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By

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On the Integral Equations of Two-Dimensional Subsonic Flutter Derivative Theory

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Summary.—This note gives the result of an attempt to find an analytical solution of Possio's integral equation—the equation which connects the downwash and the pressure distribution on an aerofoil oscillating in two-dimensional subsonic compressible flow. A method is given for solving this problem and for solving the corresponding problem in incompressible flow (the solution of Birnbaum's integral equation).

1. *Introduction*—The problem of calculating the pressure distribution on an oscillating aerofoil in two-dimensional subsonic compressible flow can be approached in two ways. The first approach is to solve the boundary-value problem for the partial differential equation of the flow. This has been done by Timman and van de Vooren[1], and they have obtained an analytical solution of the problem. The second approach is to solve the equivalent integral equation which connects the pressure distribution over the aerofoil with the known downwash distribution, that is, the Possio integral equation. This approach has been used to obtain the analytical solution presented in this note. The solution is the same as that given by Timman and van de Vooren but it is of interest to obtain it from the integral equation.

As a guide in devising a suitable method for solving Possio's equation, a new method of solution is first derived for Birnbaum's equation, the equation to which Possio's equation reduces when the flow is incompressible. Although there are a number of existing methods of solution, this new method is of interest and is described in section 4. It is then used to obtain the solution to Possio's equation described in section 5.

The method given in this note is possible because each of these integral equations can be split up into two integral equations, one of which is of a very simple type and the other an integral equation of the first kind with a symmetric nucleus. The integrand in the second integral equation has a second order singularity and the integral must be understood to take its principal value. This singularity causes no difficulty and the integral equation can be solved by the standard Hilbert-Schmidt[2] method.

No singular solution of this second integral equation could at first be found for Possio's integral equation. However, when the note was in draft a paper by H. G. Küssner[3] appeared in which he solved the problem by a combination of the differential and integral equation approaches, and gave the singular solution in a simpler form than that given by Timman and van de Vooren. It was then found possible to derive the singular solution in Küssner's form by a method consistent with the general method of this report.

* R.A.E. Report Structures 181, received 11th October, 1955.

2. *List of Symbols.*—All lengths have been made non-dimensional with respect to the semi-chord l so that the wing covers the interval $(-1, 1)$ of the x-axis.

p		Frequency of oscillation in radn/sec
V		Velocity of the steady stream
ω		Reduced frequency parameter
	$=$	$(pl)/V$
M		Mach number
Ω	$=$	$\omega/(1-M^2)$
ρ		Air density
Π		Pressure jump over the aerofoil
	$=$	$\rho V^2 \Gamma \, e^{ipt}$
w		Downwash on the aerofoil
	$=$	$VW \, e^{ipt}$
a		Normal acceleration over the aerofoil
	$=$	$(V^2/l)A \, e^{ipt}$
$se_n\theta$		The odd Mathieu function of order n and period 2π
$Ne_n^{(2)}(t)$		Modified Mathieu function of the second kind and order n
$H_1^{(2)}(x)$		Hankel function
$K_0(x)$, $K_1(x)$		Modified Bessel functions of the second kind

3. *Principal Values of Improper Integrals.*—All the improper integrals in this note can be expressed in terms of the improper integrals.

$$\int_a^b \frac{f(y)}{(x-y)^{n+1}}\,dy\,.$$

The principal value of this integral is defined by Mangler[4] to be

$$P\int_a^b \frac{f(y)}{(x-y)^{n+1}}\,dy = \lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \frac{f(y)}{(x-y)^{n+1}}\,dy + (-)^n K_n(x,\varepsilon)\right]$$

where

$$K_0(x,\varepsilon) = 0$$

and

$$K_n(x,\varepsilon) = \sum_{r=0}^{n-1} \frac{1}{r!\,(n-r)} \frac{1-(-1)^{n-r}}{\varepsilon^{n-r}} \left(\frac{d^r f}{dy^r}\right)_{y=x}.$$

For $n = 0$ we get

$$P\int_a^b \frac{f(y)}{x-y}\,dy = \lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \frac{f(y)}{x-y}\,dy\right],$$

which is the usual definition for the Cauchy principal value, and for $n = 1$,

$$P\int_a^b \frac{f(y)}{(x-y)^2}\,dy = \lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \frac{f(y)}{(x-y)^2}\,dy - \frac{2}{\varepsilon} f(x)\right].$$

An important property of these integrals is that we can differentiate under the integral sign, *i.e.*,

$$\frac{d}{dx} P\int_a^b \frac{f(y)}{(x-y)^n}\,dy = -\,nP\int_a^b \frac{f(y)}{(x-y)^{n+1}}\,dy\,.$$

This fact will be used repeatedly.

4. *Solution for Incompressible Flow.*—4.1. For a two-dimensional aerofoil in incompressible flow, the downwash velocity $VW(x)$ is related to the pressure $\rho V^2\Gamma(x)$ by means of Birnbaum's integral equation[5, 6], *viz.*,

$$2\pi W(x)\,e^{i\omega x} = -\int_{-\infty}^{x} e^{i\omega u}\,du\int_{-1}^{+1}\frac{\Gamma(v)}{(u-v)^2}\,dv\,, \qquad (4.1)$$

where $W(x)$ is known for $|x| < 1$ and Γ must satisfy the Kutta-Joukowski condition at the trailing edge, i.e., $\Gamma(1) = 0$.

The integral equation can be split up into two integral equations:

$$W(x)\,e^{i\omega x} = -\int_{-\infty}^{x} e^{i\omega u}\,G(u)\,du \qquad (4.2)$$

and

$$2\pi G(u) = \int_{-1}^{+1}\frac{\Gamma(v)}{(u-v)^2}\,dv\,. \qquad (4.3)$$

From equation (4.2) we get

$$\begin{aligned} e^{i\omega x}\,G(x) &= -\frac{d}{dx}\{W(x)\,e^{i\omega x}\} \\ &= -\,e^{i\omega x}\left[\frac{dW}{dx} + i\omega W\right] \\ &= -\,e^{i\omega x}\,A(x)\,, \end{aligned} \qquad (4.4)$$

where $A(x)$ is the normal acceleration.

4.2. *Solution of Equation* (4.3) *for* $|u| < 1$.—When we put $u = -\cos\theta$, $v = -\cos\varphi$, equation (4.3) becomes

$$2\pi G(\theta) = \int_0^{\pi}\frac{\Gamma(\varphi)\sin\varphi\,d\varphi}{(\cos\theta - \cos\varphi)^2}\,. \qquad (4.5)$$

To solve this we use the integral relation

$$\sin n\theta = \left(-\frac{1}{n\pi}\right)\int_0^{\pi}\frac{\sin\theta\,\sin\varphi}{(\cos\theta-\cos\varphi)^2}\sin n\varphi\,d\varphi\,, \qquad (4.6)$$

which is proved in Appendix I, section 1.1.

Let

$$\Gamma = \sum_1^{\infty} a_n \sin n\varphi\,. \qquad (4.7)$$

$\Gamma(\pi) = 0$ and so the trailing-edge condition is satisfied.

From equations (4.5) and (4.7) using equation (4.6) we get

$$2\pi G(\theta)\sin\theta = \sum_1^{\infty} a_n(-\,n\pi)\sin n\theta$$

and so

$$\frac{\pi}{2}a_n = -\frac{2}{n}\int_0^{\pi} G(\theta)\sin\theta\,\sin n\theta\,d\theta\,.$$

Therefore

$$\Gamma(\varphi) = -\frac{4}{\pi}\sum_1^\infty \frac{\sin n\varphi}{n}\int_0^\pi G(\theta)\sin\theta\ \sin n\theta\ d\theta$$

$$= -\frac{1}{\pi}\int_0^\pi G(\theta)\sin\theta\left[4\sum_1^\infty\frac{\sin n\theta\ \sin n\varphi}{n}\right]d\theta$$

$$= -\frac{1}{\pi}\int_0^\pi G(\theta)\sin\theta\ \log\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\,d\theta$$

$$= -\frac{2}{\pi}\int_0^\pi G(\theta)\sin\theta\ L(\theta,\varphi)\,d\theta \quad \ldots \quad (4.8)$$

$$= \Gamma_1 \text{ say}$$

where

$$L(\theta,\varphi) = \tfrac{1}{2}\log\frac{1-\cos(\varphi+\theta)}{1-\cos(\varphi-\theta)}\,. \quad \ldots \quad (4.9)$$

It will be shown in section 6 and Appendix I, section 1.2 that $\Gamma_0 = \cot\frac{1}{2}\varphi$ is a singular solution of equation (4.5) which satisfies the trailing-edge condition. The complete solution of equation (4.5) is therefore

$$\Gamma = \alpha\Gamma_0 + \Gamma_1$$

$$= \alpha\cot\tfrac{1}{2}\varphi - \frac{2}{\pi}\int_0^\pi G(\theta)\sin\theta\ L(\theta,\varphi)\,d\theta\,, \quad \ldots \quad (4.10)$$

where α is an arbitrary constant.

4.3. We have not yet used all the given data. To find Γ we have used, not the given downwash W, but only the given normal acceleration over the wing, A. Since $A = (dW/dx) + i\omega W$, any downwash of the form $W + C\,e^{-i\omega x}$, where C is independent of x will give the same acceleration A. We shall use the fact that the downwash over the wing is W and not $W + C\,e^{-i\omega x}$ to find the value of α.

To do this the normal acceleration A induced ahead of the wing by the pressure distribution (equation (4.10)) will be calculated by using equation (4.3). The acceleration will be linearly dependent on α. Equation (4.2) will then be used to calculate the induced downwash over the wing and α will be chosen so that the induced downwash is equal to the given downwash over the wing.

4.4. *Determination of the Constant* α.—

Let

$$2\pi\bar{G}(u) = \int_{-1}^{+1}\frac{\Gamma(v)}{(u-v)^2}\,dv$$

For $|u| < 1$, $\bar{G}(u) = G(u)$.

For $u < -1$, if we write $u = -\cosh t$ and $v = -\cos\varphi$ we have

$$2\pi\,\bar{G}(u) = \int_0^\pi \frac{\alpha\Gamma_0 + \Gamma_1}{(\cosh t - \cos\varphi)^2}\sin\varphi\,d\varphi$$

$$= 2\pi(\alpha f + g) \text{ say.}$$

The acceleration ahead of the aerofoil $A = -\bar{G}$ is now known. We must now calculate the downwash $\bar{W}$ induced by Γ.

Let

$$\bar{W}(x)\,e^{i\omega x} = -\int_{-\infty}^{x} e^{i\omega u}\,\bar{G}(u)\,du\,.$$

Then for $|x| < 1$

$$\bar{W}(x)\, e^{i\omega x} = -\int_{-1}^{x} e^{i\omega u}\, G(u)\, du - \int_{-\infty}^{-1} e^{i\omega u}\, \bar{G}(u)\, du$$

$$= -\int_{-1}^{x} \frac{d}{du}\{e^{i\omega u}\, W(u)\}\, du$$

$$-\alpha \int_{-\infty}^{-1} e^{i\omega u} f(u)\, du - \int_{-\infty}^{-1} e^{i\omega u}\, g(u)\, du\,. \quad \ldots \quad \ldots \quad \ldots (4.11)$$

To evaluate the second and third integrals we shall use the integral relation (*see* Appendix I, section 1.3)

$$\int_0^\pi \frac{\sin\theta \sinh t}{(\cosh t - \cos\theta)^2} \sin n\theta\, d\theta = n\pi\, e^{-nt}\,. \quad \ldots \quad \ldots \quad \ldots (4.12)$$

In section 6.2 it is shown that

$$\Gamma_0 = \cot \tfrac{1}{2}\theta$$

may be replaced by the divergent series

$$2 \sum_1^\infty \sin n\theta\,.$$

Using this result we have for the second integral on the right-hand side of equation (4.11)

$$2\pi \int_{-\infty}^{-1} e^{i\omega u} f(u)\, du = \int_{-\infty}^{-1} e^{i\omega u}\, du \int_{-1}^{+1} \frac{\Gamma_0(v)}{(u-v)^2}\, dv$$

$$= \int_0^\infty e^{-i\omega \cosh t} . \sinh t\, dt \int_0^\pi \frac{\sin\varphi}{(\cosh t - \cos\varphi)^2}\, \Gamma_0(\varphi)\, d\varphi$$

$$= 2\int_0^\infty e^{-i\omega \cosh t} \sinh t\, dt \int_0^\pi \frac{\sin\varphi}{(\cosh t - \cos\varphi)^2}\, d\varphi \sum_1^\infty \sin n\varphi$$

$$= 2\int_0^\infty e^{-i\omega \cosh t}\, dt \sum_{n=1}^\infty \int_0^\pi \frac{\sinh t \sin\varphi}{(\cosh t - \cos\varphi)^2} \sin n\varphi\, d\varphi$$

$$= 2\pi \int_0^\infty e^{-i\omega \cosh t} \sum_1^\infty n\, e^{-nt}\, dt$$

$$= -\pi i\omega\, \{K_0(i\omega) + K_1(i\omega)\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (4.13)$$

(*see* Appendix II) where $K_0(i\omega)$ and $K_1(i\omega)$ are modified Bessel functions of the second kind.

Similarly for the third integral on the right-hand side of equation (4.11) we have

$$2\pi \int_{-\infty}^{-1} e^{i\omega u}\, g(u)\, du = \int_{-\infty}^{-1} e^{i\omega u}\, du \int_{-1}^{+1} \frac{\Gamma_1(v)}{(u-v)^2}\, dv$$

$$= \int_0^\infty e^{i\omega \cosh t} \sinh t\, dt \int_0^\pi \frac{\sin\varphi}{(\cosh t - \cos\varphi)^2}\, \Gamma_1(\varphi)\, d\varphi$$

$$= -\frac{4}{\pi}\int_0^\infty e^{i\omega \cosh t} \sinh t\, dt \int_0^\pi \frac{\sin\varphi}{(\cosh t - \cos\varphi)^2}\, d\varphi$$

$$\times \int_0^\pi G(\theta) \sin\theta \sum_1^\infty \frac{\sin n\theta\, \sin n\varphi}{n}\, d\theta$$

$$= -4 \sum_{n=1}^\infty \int_0^\pi G(\theta) \sin\theta\, \sin n\theta\, d\theta \int_0^\infty e^{i\omega \cosh t - nt}\, dt$$

$$= 4 \sum_{n=1}^\infty \int_0^\pi A(\theta) \sin\theta\, \sin n\theta\, d\theta \int_0^\infty e^{-i\omega \cosh t - nt}\, dt\,, \quad \ldots \quad \ldots (4.14)$$

where $A(\theta)$ is the normal acceleration.

It is shown in Appendix III that this result can be simplified and we get

$$2\pi \int_{-\infty}^{-1} e^{i\omega u}\, g(u)\, du = 4\left\{\frac{i\omega}{2}\int_0^\pi W(\theta)\Big(K_1(i\omega) + K_0(i\omega)\cos\theta\Big)\, d\theta - \frac{\pi}{2} W(0)\, e^{-i\omega}\right\}.$$

Therefore with these results, equation (4.11) becomes

$$\bar{W}(x)\, e^{i\omega x} = W(x)\, e^{i\omega x} - W(-1)\, e^{-i\omega}$$

$$+ \alpha\frac{i\omega}{2}\{K_0(i\omega) + K_1(i\omega)\} + W(-1)\, e^{-i\omega}$$

$$- \frac{i\omega}{\pi}\int_0^\pi W(\theta)\{K_1(i\omega) + K_0(i\omega)\cos\theta\}\, d\theta$$

and so since $\bar{W}(x) = W(x)$ we must have

$$\alpha = \frac{2}{\pi}\int_0^\pi W(\theta)\{C + (1 - C)\cos\theta\}\, d\theta$$

where

$$C = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)}.$$

The solution of Birnbaum's integral equation from equation (4.10) is then

$$\frac{\pi}{2}\Gamma(\varphi) = + \left[\int_0^\pi W(\theta)\{C + (1 - C)\cos\theta\}\, d\theta\right]\cot\tfrac{1}{2}\varphi$$

$$+ \tfrac{1}{2}\int_0^\pi \left\{\frac{dW}{d\theta} + i\omega\sin\theta\, W\right\}\log\frac{1 - \cos(\theta + \varphi)}{1 - \cos(\theta - \varphi)}\sin\theta\, d\theta\,.$$

This is the standard solution obtained by other methods (*see* Neumark[5] and Küssner[6]).

5. *Solution for Subsonic Compressible Flow.*—5.1. *Possio's Integral Equation.*—The analytical method developed in section 4 will now be used to solve the corresponding problem in subsonic compressible flow. The pressure $\rho V^2\Gamma(x)$ is now related to the downwash velocity $VW(x)$ by means of Possio's integral equation[4],

$$2\pi W(x)\, e^{i\omega x} = -\frac{\pi M\Omega\sqrt{(1 - M^2)}}{2i}\int_{-\infty}^{x} e^{i\Omega u}\, du\int_{-1}^{+1}\{\Gamma(v)\exp(-iM^2\Omega v)\}\frac{H_1^{(2)}(M\Omega|u - v|)}{|u - v|}dv, \quad (5.1)$$

where $W(x)$ is known for $|x| < 1$ and $\Gamma(1)$ must be zero.

This form of Possio's equation has been derived from that given by Küssner[4]; it can be obtained directly in this form by the use of the acceleration potential method.

This equation, like Birnbaum's equation (section 4), can be split up into two integral equations:

$$e^{i\omega x}\, W(x) = -\int_{-\infty}^{x} e^{i\Omega u}\, G(u)\, du \quad \ldots \quad (5.2)$$

and

$$2\pi\, G(u) = \lambda\int_{-1}^{+1}\Gamma^*(v)\,\frac{H_1^{(2)}\{M\Omega|u - v|\}\, dv}{|u - v|}, \quad \ldots \quad (5.3)$$

where

$$\Gamma^*(v) = \exp(-iM^2\Omega v)\,\Gamma(v) \quad \ldots \quad (5.4)$$

and

$$\lambda = \frac{\pi M\Omega\sqrt{(1 - M^2)}}{2i}. \quad \ldots \quad (5.5)$$

From equation (5.4) it follows that $\Gamma^*(1) = 0$.

By differentiating equation (5.2) we see that

$$e^{i\Omega x} G(x) = - \frac{d}{dx}\left[e^{i\omega x} W(x)\right]$$

$$= - e^{i\omega x} A(x) . \quad \ldots \quad (5.6)$$

5.2. *The Solution of Equation* (5.3) *for* $|u| < 1$.—When we put $u = -\cos\theta$, $v = -\cos\varphi$, equation (5.3) becomes

$$2\pi\, G(\theta) = \lambda \int_0^\pi \Gamma^*(\varphi) \sin\varphi \frac{H_1^{(2)}\{M\Omega\,|\cos\theta - \cos\varphi|\}}{|\cos\theta - \cos\varphi|} d\varphi . \quad \ldots \quad (5.7)$$

To solve this we use the integral relation, proved in Appendix IV,

$$se_n\,\theta = \lambda_n \int_0^\pi \sin\theta\ \sin\varphi \frac{H_1^{(2)}\{M\Omega\,|\cos\theta - \cos\varphi|\}}{|\cos\theta - \cos\varphi|} se_n\,\varphi\, d\varphi , \quad \ldots \quad (5.8)$$

where the λ_n are functions of $M\Omega$ and $se_n\,\theta$ is the odd Mathieu function of order n. These functions are orthogonal and are chosen so that

$$\int_0^\pi se_n\,\theta\ se_m\,\theta\, d\theta = \begin{cases} \frac{1}{2}\pi & m = n \\ 0 & m \neq n . \end{cases}$$

Let

$$\Gamma^* = \sum_1^\infty a_n\, se_n\,\varphi . \quad \ldots \quad (5.9)$$

The Mathieu functions are such that $se_n(\pi) = 0$ and so the trailing-edge condition is satisfied. From equation (5.7)

$$2\pi\, G(\theta)\ \sin\theta = \lambda \sum_1^\infty \frac{a_n}{\lambda_n} se_n\,\theta$$

and so

$$a_n = \frac{4\lambda_n}{\lambda} \int_0^\pi G(\theta) \sin\theta\ se_n\,\theta\, d\theta .$$

Therefore

$$\Gamma^* = \sum_1^\infty a_n\, se_n\,\varphi$$

$$= \frac{4}{\lambda} \sum_1^\infty \lambda_n\, se_n\,\varphi \int_0^\pi G(\theta) \sin\theta\ se_n\,\theta\, d\theta$$

$$= - \frac{4}{\lambda} \sum_1^\infty \lambda_n\, se_n\,\varphi \int_0^\pi \exp(- iM^2\Omega \cos\theta) \sin\theta\ se_n\,\theta\ A(\theta)\, d\theta$$

$$= - \frac{4}{\lambda} \sum_1^\infty \lambda_n A_n\, se_n\,\varphi$$

$$= \Gamma_1^* \text{ say,} \quad \ldots \quad (5.10)$$

where

$$A_n = \int_0^\pi \exp(- iM^2\Omega \cos\theta) \sin\theta\ se_n\,\theta\ A(\theta)\, d\theta . \quad \ldots \quad (5.11)$$

There is good reason to believe that the singular solution which gives the correct singularity at the leading edge is

$$\Gamma_0^* = \sum_1^\infty \lambda_n\, se_n{}'\, 0\ se_n\,\varphi$$

(*see* section 6).

The complete solution of the integral equation is then

$$\Gamma^* = \alpha \Gamma_0^* + \Gamma_1^*$$

$$= \alpha \sum_1^\infty \lambda_n \, se_n' \, 0 \; se_n \, \varphi - \frac{4}{\lambda} \sum_1^\infty \lambda_n \, A_n \, se_n \, \varphi \; . \quad .. \quad .. \quad .. \quad .. \quad (5.12)$$

As before the pressure Γ^* may induce, not the downwash, W, but a downwash $W + C\,e^{-i\omega x}$ where C is a constant independent of α. We must choose α so that $C = 0$.

5.3. *Determination of* α.—

Let

$$2\pi \, \bar{G}(u) = \lambda \int_{-1}^{+1} \Gamma^*(v) \, \frac{H_1^{(2)}\{M\Omega \, | u - v | \} \, dv}{| u - v |} \; .$$

For $|u| < 1$, $\qquad \bar{G}(u) = G(u)$.

For $u < -1$

$$2\pi \, \bar{G}(u) = \alpha\lambda \int_{-1}^{+1} \Gamma_0^*(v) \, \frac{H_1^{(2)}\{M\Omega \, | u - v | \}}{| u - v |} \, dv$$

$$+ \lambda \int_{-1}^{+1} \Gamma_1^*(v) \, \frac{H_1^{(2)}\{M\Omega \, | u - v | \}}{| u - v |} \, dv \; .$$

If we put $u = -\cosh t$, $v = -\cos \varphi$, this becomes

$$2\pi \, \bar{G}(t) = \alpha\lambda \sum_1^\infty \lambda_n \, se_n' \, 0 \int_0^\pi \sin \varphi \;\, se_n \, \varphi \, \frac{H_1^{(2)}\{M\Omega(\cosh t - \cos \varphi)\}}{(\cosh t - \cos \varphi)} \, d\varphi$$

$$- 4 \sum_1^\infty \lambda_n \, A_n \int_0^\pi \sin \varphi \;\, se_n \, \varphi \, \frac{H_1^{(2)}\{M\Omega(\cosh t - \cos \varphi)\}}{(\cosh t - \cos \varphi)} \, d\varphi \; .$$

Let

$$\bar{W}(x) \, e^{i\omega x} = - \int_{-\infty}^{x} e^{i\Omega u} \, \bar{G}(u) \, du \; .$$

Then for $|x| < 1$

$$\bar{W}(x) \, e^{i\omega x} = - \int_{-1}^{x} e^{i\omega u} \, \bar{G}(u) \, du - \int_{-\infty}^{-1} e^{i\Omega u} \, \bar{G}(u) \, du$$

$$= W(x) \, e^{i\omega x} - W(-1) \, e^{-i\omega} - \int_{-\infty}^{-1} e^{i\Omega u} \, \bar{G}(u) \, du \; .$$

Therefore $\bar{W}(x) = W(x)$ for $|x| < 1$ if

$$W(-1) \, e^{-i\omega} + \int_{-\infty}^{-1} e^{i\Omega u} \, \bar{G}(u) \, du = 0 \; ,$$

i.e., if

$$2\pi W(-1) \, e^{-i\omega} + \alpha\lambda \sum_1^\infty \lambda_n \, se_n' \, 0 \int_0^\infty e^{-i\Omega \cosh t} \, \sinh t \, dt \int_0^\pi \sin \varphi \;\, se_n \, \varphi \, \frac{H_1^{(2)}\{M\Omega(\cosh t - \cos \varphi)\}}{(\cosh t - \cos \varphi)} \, d\varphi$$

$$- 4 \sum_1^\infty \lambda_n A_n \int_0^\infty e^{-i\Omega \cosh t} \, \sinh t \, dt \int_0^\pi \frac{\sin \varphi \;\, se_n \, \varphi \; H_1^{(2)}\{M\Omega(\cosh t - \cos \varphi)\}}{(\cosh t - \cos \varphi)} \, d\varphi = 0 \; .$$

If we use the relation

$$Ne_n^{(2)}(t) = \mu_n \int_0^\pi \sinh t \, \sin \varphi \, \frac{H_1^{(2)}\{M\Omega(\cosh t - \cos \varphi)\}}{(\cosh t - \cos \varphi)} \, se_n \, \varphi \, d\varphi \; , \quad .. \quad (5.13)$$

where μ_n is a function of $M\Omega$ (*see* Appendix IV) and write

$$\int_0^\infty e^{-i\Omega \cosh t}\; Ne_n^{(2)}(t)\, dt = N_n ,$$

we get

$$2\pi W(-1)\, e^{-i\omega} + \alpha\lambda \sum_1^\infty \frac{\lambda_n N_n}{\mu_n} se_n{}'\, 0 - 4 \sum_1^\infty \frac{\lambda_n N_n}{\mu_n} A_n = 0 ,$$

i.e.,

$$\lambda\alpha = \frac{4 \sum_1^\infty \frac{\lambda_n N_n}{\mu_n} A_n - 2\pi W(-1)\, e^{-i\omega}}{\sum_1^\infty \frac{\lambda_n N_n}{\mu_n} se_n{}'\, 0} .$$

The complete solution is therefore

$$\exp\,(iM^2\Omega \cos\varphi)\; \Gamma(\varphi) = \frac{1}{\lambda} \frac{\left[4 \sum_1^\infty \frac{\lambda_n A_n N_n}{\mu_n} - 2\pi\, e^{-i\omega}\, W(-1)\right]}{\sum_1^\infty \frac{\lambda_n N_n}{\mu_n} se_n{}'\, 0} \sum_1^\infty \lambda_n\, se_n{}'\, 0\; se_n\, \varphi$$
$$- \frac{4}{\lambda} \sum_1^\infty \lambda_n\, A_n\, se_n\, \varphi ,$$

where the coefficients A_n are determined from the known downwash velocity by the use of equations (5.6) and (5.11).

Since

$$\lambda_n = \frac{\pi}{2i} M\Omega \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} \quad\text{and}\quad \mu_n = \frac{\pi}{2i} M\Omega \frac{Ne_n^{(2)}(0)}{se_n{}'\, 0} ,$$

the result becomes

$$\sqrt{(1-M^2)}\; \exp\,(iM^2\Omega\cos\varphi)\; \Gamma(\varphi) =$$
$$= \left[\frac{4 \sum_1^\infty \frac{se_n{}'\, 0}{Ne_n^{(2)\prime}(0)} A_n N_n - 2\pi W(-1)\, e^{-i\omega}}{\sum_1^\infty \frac{[se_n{}'0]^2}{Ne_n^{(2)\prime}(0)} N_n}\right] \sum_1^\infty \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} se_n{}'\, 0\; se_n\, \varphi$$
$$- 4 \sum_1^\infty \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} se_n\, \varphi .$$

This is not quite the same form of the solution as that given by Timman and van de Vooren because they have introduced incompressible-flow results to ensure the convergence of the series which occur.

The series given here for the singular part of the pressure

$$\sum_1^\infty \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} se_n{}'\, 0\; se_n\, \varphi$$

is divergent but it has a $(C, 1)$ sum. The integral of the series will probably converge. The divergence of the series

$$\sum_1^\infty \frac{[se_n{}'\, 0]^2}{Ne_n^{(2)\prime}(0)} N_n$$

is more serious. A method of finding its sum is suggested in Appendix II.

6. *Singular Solutions of the Integral Equations.*—6.1. In order to solve completely the integral equation

$$g(x) = \int_a^b K(x,y)\, f(y)\, dy, \qquad \qquad (6.1)$$

we need to know the singular solutions, functions $f(y)$ for which

$$\int_a^b K(x,y)\, f(y)\, dy = 0\,. \qquad \qquad (6.2)$$

Let $K(x,y) = K(y,x)$ and let the eigenvalues and eigenfunctions of the homogeneous integral equation

$$\phi(x) = \lambda \int_a^b K(x,y)\, \phi(y)\, dy \qquad \qquad (6.3)$$

be λ_n and ϕ_n respectively. The functions $\phi_n(x)$ are orthogonal and are assumed to be normalised so that

$$\int_a^b [\phi_n(x)]^2\, dx = 1\,.$$

We shall show formally that the functions

$$\Gamma_{01} = \sum_{n=1}^{\infty} \lambda_n\, \phi_n(y)\, \phi_n(z) \qquad \qquad (6.4)$$

and

$$\Gamma_{02} = \sum_{n=1}^{\infty} \lambda_n\, \phi_n(y)\, \phi_n{}'(z) \qquad \qquad (6.5)$$

are singular solutions of equation (6.1).

We shall first show that

$$\sum_1^{\infty} \phi_n(x)\, \phi_n(z) = \delta(x - z)\,, \qquad \qquad (6.6)$$

where $\delta(x)$, the Dirac delta function, is zero for $x \neq 0$ and at $x = 0$ behaves in such a way that

$$\int_{-\infty}^{\infty} \delta(x)\, f(x) = f(0)\,.$$

$\delta(x)$ is the 'derivative' of the Heaviside step function†

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

---

† Let

$$g(x) = \int_{-\infty}^{\infty} H(y - x)\, f(y)\, dy\,,$$

then formally

$$g'(x) = -\int_{-\infty}^{\infty} H'(y - x)\, f(y)\, dy\,.$$

But

$$g(x) = \int_x^{\infty} f(y)\, dy$$

and so

$$g'(x) = -f(x)\,.$$

Therefore

$$\int_{-\infty}^{\infty} H'(y - x)\, f(y)\, dy = f(x)\,.$$

If we write

$$H'(y - x) = \delta(y - x)$$

we get

$$\int_{-\infty}^{\infty} \delta(y - x)\, f(y)\, dy = f(x)\,.$$

Differentiating this with respect to x we get

$$\int_{-\infty}^{\infty} \delta'(y - x)\, f(y)\, dy = -f'(x)$$

and putting $x = 0$ in these last two equations we get the results quoted in the main text.

Let x be an interior point of the interval (a, b) and let

$$H(x-z) = \sum_1^\infty a_n \, \phi_n(z)$$

Then since

$$\int_a^b \phi_n(x) \, \phi_m(x) \, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m , \end{cases}$$

we have

$$a_n = \int_a^b H(x-z) \, \phi_n(z) \, dz$$

$$= \int_a^x \phi_n(z) \, dz \, .$$

Therefore

$$H(x-z) = \sum_1^\infty \phi_n(z) \int_a^x \phi_n(z) \, dz$$

and so

$$\delta(x-z) = H'(x-z) = \sum_1^\infty \phi_n(x) \, \phi_n(z) \, .$$

If we differentiate this equation with respect to x we get

$$\delta'(x-z) = \sum_1^\infty \phi_n'(x) \, \phi_n(z) \, . \qquad \ldots \qquad (6.7)$$

$\delta'(x)$ is zero for $x \neq 0$ and behaves at $x = 0$ in such a way that†

$$\int_{-\infty}^{\infty} \delta'(x) \, f(x) \, dx = -f'(0) \, .$$

The relations (6.6) and (6.7) can be proved rigorously by using the Schwartz theory of distributions[9, 10].

We can now show that Γ_{01} is a singular solution, for

$$\int_a^b \Gamma_{01}(y) \, K(x,y) \, dy = \int_a^b K(x,y) \, \sum_1^\infty \lambda_n \phi_n(y) \, \phi_n(z) \, dy$$

$$= \sum_1^\infty \phi_n(x) \, \phi_n(z)$$

and so is zero for $x \neq z$.

Similarly Γ_{02} is a singular solution.

In section 6.2 these results will be verified for incompressible flow by using the concept of a Cesaro sum. This is defined as follows:

Let

$$s_n(x) = \sum_1^n a_n(x) \quad \text{and let} \quad \sigma_n(x) = \frac{1}{n} \sum_1^n s_n(x) \, .$$

Then if as $n \to \infty$, $\sigma_n(x) \to \sigma(x)$, $\sigma(x)$ is said to be the first Cesaro sum, the $(C, 1)$ sum of the series. Even though a series is divergent it may have a Cesaro sum. If a series is convergent and has a sum $s(x)$ then it has a $(C, 1)$ sum $\sigma(x)$, and $\sigma(x) = s(x)$.

It may be possible to prove that for $x \neq z$ the $(C, 1)$ sums of the series (6.6) and (6.7) are zero. It is not known what connection, if any, there is between the theory of distributions and the theory of divergent series.

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† *See* footnote to section 6.1, para. 4.

6.2. *Singular Solutions for Incompressible Flow.*—For incompressible flow the eigenvalues are $\{-1/(n\pi)\}$ and the eigenfunctions are $\sin n\theta$. The series (6.6, 6.7) become

$$(a)\qquad \sum_1^\infty n \sin n\theta \ \cos n\psi = \tfrac{1}{2}\sum_1^\infty n\{\sin n(\theta+\psi) + \sin n(\theta-\psi)\}$$

$$= \tfrac{1}{2}\frac{d}{d\theta}\sum_1^\infty \{\cos n(\theta+\psi) + \cos n(\theta-\psi)\}$$

and

$$(b)\qquad \sum_1^\infty \sin n\theta \ \sin n\psi = -\tfrac{1}{2}\sum_1^\infty \{\cos n(\theta+\psi) - \cos n(\theta-\psi)\}$$

and because of the relation

$$\tfrac{1}{2} + \sum_1^\infty \cos n\theta = 0 \qquad (C, 1)$$

the $(C, 1)$ sums of both series are zero.

The singular solution corresponding to equation (6.5) is

$$\Gamma_{02} = -\frac{1}{\pi}\sum_1^\infty \sin n\theta \ \cos n\psi$$

$$= -\frac{1}{2\pi}\sum_1^\infty \{\sin n(\theta+\psi) + \sin n(\theta-\psi)\}$$

and since

$$\sum_1^\infty \sin nx = \tfrac{1}{2}\cot \tfrac{1}{2}x \qquad (\mathrm{C}, 1)\ ,$$

we have

$$\Gamma_{02} = -\frac{1}{4\pi}\{\cot \tfrac{1}{2}(\theta+\psi) + \cot \tfrac{1}{2}(\theta-\psi)\}\ .$$

If this function is to give the correct singularity at the leading edge we must have $\psi = 0$ and

$$\Gamma_{02} = -\frac{1}{2\pi}\cot \tfrac{1}{2}\theta\ .$$

The singular solution corresponding to equation (6.4) is

$$\Gamma_{01} = -\frac{1}{\pi}\sum_1^\infty \frac{\sin n\theta \ \sin n\psi}{n}$$

$$= -\frac{1}{4\pi}\log\frac{1-\cos(\theta+\psi)}{1-\cos(\theta-\psi)}\ .$$

This function gives the correct singularity at the leading edge $\theta = \psi$ of a control surface.

6.3. *Singular Solutions for Compressible Flow.*—For compressible flow the eigenvalues are

$$\lambda_n = \frac{\pi}{2i} M\Omega \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\ ,$$

where $Ne_n^{(2)}(z)$ is the modified Mathieu function of the second kind and order n, and the eigenfunctions are the odd Mathieu functions $se_n\theta$ (*see* Appendix IV). By analogy with the incompressible solution it appears that the singular solutions for the compressible case are

$$\Gamma_{01}^* = \sum_1^\infty \lambda_n\, se_n\,\theta \ \ se_n\psi$$

and

$$\Gamma_{02}^* = \sum_1^\infty \lambda_n\, se_n\,\theta \ \ se_n{}'\,0$$

but no verification can be given.

6.4. Although the series for the singular part of the pressure are divergent they are still useful because the series for the lift or moment may be convergent. But even if the final series are not convergent they will have Cesaro $(C, 1)$ sums and these are not much harder to calculate numerically than the sums of convergent series.

7. *General Discussion.*—All the integral equations of subsonic flutter derivative theory are of the form

$$e^{i\omega x}\, W(x,y) = -\int_{-\infty}^{x} e^{i\Omega x}\, dx \iint_S K(x,\xi\,;\, y,\eta)\ \Gamma(\xi,\eta)\ d\xi\, d\eta\ ,$$

where $K(x,\xi\,;y,\eta) = K(\xi,x\,;\,\eta,y)$ and S is the wing area. This integral equation can be solved by the method given in this note if the eigenvalues and the eigenfunctions of the homogeneous integral equation

$$\phi(x,y) = \lambda \iint_S K(x,\xi\,;\, y,\eta)\ \phi(\xi,\eta)\ d\xi\, d\eta$$

are known. These functions will exist for any plan-form, but except for simple plan-forms like the elliptic or possibly the rectangular wing there is little hope of finding these functions analytically. However, a method could be devised for calculating these functions numerically for any plan-form. The work of this note is, however, mainly of interest because of the mathematical topics which have arisen.

It is shown that these integral equations whose kernels have a second-order singularity can be solved by the standard Hilbert-Schmidt method because the integrals are principal-value integrals. The singularity can of course be reduced to a first-order singularity by carrying out the integration with respect to x but if this is done the resulting kernel will no longer be symmetric and there is no longer an obvious method of solution. If this is done for Possio's equation the resulting kernel is extremely complicated. For Birnbaum's equation in steady flow,

$$2\pi W(x) = -\int_{-\infty}^{x} dx \int_{-1}^{+1} \frac{\Gamma(y)}{(x-y)^2}\, dy\ ,$$

the equation which results is the Prandtl-Glauert integral equation

$$2\pi W(x) = \int_{-1}^{+1} \frac{\Gamma(y)}{x-y}\, dy\ .$$

The kernel is simple but the integral equation is no longer of a standard type and it is better to leave the equation in its original form.

Prandtl's lifting-line equation

$$k(x) = \tfrac{1}{2}a_0(x)\ c(x) \left\{ V\alpha(x) + \frac{1}{4\pi}\int_{-s}^{s} \frac{k(\xi)}{(x-\xi)^2}\, d\xi \right\}$$

can similarly best be solved by keeping the second-order singularity. The equation can be written

$$\sqrt{\{\beta(x)\}}\ k(x) = \frac{V\alpha(x)}{\sqrt{\{\beta(x)\}}} + \frac{1}{4\pi}\int_{-s}^{s} \frac{1}{\sqrt{\{\beta(x)\ \beta(\xi)\}}}\ \frac{k(\xi)\ \sqrt{\{\beta(\xi)\}}}{(x-\xi)^2}\, d\xi$$

where

$$\beta(x) = \frac{2}{a_0(x)\ c(x)}\,.$$

This can be written as

$$K(x) = A(x) + \int_{-s}^{s} H(x,\xi) \; K(\xi) \, d\xi$$

where

$$K(x) = \sqrt{\{\beta(x)\}} \, k(x)$$

and

$$H(x,\xi) = \frac{1}{\sqrt{\{\beta(x) \; \beta(\xi)\}}} \frac{1}{(x - \xi)^2} = H(\xi,x) \, .$$

The equation can be solved if the eigenvalues and eigenfunctions of the equation

$$\phi(x) = \lambda \int_{a}^{b} H(x,\xi) \; \phi(\xi) \, d\xi$$

are known. If $\beta(x)$ is constant the eigenfunctions are those given in section 4. If $\beta(x)$ is not constant the solution will probably have to be determined numerically, for though the eigenfunctions exist it may not be possible to determine them analytically.

The note gives a method of finding singular solutions which does not seem to be well known. The method depends on the expansion

$$\sum_{1}^{\infty} \phi_n(x) \, \phi_n(y) = \delta(x - y)$$

of the Dirac delta function in terms of a set of orthonormal functions $\{\phi_n(x)\}$. This equation has a meaning when $\delta(x)$ and $\phi_n(x)$ are distributions as defined by Schwartz. The singular solutions are given as a series of distributions but if these series are considered to be series of functions the sums or the Cesaro sums give the correct singular solutions.

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APPENDIX I

In this appendix, we shall prove the relations (4.6) and (4.12).

1.1. Integrating by parts Glauert's integral

$$\int_0^\pi \frac{\cos n\varphi}{\cos\varphi - \cos\theta}\, d\varphi = \frac{\pi \sin n\theta}{\sin\theta}$$

we get

$$-\frac{1}{n}\int_0^\pi \frac{\sin\varphi \sin n\varphi}{(\cos\varphi - \cos\theta)^2}\, d\varphi = \frac{\pi \sin n\theta}{\sin\theta}$$

i.e.,

$$\left(-\frac{1}{\pi n}\right)\int_0^\pi \frac{\sin\theta \sin\varphi}{(\cos\varphi - \cos\theta)^2} \sin n\varphi\, d\varphi = \sin n\theta\ .$$

It is of interest that the Hilbert-Schmidt expansion of the kernel

$$\frac{\sin\theta \sin\varphi}{(\cos\theta - \cos\varphi)^2}$$

is the divergent series

$$-2\pi \sum_1^\infty n \sin n\theta \sin n\varphi\ .$$

1.2. It is easily verified that $\cot \frac{1}{2}\varphi$ is a singular solution of the integral equation, for

$$\int_0^\pi \frac{\sin\theta \sin\varphi}{(\cos\theta - \cos\varphi)^2} \cot \tfrac{1}{2}\varphi = \frac{d}{d\theta}\int_0 \frac{1+\cos\varphi}{\cos\varphi - \cos\theta}\, d\varphi$$

$$= \frac{d}{d\theta}[\pi] = 0.$$

1.3. If we integrate the integral

$$\int_0^\pi \frac{\cos n\theta}{\cosh t - \cos\theta}\, d\theta = \frac{\pi\, \mathrm{e}^{-nt}}{\sinh t}$$

by parts and rearrange, we get

$$\int_0^\pi \frac{\sinh t \sin\theta}{(\cosh t - \cos\theta)^2} \sin n\theta\, d\theta = \pi n\, \mathrm{e}^{-nt}\ .$$

APPENDIX II

Evaluation of the Integral

$$I = \int_0^\infty \mathrm{e}^{-i\omega \cosh t} \sum_1^\infty n\, \mathrm{e}^{-nt}\, dt \quad (\textit{see} \text{ section 4.4, equation (4.13)}).$$

The sum of the series is $\mathrm{e}^{-t}/(1-\mathrm{e}^{-t})^2$ and so the integral becomes

$$I = \int_0^\infty \frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^2} \mathrm{e}^{-i\omega\cosh t}\, dt$$

$$= \tfrac{1}{2}\int_0^\infty \frac{\mathrm{e}^{-i\omega\cosh t}}{\cosh t - 1}\, dt$$

$$= \tfrac{1}{2}\int_1^\infty \frac{\mathrm{e}^{-i\omega x}}{(x-1)\sqrt{(x^2-1)}}\, dx \text{ where } x = \cosh t\ .$$

To evaluate the improper integral

$$\int_1^\infty \frac{e^{-i\omega x}}{(x-1)\sqrt{(x^2-1)}}\,dx$$

we write it as

$$\int_1^\infty \frac{e^{-i\omega x}-e^{-i\omega}}{(x-1)\sqrt{(x^2-1)}}\,dx + e^{-i\omega}\int_1^\infty \frac{dx}{(x-1)\sqrt{(x^2-1)}}\,.$$

For the first integral we have, integrating by parts,

$$\begin{aligned}\int_1^\infty \frac{e^{-i\omega x}-e^{-i\omega}}{(x-1)\sqrt{(x^2-1)}}\,dx &= \left[(e^{-i\omega x}-e^{-i\omega})\left\{1-\sqrt{\left(\frac{x+1}{x-1}\right)}\right\}\right]_1^\infty \\ &\quad + i\omega\int_1^\infty e^{-i\omega x}\left\{1-\sqrt{\left(\frac{x+1}{x-1}\right)}\right\}dx \\ &= i\omega\int_1^\infty e^{-i\omega x}\left\{1-\sqrt{\left(\frac{x+1}{x-1}\right)}\right\} \\ &= -\,i\omega\{K_0(i\omega)+K_1(i\omega)\}+e^{-i\omega}\,.\end{aligned}$$

The principal value of the second integral

$$\int_1^\infty \frac{dx}{(x-1)\sqrt{(x^2-1)}}$$

is -1. This can easily be found by using Mangler's formula for integration by parts, or since the indefinite integral of the integrand is $[1-\sqrt{\{(x+1)/(x-1)\}}]$, by taking the finite part.

Therefore

$$\begin{aligned}I &= \tfrac{1}{2}[-\,i\omega\{K_0(i\omega)+K_1(i\omega)\}+e^{-i\omega}-e^{-i\omega}] \\ &= -\,\tfrac{1}{2}i\omega\{K_0(i\omega)+K_1(i\omega)\}\,.\end{aligned}$$

This method of evaluation is due to Neumark.

This method cannot be used to evaluate the corresponding integral in compressible flow

$$J = -\int_0^\infty e^{-i\Omega\cosh t}\sum_1^\infty \frac{[se_n{}'\,0]^2}{Ne_n{}^{(2)}(0)}\,Ne_n{}^{(2)}(t)\,dt\,,$$

because of the difficulty in summing the series and so it is important to find some other method of evaluating I which will also apply to J.

It might be thought that if the integral I were evaluated by integrating term by term that the final series would have a Cesaro sum, but this is not so. Some idea of the difficulties inherent in this method of evaluating the integral can be gained by evaluating the integral I for $\omega = 0$.

We have

$$I(0) = \int_0^\infty \sum_1^\infty n\,e^{-nt}\,dt\,.$$

We have seen that the value of this integral is $-\tfrac{1}{2}$. If we integrate term by term, we get

$$I(0) = \sum_1^\infty 1\,.$$

This series has no Cesaro sum but it has an Euler-Maclaurin constant C or a Ramanujan sum $(R, 0)$ of $-\tfrac{1}{2}$ (*see* Hardy[8], Ch. XIII). Therefore

$$I(0) = -\,\tfrac{1}{2}(R, 0)\,.$$

It can be seen that $-\frac{1}{2}$ is a natural sum of the series $1 + 1 + 1 +$ if we consider the series

$$\tfrac{1}{2} + \sum_1^\infty \cos n\theta \ \cos n\varphi + \sum_1^\infty \sin n\theta \ \sin n\varphi = \pi \ \delta(\theta - \varphi) ,$$

which is obtained from equation (6.6) by taking the functions $\{\varphi_n\}$ to be the usual trigonometric functions. If we put $\varphi = 0$ we get

$$\tfrac{1}{2} + \sum_1^\infty \cos n\theta = \pi \ \delta(\theta)$$

and so in some sense the ' finite part ' of the series $\frac{1}{2} + \sum_1^\infty 1$ is zero. Again the Riemann Zeta function $\zeta(s)$ is given for $R(z) > 1$ by the series $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$. But $\zeta(s)$ is analytic throughout the s plane except for a pole at $s = 1$. The series $1 + 1 +$ may then be considered to have as sum, in some sense, $\zeta(0)$, *i.e.*, $-\frac{1}{2}$.

The integral I can now be evaluated. We have

$$I = \int_0^\infty \mathrm{e}^{-z \cosh t} \sum_1^\infty n \ \mathrm{e}^{-nt} \, dt \qquad z = i\omega$$

$$= \sum_1^\infty n b_n \text{ say,}$$

where

$$b_n = \int_0^\infty \mathrm{e}^{-z \cosh t - nt} \, dt .$$

Now

$$\tfrac{1}{2}(b_{n-1} - b_{n+1}) = \int_0^\infty \mathrm{e}^{-z \cosh t - nt} \sinh t \, dt$$

$$= \left[-\frac{1}{z} \mathrm{e}^{-z \cosh t - nt} \right]_0^\infty - \frac{n}{z} \int_0^\infty \mathrm{e}^{-z \cosh t - nt} \, dt$$

$$= \frac{\mathrm{e}^{-z}}{z} - \frac{n}{z} b_n ,$$

i.e.,

$$n b_n = \mathrm{e}^{-z} - \tfrac{1}{2} z (b_{n-1} - b_{n+1})$$

and so if we let

$$I_N = \sum_1^N n b_n$$

we get

$$I_N = \mathrm{e}^{-z} \sum_1^N 1 - \tfrac{1}{2} z (b_0 + b_1 - b_N - b_{N+1}) .$$

As $N \to \infty$

$$\sum_1^N 1 \to -\tfrac{1}{2} \qquad (R, 0)$$

$$b_N \text{ and } b_{N+1} \to 0$$

and so since

$$b_0 = \int_0^\infty \mathrm{e}^{-z \cosh t} \, dt = K_0(z)$$

and

$$b_1 = \int_0^\infty \mathrm{e}^{-z \cosh t - t} \, dt = K_1(z) - \frac{\mathrm{e}^{-z}}{z} ,$$

we have

$$I_N \to -\tfrac{1}{2} i\omega \{ K_0(i\omega) + K_1(i\omega) \} \qquad (R, 0) .$$

It will be difficult in practice to apply the Euler-Maclaurin sum formula to the series

$$I = \sum_1^\infty n \int_0^\infty e^{-z \cosh t - nt}\, dt$$

without first bringing it into the form which includes the series $\sum_1^\infty 1$, and it will be even more difficult to apply it to the series

$$J = -\sum_1^\infty \frac{[se_n{}'\, 0]^2}{Ne_n{}^{(2)}(0)} \int_0^\infty e^{-i\Omega \cosh t}\, Ne_n{}^{(2)}(t)\, dt\,.$$

The numerical value of the series J can however be found by using the previous results for I and $I(0)$. We can write J in either of the two forms

$$J = \{J - I\} - \tfrac{1}{2} i\omega\{K_0(i\omega) + K_1(i\omega)\}$$

or

$$J = \{J - I(0)\} - \tfrac{1}{2}\,.$$

The series $\{J - I\}$, $\{J - I(0)\}$ should be convergent; they should at least have a Cesaro sum.

APPENDIX III

In this appendix a simpler form will be found for the series

$$F(z) = \sum_{n=1}^\infty \int_0^\pi A(\theta) \sin\theta\, \sin n\theta\, d\theta \int_0^\infty e^{-z \cosh t - nt}\, dt \qquad z = i\omega$$

(*see* section 4.4, equation (4.14).)

Let

$$W(\theta) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos n\theta$$

where

$$\tfrac{1}{2}\pi a_n = \int_0^\pi W(\theta) \cos n\theta\, d\theta$$

and let

$$b_n = \int_0^\infty e^{-z \cosh t - nt}\, dt\,.$$

Since

$$A(\theta) = \frac{1}{\sin\theta} \frac{dW}{d\theta} + zW\,,$$

we have

$$\int_0^\pi A(\theta) \sin\theta\, \sin n\theta\, d\theta = \int_0^\pi \left[\frac{dW}{d\theta} + z \sin\theta W\right] \sin n\theta\, d\theta$$

$$= \int_0^\pi W[z \sin\theta\, \sin n\theta - \cos n\theta]\, d\theta$$

$$= \tfrac{1}{2}\pi\{\tfrac{1}{2}z(a_{n-1} - a_{n+1}) - na_n\}$$

and so

$$\frac{2}{\pi} F(z) = \sum_{n=1}^\infty \{\tfrac{1}{2}z(a_{n-1} - a_{n+1}) - na_n\} b_n\,.$$

Now

$$nb_n = e^{-z} - \tfrac{1}{2}z(b_{n-1} - b_{n+1})$$

and so

$$\frac{2}{\pi} F(z) = \tfrac{1}{2}z \sum_1^\infty \{(a_{n-1} - a_{n+1})b_n + a_n(b_{n-1} - b_{n+1})\} - e^{-z} \sum_1^\infty a_n$$
$$= \tfrac{1}{2}z(a_0 b_1 + a_1 b_0) - e^{-z} \sum_1^\infty a_n$$
$$= \tfrac{1}{2}z\{a_0 K_1(z) + a_1 K_0(z)\} - e^{-z} \left\{\tfrac{1}{2}a_0 + \sum_1^\infty a_n\right\},$$

i.e.,

$$F(z) = \tfrac{1}{2}z \int_0^\pi W(\theta)\{K_1(z) + \cos\theta\, K_0(z)\}\, d\theta - \tfrac{1}{2}\pi\, e^{-z}\, W_{\theta=0}\,.$$

APPENDIX IV

Basic Integral Relations for Compressible Flow

In this appendix we shall prove the relations (5.8) and (5.13).

It can be shown that for $0 < \theta < \varphi < \pi$ and $0 < \varphi < \theta < \pi$ the function defined by

$$H(\theta, \varphi) = \sin\theta\ \sin\varphi\ \frac{H_1^{(2)}(\alpha\,|\cos\theta - \cos\varphi|)}{|\cos\theta - \cos\varphi|}$$

satisfies the differential equation

$$\frac{\partial^2 H}{\partial\theta^2} - \alpha^2 \cos^2\theta\, H = \frac{\partial^2 H}{\partial\varphi^2} - \alpha^2\cos^2\varphi\, H\,.$$

Let

$$f(\theta) = \int_0^\pi H(\theta,\varphi)\, g(\varphi)\, d\varphi\,.$$

Then since differentiation under the integral sign is permissible, *i.e.*,

$$\frac{d^2 f}{d\theta^2} = \frac{d^2}{d\theta^2}\int_0^\pi H(\theta,\varphi)\, g(\varphi)\, d\varphi = \int_0^\pi \frac{\partial^2 H(\theta,\varphi)}{\partial\theta^2}\, g(\varphi)\, d\varphi\,,$$

we get

$$\frac{d^2 f}{d\theta^2} + (a - \alpha^2\cos^2\theta) f = \int_0^\pi \left[\frac{\partial^2 H}{\partial\theta^2} + (a - \alpha^2\cos^2\theta) H\right] g(\varphi)\, d\varphi$$
$$= \int_0^\pi \left[\frac{\partial^2 H}{\partial\varphi^2} + (a - \alpha^2\cos^2\varphi) H\right] g(\varphi)\, d\varphi$$
$$= \left[\frac{\partial H}{\partial\varphi}\, g - H\frac{\partial g}{\partial\varphi}\right]_0^\pi$$
$$+ \int_0^\pi \left[\frac{d^2 g}{d\varphi^2} + (a - \alpha^2\cos^2\varphi) g\right] H\, d\varphi\,.$$

Now $H(\theta,0) = H(\theta,\pi) = 0$ and so if

$$\frac{d^2 g}{d\varphi^2} + (a - \alpha^2\cos^2\varphi) g = 0$$

and $g(0) = g(\pi) = 0$ we have

$$\frac{d^2 f}{d\theta^2} + (a - \alpha^2\cos^2\theta) f = 0\,.$$

Now the operator

$$\frac{d^2}{d\varphi^2} + (a - \alpha^2 \cos^2 \varphi)$$

is equal to

$$\frac{d^2}{d\varphi^2} + \{(a - \tfrac{1}{2}\alpha^2) - \tfrac{1}{2}\alpha^2 \cos 2\theta\}$$

and so g is a multiple of one of the odd Mathieu functions $se_n\, \varphi$ if a is suitably chosen.

The function f satisfies the same differential equation with the same eigenvalue a, and since $H(0,\varphi) = H(\pi,\varphi) = 0$ we have $f(0) = f(\pi) = 0$. Therefore if g is a multiple of $se_n\, \varphi$, f must be a multiple of $se_n\, \theta$. Therefore

$$se_n\, \theta = \lambda_n \int_0^\pi \sin\theta\, \sin\varphi\, \frac{H_1^{(2)}(\alpha\,|\cos\theta - \cos\varphi\,|)}{|\cos\theta - \cos\varphi\,|}\, se_n\, \varphi\, d\varphi\,,$$

where λ_n is a function of α.

It can also be shown that the function

$$h(t,\varphi) = \sinh t\, \sin\varphi\, \frac{H_1^{(2)}\{\alpha(\cosh t - \cos\varphi)\}}{(\cosh t - \cos\varphi)}$$

satisfies the differential equation

$$\frac{\partial^2 h}{\partial\varphi^2} - \alpha^2 \cos^2\varphi\; h = -\frac{\partial^2 h}{\partial t^2} - \alpha^2 \cosh^2 t\; h\,.$$

Let

$$f(t) = \int_0^\pi h(t,\varphi)\, g(\varphi)\, d\varphi\,,$$

then

$$\begin{aligned} -\left[\frac{d^2 f}{dt^2} - (a - \alpha^2\cosh^2 t) f\right] &= -\int_0^\pi \left[\frac{\partial^2 h}{\partial t^2} - (a - \alpha^2 \cosh^2 t)h\right] g(\varphi)\, d\varphi \\ &= \int_0^\pi \left[\frac{\partial^2 h}{\partial\varphi^2} + (a - \alpha^2\cos^2\varphi)h\right] g(\varphi)\, d\varphi \\ &= \left[\frac{\partial h}{\partial\varphi} g - h\frac{\partial g}{\partial\varphi}\right]_0^\pi \\ &\quad + \int_0^\pi \left[\frac{d^2 g}{d\varphi^2} + (a - \alpha^2\cos^2\varphi) g\right] h\, d\varphi\,. \end{aligned}$$

Now $h(t,0) = h(t,\pi) = 0$ and so if $g(0) = g(\pi) = 0$ and

$$\frac{d^2 g}{d\varphi^2} + (a - \alpha^2\cos^2\varphi) g = 0\,,$$

i.e., if g is an odd Mathieu function then

$$\frac{d^2 f}{dt^2} - (a - \alpha^2\cosh^2 t) f = 0\,.$$

If g is a multiple of $se_n\, \varphi$ then since the eigenvalue a is unchanged, the solutions of this equation are the modified Mathieu functions $Ne_n^{(1),(2)}(t)$. From the known behaviour of $h(\varphi,t)$ for large t, we see that f must be a multiple of $Ne_n^{(2)}(t)$ and so

$$Ne_n^{(2)}(t) = \mu_n \int_0^\pi \sinh t\, \sin\varphi\, \frac{H_1^{(2)}\{\alpha(\cosh t - \cos\varphi)\}}{(\cosh t - \cos\varphi)}\, se_n\, \varphi\, d\varphi\,,$$

where μ_n is a function of α.

APPENDIX V

Calculation of the Eigenvalues λ_n, μ_n

5.1. *Calculation of the eigenvalue* μ_n.—

Let

$$Ne_n^{(2)}(t) = \mu_n \int_0^\pi \sinh t \, \sin \varphi \, \frac{H_1^{(2)}\{M\Omega(\cosh t - \cos \varphi)\}}{(\cosh t - \cos \varphi)} se_n \, \varphi \, d\varphi \, .$$

Now

$$H_1^{(2)}(r) = \frac{2i}{\pi r} + r\{P(r) \log r + Q(r)\} \, ,$$

where $P(r)$ and $Q(r)$ are power series in r. Therefore

$$\frac{1}{\mu_n} Ne_n^{(2)}(t) = \frac{2i}{\pi M\Omega} \int_0^\pi \frac{\sinh t \, \sin \varphi}{(\cosh t - \cos \varphi)^2} se_n \, \varphi \, d\varphi + R_n(t) \text{ say.}$$

If $\qquad se_n \, \varphi = \sum_1^\infty a_r \sin r\varphi$, then

$$\int_0^\pi \frac{\sinh t \, \sin \varphi}{(\cosh t - \cos \varphi)^2} se_n \, \varphi \, d\varphi = \sum_1^\infty a_r \int_0^\pi \frac{\sinh t \, \sin \varphi}{(\cosh t - \cos \varphi)^2} \sin r\varphi \, d\varphi$$

$$= \sum_1^\infty r \, a_r \, e^{-rt} \, .$$

It can easily be seen that $R_n(t) \to 0$ as $t \to 0$ and so in the limit we have

$$\frac{1}{\mu_n} Ne_n^{(2)}(0) = \frac{2i}{\pi M\Omega} \sum_1^\infty r \, a_r$$

$$= \frac{2i}{\pi M\Omega} se_n' \, 0 \, ,$$

i.e.,

$$\mu_n = \frac{\pi}{2i} M\Omega \frac{Ne_n^{(2)}(0)}{se_n' \, 0} \, .$$

5.2. *Calculation of the Eigenvalue* λ_n.—

Let

$$se_n \, \theta = \lambda_n \int_0^\pi \sin \theta \, \sin \varphi \, \frac{H_1^{(2)}(M\Omega \, |\cos \theta - \cos \varphi|)}{|\cos \theta - \cos \varphi|} se_n \, \varphi \, d\varphi \, . \qquad \text{(5.2.1)}$$

If we let $\theta \to 0$ we get

$$\lambda_n \int_0^\pi \sin \varphi \, \frac{H_1^{(2)}(M\Omega(1 - \cos \varphi))}{(1 - \cos \varphi)} se_n \, \varphi \, d\varphi = \lim_{\theta \to 0} \frac{se_n \, \theta}{\sin \theta} = se_n' \, 0 \, . \qquad \text{(5.2.2)}$$

The integrals in 5.2 are principal-value integrals but the integral in 5.1 is a proper integral which becomes improper at $t = 0$. In order, therefore, to equate the limit as $t \to 0$ of the integral 5.1 with the integral in (5.2.2) we must take the principal value or finite part of 5.1 at $t = 0$. The finite part of $\{Ne_n^{(2)}(t)/\}\sinh t$ at $t = 0$ is $Ne_n^{(2)\prime}(0)$ and so equating the two limit integrals we get

$$\lambda_n \, Ne_n^{(2)}(0) = \mu_n \, se_n' \, 0 \, ,$$

i.e.,

$$\lambda_n = \frac{\pi}{2i} M\Omega \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} \, .$$

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