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Parts I and II

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COMMUNICATED BY THE DIRECTOR-GENERAL OF SCIENTIFIC RESEARCH (AIR), MINISTRY OF SUPPLY

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PART I

Summary.—A general method (requiring the aid of a digital computer) is described for deriving the influence coefficients of any type of wing, and hence for evaluating its strength and stiffness characteristics. The method allows for shear deflections, and hence implicitly takes account of effects like shear lag and warping of wing cross-sections. A rapid method accurate enough to serve as a basis for dynamical calculations is first described, and secondly a more rigorous method on which to base final stressing of the structure.

1. *Introduction.*—In a recent lecture[1] the writer discussed the profound impact of the electronic digital computer on the problem of computing the strength and dynamical characteristics of aeroplane structures. In the same lecture he put forward a method of deriving the influence coefficients of thin wings of any shape by exploiting three simplifying factors: plate theory, the matrix notation and the digital computer. It will be appreciated that, once a comprehensive enough set of influence coefficients are obtained for a structure, its stiffness and strength under any kind of external loading can at once be deduced. The same set also constitutes the essential data for computing the natural frequencies and modes of the structure. By seeking to establish a sound and rapid method of deriving a comprehensive set of influence coefficients for the complete aeroplane, one is in effect seeking a royal road to the solution of all strength, stiffness and vibration problems of an aeroplane structure. With regard to wing structures, to the consideration of which the present report is confined, a general method of derivation, once established, has the great advantage of making the most unorthodox type of wing amenable to what may be described as a stereotyped approach.

A major assumption made in the method above alluded to was that shear deflections could be neglected in the expression for the curvature of the wing. This is a legitimate assumption when treating thin near-solid wings, but open to question for more orthodox designs. The main purpose of the present report is therefore to modify the original method so as to be applicable to any kind of wing. At the same time the opportunity is taken to put on record the original method, which up to the present appears only as an appendix to a lecture embracing a much wider thesis.

* R.A.E. Report Struct. 168, received 7th March, 1955.
R.A.E. Report Struct. 209, received 8th November, 1956.

2. *Description of Original Method—Shear Deflections Neglected.*—The essence of the original method above referred to is to use elementary plate theory for calculating, by means of a digital computer, the stiffness coefficients associated with a comprehensive set of discrete stations well distributed over the wing. A further essential factor in the method is the inversion, again by the help of the digital computer, of the stiffness matrix thus obtained to form the flexibility matrix, whose elements consist of the influence coefficients finally required.

Now elementary plate theory cannot be legitimately applied to a wing unless two major conditions are satisfied. These are:

(*a*) that curvatures produced by shear deflections are negligible compared with bending curvatures

(*b*) that deflections (as calculated by 'small deflections theory') relative to a developable surface are small compared with the thickness of the wing.

The thinner the wing the more nearly is condition (*a*) satisfied, but the more doubtful is the satisfaction of condition (*b*).

For nearly solid wings with thickness/chord ratio of 5 per cent or less condition (*a*) is approximately enough satisfied, and may even be satisfied for considerably thicker wings.

The satisfaction of condition (*b*) naturally depends on the severity of the load applied. For a light-alloy wing with a spanwise curvature corresponding to a bending stress of 40,000 lb/in.^{2}, the deflections due to anti-elastic curvature are still approximately within the limitations of the 'elementary' or 'small deflections' theory when the thickness/chord ratio is only 5 per cent. When part of the chord is taken up by ailerons or flaps this figure could drop to 4 per cent, or even 3 per cent, without violating the conditions under which elementary plate theory is applicable. For a steel wing, of course, the figure of 40,000 lb/in.^{2} goes up in the ratio of the Young's Modulus to 120,000 lb/in.^{2}.

2.1. *Basis of Method.*—It is hardly necessary perhaps to point out that, in expressing the deformation of a highly redundant structure under load by means of influence coefficients, good accuracy requires that an adequate number of stations should be used. For, the greater the number, the more nearly can the discontinuous station deflections represent the essentially continuous true deflection shape. Since, however, the amount of computational work goes up somewhat faster than the cube of the number of stations, a digital computer is an indispensable adjunct to the proposed method.

As shown in Ref. 1, it is now accepted that it is much easier to derive the influence coefficients of a highly redundant structure indirectly, by first deriving the stiffness coefficients, rather than directly. By using the machine (*i.e.*, the digital computer), the stiffness matrix so obtained can then be inverted to give the corresponding flexibility (or influence) matrix, from which the deflections and stresses in the structure can be obtained for any applied load.

To look at the matter in a slightly different way, we may consider the differential equation (or set of equations) that relates the deflections of a structure to the loads applied to it. Always on the left-hand side of the equation is a function of the deflection and its derivatives with respect to the co-ordinates of the system, and on the right-hand side appears the arbitrary external loading. What the engineer is asked to do is to find, from a given external loading, the consequent deformation. He is never given the deformation of the structure and requested to determine the loading that produced it, because such a problem has no interest. It is only since the advent of the digital computer that the facility with which this inverse problem can be solved has become open to exploitation. For, after the loading associated with a given set of hypothetical deflections has been determined *in general terms*, the machine is capable of solving with great facility the set of linear equations by which the loads are thus expressed in terms of the deflections. In other words the machine solves the real problem whereas we solve only the easy inverse problem.

2.2. *Derivation of Stiffness Coefficients.*—The problem of finding the stiffness coefficients of a wing is that of evaluating the reaction at each station necessary to maintain an arbitrary set of wing deflections. For this purpose, the wing cannot be regarded as isolated, but must be considered as an integral part of the complete aircraft. This means that the influence coefficients must relate all wing deflections to some convenient three-point datum as described in Ref. 1.

One of the advantages of dealing with stiffness, rather than influence, coefficients is that the number of coefficients is greatly reduced. This is readily seen from the fact that, taking a simple example, the force at a wing-tip station due to a unit displacement at a root station (the displacements at all other stations being zero) is vanishingly small, but the displacement at a tip station due to a unit force at the root station (with all other forces zero) is not. That is why the reaction (and hence stiffness coefficients) for a wing station can be expressed in terms of the deflections of only those stations immediately adjacent to it. As a consequence, in illustrating the application of plate theory to the wing, we can confine attention to a restricted area of wing.

2.2.1. *Choice of stations.*—Since the shear stiffness of ribs and shear webs is, in this preliminary approach, assumed infinite, the choice of location for the stations need have no relation to the disposition of the shear-carrying members inside the wing. If, in addition, reliance is placed on a thick skin alone to resist bending stresses, location of the stations becomes a matter of choice, to be decided largely by computational convenience.

In order to demonstrate the essential character of the method without introducing unwieldy formulae, the wing is assumed to be of the thick-skin type, so that, in any particular station, the I of the wing section per unit width is the same for a chordwise as for a spanwise section. On this basis it is legitimate to distribute the stations over the wing surface in a regular pattern; for choice a chess-board pattern with a station at the corner of each square. Fig. 1 shows such a square mesh of stations suitably numbered from 0 to 12.

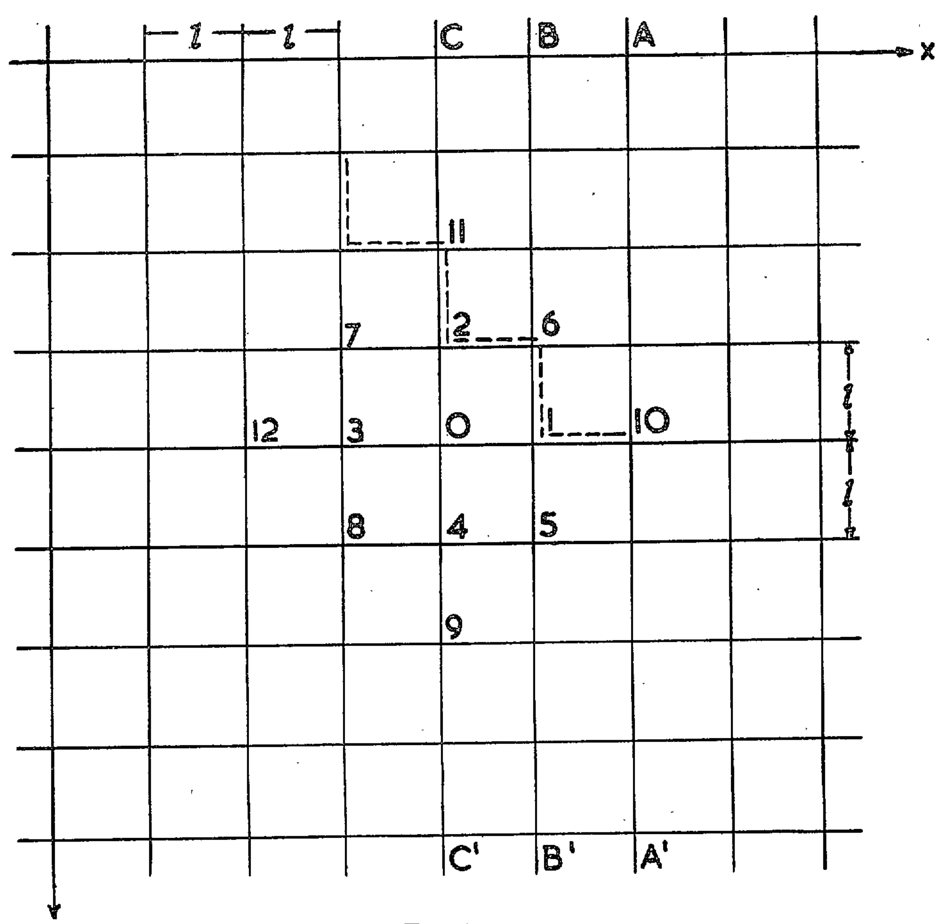


FIG. 1.

2.2.2. *Adaptation of plate theory.*—We can now show that the reaction R_0 at the central station 0 can be expressed in terms of the 13 deflections $w_0, w_1, w_2, \ldots w_{12}$ and the I per unit width of the spanwise and chordwise cross-sections, *i.e.*, per unit length along x and y in the figure. This I will vary with the skin thickness and the depth of the wing section, and there is no difficulty in taking account of its different values at different stations. It is only for convenience of writing therefore that the I's at stations 1, 2, 3 and 4 are assumed equal.

Following the standard notation for plate theory, let:

M_x = bending moment per unit section parallel to the y-axis

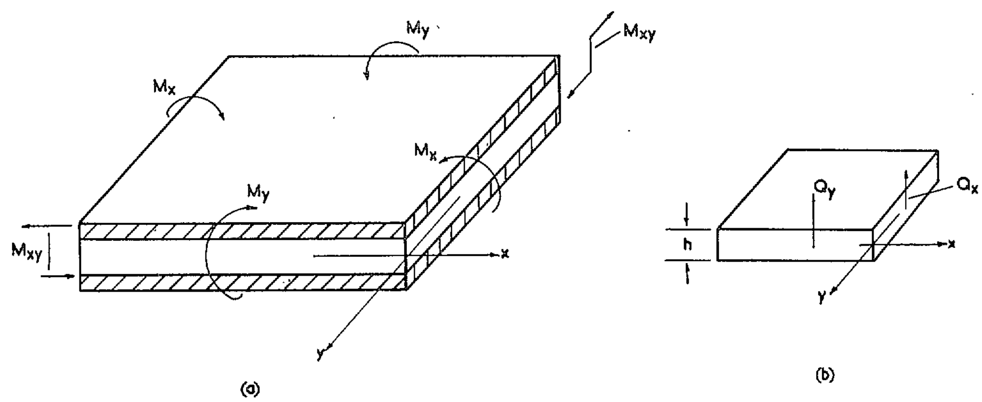
M_y = bending moment per unit section parallel to the x-axis

M_{xy} = $-M_{yx}$ = twisting moment per unit section parallel to the y-axis

Q_x = shear per unit section parallel to the y-axis

Q_y = shear per unit section parallel to the x-axis.

The positive directions of these moments and shears are shown in Figs. 2a and 2b.



FIGS. 2a and 2b.

From standard formulae[3,4]:

$$M_x = \frac{EI}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad (1)$$

$$M_y = \frac{EI}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad (2)$$

$$M_{xy} = 2GI\frac{\partial^2 w}{\partial x\,\partial y} = \frac{EI}{(1+\nu)}\frac{\partial^2 w}{\partial x\,\partial y}, \qquad (3)$$

where w is the deflection normal to the plane x, y, reckoned positive upward.

For equilibrium:

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} + Q_y = 0 \qquad (4)$$

$$-\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} + Q_x = 0\,. \qquad (5)$$

Similarly:

$$(R_y)_0 = D(H_4 + H_2 - 2H_0)/l\,. \qquad \qquad (16)$$

The total reaction at station 0 is therefore:

$$\begin{aligned} R_0 &= \{(R_x)_0 + (R_y)_0\}l \\ &= D(H_1 + H_2 + H_3 + H_4 - 4H_0)\,. \qquad \qquad (17) \end{aligned}$$

Expanding the H's in terms of the w's by using (9), (10) and (11) to obtain H_0, H_3 and H_1 and similar equations to obtain H_2 and H_4, we finally write:

$$\begin{aligned} R_0 = D/l^2 &\begin{bmatrix} & w_{10} + w_0 + w_5 + w_6 - 4w_1 \\ + & w_6 + w_7 + w_0 + w_{11} - 4w_2 \\ + & w_0 + w_{12} + w_8 + w_7 - 4w_3 \\ + & w_5 + w_8 + w_9 + w_0 - 4w_4 \\ - & 4(w_1 + w_3 + w_4 + w_2 - 4w_0) \end{bmatrix} \\ = \frac{D}{l^2} &\Big\{20w_0 - 8(w_1 + w_2 + w_3 + w_4) \\ &+ 2(w_5 + w_6 + w_7 + w_8) + (w_9 + w_{10} + w_{11} + w_{12})\Big\} \\ = \frac{D}{l^2} &\left[20w_0 - 8\sum_{r=1}^{4} w_r + 2\sum_{r=5}^{8} w_r + \sum_{r=9}^{12} w_r\right]. \qquad \qquad (18) \end{aligned}$$

The bracketed quantity in (18) represents a standard pattern of station deflections that can immediately be applied to write down the reaction at any other station within two pitches of the wing-plan boundary.

It will be noted from (17) that the I of the wing section comes in only for the central station 0 and the four inner stations 1, 2, 3 and 4. At each of these stations (if the bending resistance is provided by a thick skin alone, thus making the I in the x and y directions the same) the change in I with change in wing-section depth must normally be taken into account. This merely means that the appropriate constants have to be taken inside the bracket in (18) instead of being included in the stiffness D outside the bracket, as they can be when the I's are equal. The case where the I's in the x and y directions are different presents no real difficulty and is treated later.

The coefficients of the thirteen station deflections in (18) are, of course, the stiffness coefficients that give the force (or reaction) at station 0 due to any possible deformation of the wing structure. If the whole wing is represented by (say) 50 stations it means that all but 13 of them have zero stiffness coefficients.

The fact that these coefficients can be written down in such a systematic fashion, at least for the interior stations, suggests that there should be little difficulty in programming the machine to do the job.

2.2.3. *Boundary conditions.*—A main advantage of the approach *via* stiffness coefficients is that the problem of satisfying boundary conditions ceases to be a problem. This follows from the fact that the wing boundary conditions can all be expressed in terms of wing deflections or their derivatives, in terms, in other words, of a set of given quantities or quantities easily deduced from their boundary conditions.

A few typical boundary conditions are discussed in the following paragraphs but a detailed discussion is relegated to Part II of the report.

Boundary conditions at free edge.—As an indication of how to deal with boundary conditions, suppose the line AA^1 passing through station 10 in Fig. 1 to be a free edge. As the whole cluster of stations is unbroken, equation (18) is still applicable for writing down the reaction at station 0.

If BB^1 became a free edge, however, the x curvature at station 1 can no longer be expressed in terms of the wing deflections, since there is now no station 10 to give the slope between 1 and 10. But as the y curvature at 1 is still fully defined, we can use the fact that:

$$(M_x)_1 = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_1 = 0 \quad \ldots \quad (19)$$

to give $\partial^2 w/\partial x^2$ as $(-\nu \partial^2 w/\partial y^2)$. The same procedure can obviously be applied directly to any other straight boundary.

Suppose the boundary is the stepped shape defined by the stations 11, 2, 6, 1, 10 shown dotted in Fig. 1. To obtain the x and y curvatures at a re-entrant corner such as station 1, we use the standard formula, but for a projecting corner such as station 6 one needs only to use the two conditions:

$$\left.\begin{aligned}(M_x)_6 &= D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_6 = 0\\(M_y)_6 &= D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)_6 = 0\end{aligned}\right\}, \quad \ldots \quad (20)$$

to show that:

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_6 = \left(\frac{\partial^2 w}{\partial y^2}\right)_6 = 0 . \quad \ldots \quad (21)$$

Any shape of boundary can in this way be approximately represented by a suitably stepped outline.

Boundary conditions at wing root.—The conditions at the wing root are just as straightforward. They are different however for the symmetrical and the anti-symmetrical displacements.

For the symmetrical case, the x slope, by symmetry, is reversed as the plane of symmetry is crossed, a fact that, for a wing passing straight through the fuselage, enables the wing x curvature at the intersection with the plane of symmetry to be at once written down. For wing spars that bridge the fuselage by some kind of frame, the root slope at the side of such a frame is given by the deflections of the frame itself. But if the wing spar is pin-jointed to the fuselage side, the same condition as for a free edge applies, *i.e.*, zero spanwise bending moment.

For the anti-symmetrical case, straight-through spars have zero bending moment at the plane of symmetry. A spar fixed to the fuselage has its root slope defined as in the symmetrical case, while a pin-jointed root gives as before zero bending moment.

Along the y direction the curvature is, by definition, completely defined.

2.2.4. *Conversion of stiffness into influence coefficients.*—The conversion of the stiffness coefficients found by the above method into influence coefficients by matrix inversion has been fully covered in Ref. 1, and will not be discussed here.

2.2.5. *Use of machine for computing stiffness coefficients.*—When a fairly large number of stations are used on each side of the plane of symmetry of an aeroplane structure, it is essential that the procedure of deriving the stiffness coefficients should be systematised to the last degree. For this enables the machine to be programmed thoroughly enough to require feeding only with such basic wing-design quantities as:

(*a*) the spacing of the stations

(*b*) the wing-section properties at each station

(*c*) the wing plan-form.

3. *Approximate Method for Taking Account of Shear Deflections.*—Shear flexibility in bending complicates the issue because, by introducing a curvature of its own, it impairs the otherwise simple relation between resultant curvature and bending moment. Since shear flexibility is also

a governing factor in such secondary effects as shear lag and warping of wing cross-sections, no structural analysis can lay claim to accuracy without taking it into account. Only in the case of solid, or nearly solid, thin wings can it be legitimately neglected, when the method already described may be used without modification.

The effect of shear flexibility is two-fold; in the first place it reduces the wing stiffness and in consequence affects the frequencies and modes, and in the second place it modifies the stress distribution in the wing. So far as the first, or dynamic, effect is concerned, an approximate method is here put forward that appears to be quite adequate. The approximation used may not however be good enough in all cases for the purpose of the static stressing of a wing, where it is desired to take full account of such secondary effects as shear lag and the warping of wing cross-sections. For this, if the utmost accuracy is required, it is necessary to increase the number of degrees of freedom three-fold in order to allow for x and y displacements in the plane of the wing in addition to the z displacement normal to that plane*.

What makes it possible, in the rigid shear method already described, to completely define the deformation of the wing in terms only of the deflections normal to the plane of the wing, is the fact that the normal deflections, in defining the slopes of the neutral plane of the wing in the x and y directions, thereby also define the displacement of the skin in those directions. This follows from the fact that, with infinite shear stiffness (of ribs and spar webs) a point on the skin cut by a normal to the neutral plane before deformation remains on the same normal after deformation. Any method of defining the deformation of a wing by the normal displacements w alone of a number of stations distributed over it must therefore specify how the dependent displacements u and v in the x and y directions are related to the independent deflections w.

In the approximate method now to be described this is done in the following way. The wing is imagined to be rigid in shear to begin with so that the method already described is immediately applicable. In this way the flexibility matrix for bending is derived, which gives the deflection at every station in terms of any arbitrary system of applied loads. Suppose the wing to have taken up its appropriate contour of displacements under a particular external load distribution. If now, while the u and v displacements are constrained by some external agency to remain the same, full shear flexibility is restored to the ribs and webs, the deflections w will generally increase. The amount of this increase under the external load applied, since we are dealing with linear conditions, is independent of the bending displacements already in existence and these may therefore be disregarded for the purpose of obtaining the extra shear deflections. In fact we can consider the unloaded wing subjected to a system of displacements w of a kind that allows no accompanying u or v displacement for any station, and that therefore allows only the components of shear displacement represented by $\partial w/\partial x$ and $\partial w/\partial y$ ($\partial u/\partial z$ and $\partial v/\partial z$ being both zero).

The set of reactions R necessary to hold the wing in its deflected position are easily written down, because the shear in the shear-carrying internal structure can, under the special conditions visualised, be expressed in terms of the first derivatives of the station deflections and not in terms of their third derivatives as in the rigid-shear case above treated. In Fig. 1 the shears in the shear webs that meet at station 0 are given by:

$$\left.\begin{aligned} S_{0,1} &= (K)_{0,1}\,(w_1 - w_0)\,\frac{Gh(t)_{0,1}}{l} \\ S_{3,0} &= (K)_{3,0}\,(w_0 - w_3)\,\frac{Gh(t)_{3,0}}{l} \\ S_{0,4} &= (K)_{0,4}\,(w_4 - w_0)\,\frac{Gh(t)_{0,4}}{l} \\ S_{2,0} &= (K)_{2,0}\,(w_0 - w_2)\,\frac{Gh(t)_{2,0}}{l} \end{aligned}\right\}, \quad \ldots \qquad (22)$$

* *See* section 4.

where h is the thickness of the wing, $(t)_{nm}$ the thickness of the shear web between stations n and m, and the K's are the appropriate shear constants for the several panels. The external upward reaction necessary to equilibrate this system of vertical shears at station 0 is then:

$$(R_0)_S = (S_{3,0} - S_{0,1}) + (S_{2,0} - S_{0,4}) \,. \qquad \qquad (23)$$

In this case every station, whether on the boundary or not, can be treated in the same way, so that the stiffness matrix is obtained very easily. On inverting this matrix we obtain the flexibility matrix for shear, *i.e.*, the shear deflection (of the particular kind concerned—$\partial w/\partial x$ or $\partial w/\partial y$) at each station under any arbitrary external loading. The complete flexibility matrix is now at once written down since it merely requires the straightforward addition of the two matrices, the one of bending and the other of shear.

3.1. *Remarks on the Approximation.*—Certain points regarding the above approximation may be noticed. A minor but interesting point is that it would not have been feasible to superpose the shear and bending reactions at each station in turn, so as to combine bending and shear reactions in the same stiffness matrix, the inversion of which would then give the resultant flexibility matrix in one operation. Any attempt to do so introduces the reciprocals of the deflections. The situation may be summarised by saying that, whereas the bending and shear stiffness matrices cannot be directly added together, the corresponding flexibility matrices can. Since it is generally expedient to derive the flexibility matrix *via* the stiffness matrix, the only approach here is to derive the bending and shear stiffness matrices separately, to invert them separately so as to obtain the corresponding two flexibility matrices, and finally to add these together to give the resultant flexibility matrix.

A major point is the fact that the deflections obtained are correct only so long as the constraints needed to prevent u and v displacements during the shear deformation are maintained. Their final removal will produce not only u and v displacements additive to those induced by the initial bending deflections but also further w deflections, with of course an attendant modification of the stresses.

It will be noticed that the problem remaining to be solved, *i.e.*, the effect of removing the constraints, no longer involves forces normal to the wing plane but only forces in that plane. This is equivalent to saying that the resultant of the shears in the four shear-carrying panels that meet at each station vanishes.

3.2. *Approximate Method Applied to Single Box Cell.*—Obviously, the greater the shear flexibility, of a structure in relation to its bending flexibility, the more important become the constraining forces. It was therefore thought desirable to see how the method works under extremely unfavourable conditions, where the shear deflections constitute the major part of the total. For this purpose a single-cell torsion box was chosen and assumed fixed along one side ADA′D′ and loaded by a couple consisting of two vertical forces P as shown in Fig. 3.

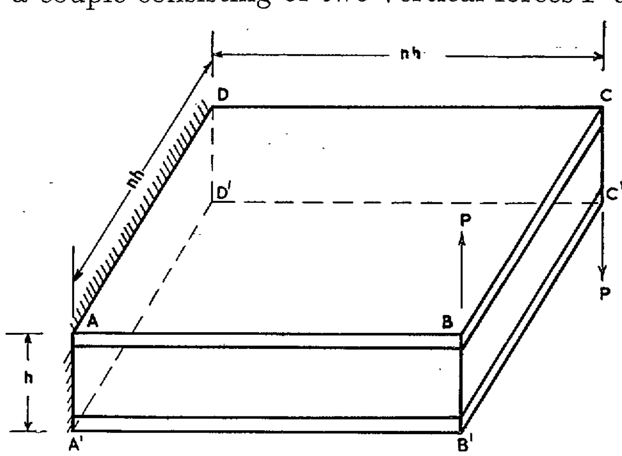


FIG. 3.

Bending stiffness is introduced by edge booms, which are connected by panels capable only of resisting shear. The box is square in plan and has a length n times greater than its depth h.

The equal and opposite deflections at B and C were calculated:

(*a*) correctly by any of the various methods available

(*b*) assuming the sides and end infinitely stiff in shear

(*c*) assuming the four end corners to be constrained to move only vertically, in other words assuming infinite stiffness for the booms and the two horizontal panels

(*d*) under forces equal and opposite to the resultant constraints required by (*b*) and (*c*).

Deflections (*b*) and (*c*) are the constrained bending and shear deflections above discussed.

Deflections (*d*) arise from the unbalanced forces brought into action by removal of the constraints and must always take the form, shown in Fig. 4, of tractive forces along the horizontal edges of constant amount S per unit length. The structural analysis of this box is given in Appendix I to this Part, where it is shown that if:

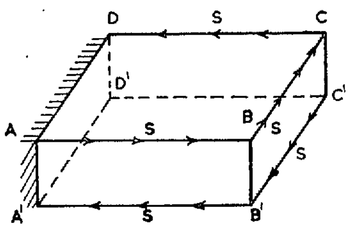


FIG. 4.

t_1 = thickness of side panels

t_2 = thickness of top and bottom panels

t_3 = thickness of end panel

A = cross-sectional area of edge booms,

the unbalanced residual edge force S per unit length has the value:

$$S = \frac{P}{l}\left[\frac{2g + 1\cdot067}{2g + 2\cdot133} - \frac{t_1/t_3}{1 + 2t_1/t_3}\right], \qquad (24)$$

where

$$g = \lambda A/(t_2 nh) \qquad (25)$$

and the ratio G/E of the elastic moduli has been taken to be equal to $0\cdot4$. This shows that for certain values of A and of the ratio (t_1/t_3) the edge force S vanishes and the approximation is exact.

Perhaps the best way of showing how effectively the approximation takes account of the shear flexibility of the vertical panels is to quote numerical values for various combinations of relative panel thicknesses and boom cross-sections. This is done in Table 1 in which the approximate deflection $(w_b + w_s)$ is shown against the true deflection.

In looking at the figures in Table 1,

TABLE 1

g	n	t_1/t_2	t_3/t_2	By approximation			w_{true}
				w_b	w_s	$(w_b + w_s)$	
1	2	1	0·2	0·517	1·43	1·947	1·95
1	2	1	0·0	0·517	2·0	2·517	3·07
1	2	1	1	0·517	0·67	1·18	1·55
1	2	1	10	0·517	1·67	2·18	2·22
0·5	2	0·1	0·05	0·7	10	10·7	10·9
0·5	2	0·05	0·1	0·7	8	8·7	10·78
0·5	4	1	0·2	2·73	2·86	5·59	5·63
1·0	4	1	0·2	2·07	2·86	4·93	4·93
1·0	4	1	0·0	2·07	4	6·07	8·37

one must remember that the box structure here considered and the kind of loading assumed represent an extreme case, in that the shear deflection accounts for much the greater part of the total deflection. Even in this highly unfavourable example the addition of the shear deflection w_s to the bending deflection w_b brings the total $(w_s + w_b)$ well into line with the true value, whereas the neglect of w_s gives a hopelessly inaccurate result.

If, instead of the two forces P being applied in opposite directions, they are applied in the same direction, thus changing the torque into a transverse load, the approximation becomes exact. It follows that, since any two upward forces at the outer corners B and C of the box can be represented by a transverse force and a couple, the approximation approaches closer to the truth the more nearly equal the two forces become. It is to be expected therefore that in most practical cases this approximate method should be satisfactory.

A final point to notice is that, if the deflections obtained by the approximate method are used for stressing the wing, the stresses in the skin and its reinforcements (including spar flanges, etc.) are obtained directly from the bending deflections (*i.e.*, w_b) since the superposed shear deflections (w_s), by virtue of the constraints, have no effect on the skin stresses. This means that, although the allowance made for shear flexibility makes a valuable adjustment to the deflections, it has no effect on the skin stresses. To obtain the latter with sufficient accuracy a more rigorous method, now to be described, is needed in some cases.

4. *More Rigorous Method—Suitable for Both Wing Stressing and Dynamic Calculations.*—The above approximate method depends essentially on assuming an implicit and arbitrary relation between the normal displacement w and the displacements u and v in the plane of the wing. This means that the u and v displacements are subject to an external constraint that has later to be liquidated. To remove this source of inaccuracy it is proposed to use the rigorous method of regarding the three displacements u, v and w at each station as independent variables. The disadvantage of this course is the three-fold increase in the number of variables it entails, since three independent displacements are now associated with each station.

There are compensations, however, that mitigate this disadvantage. One of these is the fact that, consequent on the relevant difference equations being only of the second order, the reaction at any station can now be expressed in terms of only eight of the adjoining station deflections instead of the 12 previously necessary. It is also to be noted that, once the influence coefficients connecting normal loads and displacements are found, the dynamical matrix is of the same order as before, the number of elements being unaltered. It follows that the task of matrix iteration for the natural frequencies and modes and all aero-elastic calculations can proceed just as if the u and v displacements never entered the problem.

4.1. *Derivation of Stiffness Coefficients in Terms of Station Displacements u, v and w.*—

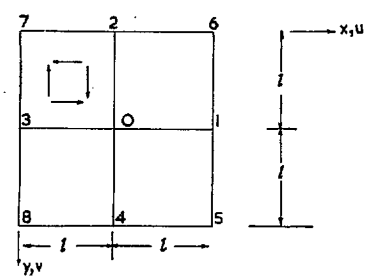
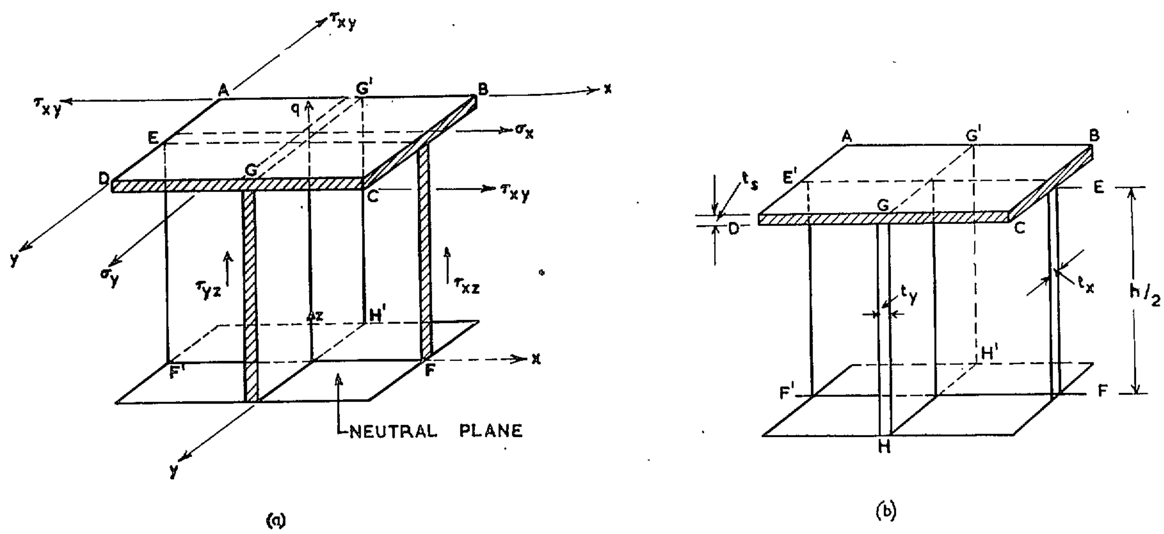


FIG. 5.

The reactions X, Y and Z in directions x, y and z at any interior station 0 (Fig. 5) can be completely expressed in terms of the u, v and w displacements at the eight surrounding stations 1, 2, . . . 8.

Relevant equations in differential form.—In order to facilitate the writing down of the appropriate difference equations, the three equations of equilibrium that govern the relation between the three displacements u, v, w and the applied forces are first expressed in differential form.



FIGS. 6a and 6b.

In Figs. 6a and 6b let ABCD represent a square element of area of the top skin of unit side. Assuming that there is an identical bottom skin symmetrically situated relative to the neutral plane FF′HH′, we need only consider the equilibrium of the top skin element. The stress symbols used in Fig. 6 follow standard practice and need not be defined here, except to note that the direct stresses in the booms or stringers are distinguished from those in the adjacent skin by having the symbol σ' as against σ for the skin. We assume here that stringers and ribs are parallel to the co-ordinate axes. When they are not the method described in Part II must be used.

Let

A_x = average or equivalent flange area per unit width of y due to booms and stringers in direction x at station 0

A_y = average or equivalent flange area per unit width of x due to booms and stringers in direction y at station 0

A_s = equivalent flange area of skin in x and y direction at station 0 per unit width

t_s = thickness of skin

t_x = equivalent web thickness in direction x per unit width along y

t_y = equivalent web thickness in direction y per unit width along x

l = pitch of stations

h = depth of wing section

X = applied force in direction x per unit area of skin

Y = applied force in direction y per unit area of skin

Z = applied force in direction z per unit area of skin.

The three equations of equilibrium of forces in the x, y and z directions may now be written down at once.

In the x direction :

$$A_x \frac{\partial \sigma_x'}{\partial x} + t_s \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) - t_x \tau_{xz} = - X \, , \quad \ldots \qquad (26)$$

where the stress σ_x' in the boom (not being subject to Poisson's Ratio) is different from σ_x in the adjacent skin.

In the y direction, similarly:

$$A_y \frac{\partial \sigma_y'}{\partial y} + t_s \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right) - t_y \tau_{yz} = - Y \, . \quad \ldots \qquad (27)$$

In the z direction:

$$t_x h \frac{\partial \tau_{xz}}{\partial x} + t_y h \frac{\partial \tau_{yz}}{\partial y} = - Z \, . \quad \ldots \qquad (28)$$

In a wing the forces X and Y are always zero.

Putting:

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) \\ \sigma_x' &= E \frac{\partial u}{\partial x} \\ \sigma_y &= \frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) \\ \sigma_y' &= E \frac{\partial v}{\partial y} \\ \tau_{xy} &= G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{xz} &= G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \tau_{yz} &= G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \end{aligned}\right\} , \quad \ldots \qquad (29)$$

we write (26), (27) and (28) in the form:

$$EA_x \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + t_s \left\{\frac{E}{1-\nu^2} \; \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) + G \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} - t_x G \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = - X \, , \quad (30)$$

$$EA_y \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) + t_s \left\{\frac{E}{1-\nu^2} \; \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) + G \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} - t_y G \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = - Y \, , \quad (31)$$

$$Ght_x \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + Ght_y \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = - Z \, . \quad (32)$$

Since, in all practical cases of wing loading, both forces X and Y disappear, it is theoretically possible, to eliminate u and v and so express the displacement w in terms of the loads Z applied normal to the surface. This, however, would introduce differential equations of the sixth order, as well as the necessity of subsequently evaluating u and v in order to obtain the stresses. Such raising of the order of the differential equations appears to be regarded by Benscoter and MacNeal[5] as unavoidable if a digital computer is used for their solution, in contrast with their own method of solution by ' analog computer ', in which first-order difference equations are alone used. The method presented here of taking the three displacements at each station as independent variables shows that this difficulty need not arise.

4.2. *Boundary Conditions.*—The boundary conditions in this method of approach are similar to those discussed in section 2.2.3 in so far as the direct stresses are concerned. For example, in Fig. 1, if CC' is a free edge, the direct edge load must be zero, *i.e.*,

$$\left.\begin{aligned} &El\left\{A_x\frac{\partial u}{\partial x}+\frac{A_s}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)\right\}=0 \\ \text{or}\quad &\frac{\partial u}{\partial x}=\frac{-\nu A_s}{A_s+(1-\nu^2)A_x}\;\frac{\partial v}{\partial y}\,, \end{aligned}\right\} \qquad (38)$$

since the shears τ_{xy} and τ_{xz} must each be zero.

As regards boundary shear displacements, for a free edge the shear stress, as just noted, is zero, so that at a free edge parallel to the y-axis,

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\,,\qquad \frac{\partial w}{\partial x}=-\frac{u}{h/2}\,. \qquad (39)$$

At a fixed edge, such as the wing intersection with the plane of symmetry, the boundary conditions, both for direct and shear displacements, are even more simple, in that the wing may be imagined joined on to its symmetrical, or antisymmetrical, image as already discussed in section 2.2.3.

4.3. *Form Taken by Stiffness and Flexibility Matrices.*—When the X, Y and Z reactions have been obtained for each of the stations concerned in terms of the displacements, we can write down the complete set of $3n$ equations as follows, where, for example,

$\overline{xx}_{rs}$ denotes the reaction in direction x at station r due to unit displacement in direction x at station s

$\overline{yx}_{rs}$ denotes the reaction in direction y at station r due to unit displacement in direction x at station s

$\overline{zy}_{rs}$ denotes the reaction in direction z at station r due to unit displacement in direction y at station s,

etc.

$$\left.\begin{aligned}
&\left\{\begin{aligned}
X_1 &= (\overline{xx}_{11}u_1+\overline{xx}_{12}u_2+\ldots+\overline{xx}_{1n}u_n)+(\overline{xy}_{11}v_1+\overline{xy}_{12}v_2+\ldots+\overline{xy}_{1n}v_n)+(\overline{xz}_{11}w_1+\ldots)\\
X_2 &= (\overline{xx}_{21}u_1+\overline{xx}_{22}u_2+\ldots+\overline{xx}_{2n}u_n)+(\overline{xy}_{21}v_1+\overline{xy}_{22}v_2+\ldots+\overline{xy}_{2n}v_n)+(\overline{xz}_{21}w_1+\ldots)\\
&\cdots\cdots\\
X_n &= (\overline{xx}_{n1}u_1+\overline{xx}_{n2}u_2+\ldots+\overline{xx}_{nn}u_n)+(\overline{xy}_{n1}v_1+\overline{xy}_{n2}v_2+\ldots+\overline{xy}_{nn}v_n)+(\overline{xz}_{n1}w_1+\ldots)
\end{aligned}\right.\\
&\left\{\begin{aligned}
Y_1 &= (\overline{yx}_{11}u_1+\overline{yx}_{12}u_2+\ldots+\overline{yx}_{1n}u_n)+(\overline{yy}_{11}v_1+\overline{yy}_{12}v_2+\ldots+\overline{yy}_{1n}v_n)+(\overline{yz}_{11}w_1+\ldots)\\
Y_2 &= (\overline{yx}_{21}u_1+\overline{yx}_{22}u_2+\ldots+\overline{yx}_{2n}u_n)+(\overline{yy}_{21}v_1+\overline{yy}_{22}v_2+\ldots+\overline{yy}_{2n}v_n)+(\overline{yz}_{21}w_1+\ldots)\\
&\cdots\cdots\\
Y_n &= (\overline{yx}_{n1}u_1+\overline{yx}_{n2}u_2+\ldots+\overline{yx}_{nn}u_n)+(\overline{yy}_{n1}v_1+\overline{yy}_{n2}v_2+\ldots+\overline{yy}_{nn}v_n)+(\overline{yz}_{n1}w_1+\ldots)
\end{aligned}\right.\\
&\left\{\begin{aligned}
Z_1 &= (\overline{zx}_{11}u_1+\overline{zx}_{12}u_2+\ldots+\overline{zx}_{1n}u_n)+(\overline{zy}_{11}v_1+\overline{zy}_{12}v_2+\ldots+\overline{zy}_{1n}v_n)+(\overline{zz}_{11}w_1+\ldots)\\
Z_2 &= (\overline{zx}_{21}u_1+\overline{zx}_{22}u_2+\ldots+\overline{zx}_{2n}u_n)+(\overline{zy}_{21}v_1+\overline{zy}_{22}v_2+\ldots+\overline{zy}_{2n}v_n)+(\overline{zz}_{21}w_1+\ldots)\\
&\cdots\cdots\\
Z_n &= (\overline{zx}_{n1}u_1+\overline{zx}_{n2}u_2+\ldots+\overline{zx}_{nn}u_n)+(\overline{zy}_{n1}v_1+\overline{zy}_{n2}v_2+\ldots+\overline{zy}_{nn}v_n)+(\overline{zz}_{n1}w_1+\ldots)
\end{aligned}\right.
\end{aligned}\right\} \qquad (40)$$

or, in partitioned matrix form:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \overline{xx} & \overline{xy} & \overline{xz} \\ \overline{yx} & \overline{yy} & \overline{yz} \\ \overline{zx} & \overline{zy} & \overline{zz} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \ldots \ldots \ldots \ldots \ldots (40a)$$

where, for example

$$\left.\begin{aligned} X &= \begin{bmatrix} X_1 \\ X_2 \\ \cdot\cdot \\ X_n \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ u_n \end{bmatrix} \\ \overline{xx} &= \begin{bmatrix} \overline{xx}_{11} & \overline{xx}_{12} \ldots \overline{xx}_{1n} \\ \overline{xx}_{21} & \overline{xx}_{22} \ldots \overline{xx}_{2n} \\ \cdot\cdot & \cdot\cdot \;\; \cdot\cdot\cdot\cdot\cdot \\ \overline{xx}_{n1} & \overline{xx}_{n2} \ldots \overline{xx}_{nn} \end{bmatrix} \end{aligned}\right\} \qquad \ldots \ldots \ldots \ldots \ldots (41)$$

By virtue of the Reciprocal Theorem, the n-sided square sub-matrices $\overline{xx}$, $\overline{yy}$ and $\overline{zz}$ are symmetrical, and the square sub-matrices $\overline{yx}$, $\overline{zx}$ and $\overline{zy}$ are respectively the transposed forms of $\overline{xy}$, $\overline{xz}$ and $\overline{yz}$.

Inversion of the $3n$-sided square stiffness matrix of (40a) gives the corresponding flexibility matrix, which allows the displacements to be expressed in terms of the applied loads in a set of $3n$ equations similar in form to (40). The partitioned-matrix form of this set may be written as:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \begin{bmatrix} \overline{\xi\xi} & \overline{\xi\eta} & \overline{\xi\zeta} \\ \overline{\eta\xi} & \overline{\eta\eta} & \overline{\eta\zeta} \\ \overline{\zeta\xi} & \overline{\zeta\eta} & \overline{\zeta\zeta} \end{bmatrix} \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad \ldots \ldots \ldots \ldots \ldots (42)$$

where, for example

$$\overline{\xi\xi} = \begin{bmatrix} \overline{\xi\xi}_{11} & \overline{\xi\xi}_{12} & \ldots & \overline{\xi\xi}_{1n} \\ \overline{\xi\xi}_{21} & \overline{\xi\xi}_{22} & \ldots & \overline{\xi\xi}_{2n} \\ \ldots & \ldots & & \ldots \\ \overline{\xi\xi}_{n1} & \overline{\xi\xi}_{n2} & \ldots & \overline{\xi\xi}_{nn} \end{bmatrix}, \qquad \overline{\xi\eta} = \begin{bmatrix} \overline{\xi\eta}_{11} & \overline{\xi\eta}_{12} & \ldots & \overline{\xi\eta}_{1n} \\ \overline{\xi\eta}_{21} & \overline{\xi\eta}_{22} & \ldots & \overline{\eta\xi}_{2n} \\ \ldots & \ldots & & \ldots \\ \overline{\xi\eta}_{n1} & \overline{\xi\eta}_{n2} & \ldots & \overline{\xi\eta}_{nn} \end{bmatrix} \quad \ldots (43)$$

Here $\overline{\xi\xi}_{rs}$ = displacement in direcrion x at station r due to unit load in direction x at station s

$\overline{\xi\eta}_{rs}$ = displacement in direction x at station r due to unit load in direction y at station s, etc.

The consequences of the Reciprocal Theorem noted above for the stiffness coefficients are equally applicable to the flexibility (or influence) coefficients.

4.3.1. *Consequence of external loads in wing plane being zero.*—In practice there are no X and Y external loadings at the wing stations, and therefore the first two columns of the square matrix of (42) disappear. That equation then takes the form:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \overline{\xi\zeta} \\ \overline{\eta\zeta} \\ \overline{\zeta\zeta} \end{bmatrix} [Z] \qquad \ldots \ldots \ldots \ldots \ldots \ldots (44)$$

which gives u, v and w at each station in terms of the applied loads Z. To visualise the final result better, we may expand the three component parts of (44) to give the u's, v's and w's separately as follows:

$$\left.\begin{array}{l}
\begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ \cdot\cdot \\ u_n \end{bmatrix} = \begin{bmatrix} \overline{\xi\zeta}_{11} & \overline{\xi\zeta}_{12} & \ldots & \overline{\xi\zeta}_{1n} \\ \overline{\xi\zeta}_{21} & \overline{\xi\zeta}_{22} & \ldots & \overline{\xi\zeta}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \overline{\xi\zeta}_{n1} & \overline{\xi\zeta}_{n2} & \ldots & \overline{\xi\zeta}_{nn} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \cdot\cdot \\ \cdot\cdot \\ Z_n \end{bmatrix} \\[2ex]
\begin{bmatrix} v_1 \\ v_2 \\ \cdot\cdot \\ \cdot\cdot \\ v_n \end{bmatrix} = \begin{bmatrix} \overline{\eta\zeta}_{11} & \overline{\eta\zeta}_{12} & \ldots & \overline{\eta\zeta}_{1n} \\ \overline{\eta\zeta}_{21} & \overline{\eta\zeta}_{22} & \ldots & \overline{\eta\zeta}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \overline{\eta\zeta}_{n1} & \overline{\eta\zeta}_{n2} & \ldots & \overline{\eta\zeta}_{nn} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \cdot\cdot \\ \cdot\cdot \\ Z_n \end{bmatrix} \\[2ex]
\begin{bmatrix} w_1 \\ w_2 \\ \cdot\cdot \\ \cdot\cdot \\ w_n \end{bmatrix} = \begin{bmatrix} \overline{\zeta\zeta}_{11} & \overline{\zeta\zeta}_{12} & \ldots & \overline{\zeta\zeta}_{1n} \\ \overline{\zeta\zeta}_{21} & \overline{\zeta\zeta}_{22} & \ldots & \overline{\zeta\zeta}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \overline{\zeta\zeta}_{n1} & \overline{\zeta\zeta}_{n2} & \ldots & \overline{\zeta\zeta}_{nn} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \cdot\cdot \\ \cdot\cdot \\ Z_n \end{bmatrix}
\end{array}\right\} \quad \ldots \quad (45)$$

With all the displacements found, the direct and shear stresses everywhere can at once be evaluated.

4.4. *Shear-lag Problem Implicitly Solved.*—The shear-lag problem is automatically solved in this method of approach, as may be seen by considering the simple structure of Fig. 7.

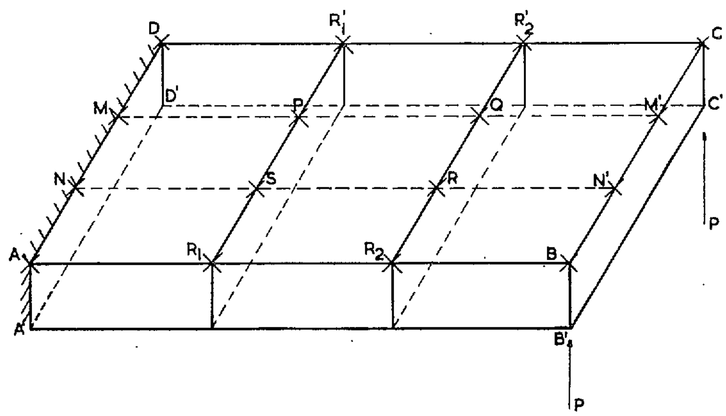


FIG. 7.

This represents a closed box having two internal ribs R_1R_1', R_2R_2', but no spanwise webs other than the sides of the box. If, further, the box has no corner flanges and the skin is heavily reinforced with stringers, we have a structure in which, under transverse loads P, shear-lag effects must dominate the picture. However, the fact that the surface has a number of stations, such as those marked in the figure with a cross, distributed over it ensures that full account of shear lag and section-warping is automatically taken by the above 'three variable per station' method.

5. *Concluding Remarks*.—Three methods are described in Part I of this report for deriving the influence coefficients of wing structures on the supposition that a digital computing machine is available.

The first (described in section 2) applies to heavy-skinned thin wings for which the shear flexibility of the vertical shear-carrying ribs and spar webs can be legitimately neglected. This involves a straightforward adaptation of elementary plate theory as originally described in the appendix to Ref. 1, and needs no particular comment.

In the second method (described in section 3) shear flexibility is taken account of in an approximate way, in accordance with which shear deflections are allowed to take place under a certain amount of external constraint. The deflections of the wing normal to its plane obtained by this method are accurate enough for use in all dynamic and aero-elastic calculations whatever the type of wing construction. They are also probably accurate enough for stressing purposes in the case of thin wings, and certainly good enough for the preliminary stressing of any kind of wing.

For the meticulous stressing of any kind of wing structure the more rigorous method of Section 4 can be brought in. This unfortunately does mean a threefold increase in the number of degrees of freedom if every station is given three instead of one independent displacement. It can well happen, however, that in many wing structures only those stations that are situated in parts of the structure that are ordinarily difficult to stress, need be given the added two degrees of freedom. Moreover, it is to be remembered that the machine is supposed to do the work, a fact that makes a substantial increase in the number of degrees of freedom not a very serious matter. The important point is that this method of deriving influence coefficients does not require a stressman to possess a profound knowledge of structural theory to enable him to deal satisfactorily with wings of any type.

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APPENDIX TO PART I

Deflections of a Simple Box under Various Conditions

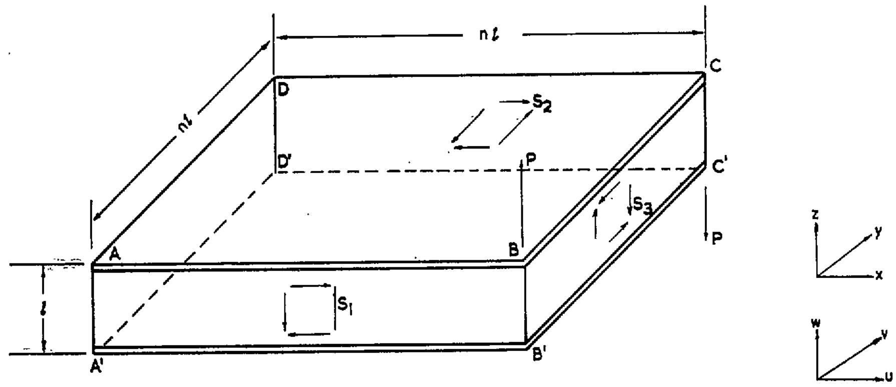

FIG. 1A.

Let l = depth of box
nl = length and breadth of box
A = cross-sectional area of booms
t_1 = thickness of side panels
t_2 = thickness of top and bottom panels
t_3 = thickness of end panel (or rib)
S_1 = shear per unit length in side panels
S_2 = shear per unit length in top and bottom panels
S_3 = shear per unit length in end panel
x, y, z = co-ordinates as shown
u, v, w = displacements as shown
w_b = bending deflection
w_s = shear deflection
μ = G/E, the ratio of the elastic moduli
P = applied loads.

Equilibrium requires that

$$S_3 = S_2 \quad \text{(1A)}$$

and

$$l(S_1 + S_3) = l(S_1 + S_2) = P\,. \quad \text{(2A)}$$

Neglecting shear deflection, we have for side AB′:

$$\frac{\partial u}{\partial z} + \frac{\partial w_b}{\partial x} = 0$$

so

$$\frac{\partial w_b}{\partial x} = -\frac{u}{l/2} = -\frac{2}{l}\left(\frac{S_1 - S_2}{AE}\right)\left(\frac{x^2}{2} - nlx\right),$$

giving for $x = nl$:

$$w_b = \frac{2}{3} n^3 l^2 \left(\frac{S_1 - S_2}{AE}\right). \quad \text{(3A)}$$

The corresponding shear deflection:

$$w_s = \frac{S_1 nl}{t_1 G}, \quad \ldots \qquad (4A)$$

and the total deflection:

$$(w)_{AB'} = w_b + w_s = \frac{S_1 nl}{t_1 G} + \frac{2}{3} n^3 l^2 \left(\frac{S_1 - S_2}{AE}\right). \qquad (5A)$$

In similar fashion for the top (and bottom) panel:

$$(v)_{AC} = v_b + v_s = \frac{S_2 nl}{t_2 G} - \frac{2}{3} n^2 l^2 \left(\frac{S_1 - S_2}{AE}\right). \qquad (6A)$$

Shear strain in end panel (or rib):

$$\gamma = \frac{S_2}{t_3 G} \qquad (7A)$$

and for compatibility of displacement:

$$\frac{S_2}{t_3 G} = \frac{2}{l}\left\{\frac{(w)_{AB'}}{n} - (v)_{AC}\right\}. \qquad (8A)$$

Substituting for w and v from (5A) and (6A) gives

$$S_2 = \left(\frac{\dfrac{2}{t_1} + \dfrac{8n^2 l\mu}{3A}}{\dfrac{1}{t_3} + \dfrac{2n}{t_2} + \dfrac{8n^2 l\mu}{3A}}\right) S_1. \qquad (9A)$$

Using (2A) and (5A), we obtain the upward deflection at B in the form:

$$w_B = \frac{P}{G}\left[\frac{2n\alpha\left(\dfrac{1}{t_3} + \dfrac{2n}{t_2} + \dfrac{2}{t_1}\right) + \dfrac{n}{t_1}\left(\dfrac{1}{t_3} + \dfrac{2n}{t_2}\right)}{\left(\dfrac{1}{t_3} + \dfrac{2n}{t_2} + \dfrac{2}{t_1}\right) + 16\alpha}\right] \qquad (10A)$$

where

$$\alpha = \frac{n^2 l\mu}{3A}. \qquad (11A)$$

What is called the bending deflection w_b is at once obtained by putting $t_3 = t_1 = \infty$ in (10A) whence:

$$w_b = \frac{P}{G}\left(\frac{4n^2\alpha}{2n + 16t_2\alpha}\right). \qquad (12A)$$

The shear deflection w_s is similarly obtained by making $t_2 = A = \infty$, whence:

$$w_s = \frac{P}{G}\left(\frac{n}{t_1 + 2t_3}\right). \qquad (13A)$$

It follows that under the conditions of bending deflection:

$$(S_1)_b = \frac{P}{l}\left(\frac{\frac{2n}{t_2} + 8\alpha}{\frac{2n}{t_2} + 16\alpha}\right) \qquad \text{.. .. .. (14A)}$$

and under the conditions of shear deflection:

$$(S_1)_s = \frac{P}{l}\left(\frac{t_1}{t_1 + 2t_3}\right). \qquad \text{.. .. .. (15A)}$$

In either case, from (2*A*):

$$S_2 = \frac{P}{l} - S_1 .$$

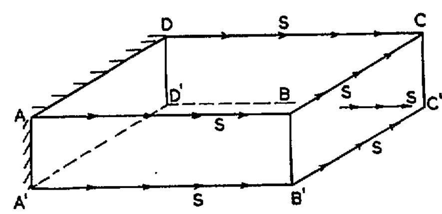

FIG. 2A.

The out-of-balance forces induced by the removal of the constraints are now seen to be a constant tractive force per unit length of edge of amount:

$$S_x = (S_1)_b - (S_1)_s \qquad \text{.. .. .. (16A)}$$

along the longitudinal edges, and:

$$S_y = -(S_2)_b + (S_2)_s \qquad \text{.. .. .. (17A)}$$

along the cross edges.

It follows at once from (2*A*) that:

$$S_x = S_y = S \text{ (say)}, \qquad \text{.. .. .. (18A)}$$

the distribution of which is shown in Fig. 2A.

PART II

Notes on the practical application of the method described in Part I

Summary.—In Part I a method for deriving the influence coefficients of any type of wing, and hence of deriving the deflection and stresses, was described in outline. The practical application of the method, however, raises a number of minor problems (mostly concerned with boundary values), which have all to be overcome if the method is to become popular among stressmen. Some of the more important of these problems are treated in this part.

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1. *Introduction.*—In Part I of this report a general method of calculating the deflections and stresses in any type of wing has been outlined. Only a brief reference was made to the boundary problems involved and to the problem of adequately taking account of such reinforcing members as stringers, spar and rib booms. These problems are now discussed here in detail.

Boundary conditions.—A more convenient method of treating the boundary conditions at free edges and corners than that normally used is described in the first few paragraphs. This method has two advantages; it derives the relevant reactions more directly than does the classical method, and is at the same time a physically more obvious approach in terms of finite differences than the standard approach associated with the name of Kirchhoff. Applied to the problem of the plate of constant thickness, this suggested alternative approach is shown to give identical formulae for the reactions as those derived by the more familiar standard method. This may be taken as inferential proof that the boundary formulae derived for reactions in terms of displacements for more complicated cases, such as wings with oblique stringers and shear webs, are equally sound.

The boundary conditions discussed here, ostensibly applicable to wing structures, are in fact applicable to any flat or plane structure under transverse loading. Such a structure, if held all round its periphery, either encastré or simply supported, offers little difficulty in the matter of boundary conditions. If, however, it is held cantilever-like along one edge only, as an aeroplane wing is held, the rest of the boundary constitutes a free edge, frequently of irregular plan-form.

In the case of the wing the supported edge may be encastré at the plane of symmetry, or it may be simply supported where it meets the fuselage sides. In either case the boundary conditions are straightforward and do not require the careful consideration they do over the free edges.

A main difficulty in dealing with the free edges of a wing arises from the often irregular plan-form. The view has been taken that the most convenient way of approximately representing the true plan-form is to mark off the boundary in a series of steps as described in para. 3 and indicated in Fig. 5. In this way every free edge is parallel either to the x or y co-ordinate axis, so that every station located on the boundary must lie either on one or other of these free edges or on a corner common to both. It follows that, in deriving standard formulae for the station reactions, it becomes necessary to consider four distinct types of boundary stations:

(*a*) A station located on a free edge parallel to the x-axis

(*b*) A station located on a free edge parallel to the y-axis

(*c*) A station located on a free projecting corner

(*d*) A station located on a re-entrant corner.

In point of fact, the boundary conditions appropriate to a station of type (*a*) can easily be applied to derive those appropriate to the other three types.

The boundary conditions for each type of structure are discussed in the main text, but it was considered better to relegate the derivation of the formulae appropriate to the varous types (*a*) . . . (*d*) of boundary stations to a series of appendices. These not only give the formulae, but also their derivation.

Particular cases considered.—The first part of Part II of this report describes the alternative method advocated for deriving the reactions at boundary stations, and applies it to the case of the constant-thickness flat plate, in which it is legitimate to neglect transverse shear deflections.

The second part considers the case of the hollow wing reinforced by one or more sets of parallel stringers. Shear deflections are again neglected.

The third part describes an approximate method of deriving shear deflections when the shear is carried by one or more sets of shear webs.

For thin wings a combination of the above two methods should give resultant deflections, the one due to bending and the other due to shear, that are satisfactorily accurate for the purpose of aero-elastic calculations. The bending stresses should also be accurate enough, although they are unaffected by the shear deflections obtained by the approximate method.

The fourth part describes a more accurate method of deriving the stresses and deflections of a wing. This method may be necessary for obtaining the wing stresses for the thicker type of wing though not for obtaining the wing deflections required for aero-elastic calculations. Neither for deriving stresses nor deflections is it necessary to use this more refined approach in dealing with thick-skinned wings of low thickness-chord ratio.

Appendices A, B, and C follow and are concerned with the detailed derivation of various formulae. Appendix D describes the application of the method to the practical problem of deriving the influence coefficients of a thin cantilever plate of constant thickness and square plan-form. Although the plan-form is simple and the thickness constant, it seems a fair assumption that the degree of accuracy obtained should be representative of what is attainable in a thin wing of variable thickness.

Relative accuracy of stresses and deflections derived by the present method.—In discussing the relative accuracy of stresses and deflections, it may be well to summarise briefly the basis of the present method of approach. In essence, the method consists in defining the deflected shape of a wing by the deflections of a fairly large number of stations uniformly distributed over its surface. Under a hypothetical set of displacements these stations are held in their displaced positions by reactions that can at once be written down. The set of linear equations connecting displacements and reactions can now be solved to give the displacements in terms of any set of reactions or applied load. The method has been criticised on the grounds that, whereas the displacements are obtained directly, the stresses are derived only indirectly by differentiation of the displacements. At first sight this seems a valid criticism but an examination of the special factors involved has demonstrated (Ref. 1) that in the present case it is not valid, and that stresses are unlikely to be less accurate in general than displacements; indeed in many practical examples they are found to be more accurate.

2. *An Alternative and more Convenient way of Expressing the Boundary Conditions at a Free Edge or Corner.*—At the end of this Part II of the report (Appendix D) a numerical example is worked out to illustrate the kind of accuracy obtained by the method of the report. The problem considered is that of the square plate mounted along one edge as a cantilever. The boundary conditions assumed for the other three edges express the fact that the bending moment across a free edge must be zero and that the resultant shear, as first expressed by Kirchhoff, must also be zero. In addition, special consideration has to be given to the conditions at the two free corners.

The method here put forward short-circuits the Kirchhoff condition and, instead of the conditions at a free-corner station being more complicated, they become even simpler than for an internal station. Moreover, the whole procedure becomes straightforward and no subtle reasoning of any kind is required. All that is done is to discard the notion of a free edge and, instead, to imagine the plate (or wing) to extend beyond the free edge in the form of a real plate of zero stiffness.

To appreciate the simplicity of this method of approach, we may consider the case of a free-corner station in the square plate of Appendix II (Case (*d*)) of Ref. 1.

2.1. *Reaction at Free-corner Station by New Method.*—We first write equation (1) of Appendix II[2] in the form:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} = q_0\,, \quad \ldots \quad (1)$$

which gives the reaction q_0 per unit area at the corner station 0 in terms of the moments per unit width of plate. The usual 12 stations, surrounding the station for which the reaction is required, are marked in Fig. 1. Supplementary stations *a*, *b*, *c*, *d*, are also introduced, later to disappear.

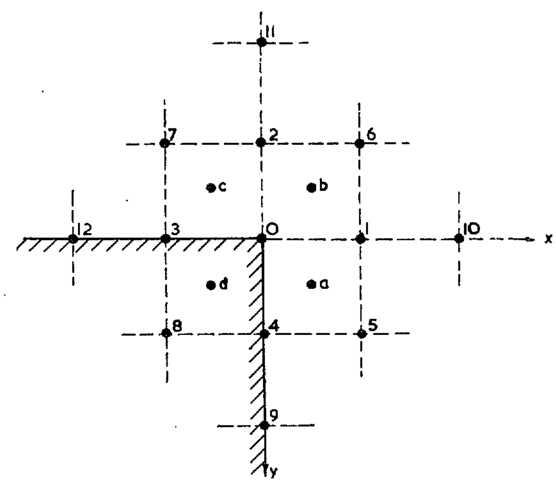


FIG. 1.

We write:

$$l^2 \frac{\partial^2 M_x}{\partial x^2} = (M_x)_1 + (M_x)_3 - 2(M_x)_0 \quad \ldots \quad (2)$$

$$l^2 \frac{\partial^2 M_y}{\partial y^2} = (M_y)_2 + (M_y)_4 - 2(M_y)_0 \quad \ldots \quad (3)$$

$$2l^2 \frac{\partial^2 M_{xy}}{\partial x\,\partial y} = 2\{(M_{xy})_a + (M_{xy})_c - (M_{xy})_b - (M_{xy})_d\}\,. \quad \ldots \quad (4)$$

In equation (2), $(M_x)_0$ is zero and $(M_x)_1$ drops out because the plate stiffness is zero leaving therefore only $(M_x)_3$. Similarly $(M_y)_2$ and $(M_y)_0$ disappear from equation (3), and in (4) only $(M_{xy})_d$ has a value. Thus

$$q_0 = \{(M_x)_3 + (M_y)_4 - 2(M_{xy})_d\}/l^2\,. \quad \ldots \quad (5)$$

It is noted that here only the second differentials of the deflection have to be considered, whereas in equation (1) the fourth differentials are involved. A direct result is that deflections w_{10} and w_{11} do not now enter.

Now a line of stations along a free edge represents a strip of plate half of whose width has the I of the plate proper and the other half the zero I of the infinitely flexible plate beyond the boundary.

The average I per unit width at a free edge station is therefore only half the actual I of the plate.

Since

$$(M_y)_3 = \frac{E(I_y)_3}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)_3 = 0$$

and

$$(M_x)_4 = \frac{E(I_x)_4}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_4 = 0 \quad \ldots \quad (6)$$

we can write:

$$(M_x)_3 = \frac{E(I_x)_3}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_3 = E(I_x)_3\left(\frac{\partial^2 w}{\partial x^2}\right)_3$$

and

$$(M_y)_4 = \frac{E(I_y)_4}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)_4 = E(I_y)_4\left(\frac{\partial^2 w}{\partial y^2}\right)_4 \quad \ldots \quad (7)$$

Using equation (7) in (5) we have:

$$q_0 = \left[E(I_x)_3\left(\frac{\partial^2 w}{\partial x^2}\right)_3 + E(I_y)_4\left(\frac{\partial^2 w}{\partial y^2}\right)_4 - \frac{2EI_d}{(1+\nu)}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_d\right]\Big/ l^2 . \quad \ldots \quad (8)$$

For a plate of constant thickness, we have, by the above argument:

$$I_d = I_0 \text{ and } (I_x)_3 = (I_y)_4 = \tfrac{1}{2}I_0 , \quad \ldots \quad (9)$$

where I_0 is the constant I per unit width of plate. For such a plate the reaction per unit area at the corner station, after substituting B for $EI/(1-\nu^2)$:

$$q_0 = \frac{B}{l^4}(1-\nu)\left[\left(\frac{1+\nu}{2}\right)(w_0 + w_{12} - 2w_3) + \left(\frac{1+\nu}{2}\right)(w_0 + w_9 - 2w_4) - 2(w_4 - w_0 - w_8 + w_3)\right] . \quad \ldots \quad (10)$$

To obtain the total reaction at station 0, we now multiply the reaction q_0 per unit area by l^2 in order to take account of the complete square of side l surrounding station 0. The fact that three-quarters of that square is taken up by a plate of zero stiffness has already been allowed for by making the I over that area equal to zero. The total reaction is therefore:

$$R_0 = q_0 l^2 = \frac{B(1-\nu)}{l^2}\left[(3+\nu)(w_0 - w_3 - w_4) + 2w_8 + \left(\frac{1+\nu}{2}\right)(w_9 + w_{12})\right] \quad \ldots \quad (11)$$

which is identical with that given by equation (15) of Ref. 1, Appendix II, after the substitution of the solid plate bending stiffness D for B.

2.2. *Reaction at Free-Edge Station by New Method.*—For the free edge shown in Fig. 2, the corresponding procedure is as follows:

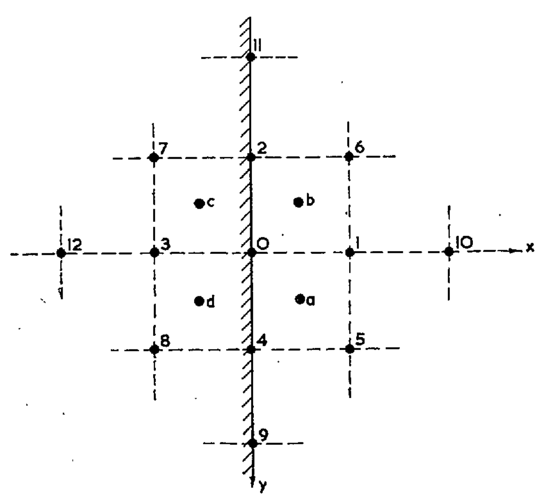


FIG. 2.

The basic equations (2), (3) and (4) again apply. Using the fact that $(M_x)_2$, $(M_x)_0$, $(M_x)_4$ are each zero by virtue of the free edge, and that $(M_x)_1$, $(M_{xy})_a$, $(M_{xy})_b$ are each zero because of zero plate stiffness, we have:

$$q_0 = \frac{1}{l^2}\left[(M_x)_3 + (M_y)_2 + (M_y)_4 - 2(M_y)_0 + 2\{(M_{xy})_c - (M_{xy})_d\}\right] . \quad \ldots \quad (12)$$

Since, at stations (2), (0) and (4),

$$\frac{\partial^2 w}{\partial x^2} = - \nu \frac{\partial^2 w}{\partial y^2} .$$

and since the average (I_y) at these stations is only half that for interior stations, (12) becomes:

$$q_0 = \frac{E}{l^4}\left[\frac{I}{1-\nu^2}\{(w_0 + w_{12} - 2w_3) + \nu(w_7 + w_8 - 2w_3)\}\right.$$
$$+ \frac{I}{(1-\nu^2)2}\{(1-\nu^2)(w_0 + w_{11} - 2w_2)\} + \frac{I}{2}\{(1-\nu^2)(w_0 + w_9 - 2w_4)\}$$
$$- \frac{2I}{(1-\nu^2)2}\{(1-\nu^2)(w_2 + w_4 - 2w_0)\}$$
$$\left. + \frac{2I}{(1+\nu)}\{(w_0 + w_7 - w_2 - w_3) - (w_4 + w_3 - w_0 - w_8)\}\right] . \quad \ldots \quad \ldots \quad (13)$$

The total reaction is therefore, after putting B for $EI/(1-\nu^2)$:

$$R_0 = q_0 l^2 = \frac{B}{l^2}\left[(8 - 4\nu - 3\nu^2)\, w_0 - (4 - 2\nu - 2\nu^2)(w_2 + w_4) - (6 - 2\nu)w_3\right.$$
$$\left. + (2-\nu)(w_7 + w_8) + \frac{1-\nu^2}{2}(w_9 + w_{11}) + w_{12}\right], \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

which is identical with that given by equation (6) of Ref. 1, Appendix II, derived by using the Kirchhoff condition for shear.

Free edge parallel to x-axis.—The corresponding formula for a station 0 located on a free edge parallel to the x-axis is at once written down by turning the pattern of stations through a right angle. Thus:

$$R_0 = \frac{B}{l^2}\left[(8 - 4\nu - 3\nu^2)w_0 - (4 - 2\nu - 2\nu^2)(w_3 - w_1) - (6 - 2\nu)w_4\right.$$
$$\left. + (2-\nu)(w_5 + w_8) + \left(\frac{1-\nu^2}{2}\right)(w_{10} + w_{12}) + w_9\right] . \quad \ldots \quad \ldots \quad \ldots \quad (14a)$$

To obtain the corresponding formula for the other free edge parallel to the x-axis, it is only necessary to turn the pattern of stations round through 180 deg, and then re-number the stations.

2.3. *Reaction at Re-entrant Corner.*

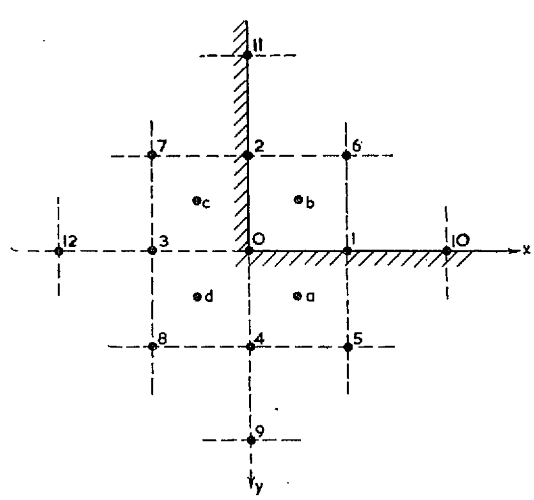


FIG. 3.

In equations (2), (3) and (4) (which are universally applicable) we have this time

$$(M_x)_1 = \frac{E}{2}(I_x)_1 \left(\frac{\partial^2 w}{\partial x^2}\right)_1$$

$$(M_x)_3 = \frac{E(I_x)_3}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_3$$

$$(M_x)_0 = \frac{3E(I_x)_0}{4(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_0$$

(*Note:* The proper I at station 0 along the line of stations 4, 0, 2 is greater than its value $\frac{1}{2}(I_x)_0$ between 0 and 2 and smaller than the full $(I_x)_0$ that is its value between 4 and 0. The value $\frac{3}{4}(I_x)_0$ seems therefore the best average value to take).

$$(M_y)_2 = \tfrac{1}{2}E(I_y)_2\left(\frac{\partial^2 w}{\partial y^2}\right)_2,$$

$$(M_y)_4 = \frac{E(I_y)_4}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)_4,$$

$$(M_y)_0 = \frac{3}{4}\frac{E(I_y)_0}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)_0,$$

$$(M_{xy})_b = 0$$

$$(M_{xy})_d = \frac{EI_d}{1+\nu}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_d,$$

$$(M_{xy})_a = \frac{EI_a}{1+\nu}\left(\frac{\partial^2 w}{dx\,\partial y}\right)_a,$$

$$(M_{xy})_c = \frac{EI_c}{1+\nu}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_c. \qquad (15)$$

Using these values in equations (2), (3) and (4) and expressing the second derivatives of w in finite difference form, we obtain from (1) the reaction at station 0 for a plate of constant thickness:

$$R_0 = q_0 l^2 = \frac{B}{l^2}\Big[(15-\nu^2)w_0 - (4\tfrac{1}{2} - \tfrac{1}{2}\nu - \nu^2)(w_1 + w_2) - (7\tfrac{1}{2} - \nu/2)(w_3 + w_4)$$
$$+ (2-\nu)(w_5 + w_7) + (w_9 - w_{12}) + 2w_8 + \left(\frac{1-\nu^2}{2}\right)(w_{10} + w_{11})\Big]. \qquad (16)$$

If the plate is not of constant thickness the appropriate I's at the various stations must be used as shown in (15).

2.4. *Re-entrant Corner in Stepped Boundary.*—If the re-entrant corner is situated on a ' stepped ' boundary as station 0 would be in Fig. 2 if the free boundary were marked by the station sequence 7, 2, 0, 1, 5, we should have $(M_x)_1$, $(M_y)_2$ and $(M_{xy})_b$ dropping out of equations (2), (3) and (4), and the reaction (with local variation of I neglected) given by:

$$R_0 = (M_x)_3 - 2(M_x)_0 + (M_y)_4 - 2(M_y)_0 + 2\{(M_{xy})_a + (M_{xy})_c - (M_{xy})_d\}$$
$$= \frac{EI}{1-\nu^2}\Big[w_x'' + \nu w_y'')_3 + (w_y'' + \nu w_x'')_4 - \frac{3}{2}(1+\nu)(w_x'' + w_y'')_0$$
$$+ 2(1-\nu)\{(w_{xy}'')_a + (w_{xy}'')_c - (w_{xy}'')_d\}\Big]$$

or

$$R_0 = \frac{B}{l^2}\Big[14w_0 - \tfrac{1}{2}(7+\nu)(w_1 + w_2) - \tfrac{1}{2}(15-\nu)(w_3 + w_4)$$
$$+ (2-\nu)(w_5 + w_7) + 2w_8 + (w_9 + w_{12})\Big]. \qquad (17)$$

2.5. *Standard Form of Reaction at an Internal Station.*—When the plate (or wing) moment of inertia of cross-section I is constant the formula quoted in equation (17) of Ref. 1 is applicable.

If the I varies appreciably within a single pitch length of station 0 the appropriate I for each station may need to be taken. In practical cases this will rarely be necessary. The procedure (Fig. 4) is briefly as follows:

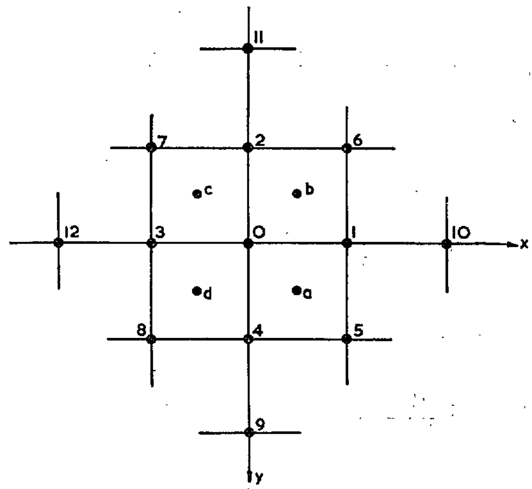


FIG. 4.

Using equations (1), (2), (3) and (4) above we have:

$$q_0 l^2 = \{(M_x)_1 + (M_x)_3 - 2(M_x)_0\} + \{(M_y)_2 + (M_y)_4 - 2(M_y)_0\} + 2\{(M_{xy})_a + (M_{xy})_c - (M_{xy})_b - (M_{xy})_d\} .$$

On putting w_x'' for $\partial^2 w/\partial x^2$, etc., we obtain:

$$R_0 = q_0 l^2 = \frac{1}{l^2}\Big[B_1(w_x'' + \nu w_y'')_1 + B_3(w_x'' + \nu w_y'')_3 - 2B_0(w_x'' + \nu w_y'')_0 + B_1(w_y'' + \nu w_x'')_2 + B_4(w_y'' + \nu w_x'')_4 - 2B_0(w_y'' + \nu w_x'')_0 + 2(1 - \nu)\{B_a(w_{xy}'')_a + B_c(w_{xy}'')_c - B_b(w_{xy}'')_b - B_d(w_{xy}'')_d\}\Big] . \quad .. \quad (18)$$

All that now remains is to write down the second differentials in finite difference form. The first term in the square bracket for example becomes:

$$B_1\{(w_0 + w_{10} - 2w_1) + \nu(w_5 + w_6 - 2w_1)\}/l^2$$

and the first element in the last term $B_a(w_{xy}'')_a$ becomes:

$$B_a\{w_1 + w_2 - w_0 - w_6\}/l^2 .$$

The reaction $q_0 l^2$ at station 0 is thus obtained in terms of the deflections at the 12 surrounding stations and the I at stations a, b, c, d, 1, 2, 3, 4.

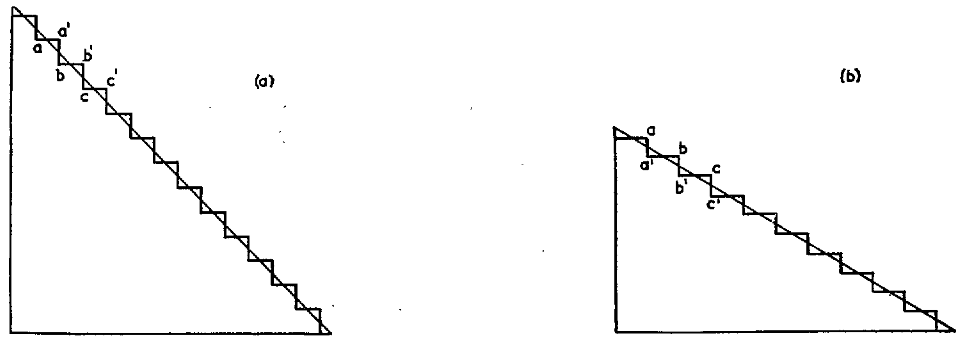
It is to be noted that, in the above, it is assumed that I_x and I_y are equal at each station. If they are unequal due to stringer reinforcement of the skin of a wing, for example, the effect of such stringers on the required reaction is best determined separately from the skin. This is discussed in section 6.

When, as in most practical cases, the I can be regarded as constant over the area included by the square enclosed by stations 1, 2, 3, 4, the reaction reduces to that given by equation (17) of Ref. 1, *i.e.*,

$$R_0 = \frac{EI}{l^2(1 - \nu^2)}\left[20w_0 - 8\sum_{r=1}^{4} w_r + 2\sum_{r=5}^{8} w_r + \sum_{r=9}^{12} w_r\right] . \quad .. \quad (18a)$$

3. *Representation of Wings with Edges Oblique to the Co-ordinate Axes.*—In most practical cases the wing plan-form will entail leading and trailing edges oblique to the co-ordinate axes.

The easiest way of allowing for such boundary conditions is to replace the actual boundary by a stepped boundary. Take for example the 45-deg delta wing of Fig. 5a and the $26\frac{1}{2}$-deg swept leading edge of Fig. 5b.



FIGS. 5a and 5b.

The square grid is here easily arranged so that in both cases the straight leading edge is replaced by a stepped edge. The projecting and re-entrant corners of the successive steps are arranged so that the actual edge lies midway between them, *i.e.*, the true edge lies midway between the straight lines a, b, c, and a', b', c' . . . .

To appreciate the degree of approximation introduced by this, we need only to imagine how the behaviour of the wing (the deflections and stresses) would change if the edge were physically cut up into steps in the way suggested.

One expects, by the St. Venant's Principle, that whereas the stresses and deflections very near to the edge may depart somewhat from the correct values those not adjacent to the edge should have satisfactory accuracy.

An important point to note is that the loss of accuracy at the stepped edge is likely to be of much less significance for a wing, and particularly for a thin wing, than for a plate of uniform thickness. This is because the imaginary modification to the wing plan-form, *i.e.*, the steps cut in the oblique edge, is made only at the leading (and trailing) edges where, owing to the tapered cross-section of the aerofoil, the wing section is very shallow, and where, therefore, any slight deviation from the true plan-form does not matter. One is confirmed in this view when one remembers that the aerodynamic loading also drops to zero at the leading and trailing edges.

It is seen that this method of representing oblique edges by a series of steps enables all boundary stations to be regarded as situated either on a free edge, a free corner or a re-entrant corner. The expressions for the reactions at such stations have been given by the above formulae.

It is to be remembered that bending moments M_x for a station located on a free edge parallel to the x-axis and moments M_y along a free edge parallel to the y-axis are associated respectively with moments of inertia I_x and I_y per unit width that have only half the corresponding values for an interior station. This, as already explained, is because an edge station is associated with half a pitch of real plate and half a pitch of the infinitely flexible plate we imagine to extend beyond the free edge. At a re-entrant–corner station the appropriate I along both x and y directions is three-quarters of what it would be for an interior station.

4. *Hinge Moments Due to Ailerons, Flaps or Wing-tip Fins.*—The moments that are applied to what would otherwise be a free edge by ailerons, flaps or wing-tip fins would seem at first sight to vitiate the condition of zero bending moment along such an edge. This difficulty is readily overcome by imagining the couple introduced by such control surfaces to be applied as up and down forces at the appropriate set of edge stations and the set immediately inboard parallel to the edge.

5. *Connection of Wing and Fuselage.*—The root rib of the wing may be regarded just as much a part of the fuselage as of the wing itself and any deformation of the rib in its own plane is resisted partly by the chordwise bending stiffness of the wing and partly by the bending stiffness of the fuselage. A reasonable practical procedure is to consider two stations on the wing-root chord as being fixed in space. The reactions at these two stations and at the other stations on the root chord-line can then be derived on the basis of the wing and fuselage bending stiffnesses.

6. *Treatment of Oblique Stringers and Shear Webs.*—In a straight wing the stringers and shear webs are usually parallel to the co-ordinate axes (x being normal and y parallel to the plane of symmetry of the aircraft), and then the method described in Part I, section 4, can be applied. For swept wings and deltas the stringers and webs are usually oblique to the axes and need special consideration.

When considering thin wings, and other cases where flexibility in shear can be neglected, the problem of oblique shear webs naturally does not arise and only the problem of oblique stringers, spanwise and chordwise, need be treated.

7. *Wings, Infinitely Stiff in Shear, Reinforced by Oblique Stringers.*—It is clear that a set of hypothetical deflections determines the curvatures not only of the wing skin but also of the stringers attached to it. At an internal station, therefore, the reaction necessary to hold the wing contour is made up of two independent parts: that due to the wing minus stringers, and that due to the stringers themselves.

At a station located on a free edge the reactions due to skin and stringers can still be expressed separately but the boundary condition for each depends on the interaction between the two brought about by the free edge. Subject to taking account of this interaction it is possible to form the stiffness matrix for skin and stringers independently and the two matrices then added before inversion. Alternatively, the two reactions (the one due to the skin and the other to the stringers) can be added together at each station as they are obtained.

7.1. *Reaction Forces due to Stringers Alone.*—Given the contour of the transverse displacement of the wing, one can express the curvature of the stringers along their own direction. This gives the bending moments and hence the reactions at the several stations due to the stringers.

Let n = direction of stringers

α = angle between the n and x directions

I_n = I of stringers (about wing neutral axis) per unit width normal to n

w_n'' = second differential of w with respect to n.

Then

$$M_n = EI_n w_n'' \qquad (19)$$

and, using the standard formula for curvature in the n direction, we have:

$$M_n = EI_n w_n'' = EI_n(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2s_\alpha c_\alpha w_{xy}'') \; , \qquad (20)$$

where c_α and s_α are written for cosine α and sine α.

Having obtained M_n in terms of curvatures and twists along the x and y axis, we can resolve it into its components M_x, M_y and M_{xy}. Thus:

$$M_x = c_\alpha^2 M_n \; , \qquad M_y = s_\alpha^2 M_n \; , \qquad M_{xy} = s_\alpha c_\alpha M_n \; . \qquad (21)$$

We can now write down the reaction per unit area at any point of the wing in the standard form:

$$q = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + \frac{2\partial^2 M_{xy}}{\partial x\, \partial y}$$

$$= \left(c_\alpha^{\,2} \frac{\partial^2}{\partial x^2} + s_\alpha^{\,2} \frac{\partial^2}{\partial y^2} + 2 s_\alpha c_\alpha \frac{\partial^2}{\partial x\, \partial y}\right) M_n \quad \ldots \quad (22)$$

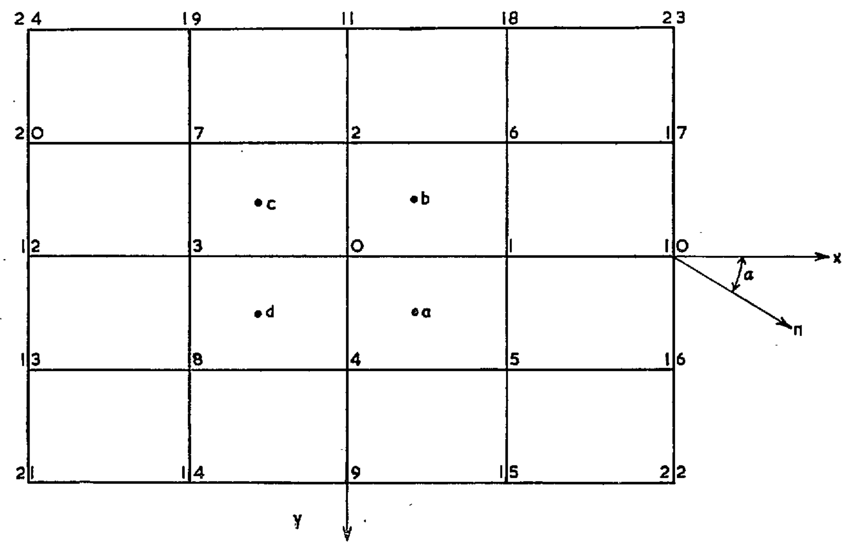


FIG. 6.

7.1.1. *Typical internal station.*—For the internal station 0 of Fig. 6 we write (22) in finite difference form to give the reaction $(R_0)_{\rm str}$ as:

$$(R_0)_{\rm str} = l^2(q_0)_{\rm str} = c_\alpha^{\,2}\{(M_n)_1 + (M_n)_3 - 2(M_n)_0\} + s_\alpha^{\,2}\{(M_n)_2 + (M_n)_4 - 2(M_n)_0\}$$

$$+ \tfrac{1}{2} c_\alpha s_\alpha \{(M_n)_5 + (M_n)_7 - (M_n)_6 - (M_n)_8\} \quad \ldots \quad (23)$$

$$= c_\alpha^{\,2} E\{(I_n)_1(w_n'')_1 + (I_n)_3(w_n'')_3 - 2(I_n)_0(w_n'')_0\}$$

$$+ s_\alpha^{\,2} E\{(I_n)_2(w_n'')_2 + (I_n)_4(w_n'')_4 - 2(I_n)_0(w_n'')_0\}$$

$$+ \tfrac{1}{2} c_\alpha s_E \{(I_n)_5(w_n'')_5 + (I_n)_7(w_n'')_7 - (I_n)_6(w_n'')_6 - (I_n)_8(w_n'')_8\}\,. \quad \ldots \quad (23a)$$

Since $w_n'' = c_\alpha^{\,2} w_x'' + s_\alpha^{\,2} w_y'' + 2 c_\alpha s_\alpha w_{xy}''$, equation (23*a*) in finite-difference form gives the reaction R_0 in terms of the deflections of the group of stations 0, 1, 2, . . . 24, shown in Fig. 6.

If, as in practical cases, the I of the stringers can be regarded as constant over the inner square defined by stations 5, 6, 7, 8, the quantity I_n in (23*a*) becomes a common factor and we can write:

$$(R_0)_{\rm str} = l^2(q_0)_{\rm str} = l^2 E I_n \left\{ c_\alpha^{\,4} \frac{\partial^4 w}{\partial x^4} + s_\alpha^{\,4} \frac{\partial^4 w}{\partial y^4} + 6 c_\alpha^{\,2} s_\alpha^{\,2} \frac{\partial^4 w}{\partial x^2\, \partial y^2} \right.$$

$$\left. + 4 c_\alpha^{\,3} s_\alpha \frac{\partial^4 w}{\partial x^3\, \partial y} + 4 c_\alpha s_\alpha^{\,3} \frac{\partial^4 w}{\partial x\, \partial y^3} \right\} \quad \ldots \quad (24)$$

with the result that the deflections at stations 21, 22, 23, and 24 no longer enter the expression for R_0, which now takes the form

$$(R_0)_{\text{str}} = \frac{EI_n}{l^2}\Big[(6 + 12c_\alpha^2 s_\alpha^2)w_0 - 4c_\alpha^2(1 + 2s_\alpha^2)(w_1 + w_3) - 4s_\alpha^2(1 + 2c_\alpha^2)(w_2 + w_4)$$
$$- 2c_\alpha s_\alpha(1 - 3c_\alpha s_\alpha)(w_5 + w_7) + 2c_\alpha s_\alpha(1 + 3c_\alpha s_\alpha)(w_6 + w_8)$$
$$+ s_\alpha^4(w_9 + w_{11}) + c_\alpha^4(w_{10} + w_{12}) + c_\alpha^3 s_\alpha(w_{16} + w_{20} - w_{13} - w_{17})$$
$$+ c_\alpha s_\alpha^3(w_{15} + w_{19} - w_{14} - w_{18})\Big] . \quad (25)$$

To obtain the total reaction we need to add to this the reaction due to the skin alone given by equation (18*a*).

7.2. *Boundary Conditions at a Free Edge.*—The first boundary condition at a free edge requires that the resultant bending moment, of skin and stringers, normal to the free edge must be zero.

Free edge parallel to the x-axis.—The bending moment in the skin is:

$$(M_y)_{\text{skin}} = \frac{EI}{1 - \nu^2}(w_y'' + \nu w_x'') \quad (26)$$

(where I stands for the I of the skin per unit width), and that in the stringers is:

$$(M_y)_{\text{str}} = s_\alpha^2 EI_n(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'') . \quad (27)$$

Thus

$$(M_y)_{\text{skin}} + (M_y)_{\text{str}} = 0 \quad (28)$$

or

$$\left(\frac{\nu I}{1 - \nu^2} + I_n s_\alpha^2 c_\alpha^2\right) w_x'' + \left(\frac{I}{1 - \nu^2} + s_\alpha^4 I_n\right) w_y'' + (2c_\alpha s_\alpha^3 I_n) w_{xy}'' = 0 . \quad (29)$$

The second condition requires that the resultant twisting moment M_{xy} should be zero, *i.e.*,

$$(M_{xy})_{\text{skin}} + (M_{xy})_{\text{str}} = 0 , \quad (30)$$

or

$$\frac{EI}{1 + \nu} w_{xy}'' + EI_n s_\alpha c_\alpha(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'') = 0 ,$$

or

$$I_n c_\alpha s_\alpha(c_\alpha^2 w_x'' + s_\alpha^2 w_y'') + \left(\frac{I}{1 + \nu} + 2c_\alpha^2 s_\alpha^2 I_n\right) w_{xy}'' = 0 . \quad (30a)$$

By means of (29) and (30*a*) we can express w_y'' and w_{xy}'' in terms of the known value of w_x''. Thus:

$$\left.\begin{aligned} w_y'' &= H_1 w_x'' \\ w_{xy}'' &= H_2 w_x'' \end{aligned}\right\} \quad (31)$$

where

$$H_1 = \frac{2c_\alpha^4 s_\alpha^4\left(\frac{I_n}{I}\right)^2 - \left(\frac{\nu}{1 - \nu^2} + \frac{I_n}{I}s_\alpha^2 c_\alpha^2\right)\left(\frac{1}{1 + \nu} + 2c_\alpha^2 s_\alpha^2\frac{I_n}{I}\right)}{\left(\frac{1}{1 + \nu} + 2c_\alpha^2 s_\alpha^2\frac{I_n}{I}\right)\left(\frac{1}{1 - \nu^2} + s_\alpha^4\frac{I_n}{I}\right) - 2c_\alpha^2 s_\alpha^6\left(\frac{I_n}{I}\right)^2} \quad (32)$$

$$H_2 = \frac{c_\alpha s_\alpha^3\left(\frac{I_n}{I}\right)\left(\frac{\nu}{1 - \nu^2} + \frac{I_n}{I}c_\alpha^2 s_\alpha^2\right) - s_\alpha c_\alpha^3\frac{I_n}{I}\left(\frac{1}{1 - \nu^2} + s_\alpha^4\frac{I_n}{I}\right)}{\left(\frac{1}{1 + \nu} + 2c_\alpha^2 s_\alpha^2\frac{I_n}{I}\right)\left(\frac{1}{1 - \nu^2} + s_\alpha^4\frac{I_n}{I}\right) - 2c_\alpha^2 s_\alpha^6\left(\frac{I_n}{I}\right)^2} , \quad (33)$$

both quantities depending only on the ratio I_n/I.

Free edge parallel to the y-axis.—For a station located on a free edge parallel to the y-axis the corresponding formulae (obtained by merely substituting s_α and c_α respectively for c_α and s_α) are:

$$\left.\begin{aligned} w_x'' &= H_3 w_y'' \\ w_{xy}'' &= H_4 w_y'' \end{aligned}\right\}, \qquad (34)$$

where

$$H_3 = \frac{2c_\alpha^4 s_\alpha^4 \left(\frac{I_n}{I}\right)^2 - \left(\frac{1}{1-\nu} + 2c_\alpha^2 s_\alpha^2 \frac{I_n}{I}\right)\left(\frac{\nu}{1-\nu^2} + c_\alpha^2 s_\alpha^2 \frac{I_n}{I}\right)}{\left(\frac{1}{1-\nu^2} + c_\alpha^4 \frac{I_n}{I}\right)\left(\frac{1}{1+\nu} + 2c_\alpha^2 s_\alpha^2 \frac{I_n}{I}\right) - 2c_\alpha^6 s_\alpha^2 \left(\frac{I_n}{I}\right)^2}, \qquad (35)$$

$$H_4 = \frac{\frac{I_n}{I} c_\alpha^3 s_\alpha \left(\frac{\nu}{1-\nu^2} + c_\alpha^2 s_\alpha^2 \frac{I_n}{I}\right) - c_\alpha s_\alpha^5 \left(\frac{1}{1-\nu^2} + c_\alpha^4 \frac{I_n}{I}\right)\frac{I_n}{I}}{\left(\frac{1}{1-\nu^2} + c_\alpha^4 \frac{I_n}{I}\right)\left(\frac{1}{1+\nu} + 2c_\alpha^2 s_\alpha^2 \frac{I_n}{I}\right) - 2c_\alpha^6 s_\alpha^2 \left(\frac{I_n}{I}\right)^2}, \qquad (36)$$

both again depending only on the ratio I_n/I.

7.3. *Boundary Condition at Free Corner.*—At a free corner conditions (31) and (34) are both applicable, which means that:

$$w_x'' = w_y'' = w_{xy}'' = 0 . \qquad (37)$$

7.4. *Boundary Condition at Encastré Edge.*—At an encastré edge the curvature along the edge is zero and that across the edge is obtained by imagining the structure to extend beyond the edge as a mirror image of itself in the plane that passes through the edge and is normal to the plane of the wing.

7.5. *Formulae for Reactions at Typical Stations.*—The above boundary conditions are used to derive the formulae for the reactions at typical boundary stations in Appendix A.

8. *Approximate Method of Taking Account of Shear Deflections* (*as described in Ref.* 1) *with Oblique Shear Webs.*—For most wings, and particularly for thin wings, the approximate method of taking shear deflections into account, as described in Ref. 1 should be accurate enough for all practical purposes. The essence of the method consists in ignoring the shear strains represented by $\partial u/\partial z$ and $\partial v/\partial z$ and taking account only of $\partial w/\partial x$ and $\partial w/\partial y$. It allows the shear deflections to be derived independently of the bending deflections, which therefore are those obtained on the basis of infinite stiffness in shear. This means that the bending stresses derived on the latter basis remain unaltered. For thin wings they are considered to be accurate enough, and the main purpose of bringing in the shear deflections is to provide more accurate data for aeroelastic calculations. Since the approximate shear deflections as above defined can be derived independently of the bending deflections, we here consider the transverse displacements due to shear alone.

On deriving all the shear reactions the next step is to invert the stiffness matrix so formed in order to derive the corresponding flexibility matrix, which then gives the shear deflections at all stations in terms of any given applied load. The total deflection (that due to shear and bending) is finally obtained by mere addition of the two flexibility matrices, the one for bending and the other for shear.

Referring to Fig. 7, let

s = direction of oblique shear webs

β, γ = angle between directions of s and x axes for two separate sets of webs

p = displacement of skin in direction s

t_β, t_γ = thickness of shear webs normal to s per unit width for webs at angles β and γ to the x-axis

τ_β = shear stress in webs β

c_β, s_β = symbols for cos β and sin β

w = displacement normal to wing plane (positive upward).

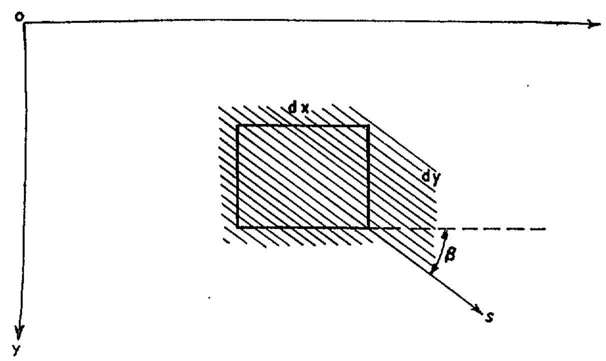


FIG. 7.

For the β webs, the upward shear per unit width normal to the direction s can be expressed in the form:

$$ht_\beta G \frac{\partial w}{\partial s} = T_\beta \text{ say}, \qquad (38)$$

since we are now neglecting the shear strain $\partial p/\partial z$.

Resolving $\partial w/\partial s$ along the co-ordinate axes, we have:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

$$= c_\beta \frac{\partial w}{\partial x} + s_\beta \frac{\partial w}{\partial y}, \qquad (39)$$

so that

$$T_\beta = ht_\beta G \left(c_\beta \frac{\partial w}{\partial x} + s_\beta \frac{\partial w}{\partial y}\right). \qquad (40)$$

8.1. *Typical Internal Station.*—The normal reaction per unit area of wing is given by:

$$q_\beta = -\frac{\partial T_\beta}{\partial s} = -\left(\frac{\partial T_\beta}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial T_\beta}{\partial y} \frac{\partial y}{\partial s}\right)$$

$$= -\left(c_\beta \frac{\partial T_\beta}{\partial x} + s_\beta \frac{\partial T_\beta}{\partial y}\right). \qquad (41)$$

Substituting for T_β from (40), we have:

$$q_\beta = -ht_\beta G(c_\beta^2 w_x'' + s_\beta^2 w_y'' + 2c_\beta s_\beta w_{xy}''), \qquad (42)$$

(where second derivatives of w are indicated by double primes).

In finite difference form, the reaction $(R)_0$ at station 0 (Fig. 1) due to shear flexibility alone is therefore given, for any internal station, by:

$$(R_0)_\beta = q_\beta l^2 = -l^2 h t_\beta G \left\{ c_\beta{}^2 \frac{(w_1 + w_3 - 2w_0)}{l^2} + s_\beta{}^2 \frac{(w_2 + w_4 - 2w_0)}{l^2} + 2c_\beta s_\beta \frac{(w_5 + w_7 - w_6 - w_8)}{4l^2} \right\}$$

$$= h t_\beta G \left\{ 2w_0 - c_\beta{}^2(w_1 + w_3) - s_\beta{}^2(w_2 + w_4) + \frac{c_\beta s_\beta}{2}(w_6 + w_8 - w_5 - w_7) \right\} . \quad (43)$$

$(R_0)_\gamma$ is given by the same expression but with γ substituted for β. The total shear reaction is then:

$$(R_0)_{\text{shear}} = (R_0)_\beta + (R_0)_\gamma . \quad (44)$$

8.2. *Boundary Condition at a Free Edge.*—The boundary condition that must be satisfied at a station located on a free edge is that the resultant shear across the edge for the two sets of shear webs must be zero. Where only one set of shear webs is present the shear carried by it at the free edge must be zero by itself.

For a free edge parallel to the x axis therefore we must have:

$$s_\beta T_\beta + s_\gamma T_\gamma = 0 \quad (45)$$

or, from (40),

$$(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma) \frac{\partial w}{\partial x} + (s_\beta{}^2 t_\beta + s_\gamma{}^2 t_\gamma) \frac{\partial w}{\partial y} = 0 . \quad (46)$$

This gives the derivative $\partial w/\partial y$ in terms of the known derivative $\partial w/\partial x$. Thus

$$\frac{\partial w}{\partial y} = -k_x \frac{\partial w}{\partial x} \quad (46a)$$

where

$$k_x = \left(\frac{c_\beta s_\beta + c_\gamma s_\gamma t_\gamma / t_\beta}{s_\beta{}^2 + s_\gamma{}^2 t_\gamma / t_\beta} \right) . \quad (47)$$

For a free edge parallel to the y-axis the corresponding equations are:

$$c_\beta T_\beta + c_\gamma T_\gamma = 0 \quad (48)$$

from which

$$\frac{\partial w}{\partial x} = -k_y \frac{\partial w}{\partial y} , \quad (49)$$

$$k_y = \left(\frac{c_\beta s_\beta + c_\gamma s_\gamma t_\gamma / t_\beta}{c_\beta{}^2 + c_\gamma{}^2 t_\gamma / t_\beta} \right) \quad (50)$$

8.3. *Boundary Conditions at a Free Corner.*—At a free corner conditions (46*a*) and (49) must both be satisfied, which can only mean that:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 . \quad (51)$$

The formulae for the reactions at stations located on the free edges, on free and re-entrant corners, etc., are derived and given in Appendix B.

9. *Procedure for Taking Account of Oblique Stringers and Shear Webs by the More Exact Method of Ref.* 1.—To take account of oblique stringers and webs by the more exact method described in Part I, it is necessary to define the resultant displacement of each station on the wing surface by three independent variables: the displacements u and v in the plane of the surface (assumed equal but opposite in sign to corresponding displacements on the opposite surface, or underskin) and a displacement w of the neutral plane normal to itself. The number of variables is thus multiplied by three, as are the sides of the relevant matrices.

For thin wings, which are necessarily covered by heavy-gauge skin, the refinement introduced by the present method is not, in the writer's view, worth seeking, the approximate approach above described being considered adequate.

For completeness, however, the method of taking account of oblique stringers and webs by the more accurate procedure is given here in detail.

9.1. *Oblique Stringers.*

Let

n	=	direction of stringers
α	=	angle between n and x directions
p	=	displacement in direction n
u, v, w	=	displacements in x, y and z directions
σ_α	=	tensile stresses in stringers
A_α	=	area of stringers per unit width across direction n
h	=	depth of wing
T_α	=	$A_\alpha \sigma_\alpha$ = tensile load in stringers per unit width across n
τ_β	=	shear stress in β webs
s_α, c_α	=	symbols for sin α and cos α
other symbols	=	those defined in section 7.

The displacement p of the stringers in their own direction α is given by:

$$p = c_\alpha u + s_\alpha v \,. \qquad (52)$$

The strain:

$$\frac{\partial p}{\partial n} = \left(\frac{\partial}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial}{\partial y}\frac{\partial y}{\partial n}\right)(c_\alpha u + s_\alpha v)$$

$$= \left(c_\alpha \frac{\partial}{\partial x} + s_\alpha \frac{\partial}{\partial y}\right)(c_\alpha u + s_\alpha v)$$

$$= c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \qquad (53)$$

Since the Poisson's ratio effect on the skin does not affect the stringers:

$$\sigma_\alpha = E\frac{\partial p}{\partial n} \qquad (54)$$

and the traction along n per unit width normal to n for stringers at angle α to the x-axis is:

$$T_\alpha = (A_\alpha \sigma_\alpha)$$

$$= EA_\alpha \left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}. \qquad (55)$$

It follows that the traction per unit width normal to the x direction is given by:

$$(T_x)_\alpha = c_\alpha^2 T_\alpha\,, \qquad (56)$$

and that normal to the y-direction by:

$$(T_y)_\alpha = s_\alpha^2 T_\alpha\,. \qquad (57)$$

The shear traction by the same convention is:

$$(T_{xy})_\alpha = c_\alpha s_\alpha T_\alpha\,. \qquad (58)$$

The resultant forces per unit area of wing skin in the x and y directions exerted by the stringers can now be written in the form:

$$\left.\begin{aligned} \frac{\partial (T_x)_\alpha}{\partial x} + \frac{\partial (T_{xy})_\alpha}{\partial y} &= X_\alpha \\ \frac{\partial (T_y)_\alpha}{\partial y} + \frac{\partial (T_{xy})_\alpha}{\partial x} &= Y_\alpha \end{aligned}\right\}. \qquad (59)$$

Substituting for T_x, etc., from equations (55) *et seq.* we have finally:

$$\left.\begin{aligned} (X_\alpha)_{\text{str}} &= EA_\alpha\left(c_\alpha^2\frac{\partial}{\partial x} + c_\alpha s_\alpha\frac{\partial}{\partial y}\right)\left\{c_\alpha^2\frac{\partial u}{\partial x} + s_\alpha^2\frac{\partial v}{\partial y} + c_\alpha s_\alpha\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} \\ (Y_\alpha)_{\text{str}} &= EA_\alpha\left(s_\alpha^2\frac{\partial}{\partial y} + c_\alpha s_\alpha\frac{\partial}{\partial x}\right)\left\{c_\alpha^2\frac{\partial u}{\partial x} + s_\alpha^2\frac{\partial v}{\partial y} + c_\alpha s_\alpha\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} \end{aligned}\right\}. \qquad (60)$$

Second set of stringers.—If there is a second set of stringers or skin reinforcements, such as rib flanges, at angle λ with the x-axis, tractions X_λ, Y_λ are given by the same equation (60) with A_λ, c_λ, s_λ merely substituted for A_α, c_α, s_α. We continue, however, on the basis of one set of stringers although clearly a second set can be brought in with no extra complication.

9.2. *Oblique Webs.*—Dealing next with the oblique webs, we write down the upward shear T_β in the webs per unit width normal to their direction s as:

$$\begin{aligned} T_\beta = ht_\beta\tau_\beta &= ht_\beta G\left(\frac{\partial p}{\partial z} + \frac{\partial w}{\partial s}\right) \\ &= ht_\beta G\left\{c_\beta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + s_\beta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\}. \end{aligned} \qquad (61)$$

The tractive force in the plane of the skin exerted by the webs in the direction s per unit area of skin is:

$$S_\beta = -t_\beta\tau_\beta\,. \qquad (62)$$

This force has components along x and y of

$$\left.\begin{aligned} (S_\beta)_x &= c_\beta S_\beta\,, \text{ and} \\ (S_\beta)_y &= s_\beta S_\beta \end{aligned}\right\}. \qquad (63)$$

Substituting in (63) the value of S_β given by (62) and (61), we have, in the x and y directions:

$$\left.\begin{aligned} (S_\beta)_x &= -t_\beta G\left\{c_\beta^2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + c_\beta s_\beta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} \\ \text{and}\qquad (S_\beta)_y &= -t_\beta G\left\{s_\beta^2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + c_\beta s_\beta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} \end{aligned}\right\}. \qquad (64)$$

Considering the corresponding vertical forces, we see that the upward force P_β exerted by the webs per unit area of skin is given by:

$$P_\beta = - h \left\{ \frac{\partial}{\partial x} (S_\beta)_x + \frac{\partial}{\partial y} (S_\beta)_y \right\}. \qquad (65)$$

Parallel equations to (65), with γ substituted for β, give the shears for a second set of shear webs at angle γ to the x-axis, and hence P_γ.

We can now write equations (26), (27) and (28) of Part I (which, as they stand, apply only to webs and stringers lying along the co-ordinate axes) in the form (assuming that there are no terms X_λ and Y_λ):

$$\left.\begin{aligned} t_s \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)_{\text{skin}} + (X_\alpha)_{\text{str}} + \left\{(S_\beta)_x + (S_\gamma)_x\right\}_{\text{webs}} &= - q_x \\ t_s \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right)_{\text{skin}} + (Y_\alpha)_{\text{str}} + \left\{(S_\beta)_y + (S_\gamma)_y\right\}_{\text{webs}} &= - q_y \\ P_\beta + P_\gamma &= - q_z \end{aligned}\right\}. \qquad (66)$$

To put equation (66) in terms of the three independent variables u, v and w, we note that, for the skin itself:

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) \\ \sigma_y &= \frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) \\ \tau_{xy} &= G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned}\right\}. \qquad (67)$$

Substituting these values, and expanding the other quantities by means of the previous equations, we have finally the following set of three equations.

9.3. *General Formulae for x, y and z Reactions (per unit area).*

In direction x:

$$\begin{aligned} & t_s \left[\frac{E}{1-\nu^2} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) + G \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\ & - t_\beta G \left[c_\beta^2 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + c_\beta s_\beta \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \\ & - t_\gamma G \left[c_\gamma^2 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + c_\gamma s_\gamma \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \\ & + EA_\alpha \left[\left(c_\alpha^2 \frac{\partial}{\partial x} + c_\alpha s_\alpha \frac{\partial}{\partial y}\right)\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}\right] = - q_x . \qquad (68a) \end{aligned}$$

In direction y:

$$\begin{aligned} & t_s \left[\frac{E}{1-\nu^2} \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) + G \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\ & - t_\beta G \left[s_\beta^2 \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + c_\beta s_\beta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\ & - t_\gamma G \left[s_\gamma^2 \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + c_\gamma s_\gamma \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\ & + EA_\alpha \left[\left(s_\alpha^2 \frac{\partial}{\partial y} + c_\alpha s_\alpha \frac{\partial}{\partial x}\right)\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}\right] = - q_y . \qquad (68b) \end{aligned}$$

In direction z:

$$hGt_\beta\left[\left(c_\beta{}^2\frac{\partial}{\partial x}+c_\beta s_\beta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\left(c_\beta s_\beta\frac{\partial}{\partial x}+s_\beta{}^2\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$+\ hGt_\gamma\left[\left(c_\gamma{}^2\frac{\partial}{\partial x}+c_\gamma s_\gamma\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+c_\gamma s_\gamma\frac{\partial}{\partial x}+s_\gamma{}^2\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]=-\,q_z\,. \qquad (68c)$$

Here

$$\frac{\partial u}{\partial z}=\frac{u}{h/2}\,; \qquad \frac{\partial v}{\partial z}=\frac{v}{h/2}\,. \qquad (69)$$

In finite difference form the above equations (68) can be written in terms of the displacements of the nine stations 0, 1, 2, . . . 8 of Fig. 6.

9.4. *Boundary Conditions.—Free edge.*—At a free edge the resultant direct stress normal to the edge, the resultant shear stress along the edge in the plane of the skin, and the resultant web shear stress must all be zero.

Resultant (skin and stringers) direct traction across a free edge parallel to direction x.—For stringers the traction per unit length of edge is given by (57):

$$(T_y)_{\alpha\,\mathrm{str}}=s_\alpha{}^2EA_\alpha\left\{c_\alpha{}^2\frac{\partial u}{\partial x}+s_\alpha{}^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}\ldots \qquad (70)$$

For skin it is:

$$(T_y)_{\mathrm{skin}}=t_s\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right). \qquad (71)$$

Therefore we must have:

$$(T_y)_{\alpha\,\mathrm{str}}+(T_y)_{\mathrm{skin}}=0\,,$$

or

$$\left(s_\alpha{}^2A_\alpha c_\alpha{}^2+\frac{\nu t_s}{1-\nu^2}\right)\frac{\partial u}{\partial x}+\left(s_\alpha{}^4A_\alpha+\frac{t_s}{1-\nu^2}\right)\frac{\partial v}{\partial y}+c_\alpha s_\alpha{}^3A_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0\,. \qquad (72)$$

(*Note:* If there is a second set of stringers another expression like (70) appears in (72) but with the difference only that a direction λ replaces direction α).

Resultant shear along edge in plane of skin.—For stringers, per unit length of edge, by (55) and (58):

$$(T_{xy})_{\mathrm{str}}=c_\alpha s_\alpha A_\alpha E\left\{c_\alpha{}^2\frac{\partial u}{\partial x}+s_\alpha{}^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}. \qquad (73)$$

For skin:

$$(T_{xy})_{\mathrm{skin}}=t_sG\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right). \qquad (74)$$

Therefore we must have $\left(\text{putting } G=\dfrac{E}{2(1+\nu)}\right)$:

$$(c_\alpha{}^3s_\alpha A_\alpha)\frac{\partial u}{\partial x}+(c_\alpha s_\alpha{}^3A_\alpha)\frac{\partial v}{\partial y}+\left(c_\alpha{}^2s_\alpha{}^2A_\alpha+\frac{t_s}{2(1+\nu)}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0\,. \qquad (75)$$

Equations (72) and (75) enable us to express the two y derivatives $\partial u/\partial y$ and $\partial v/\partial y$ in terms of the known derivative $\partial u/\partial x$. Thus:

$$\left.\begin{aligned} \frac{\partial v}{\partial y} &= U_1 \frac{\partial u}{\partial x} \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) &= U_2 \frac{\partial u}{\partial x} \end{aligned}\right\}, \qquad (76)$$

where

$$U_1 = \frac{c_\alpha^4 s_\alpha^4 \left(\frac{A_\alpha}{t_s}\right)^2 - \left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{\nu}{1-\nu^2}\right)\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{1}{2(1+\nu)}\right)}{\left(s_\alpha^4 \frac{A_\alpha}{t_s} + \frac{1}{1-\nu^2}\right)\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{1}{2(1+\nu)}\right) - c_\alpha^2 s_\alpha^6 \left(\frac{A_\alpha}{t_s}\right)^2} \qquad (77)$$

$$U_2 = \frac{c_\alpha s_\alpha^3 \frac{A_\alpha}{t_s}\left(s_\alpha^2 c_\alpha^2 \frac{A_\alpha}{t_s} + \frac{\nu}{1-\nu^2}\right) - c_\alpha^3 s_\alpha \frac{A_\alpha}{t_s}\left(s_\alpha^4 \frac{A_\alpha}{t_s} + \frac{1}{1-\nu^2}\right)}{\left(s_\alpha^4 \frac{A_\alpha}{t_s} + \frac{1}{1-\nu^2}\right)\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{1}{2(1+\nu)}\right) - c_\alpha^2 s_\alpha^6 \left(\frac{A_\alpha}{t_s}\right)^2}, \qquad (78)$$

both quantities depending therefore only on the ratio A_α/t_s of stringer to skin area.

Free edge parallel to y-axis.—The corresponding formulae for a free edge parallel to the y-axis are now at once* given by:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= U_3 \frac{\partial v}{\partial y} \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) &= U_4 \frac{\partial v}{\partial y} \end{aligned}\right\}, \qquad (79)$$

where

$$U_3 = \frac{c_\alpha^4 s_\alpha^4 \left(\frac{A_\alpha}{t_s}\right)^2 - \left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{\nu}{1-\nu^2}\right)\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{1}{2(1+\nu)}\right)}{\left(c_\alpha^4 \frac{A_\alpha}{t_s} + \frac{1}{1-\nu^2}\right)\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{1}{2(1+\nu)}\right) - c_\alpha^6 s_\alpha^2 \left(\frac{A_\alpha}{t_s}\right)^2} \qquad (80)$$

$$U_4 = \frac{c_\alpha^3 s_\alpha \frac{A_\alpha}{t_s}\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{\nu}{1-\nu^2}\right) - c_\alpha s_\alpha^3 \frac{A_\alpha}{t_s}\left(c_\alpha^4 \frac{A_\alpha}{t_s} + \frac{1}{1-\nu^2}\right)}{\left(c_\alpha^4 \frac{A_\alpha}{t_s} + \frac{1}{1-\nu^2}\right)\left(c_\alpha^2 s_\alpha^2 \frac{A_\alpha}{t_s} + \frac{1}{2(1+\nu)}\right) - c_\alpha^6 s_\alpha^2 \left(\frac{A_\alpha}{t_s}\right)^2} \qquad (81)$$

In practice U_1, U_2, U_3, U_4, being constants, are represented by a single number.

Web-shear at free edge.—In addition to the boundary conditions for direct stress and shear in the plane of the skin, we must also have zero resultant transverse shear in the shear webs.

From (45) therefore, for a free edge parallel to x:

$$s_\beta T_\beta + s_\gamma T_\gamma = 0 \qquad (82)$$

and, from (46), but substituting the full shear strains $\{(\partial v/\partial z) + (\partial w/\partial y)\}$ and $\{(\partial u/\partial z) + (\partial w/\partial x)\}$ for the approximate strains $\partial w/\partial y$ and $\partial w/\partial x$ there used:

$$(t_\beta s_\beta^2 + t_\gamma s_\gamma^2)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + (t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \,. \qquad (83)$$

* Requiring only the substitution of c by s and *vice versa*.

This gives the derivative $\partial w/\partial y$ in terms of known derivatives. More concisely,

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = V_1 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \ldots \quad (84)$$

where

$$V_1 = -(c_\beta s_\beta + c_\gamma s_\gamma t_\gamma/t_\beta)/(s_\beta^2 + s_\gamma^2 t_\gamma/t_s) . \quad \ldots \quad (85)$$

Correspondingly, for a free edge parallel to the y direction, we have, from (48), after substituting the full shear strain for the approximate strain:

$$(t_\beta c_\beta^2 + t_\gamma c_\gamma^2)\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + (t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = 0 , \quad \ldots \quad (86)$$

from which:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = V_2 \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \quad \ldots \quad (87)$$

where

$$V_2 = -\left(\frac{c_\beta s_\beta + c_\gamma s_\gamma t_\gamma/t_s}{c_\beta^2 + c_\gamma^2 t_\gamma/t_s}\right) \ldots \quad (88)$$

Boundary condition at free corner.—Since a free corner is the meeting point of the two free edges above considered, equations (76) and (79) must both be satisfied, as must equations (84) and (87). It follows that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \quad \ldots \quad (89)$$

at a corner station.

The free-edge conditions expressed by equations (76), (79), (84), (87) and (89) enable us to write down the reactions at any station located either on a free edge or a free corner, and this is done in Appendix C.

10. *Gaps or Breaks in Skin Continuity.*—There are two ways of dealing with gaps or breaks in skin continuity. If the gap is small, *i.e.*, of the order of the pitch of the stations, the most convenient method is to neglect the discontinuity while setting up the stiffness matrix and deal with the local modification in the stress distribution as a separate problem after the general distribution has been computed. If, on the other hand, the discontinuity extends over several pitches the most convenient method is probably to take account of it by regarding the edges of the break as free edges and applying the appropriate edge conditions.

11. *A Few Concluding Remarks.*—Many of the formulae derived here appear undesirably cumbrous. What should be remembered is that in applying them they reduce to a single number which in many cases, once calculated, is common to all stations with only slight modification. Typical cases are the formulae for H_1, H_2, H_3 and H_4 in section 7.2. Since the angle α of the stringers is usually constant the quantities H vary only with the ratio I_n/I of stringer to skin moment of inertia. To the extent that this tends to be constant the H's vary but little from station to station. Similar remarks apply to the formulae for k in para. 8.2, for U in para. 9.4, and for V also in section 9.4.

That the stiffness coefficients required for setting up the stiffness matrix are well adapted to computation by digital computer will be gathered from equation (25) for example. This gives the reaction at any internal station due to the stringers and it is seen that with the angle α usually constant only the factor I_n varies from station to station.

REFERENCES

No.	*Author*	*Title, etc.*
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2	R. H. MacNeal	The solution of elastic plate problems by electrical analogies. *J. App. Mech.* Vol. 18. No. 1. March, 1951.

APPENDIX A (Supplement to Section 7 of Main Text)

Stringer-Reinforced Wing of Infinite Shear Stiffness—Station Reactions at Free Edges, Free Corners, etc.

For an internal station the reaction due to the stringers alone is given by equation (25) and that due to the skin alone by (18*a*). In this appendix the reactions at stations variously situated on (or near) the boundary of the wing are derived by means of the boundary conditions expressed by equations (31), (34) and (37).

1. *Free Edge Parallel to x-Axis.*—Suppose the station 0 to be located on the edge AB of the plate (or wing) ABCD which is held along the edge AD of Fig. 1A.

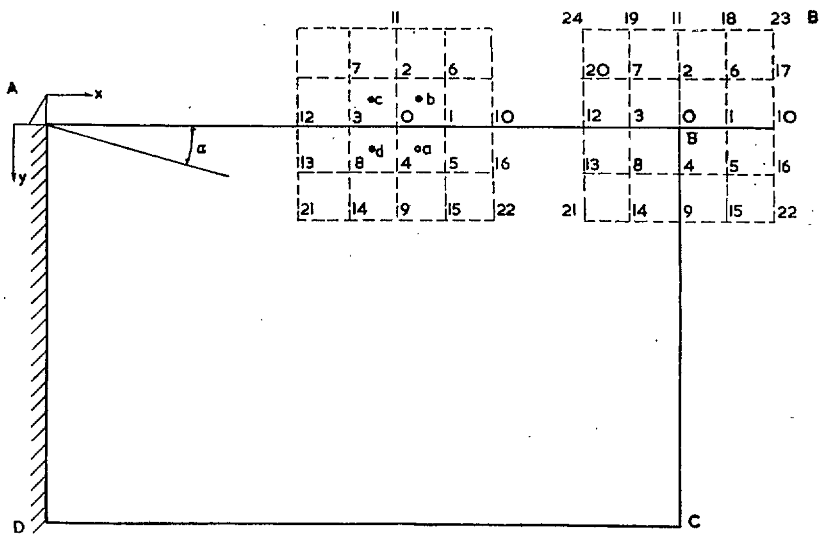

FIG. 1A.

The resultant reaction at station 0:

$$R_0 = (R_0)_{\text{skin}} + (R_0)_{\text{str}} . \qquad \qquad (1A)$$

1.1. *Reaction from Skin.*—From the general equation (1) we have:

$$(R_0)_{\text{skin}} = q_0 l^2 = l^2 \left\{ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \, \partial y} \right\} . \qquad \qquad (2A)$$

Referring to Fig. 1A and using the supplementary stations *a*, *b*, *c*, *d*, we write equation (2*A*) in finite-difference form as:

$$(R_0)_{\text{skin}} = \{(M_x)_1 + (M_x)_3 - 2(M_x)_0\} + \{(M_y)_2 + (M_y)_4 - 2(M_y)_0\}$$
$$+ 2\{(M_{xy})_a + (M_{xy})_c - (M_{xy})_b - (M_{xy})_d\} . \qquad \qquad (3A)$$

Moments $(M_y)_2$, $(M_{xy})_b$ and $(M_{xy})_c$ drop out because stations 2, *b* and *c* are outside the boundary. We can also leave out $(M_y)_0$ because, when $(R_0)_{\text{skin}}$ is later added to $(R_0)_{\text{str}}$ the resultant $(M_y)_0$ for skin and stringers is, by (28), equal to zero. In terms of twists and curvatures, (3*A*) as modified by (31) takes the form:

$$(R_0)_{\text{skin}} = \frac{E(I/2)}{1 - \nu^2} (1 + \nu H_1)\{(w_x'')_1 + (w_x'')_3 - 2(w_x'')_0\}$$
$$+ \frac{EI}{1 - \nu^2} \{(w\ '')_4 + \nu (w_x'')_4\} + 2 \frac{EI}{1 - \nu^2} (1 - \nu)\{(w_{xy}'')_a - (w_{xy}'')_d\} . \qquad \qquad (4A)$$

Converting second differentials to second differences, we have:

$$(R_0)_{\text{skin}} = \frac{EI}{l^2(1-\nu^2)}\Big[(8 + 3\nu H_1 - 4\nu)w_0 - (4 + 2\nu H_1 - 2\nu)(w_1 + w_3) - (6 - 2\nu)w_4 + (2 - \nu)(w_5 + w_8) + w_9 + \left(\frac{1 + \nu H_1}{2}\right)(w_{10} + w_{12})\Big] \quad \text{(5A)}$$

1.2. *Reaction from Stringers.*—In the general equation (23), where incidentally it is not feasible to use supplementary stations, $(M_n)_2$, $(M_n)_6$ and $(M_n)_7$ drop out, together with the term $\{-2s_\alpha^2(M_n)_0\}$, which, in combination with the term $\{-2(M_y)_0\}$ of (3A), has a zero resultant.

Thus

$$(R_0)_{\text{str}} = c_\alpha^2\{(M_n)_1 + (M_n)_2 - 2(M_n)_0\} + s_\alpha^2(M_n)_4 + \tfrac{1}{2}c_\alpha s_\alpha\{(M_n)_5 - (M_n)_8\} . \quad \text{(6A)}$$

Since, by (20),

$$M_n = EI_n(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'') ,$$

conditions (31) enable us to write, for stations 1, 0 and 3:

$$M_n = EI_n(c_\alpha^2 + H_1 s_\alpha^2 + 2H_2 c_\alpha s_\alpha)w_x'' = EI_n F_\alpha w_x'' , \quad \text{(7A)}$$

where

$$F_\alpha = (c_\alpha^2 + H_1 s_\alpha^2 + 2H_2 c_\alpha s_\alpha) . \quad \text{(8A)}$$

Using (7A) in the first term of (6A), which alone requires the application of the boundary conditions (31), we have

$$(R_0)_{\text{str}} = E\left(\frac{I_n}{2}\right) F_\alpha c_\alpha^2\{(w_x'')_1 + (w_x'')_3 - 2(w_x'')_0\} + EI_n s_\alpha^2\{c_\alpha^2(w_x'') + s_\alpha^2(w_y'') + 2c_\alpha s_\alpha w_{xy}''\}_4 + \frac{EI_n c_\alpha s_\alpha}{2}\{c_\alpha^2(w_x'') + s_\alpha^2(w_y'') + 2c_\alpha s_\alpha w_{xy}''\}_{\overline{5-8}} , \quad \text{(9A)}$$

or, in terms of the deflections w:

$$(R_0)_{\text{str}} = \frac{EI_n}{l^2}\Big[(3F_\alpha c_\alpha^2 + s_\alpha^4 + \tfrac{1}{2}c_\alpha s_\alpha)w_0 - 2F_\alpha c_\alpha^2(w_1 + w_3) - (2s_\alpha^2)w_4 - (c_\alpha s_\alpha - c_\alpha^2 s_\alpha^2)w_5 + (c_\alpha s_\alpha + c_\alpha^2 s_\alpha^2)w_8 + (s_\alpha^4 - \tfrac{1}{2}c_\alpha^2 s_\alpha^2)w_9 + (\tfrac{1}{2}F_\alpha c_\alpha^2 - \tfrac{1}{4}c_\alpha^2 s_\alpha^2)(w_{10} + w_{12}) + \tfrac{1}{2}c_\alpha^3 s_\alpha(w_{16} - w_{13}) + c_\alpha s_\alpha^3(w_{15} - w_{14}) + \tfrac{1}{4}c_\alpha^2 s_\alpha^2(w_{21} + w_{22})\Big] . \quad \text{(10A)}$$

The resultant R_0 is now obtained by adding together (5A) and (10A), to obtain:

$$R_0 = (R_0)_{\text{skin}} + (R_0)_{\text{str}} . \quad \text{(11A)}$$

2. *Opposite Free Edge.*—If the station lies on the opposite free edge DC, the above formulae, (5A) and (10A), are directly applicable so long as the pattern of stations is swung round through 180 deg and then renumbered so as to face the right way.

3. *Free Edge Parallel to y-axis.*—If the station lies on the edge BC of Fig. 8, formulae parallel to (5A) and (10A) are derived by the simple expedient of turning the pattern of stations round through a right angle. On the basis of this pattern c_α and s_α are interchanged, H_3 is substituted for H_1 in (5A) and J_α is substituted for F_α in (10A), where:

$$J_\alpha = (c_\alpha^2 H_3 + s_\alpha^2 + 2c_\alpha s_\alpha H_4) . \quad \text{(12A)}$$

Finally the stations are renumbered to face the usual way. The resulting formulae are as follows:

$$(R_0)_{\text{skin}} = \frac{EI}{l^2(1-\nu^2)}\Big\{(8+3\nu H_3-4\nu)w_0-(4+2\nu H_3-2\nu)(w_2+w_4)$$
$$-(6-2\nu)w_3+(2-\nu)(w_7+w_8)+\frac{1+\nu H_3}{2}(w_9+w_{11})+w_{12}\Big\}. \quad \text{.. (13}A\text{)}$$

$$(R_0)_{\text{str}} = \frac{EI_n}{l^2}\Big\{(3J_\alpha s_\alpha{}^2+c_\alpha{}^4+\tfrac{1}{2}c_\alpha s_\alpha)w_0-2J_\alpha s_\alpha{}^2(w_2+w_4)-2c_\alpha{}^2w_3$$
$$-(c_\alpha s_\alpha-c_\alpha{}^2s_\alpha{}^2)w_8+(c_\alpha s_\alpha+c_\alpha{}^2s_\alpha{}^2)w_7+(c_\alpha{}^4-\tfrac{1}{2}c_\alpha{}^2s_\alpha{}^2)w_{12}$$
$$+(\tfrac{1}{2}J_\alpha s_\alpha{}^2-\tfrac{1}{4}c_\alpha{}^2s_\alpha{}^2)(w_9+w_{11})+\tfrac{1}{2}s_\alpha{}^3c_\alpha(w_{14}-w_{18})$$
$$+c_\alpha{}^3s_\alpha(w_{13}-w_{20})+\tfrac{1}{4}c_\alpha{}^2s_\alpha{}^2(w_{21}+w_{21})\Big\}. \quad \text{.. (14}A\text{)}$$

4. *Station on Free Corner.*—If the station stands on a free corner (corner B, say, in Fig. 1A), the boundary conditions are given by (37). In the general equation (3A) for the skin reaction, moments $(M_x)_1$, $(M_x)_0$, $(M_y)_2$, $(M_y)_0$, $(M_{xy})_a$, $(M_{xy})_b$ and $(M_{xy})_c$ all drop out, so that:

$$(R_0)_{\text{skin}} = (M_x)_3+(M_y)_4-2(M_{xy})_d\,. \quad \text{.. (15}A\text{)}$$

Using (31) and (34), we write (15A) in the form:

$$(R_0)_{\text{skin}} = \frac{E(I/2)}{(1-\nu^2)}\Big\{(1+\nu H_1)(w_x'')_3+(1+\nu H_3)(w_y'')_4\Big\}-\tfrac{1}{2}\left(\frac{EI}{1-\nu^2}\right)(1-\nu)(w_{xy}'')_d$$

or, in terms of deflections w:

$$(R_0)_{\text{skin}} = \frac{EI}{l^2(1-\nu^2)}\Big[\Big\{\frac{3}{2}+\frac{\nu}{2}(H_1+H_3-1)\Big\}w_0-\Big\{\frac{3}{2}+\nu(H_1-\tfrac{1}{2})\Big\}w_3$$
$$-\Big\{\frac{3}{2}+\nu(H_3-\tfrac{1}{2})\Big\}w_4+\tfrac{1}{2}(1-\nu)w_8+\tfrac{1}{2}(1+\nu H_3)w_9$$
$$+\tfrac{1}{2}(1+\nu H_1)w_{12}\Big]. \quad \text{.. (16}A\text{)}$$

Correspondingly

$$(R_0)_{\text{str}} = c_\alpha{}^2(M_n)_3+s_\alpha{}^2(M_n)_4-\tfrac{1}{2}c_\alpha s_\alpha(M_n)_8.$$
$$= c_\alpha{}^2E\frac{I_n}{2}F_\alpha(w_x'')_3+s_\alpha{}^2E\left(\frac{I}{2}\right)J_\alpha(w_y'')_4$$
$$-\tfrac{1}{2}c_\alpha s_\alpha EI_n\{c_\alpha{}^2w_x''+s_\alpha{}^2w_y''-2c_\alpha s_\alpha w_{xy}''\}_8$$

or, in terms of the deflections w:

$$(R_0)_{\text{str}} = \frac{EI_n}{2l^2}\Big[(c_\alpha{}^2F_\alpha+s_\alpha{}^2J_\alpha+\tfrac{1}{2}c_\alpha s_\alpha)w_0-(2c_\alpha{}^2F_\alpha+c_\alpha s_\alpha{}^3)w_3$$
$$-(2s_\alpha{}^2J_\alpha+s_\alpha c_\alpha{}^3)w_4+(2c_\alpha s_\alpha)w_8+\left(s_\alpha{}^2J_\alpha-\frac{c_\alpha{}^2s_\alpha{}^2}{2}\right)w_9$$
$$+(c_\alpha{}^2F_\alpha+c_\alpha{}^2s_\alpha{}^2)w_{12}-(s_\alpha c_\alpha{}^3)w_{13}-(c_\alpha s_\alpha{}^3)w_{14}+\left(\frac{c_\alpha s_\alpha}{2}\right)w_{21}\Big]. \quad \text{.. (17}A\text{)}$$

The resultant reaction is now expressed by the sum of $(R_0)_{\text{skin}}$ and $(R_0)_{\text{str}}$ as in (11A).

5. *Station Located at a Re-entrant Corner.*—For a station 0 located at a re-entrant corner of the type shown in Fig. 2A, equation (37) loses the terms $(M_x)_1$, $(M_y)_2$ and $(M_{xy})_6$.

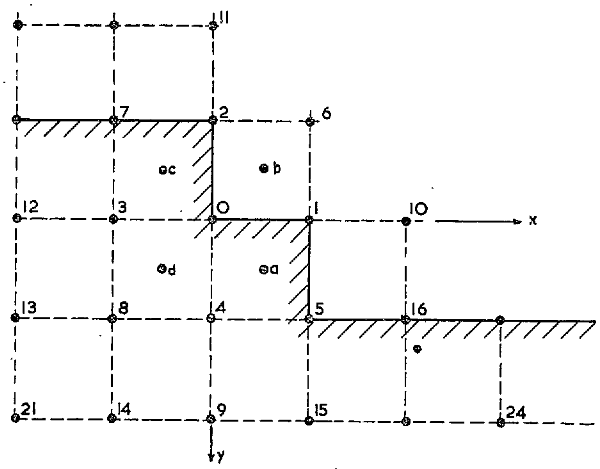

FIG. 2A.

For the skin reaction, we then have:

$$(R_0)_{\text{skin}} = \{(M_x)_3 - 2(M_x)_0\} + \{(M_y)_4 - 2(M_y)_0\} + 2\{(M_{xy})_a + (M_{xy})_c - (M_{xy})_d\} \quad .. \ (18A)$$

or

$$(R_0)_{\text{skin}} = \frac{EI}{1-\nu^2}\{(w_x'')_3 + (w_y'')_4\} - \frac{2E}{1-\nu^2}\frac{3I}{4}\{(w_x'')_0 + \nu(w_y'')_0 + (w_y'')_0 + \nu(w_x'')_0\}$$

$$+ 2\left(\frac{EI}{1-\nu^2}\right)(1-\nu)\left\{(w_{xy}'')_a + (w_{xy}'')_c - (w_{xy}'')_d\right\}, \quad .. \quad .. \quad .. \ (19A)$$

where, in the second term of the R.H.S. of (19A), $\frac{3}{4}I$ has replaced I as a better average value of I at station 0.

In finite-difference form:

$$(R_0)_{\text{skin}} = \frac{EI}{2l^2(1-\nu^2)}\Big[28w_0 - (7-\nu)(w_1 + w_2) - (15 - 5\nu)(w_3 + w_4)$$

$$+ 2(1-\nu)(w_5 + w_7 + w_8) + 2(w_9 + w_{12})\Big]. \quad .. \quad .. \quad .. \quad .. \ (20A)$$

For the stringer reaction we have, on substituting $c_\alpha^2 M_n$, $s_\alpha^2 M_n$ and $c_\alpha s_\alpha M_n$ for M_x, M_y and M_{xy} respectively by (21):

$$(R_0)_{\text{str}} = \{c_\alpha^2(M_n)_3 + s_\alpha^2(M_n)_4\} - 2\{c_\alpha^2(M_n)_0 + s_\alpha^2(M_n)_0\}$$

$$+ \frac{c_\alpha s_\alpha}{2}\{(M_n)_5 + (M_n)_7 - (M_n)_8\} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \ (21A)$$

or, using equation (20):

$$(R_0)_{\text{str}} = \frac{EI_n}{l^2}\Big[c_\alpha^2(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'')_3$$

$$+ s_\alpha^2(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'')_4$$

$$- 2(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'')_0$$

$$+ \frac{c_\alpha s_\alpha}{2}(c_\alpha^2 w_x'' + s_\alpha^2 w_y'' + 2c_\alpha s_\alpha w_{xy}'')_{\overline{5+7-8}}\Big]. \quad .. \quad .. \quad .. \quad .. \quad .. \ (22A)$$

The appropriate fraction of I_n is associated with each station located on the edge. Also, in writing down the finite difference form of $(\partial^2 w/\partial x\, \partial y)$ for station 0, for example, use is made of the fact that at station 1, by (37), this derivative has the value zero. This means that $(\partial w/\partial y)_1$ can be taken equal to $(\partial w/\partial y)_0$, *i.e.*, equal to $(w_4 - w_2)/2l$. The same argument applies to station 5.

In terms of deflections w, (22*A*) becomes:

$$(R_0)_{\text{str}} = \frac{EI_n}{l^2}\Big[\left(4 - \frac{3}{2}c_\alpha^2 s_\alpha^2\right)w_0 + \left(c_\alpha s_\alpha^3 - \frac{3}{2}c_\alpha^2 - \tfrac{1}{4}c_\alpha^2 s_\alpha^2\right)w_1$$
$$+ \left(\tfrac{1}{4}c_\alpha^2 s_\alpha^2 + c_\alpha s_\alpha + \frac{H}{2}c_\alpha s_\alpha^3 - \tfrac{1}{4}c_\alpha^3 s_\alpha - \frac{3}{2}s_\alpha^2\right)w_2$$
$$- (3\tfrac{1}{2}c_\alpha^2)w_3 - (3\tfrac{1}{2}s_\alpha^2 + c_\alpha s_\alpha - \tfrac{3}{8}c_\alpha^3 s_\alpha)w_4 + (c_\alpha^2 s_\alpha^2 - \tfrac{3}{4}c_\alpha s_\alpha)w_5$$
$$+ (c_\alpha s_\alpha + \tfrac{1}{2}c_\alpha^2 s_\alpha^2 - \tfrac{1}{2}c_\alpha^3 s_\alpha - H_1 c_\alpha s_\alpha^3)w_7 + 2(c_\alpha s_\alpha + c_\alpha^2 s_\alpha^2)w_8$$
$$+ (s_\alpha^4 - c_\alpha^2 s_\alpha^2)w_9 + (c_\alpha^2 - \tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{12} - (\tfrac{3}{4}c_\alpha^3 s_\alpha)w_{13} - (c_\alpha s_\alpha^3)w_{14}$$
$$+ (\tfrac{7}{8}c_\alpha s_\alpha^3 + \tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{15} + (\tfrac{3}{8}c_\alpha^3 s_\alpha)w_{16}$$
$$+ (\tfrac{3}{4}c_\alpha^3 s_\alpha + \tfrac{1}{2}H_1 c_\alpha s_\alpha^3 + \tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{20} + (\tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{21}\Big] \quad \ldots \quad (23A)$$

Addition of equations (20*A*) and (23*A*) gives the resultant reaction (R_0).

If, in Fig. 2A, the free edge is defined by 5, 16, 22 instead of by a continuation of the edge 5–16, the above formula is unaffected. It is affected, however, if the edge turns up at station 7 to station 19 instead of extending to station 20. Formula (23*A*) is then modified to the form:

$$(R_0)_{\text{str}} = \frac{EI_n}{l^2}\Big[\left(4 - \frac{3}{2}c_\alpha^2 s_\alpha^2\right)w_0 + \left(c_\alpha s_\alpha^3 - \frac{3}{2}c_\alpha^2 - \tfrac{1}{4}c_\alpha^2 s_\alpha^2\right)w_1$$
$$+ \left(c_\alpha s_\alpha - \frac{3}{2}s_\alpha^2 - \tfrac{1}{8}c_\alpha^3 s_\alpha\right)w_2 - (3\tfrac{1}{2}c_\alpha^2 - \tfrac{3}{8}c_\alpha s_\alpha^3 - \tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_3$$
$$- (3\tfrac{1}{2}s_\alpha^2 + c_\alpha s_\alpha - \tfrac{3}{8}c_\alpha^3 s_\alpha)w_4 + (c_\alpha^2 s_\alpha^2 - \tfrac{3}{4}c_\alpha s_\alpha)w_5$$
$$+ (c_\alpha^2 s_\alpha^2 + c_\alpha s_\alpha - \tfrac{3}{4}c_\alpha^3 s_\alpha)w_7 + 2(c_\alpha s_\alpha + c_\alpha^2 s_\alpha^2)w_8$$
$$+ (s_\alpha^4 - c_\alpha^2 s_\alpha^2)w_9 + (c_\alpha^2 - \tfrac{1}{2}c_\alpha^2 s_\alpha^2 - w_{12} - (\tfrac{3}{4}c_\alpha^3 s_\alpha)w_{13}$$
$$- (c_\alpha s_\alpha^3)w_{14} + (\tfrac{7}{8}c_\alpha s_\alpha^3 + \tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{15} + (\tfrac{3}{8}c_\alpha^3 s_\alpha)w_{16}$$
$$+ (\tfrac{3}{8}c_\alpha s_\alpha^3 - \tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{19} + (\tfrac{7}{8}c_\alpha^3 s_\alpha)w_{20} + (\tfrac{1}{4}c_\alpha^2 s_\alpha^2)w_{24}\Big]. \quad \ldots \quad (24A)$$

$(R_0)_{\text{skin}}$ is still given by formula (20*A*), so that the resultant reaction at station 0 for the slightly changed geometry is obtained by adding (20*A*) and (24*A*). It may be noted that the coefficients of the w's inside the square bracket of equation (24*A*) are constant for all similarly situated stations, only the factor I_n outside the bracket varying from station to station.

APPENDIX B (Supplement to Section 8 of Main Text)

Boundary-station Formulae for Approximate Method of Deriving Shear Deflections

The boundary conditions expressed by equations (46*a*) and (49) are here used to derive the appropriate formulae for the reactions at typical boundary stations.

1. *Free Edge Parallel to x-axis.*—Suppose station 0 to be located on the free edge AB of Fig. 1B.

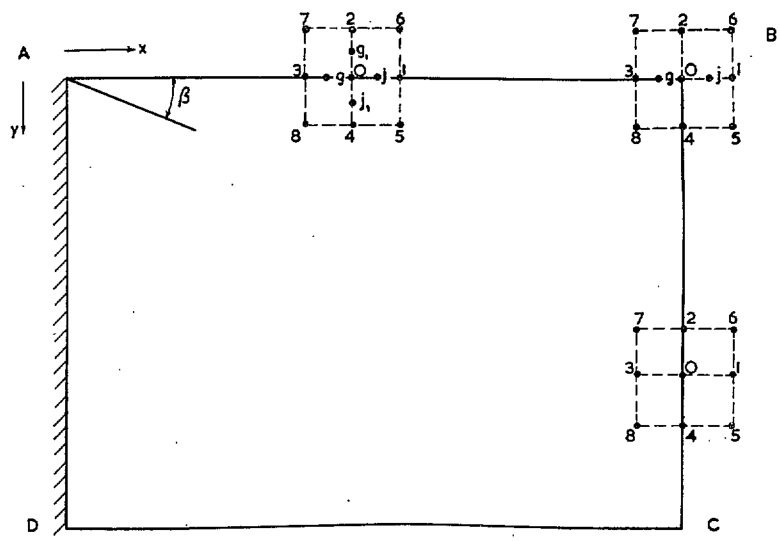

FIG. 1B.

From equation (42), we have:

$$(q_0)_\beta = -\,ht_\beta G(c_\beta^2 w_x'' + s_\beta^2 w_y'' + 2c_\beta s_\beta w_{xy}'')$$

which, by using the supplementary stations j, g, j_1, g_1 we can write in the form:

$$(q_0)_\beta = -\,ht_\beta G\left[\frac{c_\beta^2}{l}\{(w_x')_j - (w_x')_g\} + \frac{s_\beta^2}{l}\{(w_y')_{j1} - (w_y')_{g1}\} + \frac{2c_\beta s_\beta}{l}\{(w_y')_j - (w_y')_g\}\right] \quad \ldots \quad (1B)$$

Since $(w_y')_{g1}$ drops out, and since $(w_y')_g$ and $(w_y')_j$ are equal respectively to $(w_x')_g(-\,k_x)$ and $(w_x')_j(-\,k_x)$, we have:

$$(R_0)_\beta = l^2(q_0)_\beta = -\,ht_\beta G\left[\frac{c_\beta^2}{2}(w_1 + w_3 - 2w_0) + s_\beta^2(w_4 - w_0) - 2c_\beta s_\beta k_x(w_1 + w_3 - 2w)\right],$$

where the first term in the square bracket is divided by 2 to take account of the reduced average t_β in the x-direction. Rearranging the w's we have:

$$(R_0)_\beta = ht_\beta G\left\{(1 - 4c_\beta s_\beta k_x)w_0 - \left(\frac{c_\beta^2}{2} - 2c_\beta s_\beta k_x\right)(w_1 + w_3) - s_\beta^2 w_4\right\}. \quad \ldots \quad (2B)$$

The corresponding reaction for webs at an angle γ to the x-axis is given by the same expression with γ substituted for β throughout. The resultant shear reaction is then:

$$(R_0)_{\text{shear}} = (R_0)_\beta + (R_0)_\gamma \quad \ldots \quad (3B)$$

2. *Free Edge Parallel to y-axis.*—If the station is located on the free edge BC (Fig. 1B), the reaction is at once obtained by rotating the station pattern through 90 deg, interchanging c_β and s_β, and finally renumbering the pattern to make it face the right way. This gives:

$$(R_0)_\beta = ht_\beta G\left\{(1 - 4s_\beta c_\beta k_y)w_0 - \left(\frac{s_\beta^2}{2} - 2s_\beta c_\beta k_y\right)(w_2 - w_4) - c_\beta^2 w_3\right\}, \qquad \ldots \quad \ldots \quad (4B)$$

and similarly for $(R_0)_\gamma$.

3. *Free Corner.*—If the station is situated at the corner B in Fig. 1B, the terms in $(w_x')_j$, $(w_y')_{g1}$ and $(w_y')_j$ drop out of equation ($1B$) so that:

$$(R_0)_\beta = l^2(q_0)_\beta = -\frac{ht_\beta G}{2}\left\{-c_\beta^2(w_x')_g + s_\beta^2(w_y')_{j1} - 2c_\beta s_\beta(w_y')_g\right\}$$

$$= \frac{ht_\beta G}{2}\left\{(1 - 2c_\beta s_\beta k_x)w_0 - (c_\beta^2 - 2c_\beta s_\beta k_x)w_3 - s_\beta^2 w_4\right\}. \qquad \ldots \quad \ldots \quad (5B)$$

4. *Re-entrant Corner.*—For the re-entrant corner shown in Fig. 2B, we use equation ($1B$)

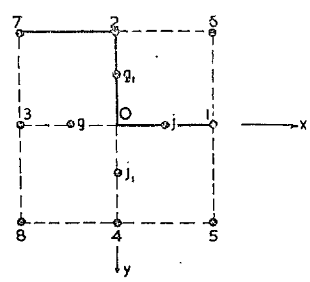

FIG. 2B.

but do not use the supplementary stations j and g for the $c_\beta s_\beta$ term.

Thus

$$(q_0)_\beta = -\frac{ht_\beta G}{l}\left[c_\beta\{(w_x')_j - (w_x')_g\} + s_\beta^2\{(w_y')_{j1} - (w_y')_{g1}\} + c_\beta s_\beta\{(w_x')_4 - (w_x')_2\}\right]. \qquad (6B)$$

Noting that, by (51), $(w_x')_2$ is zero, we write:

$$(R_0)_\beta = (q_0)_\beta l^2 = ht_\beta G\left[3w_0 - \tfrac{1}{2}c_\beta^2 w_1 - \tfrac{1}{2}s_\beta^2 w_2 + c_\beta^2 w_3 + s_\beta^2 w_4 + \tfrac{1}{2}c_\beta s_\beta(w_5 - w_8)\right]. \qquad (7B)$$

APPENDIX C (Supplement To Section 9 of Main Text)

Boundary-Station Formulae Appropriate to the More Exact Method

The boundary conditions appropriate to the more exact method have been deduced in section 9 and what is done here is to use those conditions to derive the formulae for the reactions at stations situated on various free edges and corners.

1. *Free Edge Parallel to the x-axis.*—As before, let the free edge be represented in Fig. 6 by the line of stations 12, 3, 0, 1, 10, which separates the real structure below that line from the infinitely flexible structure above it. As before, the average I of skin and stringers in the direction of the free edge is taken to be half that for an interior station, in order to take account of the half-pitch width of the strip associated with the free-edge stations.

It is to be noted that, in using equation (68) to write the down x, y and z reactions for a free-edge station, we do not need, in the case of some of the terms, to introduce the boundary conditions. For example, in (68*a*) there are two such terms. The first is:

$$t_s G \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \ldots \qquad (a)$$

and the second is:

$$EA_\alpha c_\alpha s_\alpha \frac{\partial}{\partial y}\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}. \quad \ldots \qquad (b)$$

For station 0, the first may be written in the form:

$$\frac{t_s G}{2l}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_4 - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_2\right], \quad \ldots \qquad (a')$$

and the second in the form:

$$\frac{EA_\alpha c_\alpha s_\alpha}{2l}\left[\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}_4 - \left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}_2\right]. \quad \ldots \qquad (b')$$

Since station 2 is located on the infinitely flexible extension of the real plate, expression (a') and (b') reduce respectively to:

$$\frac{t_s G}{2l}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_4 \text{ and } \frac{EA_\alpha c_\alpha s_\alpha}{2l}\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}_4,$$

and since station 4 is an interior station all the first derivatives in these expressions can be written down without reference to the boundary conditions.

The result is that equations (68) may be written in the following form, for a free edge parallel to the x axis :

$$\begin{aligned}\Bigg[\frac{t_s}{2}\frac{E}{1-\nu^2}\frac{1}{l}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right)_{\overline{j-g}} &+ \frac{t_s G}{2l}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_4 \\ &- \frac{t_\beta}{2} G\left\{c_\beta^2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_0 + c_\beta s_\beta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)_0\right\} \\ &- \frac{t_\gamma}{2} G\left\{c_\gamma^2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_0 + c_\gamma s_\gamma\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)_0\right\} \\ &+ \frac{EA_\alpha}{2}\frac{c_\alpha^2}{l}\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}_{\overline{j-g}} \\ &+ \frac{EA_\alpha c_\alpha s_\alpha}{2l}\left\{c_\alpha^2 \frac{\partial u}{\partial x} + s_\alpha^2 \frac{\partial v}{\partial y} + c_\alpha s_\alpha \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_4\right\} = -q_x. \quad \ldots \qquad (1C)a\end{aligned}$$

$$\left[\frac{t_s E}{1-\nu^2}\frac{1}{2l}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)_4+\frac{1}{l}\frac{t_s}{2}G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{\overline{j-g}}\right.$$

$$-t_\beta G\left\{s_\beta^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_0+c_\beta s_\beta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_0\right\}$$

$$-t_\gamma G\left\{s_\gamma^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_0+c_\gamma s_\gamma\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_0\right\}$$

$$+\frac{EA_\alpha c_\alpha s_\alpha}{2l}\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{\overline{j-g}}$$

$$\left.+\frac{EA_\alpha s_\alpha^2}{2l}\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_4\right]=-q_y \quad \ldots \quad \ldots \quad (1C)b$$

$$\left[\left(\frac{1}{l}hG\right)\frac{t_\beta}{2}\left\{c_\beta^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_{\overline{j-g}}+c_\beta s_\beta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_{\overline{j-g}}\right\}\right.$$

$$+\frac{t_\beta hG}{2l}\left\{c_\beta s_\beta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_4+s_\beta^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_4\right\}$$

$$+\left(\frac{1}{l}hG\right)\frac{t_\gamma}{2}\left\{c_\gamma^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_{\overline{j-g}}+c_\gamma s_\gamma\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_{\overline{j-g}}\right\}$$

$$+\frac{t_\gamma hG}{2l}\left\{c_\gamma s_\gamma\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_4+s_\gamma^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_4\right\}=-q_z \quad \ldots \quad \ldots \quad (1C)c$$

where the suffix $\overline{j-g}$ signifies the value of the relevant expression at supplementary station j minus its value at supplementary station g, and where

$$\left(\frac{\partial u}{\partial z}\right)_{\overline{j-g}}=\tfrac{1}{2}\left(\frac{\partial u}{\partial z}\right)_{\overline{1-3}}.$$

The above equations contain only the first derivatives of u, v and w with respect to the co-ordinates. Of the four stations involved, station 4 is an interior station for which, as already stated, the derivatives can be written down in finite-difference form without reference to the edge conditions. The remaining three stations (0, 1 and 3) are located on the edge and here the boundary conditions (76), (79), (84) and (87) apply in conjunction with (69). On this basis equations (68) take the form:

$$\left[\frac{t_s E}{2l(1-\nu^2)}(1+\nu U_1)\left(\frac{\partial u}{\partial x}\right)_{\overline{j-g}}+\frac{t_s G}{2l}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_4\right.$$

$$-\frac{G}{2}\left\{(t_\beta c_\beta^2+t_\gamma c_\gamma^2)+V_1(t_\beta c_\beta s_\beta+t_\gamma c_\gamma s_\gamma)\right\}\left\{\frac{2u_0}{h}+\left(\frac{\partial w}{\partial x}\right)_0\right\}$$

$$+\frac{EA_\alpha c_\alpha^2}{2l}(c_\alpha^2+U_1 s_\alpha^2+U_2 c_\alpha s_\alpha)\left(\frac{du}{dx}\right)_{\overline{j-g}}$$

$$\left.+\frac{EA_\alpha c_\alpha s_\alpha}{2l}\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_4\right]=-q_x \quad \ldots \quad \ldots \quad (2C)a$$

$$\left[\frac{t_s E}{2l(1-\nu^2)}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)_4+\frac{t_s G}{2l}U_2\left(\frac{\partial u}{\partial x}\right)_{\overline{j-g}}\right.$$

$$-G\left\{(t_\beta c_\beta s_\beta+t_\gamma c_\gamma s_\gamma)+V_1(t_\beta s_\beta^2+t_\gamma s_\gamma^2)\right\}\left\{\frac{2u_0}{h}+\left(\frac{\partial w}{\partial x}\right)_0\right\}$$

$$+\frac{EA_\alpha s_\alpha c_\alpha}{2l}(c_\alpha^2+U_1 s_\alpha^2+U_2 c_\alpha s_\alpha)\left(\frac{\partial u}{\partial x}\right)_{\overline{j-g}}$$

$$\left.+\frac{EA_\alpha s_\alpha^2}{2l}\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_4\right]=-q_y \quad \ldots \quad \ldots \quad (2C)b$$

$$\left[\frac{hG}{2l}\left\{(t_\beta c_\beta^2 + t_\gamma c_\gamma^2) + V_1(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\right\}\left\{\frac{2}{h}\frac{(U_1 - U_3)}{2} + \left(\frac{\partial w}{\partial x}\right)_{\overline{j-g}}\right\}\right.$$
$$+ \frac{hG}{2l}(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\left\{\frac{2u_4}{h} + \left(\frac{\partial w}{\partial x}\right)_4\right\}$$
$$\left. + \frac{hG}{2l}(t_\beta s_\beta^2 + t_\gamma s_\gamma^2)\left\{\frac{2v_4}{h} + \left(\frac{\partial w}{\partial y}\right)_4\right\}\right] = -q_z . \quad \ldots \quad (2C)c$$

In finite differences form this becomes:

$$\left[\left\{\frac{t_s E(1 + \nu U_1)}{2(1 - \nu^2)l^2} + E\left(\frac{A_\alpha}{2}\right)\frac{c_\alpha^2}{l^2}(c_\alpha^2 + U_1 s_\alpha^2 + U_2 c_\alpha s_\alpha)\right\}(u_1 + u_3 - 2u_0)\right.$$
$$- \frac{G}{2}\left\{(t_\beta c_\beta^2 + t_\gamma c_\gamma^2) + V_1(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\right\}\left\{\frac{2}{h}u_0 + \frac{1}{2l}(w_1 - w_3)\right\}$$
$$+ \frac{EA_\alpha c_\alpha s_\alpha}{4l^2}\left\{c_\alpha^2(u_5 - u_8) + s_\alpha^2(v_9 - v_0)\right\}$$
$$\left. + \left(\frac{EA_\alpha c_\alpha^2 s_\alpha^2}{4l^2} + \frac{t_s G}{4l^2}\right)(u_9 - u_0 + v_5 - v_8)\right] l^2 = -(q_x)_0 l^2 = -(R_x)_0 \ldots (3C)a$$

$$\left[\left\{\frac{t_s G}{2l^2}U_2 + \frac{EA_\alpha s_\alpha c_\alpha}{2l^2}(c_\alpha^2 + U_1 s_\alpha^2 + U_2 c_\alpha s_\alpha)\right\}(u_1 + u_3 - 2u_0)\right.$$
$$- G\left\{(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma) + V_1(t_\beta s_\beta^2 + t_\gamma s_\gamma^2)\right\}\left\{\frac{2}{h}u_0 + \frac{1}{2l}(w_1 - w_3)\right\}$$
$$+ \frac{E}{4l^2}\left(\frac{\nu t_s}{1 - \nu^2} + A_\alpha c_\alpha^2 s_\alpha^2\right)(u_5 - u_8) + \frac{E}{4l^2}\left(\frac{t_s}{1 - \nu^2} + A_\alpha s_\alpha^4\right)(v_9 - v_0)$$
$$\left. + \frac{EA_\alpha c_\alpha s_\alpha^3}{4l^2}(u_9 - u_0 + v_5 - v_8)\right] l^2 = -(q_y)_0 l^2 = -(R_y)_0 \quad \ldots \quad (3C)b$$

$$\left[\frac{G}{2l}\left\{(t_\beta c_\beta^2 + t_\gamma c_\gamma^2) + V_1(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\right\}\left\{(u_1 - u_3) + \frac{h}{l}(w_1 + w_3 - 2w_0)\right\}\right.$$
$$+ \frac{G}{l}(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)u_4 + \frac{G}{l}(t_\beta s_\beta^2 + t_\gamma s_\gamma^2)v_4 + \frac{hG}{4l^2}\left\{(t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)(w_5 - w_8)\right.$$
$$\left.\left. + (t_\beta s_\beta^2 + t_\gamma s_\gamma^2)(w_9 - w_0)\right\}\right] l^2 = -(q_z)_0 l^2 = -(R_z)_0 . \quad \ldots \quad \ldots \quad (3C)c$$

2. *Free Edge Parallel to y-axis.*—The corresponding formulae for a station situated on a free edge parallel to the y-axis are as follows:

$$\left[\frac{1}{2l^2}\left\{t_s G U_4 + EA_\alpha c_\alpha s_\alpha(c_\alpha^2 U_3 + s_\alpha^2 + U_4 c_\alpha s_\alpha)\right\}(v_4 + v_2 - 2v_0)\right.$$
$$- G\left\{V_2(t_\beta c_\beta^2 + t_\gamma c_\gamma^2) + (t_\beta c_\beta s_\beta + t_\gamma c_\gamma s_\gamma)\right\}\left\{\frac{2}{h}v_0 + \frac{1}{2l}(w_4 - w_2)\right\}$$
$$- \frac{E}{4l^2}\left\{\left(\frac{t_s}{1 - \nu^2} + A_\alpha c_\alpha^4\right)(u_0 - u_{12}) + \left(\frac{\nu t_s}{1 - \nu^2} + A_\alpha c_\alpha^2 s_\alpha^2\right)(v_8 - v_7)\right.$$
$$\left.\left. + A_\alpha c_\alpha^3 s_\alpha(u_8 - u_7 + v_0 - v_{12})\right\}\right] l^2 = -(q_x)_0 l^2 = -(R_x)_0 \ldots \quad \ldots (4C)a$$

$$\left[\frac{E}{2l^2}\left\{\frac{t_s}{1-\nu^2}(1+\nu U_3)+EA_\alpha s_\alpha{}^2(c_\alpha{}^2U_3+s_\alpha{}^2+c_\alpha s_\alpha U_4)\right\}(v_2+v_4-2v_0)\right.$$

$$-\frac{G}{2l}\left\{(t_\beta s_\beta{}^2+t_\gamma s_\gamma{}^2)+V_2(t_\beta c_\beta s_\beta+t_\gamma c_\gamma s_\gamma)\right\}\left\{2\frac{l}{h}v_0+\tfrac{1}{2}(w_4-w_2)\right\}$$

$$-\frac{E}{4l^2}\left\{\left(\frac{t_sG}{E}+A_\alpha s_\alpha{}^2c_\alpha{}^2\right)(v_0-v_{12}+u_8-u_7)\right.$$

$$\left.\left.+A_\alpha s_\alpha c_\alpha\left(c_\alpha{}^2\,\overline{u_0-u_{12}}+s_\alpha{}^2\,\overline{v_8-v_7}\right)\right\}\right]l^2=-(q_y)_0l^2=-(R_y)_0\quad ..\quad (4C)b$$

$$\left[\frac{G}{2l}\left\{(t_\beta s_\beta{}^2+t_\gamma s_\gamma{}^2)+V_2(t_\beta c_\beta s_\beta+t_\gamma c_\gamma s_\gamma)\right\}(v_4-v_2)+\frac{h}{l}(w_2+w_4-2w_0)\right.$$

$$-\frac{G}{l}\left\{(t_\beta c_\beta{}^2+t_\gamma c_\gamma{}^2)u_3+(t_\beta c_\beta s_\beta+t_\gamma c_\gamma s_\gamma)v_3\right\}-\frac{hG}{4l^2}\left\{(t_\beta c_\beta{}^2+t_\gamma c_\gamma{}^2)(w_0-w_{12})\right.$$

$$\left.\left.+(t_\beta c_\beta s_\beta+t_\gamma c_\gamma s_\gamma)(w_8-w_7)\right\}\right]l^2=-(q_z)_0l^2=-(R_z)_0.\quad ..\quad ..\quad (4C)c$$

3. *Reactions at a Free-corner Station.*—The boundary conditions for this case are given by equations (89). Taking the free corner to be defined as the meeting point of the lines 12, 3, 0 and 9, 4, 0 in Fig. 6, we can now write equations (68) as follows:

$$\left[-\frac{t_s}{2}\frac{E}{1-\nu^2}\left(\frac{1+\nu U_2}{l^2}\right)(u_0-u_3)+\frac{t_s}{2l^2}GU_4(v_4-v_0)\right.$$

$$-\frac{EA_\alpha c_\alpha{}^2}{2l^2}(c_\alpha{}^2+U_1s_\alpha{}^2+U_2c_\alpha s_\alpha)(u_0-u_3)$$

$$\left.+\frac{EA_\alpha c_\alpha s_\alpha}{2l^2}(U_3c_\alpha{}^2+s_\alpha{}^2+U_4c_\alpha s_\alpha)(v_4-v_0)\right]l^2=-(q_x)_0l^2=-(R_x)_0\quad (5C)a$$

$$\left[\frac{t_s}{2l^2}\frac{E}{1-\nu^2}(1+\nu U_3)(v_4-v_0)-\frac{t_s}{2l^2}GU_2(u_0-u_3)\right.$$

$$-\frac{EA_\alpha s_\alpha c_\alpha}{2l^2}(c_\alpha{}^2+U_1s_\alpha{}^2+U_2c_\alpha s_\alpha)(u_0-u_3)$$

$$\left.+\frac{EA_\alpha}{2}\frac{s_\alpha{}^2}{l^2}(U_3c_\alpha{}^2+s_\alpha{}^2+U_4c_\alpha s_\alpha)(v_4-v_0)\right]l^2=-(q_y)_0l^2=-(R_y)_0\quad (5C)b$$

$$\left[-\frac{hG}{l^2}\frac{t_\beta}{2}\left\{(c_\beta{}^2+V_1c_\beta s_\beta)\left(\frac{l}{h}u_3+w_0-w_3\right)+(V_2c_\beta s_\beta+s_\beta{}^2)\left(\frac{l}{h}v_4+w_4-w_0\right)\right\}\right.$$

$$-\frac{hG}{l^2}\frac{t_\gamma}{2}\left\{(c_\gamma{}^2+V_1c_\gamma s_\gamma)\left(\frac{l}{h}u_3+w_0-w_3\right)\right.$$

$$\left.\left.+\left(V_2c_\gamma s_\gamma+s_\gamma{}^2\right)\left(\frac{l}{h}v_4+w_4-w_0\right)\right\}\right]l^2=-(q_z)_0l^2=-(R_z)_0.\quad ..\quad (5C)c$$

4. *Reaction at Re-entrant Corner.*—For a re-entrant corner of the type shown at station 0 of Fig. 1C, we expand equations (68) in the usual way.

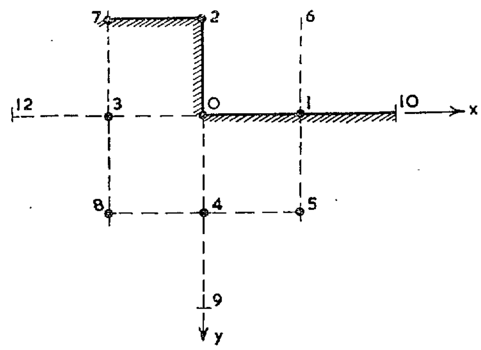

FIG. 1C.

That is, the expansion is done in two steps only, in the second of which the reduced areas and moments of inertia of section are taken into account. This is because only at that stage do the appropriate amounts of reduction become obvious.

Thus, for the reactions (per unit area of surface) in the x, y and z directions we have:

$$
\begin{aligned}
(q_x)_0 = {} & \frac{t_s E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)_{\overline{1-3}} + Gt_s\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{\overline{4-2}} \\
& - t_\beta G c_\beta^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_0 - t_\beta G c_\beta s_\beta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_0 \\
& - t_\gamma G c_\gamma^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_0 - t_\gamma G c_\gamma s_\gamma\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_0 \\
& + EA_\alpha c_\alpha^2\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{\overline{1-3}} \\
& + EA_\alpha c_\alpha s_\alpha\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{\overline{4-2}} \quad \ldots \ldots \ldots (6C)a
\end{aligned}
$$

$$
\begin{aligned}
(q_y)_0 = {} & \frac{t_s E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)_{\overline{4-2}} + Gt_s\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{\overline{1-3}} \\
& - Gt_\beta s_\beta^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_0 - Gt_\beta c_\beta s_\beta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_0 \\
& - Gt_\gamma s_\gamma^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_0 - Gt_\gamma c_\gamma s_\gamma\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_0 \\
& + EA_\alpha s_\alpha^2\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{\overline{4-2}} \\
& + EA_\alpha c_\alpha s_\alpha\left\{c_\alpha^2\frac{\partial u}{\partial x}+s_\alpha^2\frac{\partial v}{\partial y}+c_\alpha s_\alpha\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{\overline{1-3}} \quad \ldots \ldots \ldots (6C)b
\end{aligned}
$$

$$
\begin{aligned}
(q_z)_0 = {} & hGt_\beta\left[c_\beta^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_{\overline{1-3}} + c_\beta s_\beta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_{\overline{4-2}}\right. \\
& \left. + c_\beta s_\beta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_{\overline{1-3}} + s_\beta^2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)_{\overline{4-2}}\right] \quad \ldots \ldots \ldots \ldots (6C)c
\end{aligned}
$$

(where, as before the suffix $\overline{m-n}$ stands for the value of the expression concerned at station m minus its value at n).

Expressing the above in finite-difference form and using conditions (76), (79), (84) and (87), we have:

$$
\begin{aligned}
(q_x)_0 = {} & \left(\frac{t_s}{2}\right)\frac{E}{2l}(u_{10} - u_0) + \frac{t_s E}{1-\nu^2} - \frac{1}{2l}\{(u_0 - u_{12}) + \nu(v_8 - v_7)\} \\
& + \frac{Gt_s}{2l}\{(u_5 - u_0) + (v_5 - v_8)\} \\
& - (\tfrac{3}{4}t_\beta)c_\beta^2 G\left(\frac{2u_0}{h} + \frac{w_1 - w_3}{2l}\right) - (\tfrac{3}{4}t_\beta)c_\beta s_\beta G\left(\frac{2v_0}{h} + \frac{w_4 - w_2}{2l}\right) \\
& - (\tfrac{3}{4}t_\gamma)c_\gamma^2 G\left(\frac{2u_0}{h} + \frac{w_1 - w_3}{2l}\right) - (\tfrac{3}{4}t_\gamma)c_\beta s_\gamma G\left(\frac{2v_0}{h} + \frac{w_4 - w_2}{2l}\right) \\
& + \frac{EA_\alpha c_\alpha^2}{2l}\Big[\tfrac{1}{2}\{F_\alpha(u_{10} - u_0)\} \\
& - \Big\{c_\alpha^2(u_0 - u_{12}) + s_\alpha^2(v_8 - v_7) + c_\alpha s_\alpha(u_8 - u_7 + v_0 - v_{12})\Big\}\Big] \\
& + \frac{EA_\alpha c_\alpha s_\alpha}{2l}\Big\{c_\alpha^2(u_5 - u_8) + s_\alpha^2(v_9 - v_0) + c_\alpha s_\alpha(u_9 - u_0 + v_5 - v_8)\Big\} \quad \ldots \quad (7C)a
\end{aligned}
$$

$$
\begin{aligned}
(q_y)_0 = {} & \frac{t_s E}{2l(1-\nu^2)}\Big\{(v_9 - v_0) + \nu(u_5 - u_8)\Big\} + \left(\frac{t_s}{2}\right)\frac{E}{2l}(v_4 - v_2) \\
& + G\left(\frac{t_s}{2}\right)\frac{U_2}{2l}(u_{10} - u_0) - \frac{Gt_s}{2l}(u_8 - u_7 + v_0 - v_{12}) \\
& - (\tfrac{3}{4}t_\beta)s_\beta^2 G\left(\frac{2v_0}{h} + \frac{w_4 - w_2}{2l}\right) - (\tfrac{3}{4}t_\beta)c_\beta s_\beta G\left(\frac{2u_0}{h} + \frac{w_1 - w_3}{2l}\right) \\
& - (\tfrac{3}{4}t_\gamma)s_\gamma^2 G\left(\frac{2v_0}{h} + \frac{w_4 - w_2}{2l}\right) - (\tfrac{3}{4}t_\gamma)c_\gamma s_\gamma G\left(\frac{2u_0}{h} + \frac{w_1 - w_3}{2l}\right) \\
& + \frac{EAs_\alpha^2}{2l}\Big\{c_\alpha^2(u_5 - u_8) + s_\alpha^2(v_9 - v_0) + c_\alpha s_\alpha(u_9 - u_0 + v_5 - v_8)\Big\} \\
& + \frac{E}{2l}\left(\frac{A_\alpha}{2}\right)c_\alpha s_\alpha\Big\{F_\alpha(u_{10} - u_0)\Big\} \\
& - \frac{EA_\alpha c_\alpha s_\alpha}{2l}\Big\{c_\alpha^4(u_0 - u_{12}) + s_\alpha^2(v_8 - v_7) + c_\alpha s_\alpha(u_8 - u_7 + v_0 - v_{12})\Big\} \quad \ldots \quad (7C)b
\end{aligned}
$$

$$
\begin{aligned}
(q_z)_0 = hG\Big[& \tfrac{1}{2}(t_\beta c_\beta^2 + t_\gamma c_\gamma^2)\Big\{\tfrac{1}{2}\left(\frac{2u_0}{h} + \frac{w_{10} - w_0}{2l}\right) - \left(\frac{2u_3}{h} + \frac{w_0 - w_{12}}{2l}\right)\Big\} \\
& + (c_\beta s_\beta + c_\gamma s_\gamma)\left(\frac{2u_4}{h} + \frac{w_5 - w_8}{2l}\right) \\
& + (c_\beta s_\beta + c_\gamma s_\gamma)\Big\{V_1\left(\frac{2u_1}{h} + \frac{w_{10} - w_0}{2l}\right) - \left(\frac{2v_3}{h} + \frac{w_8 - w_7}{2l}\right)\Big\} \\
& + (s_\beta^2 + s_\gamma^2)\left(\frac{2v_4}{h} + \frac{w_9 - w_0}{2l}\right)\Big]. \quad \ldots \quad (7C)c
\end{aligned}
$$

It remains now only to regroup the various deflection terms so as to associate each particular displacement such as u_0 with a single coefficient.

If the free boundary of Fig. 1C is slightly modified so that from station 1 it drops to station 5 instead of continuing to station 10, station 1 becomes another free corner. The effect on equations (6C) is that all expressions having 1 as a suffix disappears as well as those with suffix 2, with a consequent simplification of equations (7C).

APPENDIX D

Solution of the Problem of the Cantilever Square Plate by the use of Influence Coefficients and a High-speed Digital Computer

1. *Choice of Example.*—The methods described in this report enable us to derive the stresses and deflections of wings (or of any flat structure transversely loaded) by the use of influence coefficients in conjunction with a digital computer. The methods vary according as the shear deflections of the structure are, or are not, negligible compared with the bending deflections. The approach is simplest naturally when shear deflections can be neglected, and an initial practical trial of the method is perhaps best made under that condition.

A suitable example seemed to be the problem of the square plate of constant thickness fixed along one edge so as to behave as a cantilever under transverse loads. This problem has not so far been solved in finite terms, and therefore a check on the accuracy of the method by comparing it with the theoretical solution is impracticable. As it happens, however, this same example was recently taken by R. H. MacNeal[2] for checking the accuracy of the 'electric analogue' method of approach. In the absence of a theoretical solution he compared his results with those obtained from experiment. It is therefore possible to compare results obtained by the present method with his experimental results and also, if desired, with those given by the 'electric analogue' method (both for a particular load distribution).

2. The square plate was divided chess-board fashion into small squares, the corners of the squares locating the stations whose deflections determine the contour of the plate. Fig. 1D shows the plate DABC (each side of which is divided into six equal lengths to accommodate six stations) encastré along the edge DA which coincides with the y axis.

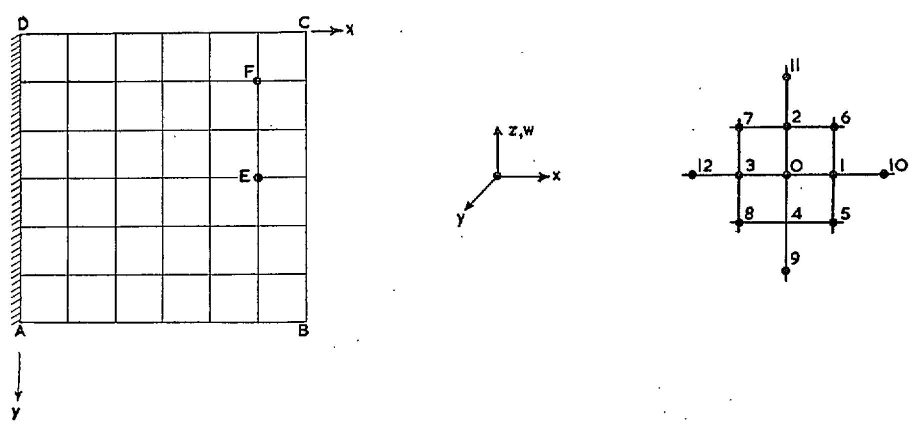

Fig. 1D.

The plate is supposed deflected into an arbitrary contour, by assigning an arbitrary deflection w_r to each station r, except of course the stations along the fixed edge DA, which by definition have no deflection. It is then possible to write down the vertical reaction that must be applied at each of the thirty-six mobile stations to hold the plate to the chosen contour. The system of loads necessary to hold any given contour is thus obtained, and can be expressed in terms of a set of simultaneous linear equations, each of which gives the reaction at a particular station in terms of the displacements of the others and of itself. By inverting the corresponding matrix, the machine (the digital computer, *i.e.*) converts this into a set of equations giving the deflections in terms of the reactions, or applied loads. Once the deflections are known the corresponding stresses are readily derived.

3. The reaction at any station 0 in a pattern of stations numbered as in the figure can be written down at once by using the formulae derived in section 2 of the main text (Part II).

3.1. *For all Stations not Nearer than Two Pitch-lengths l from an Edge* we have the Formula (18*a*), *i.e.*,

$$R_0 = \frac{D}{l^2}\left[20w_0 - 8\sum_{r=1}^{4} w_r + 2\sum_{r=5}^{8} w_r + \sum_{r=9}^{12} w_r\right], \quad \ldots \quad (1D)$$

where $$D = \frac{Eh^3}{12(1-\nu^2)} = \text{stiffness of plate}. \quad \ldots \quad (2D)$$

3.2. *For Stations on a Free Edge,* such as BC, not nearer than two pitch lengths to the corner C or B, the reaction is given by formula (14) *i.e.*,

$$R_0 = \frac{D}{l^2}\Big[(8-4\nu-3\nu^2)w_0 - (4-2\nu-2\nu^2)(w_2+w_4) - (6-2\nu)w_3$$
$$+ (2-\nu)(w_7+w_8) + \frac{1-\nu^2}{2}(w_9+w_{11}) + w_{12}\Big]. \quad \ldots \quad (3D)$$

The corresponding formula for a station on the edge DC parallel to the x-axis is obtained by simply turning the pattern of stations through 90 deg to give:

$$R_0 = \frac{D}{l^2}\Big[(8-4\nu-3\nu^2)w_0 - (4-2\nu-2\nu^2)(w_1+w_3) - (6-2\nu)w_4$$
$$+ (2-\nu)(w_5+w_8) + \frac{1-\nu^2}{2}(w_{10}+w_{12}) + w_9\Big]. \quad \ldots \quad (3D)a$$

Similarly for stations on the edge AB:

$$R_0 = \frac{D}{l^2}\Big[(8-4\nu-3^2)w_0 - (4-2\nu-2^2)(w_1+w_3) - (6-2\nu)w_2$$
$$+ (2-\nu)(w_6+w_7) + \frac{1-\nu^2}{2}(w_{10}+w_{12}) + w_{11}\Big]. \quad \ldots \quad (3D)b$$

3.3. *For Stations on a Free Corner,* such as C, the reaction is given by formula (11), *i.e.*,

$$R_0 = \frac{D}{l^2}\left[(3+\nu)(w_0 - w_3 - w_4) + 2w_8 + \left(\frac{1+\nu}{2}\right)(w_9+w_{12})\right]. \quad \ldots \quad (4D)$$

Similarly, for corner B:

$$R_0 = \frac{D}{l^2}\left[(3+\nu)(w_0 - w_2 - w_3) + 2w_7 + \left(\frac{1+\nu}{2}\right)(w_{11}+w_{12})\right]. \quad \ldots \quad (4D)a$$

3.4. *For Stations Distant one Pitch Length from a Free Edge,* such as station E in relation to the free edge CB, we see that, on superposing the standard pattern with station 0 coinciding with E, station 10 is the only station outside the edge of the plate. Thus, in the standard formula (1D) above for an interior station, all the w's are known except w_{10}. Using the fact that

$$\partial^2 w/\partial x^2 = -\nu\partial^2 w/\partial y^2$$

for the edge station 1, we find:

$$w_{10} = \{2(1+\nu)w_1 - \nu(w_5+w_6) - w_0\}. \quad \ldots \quad (5D)$$

The standard formula for an interior station may therefore be used with w_{10} replaced by expression (5D).

The same procedure is followed for stations one pitch length from either of the other two free edges.

3.5. *For a Diagonal Station such as* F *one Pitch Length from each of Two Free Edges,* if we identify the central station 0 of the regular pattern with F, stations 10 and 11 are both outside the plate edge. However, by using formula (5*D*) above for w_{10} and the parallel formula:

$$w_{11} = \{2(1 + \nu)w_2 - \nu(w_6 + w_7) - w_0\} \quad \ldots \quad (6D)$$

for w_{11}, we can again use the standard formula (1*D*) above for an interior station.

3.6. *For Stations at or Within one Pitch Length of the Fixed Edge* AD, we need only imagine any stations that spill out to the left of AD to be the mirror image of those on the right of that line as explained in the text. The standard formula, as modified by proximity to a free edge, then applies.

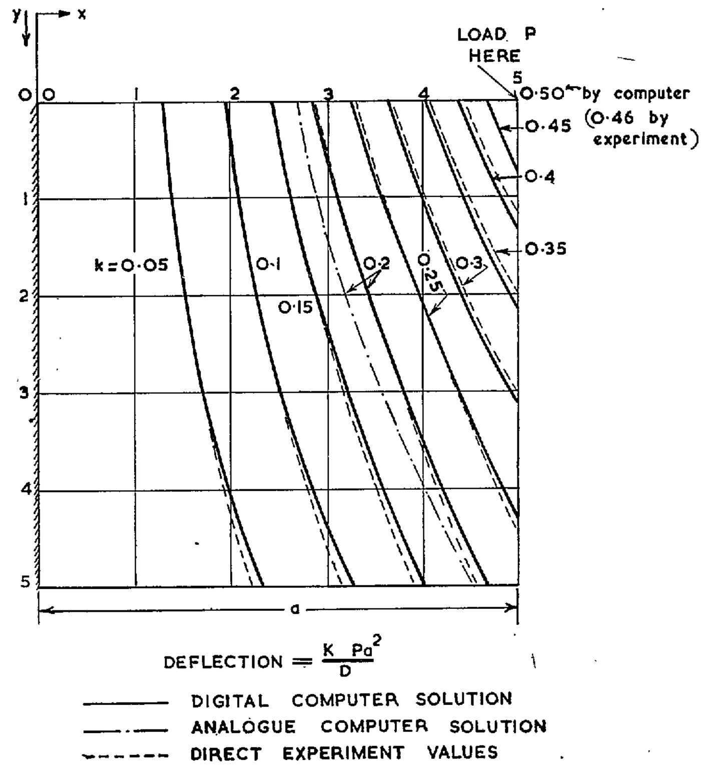

FIG. 2D.

4. *Comparison of Results Obtained by Present Method with Those Obtained by Experiment.*—In Ref. 2 MacNeal compares the deflections he obtained by the electric analogue method with measured deflections. He does this for the particular case in which the transverse load takes the form of a concentrated load at one of the two free corners—a stiff test because the plate is thereby subjected to bending and twisting at the same time. He makes the comparison by drawing the relevant two sets of contours for the deflected plate, each contour line being marked with its appropriate deflection. Apart from these contours no figures are quoted for the experimental values with the exception of the deflection at the loaded corner. One would have expected the present method to give better results than MacNeal's if only because of the finer mesh (25 sub-squares to his 9). This expectation is amply confirmed, as a glance at Fig. 2D shows.

This figure shows the contours obtained by the present method as full lines and the experimental values as dotted lines. The agreement is satisfactory on the whole; the deflection at the loaded corner works out to be $0 \cdot 49Pa^2/D$ as against $0 \cdot 46$ by experiment and $0 \cdot 52$ by the analogue method. To give an indication of the kind of accuracy obtained by the analogue method the contour line for $0 \cdot 2Pa/D^2$ is included as a chain-dotted line; the discrepancy shown by this is typical of the other contours obtained by the analogue method. No doubt a better result would have been obtained if a finer mesh had been used, but the increase in the number of stations entailed by this would have meant a corresponding increase in the amount of electrical apparatus.

5. *Further Work.*—The method shown here to be practicable for a plate of uniform thickness should be equally satisfactory for plates in which the thickness is not constant and in which the plan-form is not regular. Work on these lines is proceeding.

6. *Acknowledgments.*—The writer is indebted to Mr. P. C. Birchall for his work in setting out and programming the various formulae for the R.A.E. D.E.U.C.E. machine and for obtaining the numerical values here recorded.

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