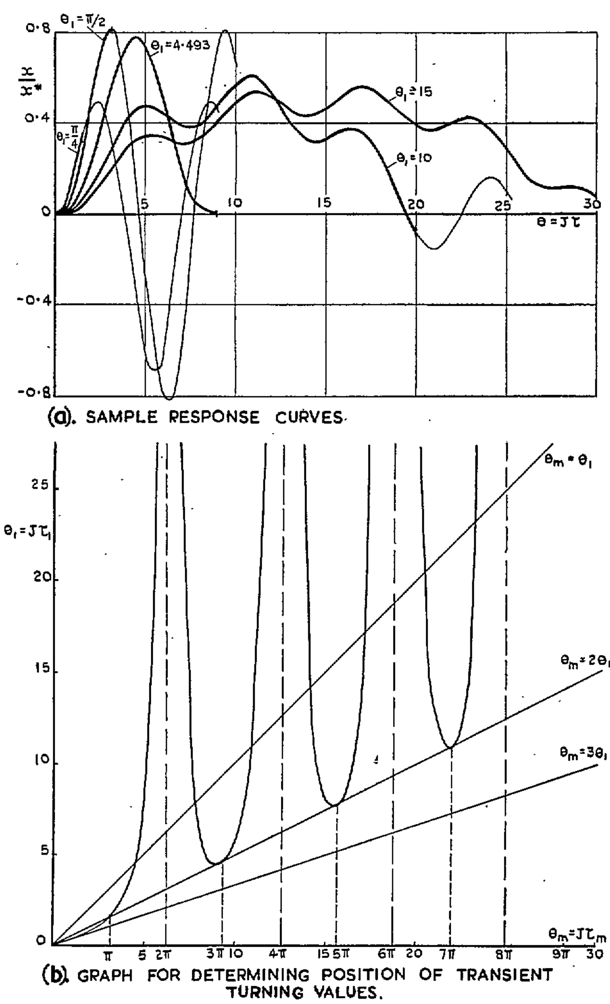
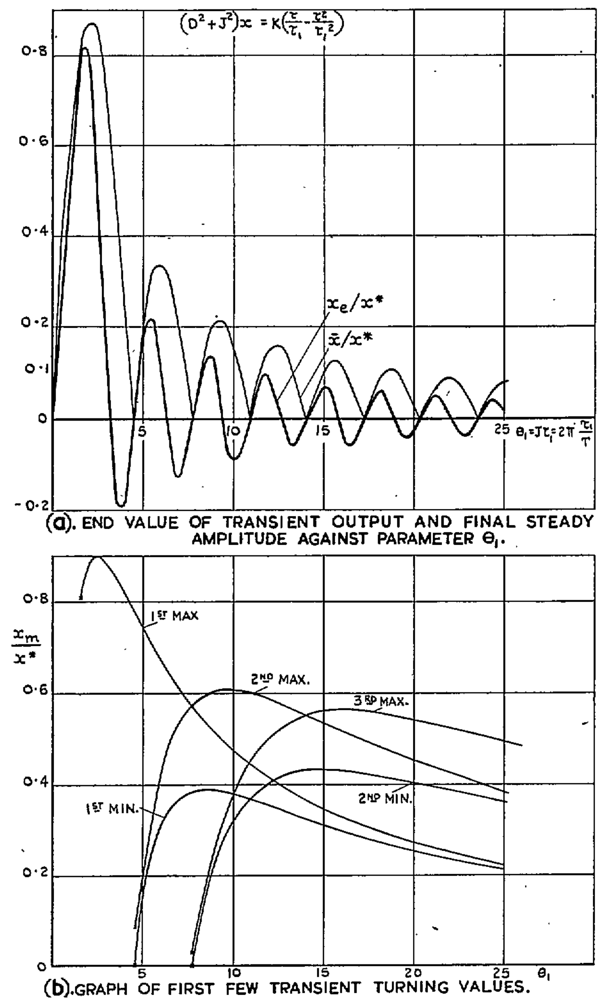


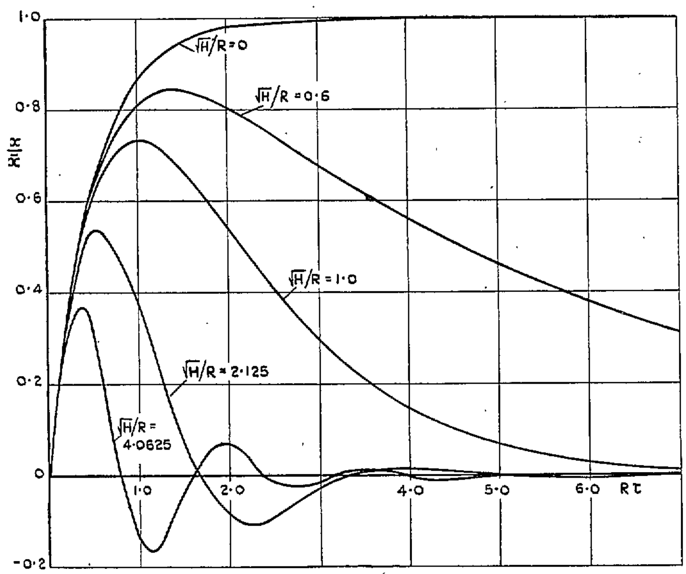
FIG. 6b. Maximum overshoot in the case of a step input into two resonant complex delay units in series, compared with that corresponding to a single unit (example 9).



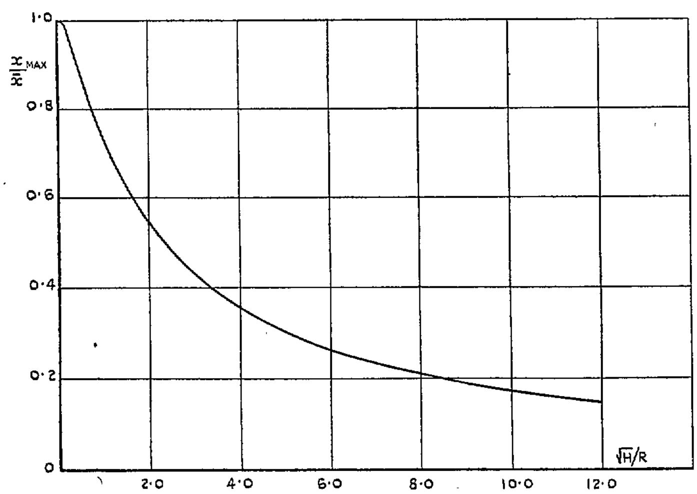
FIGS. 7a and 7b. Response of an undamped oscillator to a finite parabolic input (example 10).



FIGS. 8a and 8b. Response of an undamped oscillator to a finite parabolic input (example 10).



(a). SAMPLE RESPONSE CURVES.



(b). MAXIMUM OUTPUT DISPLACEMENT.

FIGS. 9a and 9b. Response of a damped oscillator to an instantaneous pulse (example 11).

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