R. & M. No. 3034
(18,234)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

The Matrix Force Method of Structural Analysis and Some New Applications

By

PROFESSOR J. H. ARGYRIS, D.E., AND S. KELSEY, B.Sc.(ENG.)
University of London, Imperial College of Science and Technology.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1957
TWELVE SHILLINGS NET

The Matrix Force Method of Structural Analysis and Some New Applications

By

PROFESSOR J. H. ARGYRIS, D.E., AND S. KELSEY, B.Sc.(ENG.)
University of London

COMMUNICATED BY THE DIRECTOR-GENERAL OF SCIENTIFIC RESEARCH (AIR),
MINISTRY OF SUPPLY

Reports and Memoranda No. 3034
February, 1956

Summary.—The purpose of this paper is :

(*a*) to summarise the basic principles of the matrix force method of structural analysis given in Ref. 1 and to present also some new applications of the general theory ;

(*b*) to establish and illustrate on simple examples the special method of cut-outs developed in Ref. 1. In this procedure the stresses in a structure with cut-outs are derived from the simpler analysis of the corresponding structure without cut-outs under the same loads and/or temperature distribution ;

(*c*) to present a method for the determination of the stresses in a structure, some of whose components have been modified subsequent to an initial stress analysis. This procedure, not included in Ref. 1, is, in fact, a generalisation of the cut-out method (*b*) and gives the stresses in the modified system solely in terms of the stresses of the original system subject to the same loads and temperature distribution.

The theory is illustrated on some simple examples which show clearly the extreme simplicity of the powerful techniques (*b*) and (*c*). We emphasise that the application of these methods requires only one stress analysis : that of the continuous structure under the same external loads, as the modified structure. No additional stress analysis due to other, *e.g.*, perturbation, loads is involved.

Introduction.—It is being increasingly recognised that none of the standard methods of structural analysis is really suitable for the determination of the stress distribution and flexibility of modern aircraft structures. This realisation is brought about by many concurring reasons. Thus, it is accepted that a more reliable analysis of the stresses is necessary now since many of the designs introduced at present are too complex and unusual to be analysed by the often so crude approximation of the past. It must not be forgotten, moreover, that the ever-present danger of a major catastrophy due to fatigue failure compels us to seek a more careful estimate of the stresses. But even if an accurate analysis of some of the more modern structural designs could be accomplished by any of the standard methods it would take such a long time as to be useless except as a rather belated check on the final design. In fact, quite apart from technical reasons, purely economic considerations require the completion of an accurate stress analysis at an early stage to ensure efficient design and to check it prior to production and the planning of the necessary full-scale tests.

Faced with this situation we require a radical change in our approach to the problem. This is being offered by the combination of matrix methods of structural analysis and the electronic digital computer with its enormous potentialities. However, let us stress *ab initio* that in speaking of matrix methods of analysis we do not mean, and in fact exclude, any attempt to obtain the equations in the unknowns by the usual longhand method and then solve them only by the inversion of the matrix of coefficients of the unknowns. It is in our opinion futile to seek any progress along these lines. What is required and is in fact essential is a formulation of structural analysis completely in matrix algebra, starting with the compilation of the basic data.

In Ref. 1 such a general matrix method of structural analysis has been developed with both forces and displacements as unknowns. These two methods are completely dual in character, as demonstrated there. In fact, knowing the equations in either of the two procedures we can write down by a simple ' translation ' process the equations in the other procedure. We do not intend to enter here into any lengthy arguments on which of the two methods is preferable but refer to some relevant considerations in section 1. The important fact is that in either of the two methods we require initially only three simple matrices which, in many cases, can be written down by inspection or by very simple calculations. Naturally, matrix methods of structural analysis have been given before but it is believed that none was as general and comprehensive and yet as simple as that of Ref. 1; see also Ref. 7.

In the present paper we establish first in Part 1, sections 1 to 8, the main results of structural analysis by the matrix force method. This procedure may be used to programme for one continuous operation on the digital computer the complete stress analysis for any system of loads or thermal strains, including the derivation of the flexibility matrix of the given structure. In fact, the actual programming for a particular digital computer, the medium-sized Ferranti Pegasus, has been completed and is being published (*see* Hunt[2,3]). It is of interest that this method is already being applied both in this country and abroad. However, the purpose of this paper is not only to derive some of the main results of Ref. 1 but to present also some new developments of practical importance. We refer here only to the surprisingly simple formula in section 8 giving the thermal distortion of an arbitrary structure.

One of the most important applications of the general theory of Ref. 1 is that of the stress analysis of structures with cut-outs under any given set of loads or thermal strains. This particular method is reviewed here in Part II, section 9, and we show, once more, that it is possible to find the stress distribution in a structure with cut-outs, under any load, solely in terms of the stress distribution under the same loading in a corresponding continuous structure where the cut-outs have been filled in. We emphasise that no additional stress analysis of the continuous structure, under other (perturbation) loads, is required as in techniques developed for circular fuselages by Cicala[6] and others. In these, the openings were likewise filled in, but special perturbation stress systems were used to nullify the stresses in the cut-out elements. The practical application of the present method is so simple and foolproof that there is little doubt that it is the ideal procedure, not only for finding the stresses in the structure due to the introduction of cut-outs subsequent to the analysis, which often materialise at a late stage of design, but also for the stress analysis of the structure when the cut-outs are known initially. The physical justification of the method derives from the idea that we can impose such initial strains on the filled-in elements of the continuous structure that their total stresses due to applied loads and initial strains are zero, *i.e.*, that they are effectively non-existent. It is now very simple to express the necessary initial strains in terms of the given set of loads and hence, derive the stress distribution in the cut-out structure from the initial stress analysis of the continuous system under the original loads only. Interestingly enough the same method was proposed later by Goodey[4] who also filled in the missing elements. He, however, used a purely mathematical idea to obtain the final formulae. Thus, Goodey considers in the continuous fictitious structure the variational problem of the minimum of strain energy with the additional condition of zero stress in the fictitious members. Naturally the final result is essentially identical to that of Ref. 1. For this reason we need not consider it here any more, except to point out that Goodey informs us that he has used it successfully in fuselages with large cut-outs, including the part-removal of frames, etc. The authors' attention has been drawn by Morley to a report[5] published in 1949, discussing the case of a cut-out in a circular fuselage. This report uses a method akin to that of Goodey[4] and may be considered to contain a germ of the idea developed in Ref. 1.

A generalisation of the above method on cut-outs to deal with structural modifications of components suggests itself naturally. This problem is solved in section 12 and the final formulae are, of course, similar to those in the cut-out case with only one additional term; a matrix expressing the difference of the flexibilities of the modified and original elements. Here again we find the stress distribution in the modified structure solely in terms of the stresses in the original structure under the same loads or temperature distribution, without any additional stress analysis of perturbation systems. The practical importance of the method in view of its simplicity need hardly be emphasised. In the last paragraph of Part II we derive also a formula for the flexibility matrix of the modified structure in terms of the original flexibility.

The theory reviewed in this report is illustrated on a series of examples. Their purpose is mainly to draw attention to the potentialities of the method on cut-outs and modifications. Admittedly the cases treated are simple but basically the same operations are involved in any complex structure. This is shown by the general computer programme developed by Hunt[2,3]. As an example we mention that once the three basic matrices are given, the structural calculations for a wing with a hundred redundancies, under loads and thermal strains, does not take longer on the Ferranti-Pegasus than approximately a week. This, moreover, includes the alternative stress distribution when up to 30 subsequent cut-outs or structural modifications are introduced.

In conclusion we emphasize once more that the progress of structural analysis achieved by these methods is only possible by developing the analysis *ab initio* in matrix form. With standard longhand notation it would be difficult, if not impossible, to detect many of the important new theorems. But this is not the only aspect where we must change our approach. The basic simplicity of the theory can only be immediately apparent if we free our minds from the confining strain energy considerations that obscure and complicate the mathematical derivations. By using the unit load and the unit displacement method respectively as given in Ref. 1 all the results flow out naturally from the initial idea in an immediately obvious form.

PRINCIPAL NOTATION

σ_{xx}, etc., σ_{xy}, etc.	Direct and shear stresses
ε_{xx}, etc., ε_{xy}, etc.	Direct and shear strains
q	Shear flow in sheet
$[\sigma]$	Column matrix of direct and shear stresses
$[\varepsilon]$	Column matrix of direct and shear strains
$\mathbf{R}$	Column matrix of applied (generalised) forces
$\mathbf{r}$	Column matrix of (generalised) displacements
$\mathbf{X}$	Column matrix of redundant (generalised) forces
$\mathbf{S}$	Column matrix of (generalised) stresses on structural elements
$\mathbf{v}$	Column matrix of (generalised) strains of structural elements
$\mathbf{H}$	Column matrix of (generalised) initial strains on structural elements
$\mathbf{b}$	Rectangular transformation matrix for stresses
$\mathbf{f}$	Flexibility matrix of unassembled structural elements
$\mathbf{F}$	Flexibility matrix
Θ	Temperature
$1 \ldots . j \ldots m$	Directions of external forces and displacements
$1 \ldots . i \ldots n$	Directions of redundant forces

Suffix $_h$ denotes elements to be eliminated or modified

Suffices $_m$, $_c$ denote structure with modifications and cut-outs respectively incorporated

$\mathbf{I}$	Unit matrix
$\mathbf{O}$, $\mathbf{o}$	Zero matrix
$\mathbf{A}'$, $\mathbf{A}^{-1}$	Transposed and reciprocal matrix respectively of $\mathbf{A}$
$\{\ldots .\}$	Column matrix

REFERENCES

No.	*Author*	*Title, etc.*
1	J. H. Argyris	Energy theorems and structural analysis. A generalised discourse with applications on energy principles of structural analysis, including the effects of temperature and non-linear stress-strain relations. Part I. General theory. *Airc. Eng.* Vol. XXVI. October, November, 1954. Vol. XXVII. February, March, April, May, 1955.
2	P. M. Hunt	The electronic digital computer in aircraft structural analysis. The programming of the Argyris matrix formulation of structural theory for an electronic digital computer. Part I. A description of a matrix interpretive scheme and its application to a particular example. *Airc. Eng.* Vol. XXVIII. March, 1956. A.R.C. 18,240. February, 1956. Part II. The use of preset and programme parameters with the matrix interpretive scheme and their application to general purpose programmes for the force method of analysis. *Airc. Eng.* Vol. XXVIII. April, 1956. Part III. General purpose programmes for the force and displacement methods in large structures including the use of magnetic tape storage. *Airc. Eng.* Vol. XXVIII. May, 1956.
3	P. M. Hunt	The computational procedure for the Argyris matrix method of structural analysis. Ferranti publication. List CS 72. February, 1956.
4	W. J. Goodey	Notes on a general method of treatment of structural discontinuities. *J.R. Ae. Soc.* Vol. 59. No. 538, p. 695. October, 1955.
5	L. S. D. Morley and W. K. G. Floor ..	Load distribution and relative stiffness parameters for a reinforced circular cylinder containing a rectangular cut-out. N.L.L. Amsterdam. Report S.362. 1949.
6	P. Cicala	Effects of cut-outs in semi-monocoque structures. *J. Ae. Sci.* Vol. 15. No. 3, pp. 171–179. March, 1948.
7	J. H. Argyris	Die Matrizentheorie der Statik. *Ingenieur Archiv.* Vol. XXV. No. 3, pp. 174–192. May, 1957.

PART I

The Basic Principles of the Matrix Force Method of Structural Analysis

1. *The Problem.*—Two basic methods exist for the analysis of arbitrary structures and are developed in matrix form in Ref. 1:

(*a*) the force method in which forces or stress resultants (or generalised forces) are taken as unknowns

(*b*) the displacement method in which deflections or slopes (or generalised displacements) are taken as unknowns.

The two methods are completely dual in character as demonstrated in Ref. 1, where Table 2 of the March, 1955, issue of Aircraft Engineering gives the basic theorems. For a number of reasons the force method is, in general, superior for the analysis of continuous systems like stressed skin structures. Firstly, in the force analysis of such systems there are fewer unknowns than in the displacement analysis and the equations are usually better conditioned. Furthermore, method (*a*) yields directly the flexibility matrix, which we require for aero-elastic investigations, whilst method (*b*) gives correspondingly the stiffness matrix from which the flexibility can only be obtained by inversion. Finally, the stress determination by the force analysis is always more accurate (and considerably so), since in the displacement analysis the stresses are found by what is essentially a differentiation process. On the other hand the compilation of the basic matrices may be simpler by the displacement method when the structure is irregular.

We consider here only the matrix formulation of the force method of analysis of which we give a general presentation including some important new theorems developed subsequently to Ref. 1. However, following the procedure of Table 2 of Ref. 1, all equations given may be immediately ' translated ' into the dual relations of the displacement method; see also Ref. 7.

Assume a linearly elastic structure subjected to a system of m loads or generalised forces $R_1, R_2, \ldots, R_j, \ldots, R_m$ which we denote by the column matrix:

$$\mathbf{R} = \{R_1 \, R_2 \ldots R_j \ldots R_m\} . \qquad (1)$$

Let the structure have n redundancies X_i which we form in a column matrix:

$$\mathbf{X} = \{X_1 \, X_2 \ldots X_i \ldots X_n\} . \qquad (2)$$

The particular system arising from the imposition of $\mathbf{X} = \mathbf{O}$ on our structure is called the basic system and is here statically determinate.

We seek to determine:

(*a*) the k stresses or stress resultants S_f denoted by the column matrix:

$$\mathbf{S} = \{S_1 \, S_2 \ldots S_f \ldots S_k\} \qquad (3)$$

(*b*) the flexibility matrix $\mathbf{F}$ for the points and directions of the applied forces R_j.

By definition:

$$\mathbf{r} = \mathbf{FR} \qquad (4)$$

where

$$\mathbf{r} = \{r_1 \, r_2 \ldots r_f \ldots r_m\} \qquad (5)$$

is the column matrix of the (generalised) deformations (deflections) in the directions of the forces $\mathbf{R}$ and $\mathbf{F}$ is a symmetric square matrix.

It is obvious that we can always write:

$$\mathbf{S} = \mathbf{b}_0\mathbf{R} + \mathbf{b}_1\mathbf{X} \quad \ldots \quad (6)$$

where the matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ are of dimensions $k \times m$ and $k \times n$ respectively and are determined solely by statics. Thus, the elements of the fth row of $\mathbf{b}_0$ are the (generalised) stresses at f due to each of the unit loads $R_j = 1$ applied to the basic system. Note that the stresses:

$$\mathbf{S}_0 = \mathbf{b}_0\mathbf{R} \quad \ldots \quad (7)$$

are statically equivalent to the applied loads $\mathbf{R}$.

Once $\mathbf{X}$ has been determined in terms of $\mathbf{R}$, equation (6) can be written in the form:

$$\mathbf{S} = \mathbf{b}\mathbf{R}\,. \quad \ldots \quad (8)$$

The above considerations may be applied immediately to the more general case when the basic system is itself redundant (*see* Ref. **1**).

Parallel to the applied loads $\mathbf{R}$, the system may also be subjected to initial strains H_f (*e.g.*, thermal strains) which we arrange again in a column matrix:

$$\mathbf{H} = \{H_1\, H_2 \ldots . H_f \ldots . H_k\}\,. \quad \ldots \quad (9)$$

Due to the redundancy of the system the initial strains cannot, in general, develop freely and stresses are set up. These may be calculated from:

$$\mathbf{S}_H = \mathbf{b}_1\mathbf{X}_H \quad \ldots \quad (10)$$

where $\mathbf{X}_H$ are the redundant stresses or forces due to $\mathbf{H}$.

2. *A Simple Example for the* $\mathbf{b}_0$ *and* $\mathbf{b}_1$ *Matrices.*

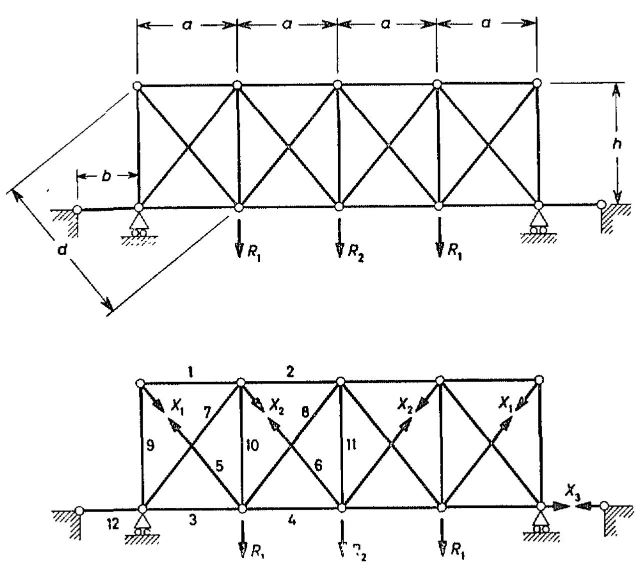

FIG. 1. Redundant pin-jointed framework to illustrate $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices.

The $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices are most easily illustrated on a conventional framework. Consider, for example, the five times redundant structure of Fig. 1, assumed symmetrical. Due to symmetry of loading and structure the system is effectively only three times redundant. As basic system we select the statically determinate structure of Fig. 1b obtained by cutting bars 5, 6 and 12. The load and redundancy matrices are:

$$\mathbf{R} = \{R_1\ R_2\} \quad \ldots \quad (1a)\,; \qquad \mathbf{X} = \{X_1\ X_2\ X_3\}\,. \quad \ldots \quad \ldots \quad (2a)$$

The $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices for half the structure, including the central vertical member (11), are found easily and are given in Table 1. The numbers over the columns refer to the external loads R_1, R_2 and the redundancies X_1, X_2, X_3 respectively and the numbers opposite the rows to the numbered bars of Fig. 1b.

TABLE 1

Basic Matrices for Framework of Fig. 1

$$\mathbf{b}_0 = \begin{array}{c} \begin{array}{cc} \mathit{1} & \mathit{2} \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ -a/h & -a/2h \\ a/h & a/2h \\ a/h & a/h \\ 0 & 0 \\ 0 & 0 \\ -d/h & -d/2h \\ 0 & -d/2h \\ 0 & 0 \\ 1 & 1/2 \\ 0 & 1 \\ 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \\ \mathit{9} \\ \mathit{10} \\ \mathit{11} \\ \mathit{12} \end{array} \quad \ldots \quad \ldots (11)$$

$$\mathbf{b}_1 = \begin{array}{c} \begin{array}{ccc} \mathit{1} & \mathit{2} & \mathit{3} \end{array} \\ \left[\begin{array}{ccc} -a/d & 0 & 0 \\ 0 & -a/d & 0 \\ -a/d & 0 & 1 \\ 0 & -a/d & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h/d & 0 & 0 \\ -h/d & -h/d & 0 \\ 0 & -2h/d & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \\ \mathit{9} \\ \mathit{10} \\ \mathit{11} \\ \mathit{12} \end{array} \quad \ldots \quad \ldots (12)$$

$$\mathbf{f} = \begin{bmatrix} 2\frac{a}{A_1E} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\frac{a}{A_2E} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\frac{a}{A_3E} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\frac{a}{A_4E} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\frac{d}{A_5E} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\frac{d}{A_6E} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\frac{d}{A_7E} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\frac{d}{A_8E} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\frac{h}{A_9E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\frac{h}{A_{10}E} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h}{A_{11}E} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\frac{b}{A_{12}E} \end{bmatrix} \quad (33a)$$

3. *The Idealised Structure.*—The problem of determining the stress distribution in a shell type of structure characteristic of major aircraft components is strictly infinitely redundant. Hence, it is necessary to introduce, for practical calculations, considerable simplifications or idealisations.

These are discussed at some length in Ref. 1. It is sufficient for the purpose of the present paper to mention the standard simplification by which a wing structure with spars and ribs (if such are used), approximately orthogonal to each other, is represented by a three-dimensional grid of flanges carrying only direct loads and walls carrying only shear flows. The cross-section of the wing may be arbitrary and the spars may taper differently in plan view and elevation but the angle of taper 2θ is assumed to be so small that $\cos \theta \simeq 1$ and $\sin \theta \simeq \theta$. The same restriction applies to the taper of ribs in plan view. Note that delta wings are included in these specifications as long as spars and ribs conform to the stipulated geometry.

In each flange of the idealised structure the direct load is assumed to vary linearly between adjacent nodal points of the grid. Furthermore, the shear flow in each field bounded by two intersecting pairs of adjacent flanges is taken as constant. The method of extracting this simplified system or idealised model from the actual structure will be found in Ref. 1, (March, 1955, p. 87) where also a more refined procedure is given for increasing the accuracy of the stress analysis.

For fuselages the method of idealisation is very similar to that in wings and need not be discussed here. Detailed procedures for more complex wing and aircraft structures will be given in subsequent publications.

The idealised structure possesses now a finite degree of redundancy, the determination of which requires more subtle considerations, especially in the presence of cut-outs, than are necessary in frameworks. This aspect is also discussed in Ref. 1. In what follows we assume that the process of idealisation has already been performed and that the degree of redundancy of the idealised structure is known. For brevity we shall use the terms ' structure ' or ' system ' for idealised structure.

4. *The Selection of the Basic System and the Redundant Forces.*—For a major aircraft component like a wing the selection of the basic system and the redundancies deserves careful attention since a skilful choice can simplify the calculations considerably. Our ideas, however, of what constitutes an appropriate choice must now be drastically revised due to the development of structural analysis in matrix form and the introduction of the electronic digital computer with its enormous potentialities. Thus, in the pre-electronic era of computations it was, in general, accepted that the choice of the basic system (which in itself might be statically indeterminate) was the more successful the less the chosen statically equivalent system,

$$\mathbf{S}_0 = \mathbf{b}_0\mathbf{R} \,, \qquad (7)$$

differed from the actual stress matrix

$$\mathbf{S} = \mathbf{b}\mathbf{R} \,. \qquad (8)$$

This argument was then correct since the calculation of a highly redundant system like a wing could not but take considerable time (if ever it was undertaken) and the complete design of the aeroplane could obviously not wait for its completion. Hence a reliable first estimate of the stresses in the structure, based solely on the statically equivalent system, was naturally indicated. However, the advent of the digital computer requires a radical change in our approach to structural analysis. To realise this inevitable development we need only consider that it is possible to programme on the digital computer, in one continuous operation, the complete structural analysis of a wing and fuselage for any loads and temperature distribution including the derivation of the ' exact ' flexibility matrix. Moreover, the time taken by the machine in performing all these calculations is strikingly small. For example, for the analysis of a typical wing structure with a hundred redundancies, a medium sized computer like the Ferranti-Pegasus will take approximately a week and give also the flexibility matrix for, say, 50 points, once all initial data of the structure like the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices are known and stored in the machine. In fact, the compilation and checking of these latter matrices and, in particular, $\mathbf{b}_1$, will, in general, take considerably longer than the actual machine operations. Actually the time for the preparation of the initial matrices will be further increased if we select the basic system on the same considerations as in the standard longhand method. Now it is obvious that the latter considerations lose their validity once we use a digital computer since the ' exact ' stress distribution can be obtained so quickly as to be available for immediate application in the design office. In place of our old ideas a new criterion of the suitability of a basic system emerges: the best choice of a basic system is that in which $\mathbf{b}_0$ may be written down either by inspection or at most by the very simplest static calculations. This not only reduces drastically the time necessary for its determination but allows also easy checking of the data. The latter point is, in fact, most important and cannot be emphasised sufficiently.

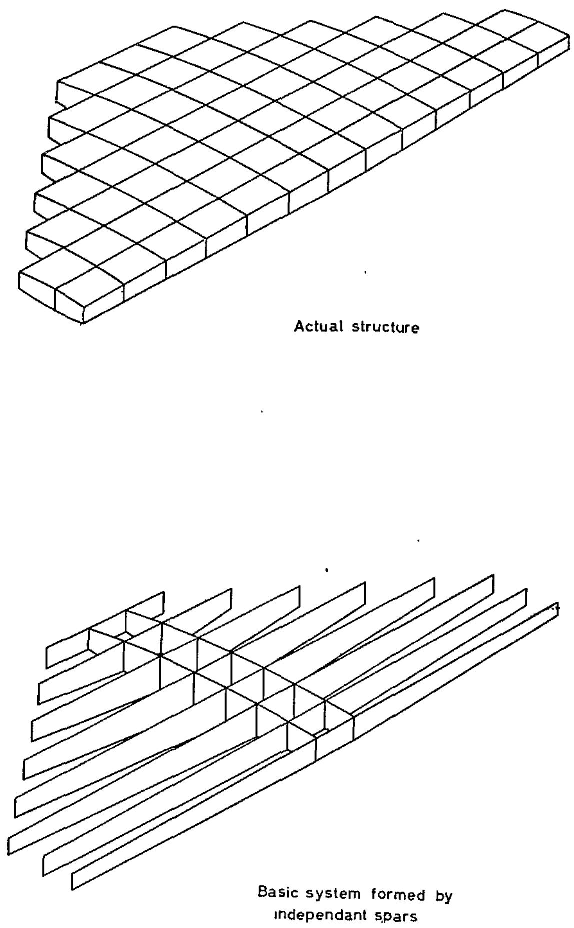

FIG. 2. Delta wing structure, illustrating selection of basic system.

To fix ideas, consider the multispar structure, in the form of a delta wing, of Fig. 2. It is obvious that by far the simplest basic system is that of the independent spars, for which $\mathbf{b}_0$ may be written down with great ease. Naturally, such a $\mathbf{b}_0$ bears no similarity to the final $\mathbf{b}$ but as stated above this is of no consequence once we accept the digital computer as the standard tool of structural analysis.

Associated with the choice of $\mathbf{b}_0$ matrix is the selection of the redundancies $\mathbf{X}$ which determine the $\mathbf{b}_1$ matrix. Here, too, for ease of compilation and checking of this basic matrix the X_i systems are preferably chosen as simple as possible; in particular, each system should affect as few elements of the structure as possible. Moreover, such a choice usually satisfies the essential requirement of well conditioned equations in the unknown redundancies. A point particularly emphasised in Ref. 1 is that we must not take the customary narrow view and consider redundancies as single forces or moments applied at actual physical cuts of the structure. A more satisfactory approach is to select as redundancies self-equilibrating systems of forces or stresses (*i.e.*, generalised forces). Besides giving a far greater flexibility in the choice of redundancies this procedure yields more symmetrical and immediately obvious expressions for the elements of the matrix. Three standard types of system for wing analysis were proposed in Ref. 1 (March, 1955). These are the X, Y and Z systems reproduced in Figs. 3, 4 and 5 of the present paper, where

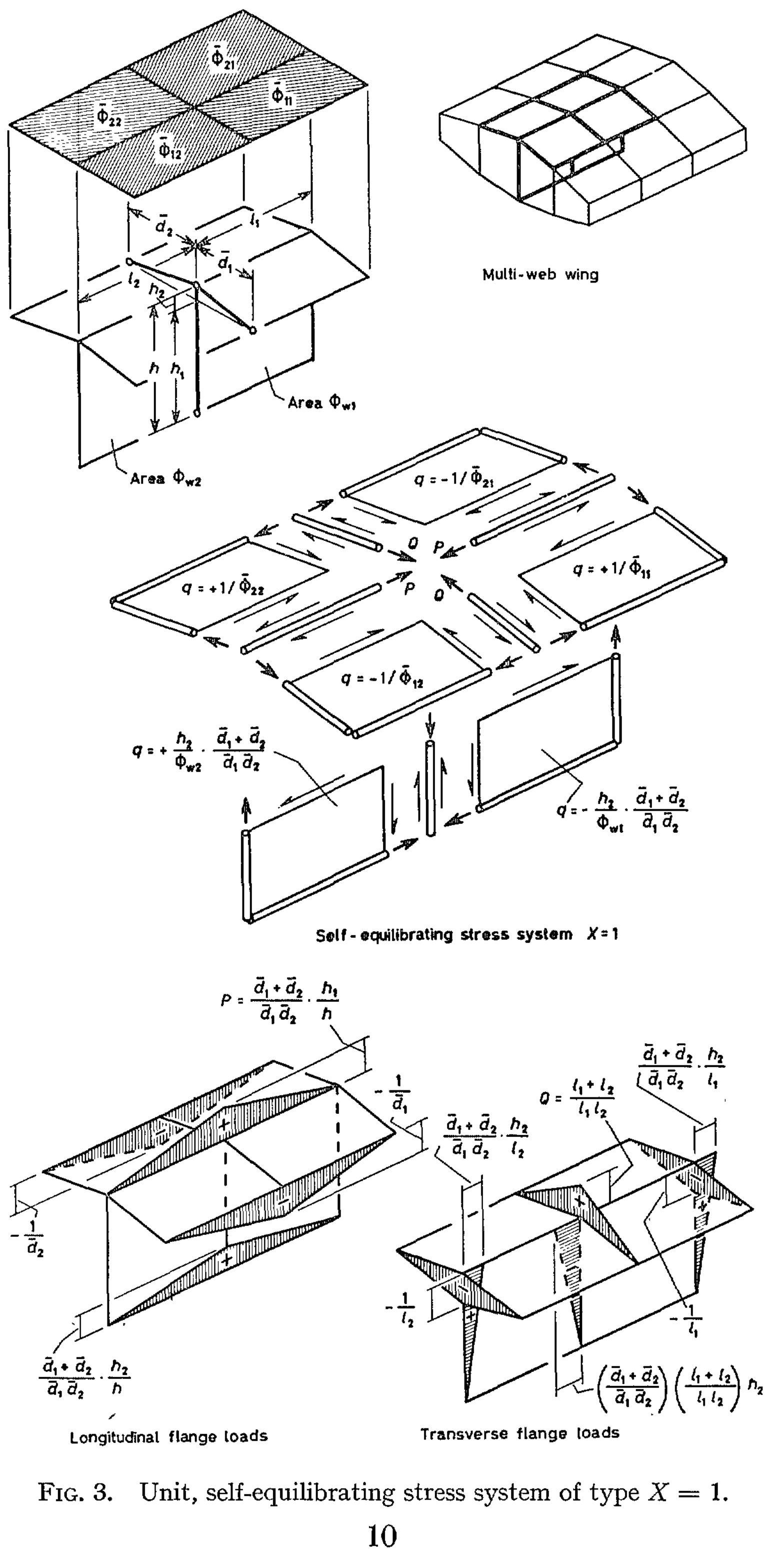

FIG. 3. Unit, self-equilibrating stress system of type $X = 1$.

full details are given of the flange loads and shear flows due to unit value of each of them. The Y-system (Fig. 4), which acts over two bays in the spanwise direction, is seen to be closely related to the boom load function P introduced by Argyris and Dunne in 1947*. The Z-system (Fig. 5) is similar to the Y-system but is applied in the chordwise direction. Finally, the X-systems (Fig. 3) are essentially diffusion systems applied over four panels in the upper or lower covers. Ref. 1 discusses in detail the selection of the best combination of X, Y, Z systems in various wing structures. In the present notation the column matrix **X** is taken to include all these systems.

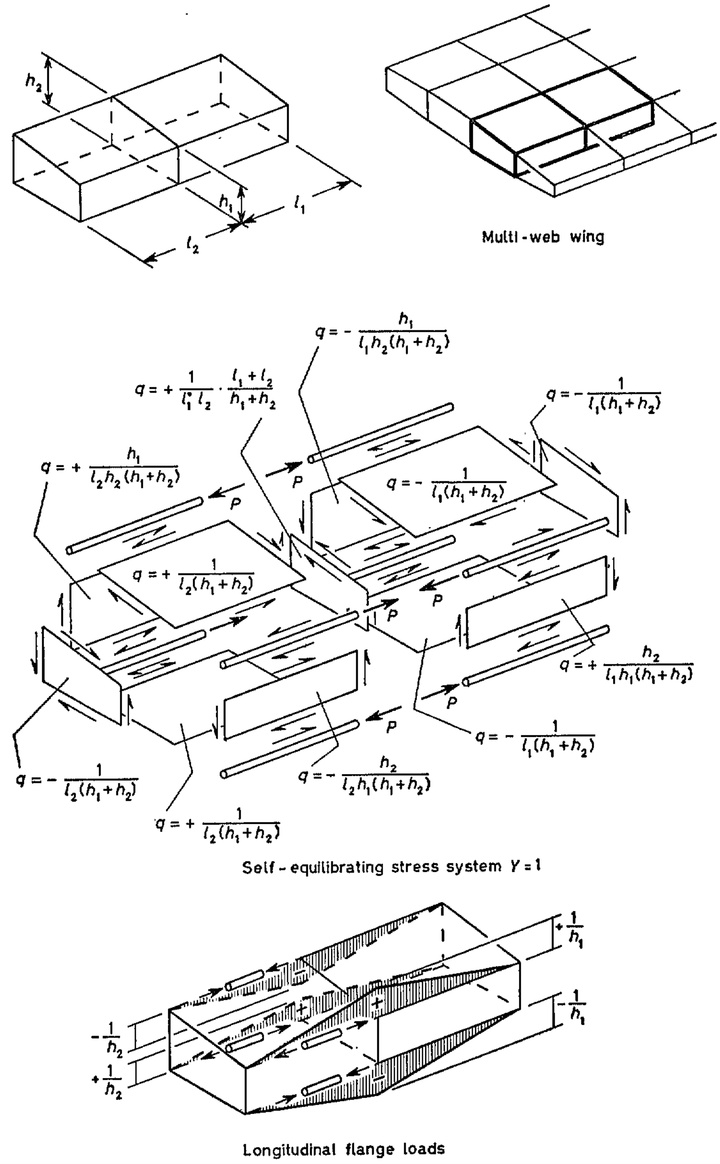

FIG. 4. Unit, self-equilibrating stress system of type $Y = 1$.
(Longitudinal four-boom tube.)

*See J. H. Argyris and P. C. Dunne, 'The General Theory, etc.' *J.R. Ae. Soc.* Vol. LI. February, September, November, 1947.

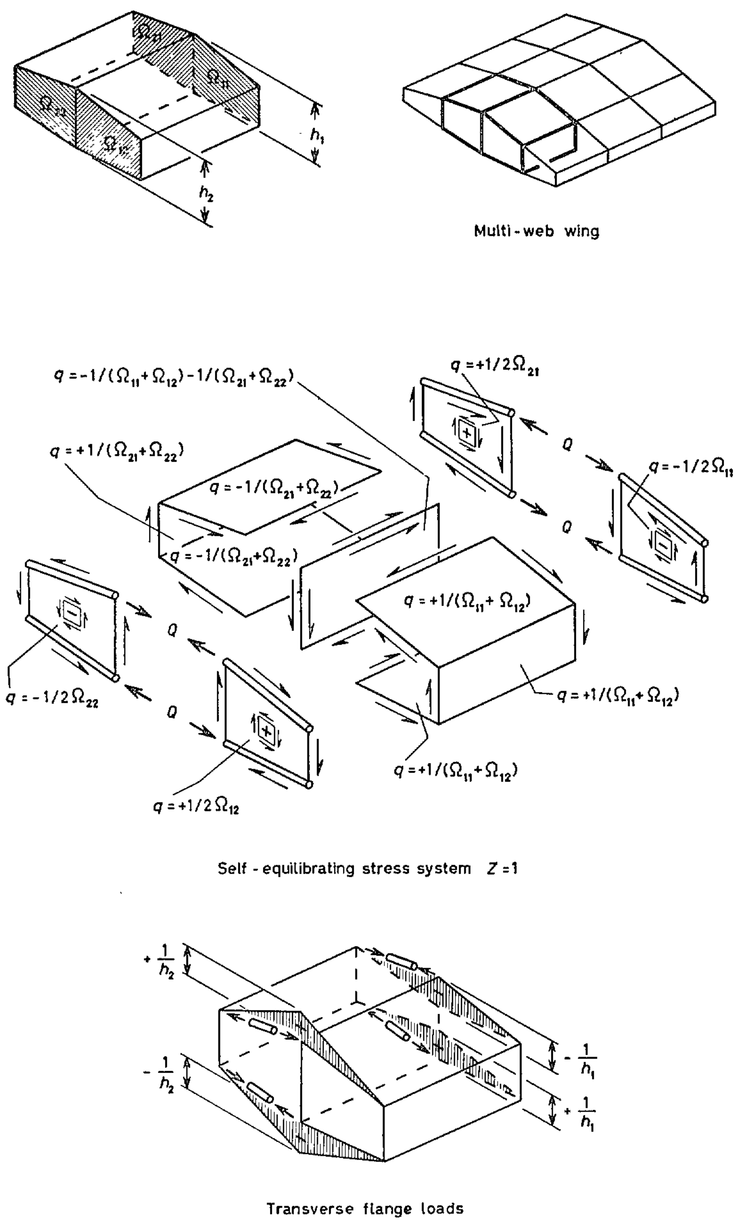

FIG. 5. Unit, self-equilibrating stress system of type $Z = 1$. (Transverse four-boom tube.)

5. *The Unit Load Method and Some Applications.*—The analysis of structures by the force method is most conveniently based on the formulation of the unit-load method given in Ref. **1**. This approach has, moreover, the great advantage that the effect of thermal or other initial strains can be included immediately without any further development of the theory. The unit-load method indicates also the procedure to be followed when the basic system is not statically determinate as in the standard analyses but is itself redundant. Finally we may apply this basic theorem to structures with non-linear elasticity.

We introduce the following definitions:

(1) Let ε_{xx}, ε_{xy}, etc., be the true direct and shear strains in a structure due to a given set of loads, prescribed displacements, thermal strains or any other initial strains, *e.g.*, those arising due to inaccurate manufacture. Also let r_j be the deformation (deflection, rotation or generalised displacement) at the point and direction j due to the same causes.

(2) Let $\bar{\sigma}_{xxj}$, $\bar{\sigma}_{xyj}$, etc., be the direct and shear stresses *statically equivaent* to a unit load (force, moment or generalised force) applied at the point and direction j. Note that $\bar{\sigma}_{xxj}$, $\bar{\sigma}_{xyj}$, etc., need satisfy merely the external and internal equilibrium conditions of the structure but not necessarily the compatibility conditions. In general, there is an infinite number of such systems of which one is the true stress system due to the unit load and consequently satisfies both equilibrium and compatibility conditions.

The unit-load theorem may now be written in the form:

$$1 . r_j = \int_V [\bar{\sigma}]_j'[\varepsilon] \, dV \qquad (13)$$

where $[\varepsilon]$, $[\bar{\sigma}]_j$ are the column matrices:

$$[\varepsilon] = \{\varepsilon_{xx} \ \varepsilon_{yy} \ \varepsilon_{zz} \ \varepsilon_{xy} \ \varepsilon_{yz} \ \varepsilon_{zx}\} \qquad (14)$$

$$[\bar{\sigma}]_j = \{\bar{\sigma}_{xxj} \ \bar{\sigma}_{yyj} \ \bar{\sigma}_{zzj} \ \bar{\sigma}_{xyj} \ \bar{\sigma}_{yzj} \ \bar{\sigma}_{zxj}\} \qquad (15)$$

and the integration extends over the volume V of the structure. An elementary illustration of this theorem is shown in Fig. 6. It is assumed that the true strains in the wing due to any external loading or initial (thermal) strains are known. To find then the deflection r_j in any given direction j we need only determine the simplest possible statically equivalent stress system $[\bar{\sigma}]_j$ corresponding to a unit load at j and apply equation (13). The most suitable choice is obviously the E.T.B. stress system in the independent spar under the unit load. Note that the structure need not be linearly elastic but may obey any non-linear stress strain relation.

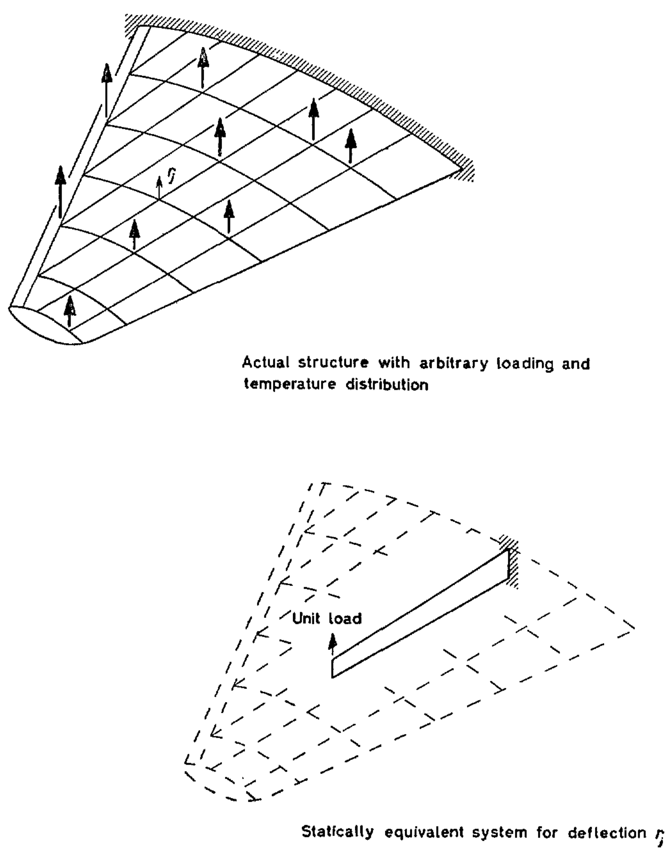

FIG. 6. Example of a statically equivalent system for deflection by the unit-load method.

An alternative form of theorem (13) is occasionally useful but is only applicable to linearly elastic structures. Thus, in such systems (13) may also be written:

$$1 . r_j = \int_V [\sigma]_j'[\bar{\varepsilon}]\, dV \quad \ldots \quad (13a)$$

where

$$[\sigma]_j = \{\sigma_{xxj}\ \sigma_{yyj}\ \sigma_{zzj}\ \sigma_{xyj}\ \sigma_{yzj}\ \sigma_{zxj}\} \quad \ldots \quad (15a)$$

is the *true* stress matrix corresponding to the unit load at j and

$$[\bar{\varepsilon}] = \{\bar{\varepsilon}_{xx}\ \bar{\varepsilon}_{yy}\ \bar{\varepsilon}_{zz}\ \bar{\varepsilon}_{xy}\ \bar{\varepsilon}_{yz}\ \bar{\varepsilon}_{zx}\} \quad \ldots \quad (14a)$$

is the strain matrix arising from the imposition of the given loads and initial strains on a system merely statically equivalent to the given structure. Thus $[\bar{\varepsilon}]$ may be found in any suitable statically determinate basic system.

The introduction of statically equivalent stress or strain systems can obviously simplify the calculations considerably.

We present now some very simple applications of theorem (13) which are helpful to subsequent developments.

(*a*) *Field with constant shear.*

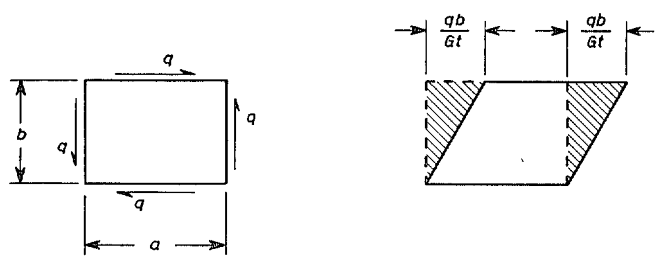

FIG. 7. Generalised strain v of a rectangular field under constant shear flow.

Consider a rectangular field under a constant shear flow (*see* Fig. 7). The true strain system is q/Gt and the selected unit-load system is the unit shear flow. Application of (13) yields:

$$1 . r = 1 . v = \iint 1 . \frac{q}{Gt}\, dx\, dy = \frac{\Phi}{Gt} q \quad \ldots \quad (16)$$

where v is a generalised displacement corresponding to the generalised force of the unit shear flow. $\Phi = ab$ is the area of the shear field. Result (16) is, of course, trivial and follows immediately from the value q/Gt of the shear strain.

We call

$$f = \frac{\Phi}{Gt} \quad \ldots \quad (17)$$

the (shear) flexibility of the field. Thus:

$$v = fq \quad \ldots \quad (16a)$$

(b) Flange under a linearly varying end load.

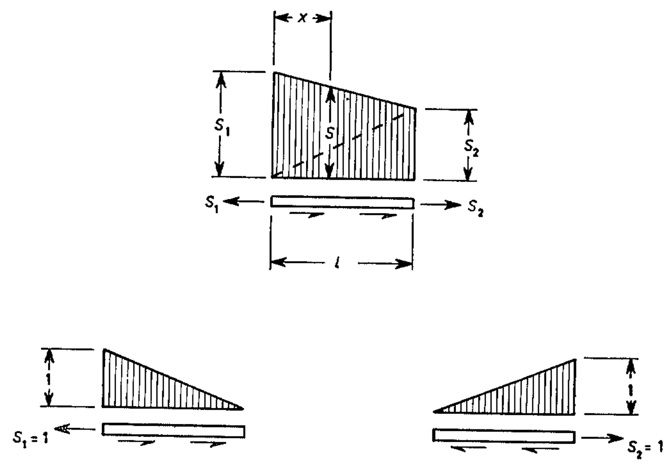

FIG. 8. Derivation of generalised strain v for a flange with linearly varying end load.

Consider a flange of constant area A, Young's modulus E and length l, subjected to an end load varying linearly from S_1 to S_2 (*see* Fig. 8). Overall equilibrium is achieved by uniform tangential shear flows applied to the flange. The direct load S at any station x may be written:

$$S = S_1\left(1 - \frac{x}{l}\right) + S_2\frac{x}{l}$$

$$= \left[\left(1 - \frac{x}{l}\right) \quad \frac{x}{l}\right]\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \quad \ldots \quad (17)$$

If we consider the matrix

$$\mathbf{S} = \{S_1\, S_2\} \quad \ldots \quad (18)$$

as defining the external load system on the flange, then equation (17) is merely a particular example of equation (8):

$$\mathbf{S} = \mathbf{bR} \quad \ldots \quad (8)$$

with

$$\mathbf{b} = \left[\left(1 - \frac{x}{l}\right) \quad \frac{x}{l}\right] \quad ; \quad \mathbf{R} = \{S_1\, S_2\}.$$

The true strains in the flange are defined by the end strains

$$[\varepsilon] = \{\varepsilon_1\, \varepsilon_2\} = \frac{\mathbf{S}}{EA}. \quad \ldots \quad (19)$$

We are interested in determining the generalised displacements:

$$\mathbf{v} = \{v_1\, v_2\}, \quad \ldots \quad (20)$$

corresponding to the load system $\mathbf{S}$. To find them we select as unit-load systems the alternative systems shown in Fig. 8. Application of (13) yields:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \int_0^l \begin{bmatrix} \left(1 - \frac{x}{l}\right) \\ \frac{x}{l} \end{bmatrix} \begin{bmatrix} \left(1 - \frac{x}{l}\right) & \frac{x}{l} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \frac{dx}{EA}.$$

Hence:

$$\mathbf{v} = \begin{bmatrix} l/3EA & l/6EA \\ l/6EA & l/3EA \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \mathbf{fS} \quad \ldots \quad (21)$$

where

$$\mathbf{f} = \begin{bmatrix} l/3EA & l/6EA \\ l/6EA & l/3EA \end{bmatrix} \quad \ldots \quad (22)$$

is called the flexibility of the uniform flange corresponding to the loading of Fig. 8. Note that v_1 and v_2 are generalised displacements and not the displacements at the ends of the flange. Similar expressions to (22) may be obtained for flanges with varying area and different end-load variation. Furthermore, the same presentation may also be applied to beams under transverse loads (*see* Ref. 1. February, 1955. Equation (136), p. 47).

In the simple case of a uniform flange under constant end load S, relation (21) reduces to the trivial form:

$$v = fS \quad \ldots \quad (21a)$$

where

$$f = \frac{l}{EA} \quad \ldots \quad (22a)$$

and v is now the elongation Δl of the flange. Form (22*a*) of the flexibility applies, for example, to the bars in a pin-jointed framework (*see* Fig. 1).

(*c*)* *Flange under linearly varying initial strain.*—Consider a flange (1, 2) of length l subjected to an initial strain η varying linearly from η_1 at nodal point 1 to η_2 at nodal point 2. η may, for example, be a thermal strain $\alpha\Theta$ imposed on the flange.

We seek now the generalised strains:

$$\mathbf{v} = \{v_1 \; v_2\}$$

arising from η and corresponding to the unit-load systems of Fig. 8. We have:

$$\begin{aligned} \eta &= \left(1 - \frac{x}{l}\right)\eta_1 + \frac{x}{l}\eta_2 \\ &= \begin{bmatrix} \left(1 - \frac{x}{l}\right) & \frac{x}{l} \end{bmatrix} [\eta] \end{aligned} \quad \ldots \quad (23)$$

where

$$[\eta] = \{\eta_1 \; \eta_2\}. \quad \ldots \quad (23a)$$

* May be omitted at first reading.

Application of (13) yields:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \int_0^l \begin{bmatrix} \left(1 - \frac{x}{l}\right) \\ \frac{x}{l} \end{bmatrix} \left[\left(1 - \frac{x}{l}\right) \quad \frac{x}{l}\right] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} dx = \mathbf{l}[\eta] \quad \ldots \quad \ldots \quad (24)$$

where

$$\mathbf{l} = \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix}. \quad \ldots \quad (25)$$

If the initial strain is uniform along the length of the element and the unit load is taken as constant, as would be the case in the bars of a pin-jointed framework under thermal strain, formula (25) reduces to the obvious:

$$v = l\eta \quad \ldots \quad (25)$$

or

$$v = l\alpha\Theta \qquad \text{for thermal extension.} \quad \ldots \quad (25)$$

v is now merely the elongation of the flange or bar.

In what follows the (generalised) strains $\mathbf{v}$ arising from initial strains are denoted by $\mathbf{H}$. We shall also use the abbreviated terms 'stress' and 'strain' to denote the matrices $\mathbf{S}$ and $\mathbf{v}$ respectively.

6. *The Formulation of the Unit-Load Method by Matrix Assembly.*—We now turn our attention to a structure consisting of an assembly of elements denoted by $a, b, c, \ldots, g, \ldots s$. In their simplest form these elements may be shear panels, flanges between nodal points, beams under transverse loads, ribs, etc. We emphasise, however, that the elements need not be the simplest constituent parts of the structure. We may select as elements suitable part assemblies of the latter components which may, in fact, form redundant sub-systems. Thus, in a fuselage we can choose as an element a complete ring and this applies even if the ring is not of uniform circular shape but is itself a complex component, say of arbitrary varying cross-section and doubly connected form. However, whatever these elements may be, we assume for the moment that their strains $\mathbf{v}_g$ are known. They may arise from external loads and/or initial strains (*see*, for example, equations (16*a*), (21), (24)). The strains of all elements can be expressed as a column matrix:

$$\mathbf{v} = \{\mathbf{v}_a \, \mathbf{v}_b \ldots \mathbf{v}_g \ldots \mathbf{v}_s\}. \quad \ldots \quad (26)$$

Similarly the stresses $\mathbf{S}$ on the elements can be written as a matrix:

$$\mathbf{S} = \{\mathbf{S}_a \, \mathbf{S}_b \ldots \mathbf{S}_g \ldots \mathbf{S}_s\}, \quad \ldots \quad (27)$$

where equation (27) is merely a re-arranged form of equation (3).

We consider now a stress system $\bar{\mathbf{S}}$ statically equivalent to the m unit loads $R_j = 1$ applied to our structure. We may write the stresses $\bar{\mathbf{S}}$ as:

$$\bar{\mathbf{S}} = \bar{\mathbf{b}}\{1 \; 1 \ldots 1 \ldots 1\}, \quad \ldots \quad (28)$$

where the matrix notation $\bar{\mathbf{b}}$ expresses the condition that the stresses $\bar{\mathbf{S}}$ need only be statically equivalent to the applied unit forces. Thus, a particular case of $\bar{\mathbf{b}}$ would be $\mathbf{b}_0$, where $\mathbf{b}_0$ may be found in the simplest and most convenient basic system of our structure. On the other hand, we can always substitute $\mathbf{b}$ (the true stress matrix corresponding to unit loads), for $\bar{\mathbf{b}}$ but the application of a suitable $\bar{\mathbf{b}}$ can often simplify the computations considerably.

We define next the displacement column matrix:

$$\mathbf{r} = \{r_1\, r_2 \ldots . r_j \ldots . r_m\}, \qquad (29)$$

where r_j are the actual deformations or generalised displacements in the m directions fixed in the previous paragraph due to the strains $\mathbf{v}$ of equation (26). Applying now equation (13) for each of the deflections r_j we find:

$$\mathbf{r} = \bar{\mathbf{b}}'\mathbf{v} = \mathbf{b}'\mathbf{v}\,. \qquad (30)$$

This is the matrix formulation of the unit-load theorem and is of fundamental importance to our theory. It must be emphasised that in equation (30) the exact cause of the strains $\mathbf{v}$ is immaterial; they may be due to loads and/or initial strains.

An alternative form of equation (30) follows immediately from equation (13*a*). Thus the column matrix $\mathbf{r}$ may also be found from:

$$\mathbf{r} = \mathbf{b}'\bar{\mathbf{v}}\,, \qquad (30a)$$

where $\mathbf{b}$ is now the true stress matrix corresponding to the m unit loads $R_j = 1$ and $\bar{\mathbf{v}}$ is a strain matrix due to the applied loads or initial strains which need be found only in a system statically equivalent to the given structure. An interesting application of equation (30*a*) arises when the stress distribution $\mathbf{b}$ due to a set of unit loads is known and we want to find the deflection in *their* directions due to another system of loads applied in different directions or due to any initial strains. Then equation (30*a*) shows that it is not necessary to solve the redundant problem for the second set of loads or the initial strains since we can determine $\bar{\mathbf{v}}$ in the statically determinate basic system.

If the strains $\mathbf{v}$ arise only from a load system,

$$\mathbf{R} = \{R_1\, R_2 \ldots . R_j \ldots . R_m\}, \qquad (1)$$

acting on a linearly elastic structure, then $\mathbf{v}$ can always be written in the form:

$$\mathbf{v}_g = \mathbf{f}_g \mathbf{S}_g\,, \qquad (21)$$

where $\mathbf{f}_g$ is the flexibility of the gth element.

Hence:

$$\mathbf{v} = \mathbf{f}\mathbf{S} \qquad (32)$$

where $\mathbf{S}$, the stress matrix equation (27), is

$$\mathbf{S} = \mathbf{b}\mathbf{R} \qquad (8)$$

and $\mathbf{f}$, the flexibility of the *unassembled* elements of the structure, is:

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_a & 0 \ldots\ldots 0 \ldots\ldots 0 \\ 0 & \mathbf{f}_b \ldots\ldots 0 \ldots\ldots 0 \\ & \ddots \\ 0 & 0 \ldots\ldots \mathbf{f}_g \ldots\ldots 0 \\ & \quad\ddots \\ 0 & 0 \ldots\ldots 0 \ldots\ldots \mathbf{f}_s \end{bmatrix} \quad \ldots \quad (33)$$

Note that the diagonal elements of $\mathbf{f}$ are scalar numbers for shear fields but 2×2 matrices for flanges under a varying end load. For a pin-jointed framework, $\mathbf{f}_g$ is simply given by equation (22*a*) and a corresponding typical flexibility matrix $\mathbf{f}$ is shown in equation (33*a*), Table 1, for the system of Fig. 1; the factor 2 in all flexibilities, but for bar (11), arises from the use of symmetry of the structure in writing $\mathbf{b}_0$ and $\mathbf{b}_1$ for only half the system including bar (11). For other types of elements $\mathbf{f}_g$ may be determined from equation (13).

Substituting equations (32) and (8) into equation (30) we find the deflections r_j due to the m loads, as

$$\mathbf{r} = \bar{\mathbf{b}}'\mathbf{f}\mathbf{b}\mathbf{R} = \mathbf{b}'\mathbf{f}\mathbf{b}\mathbf{R} . \quad \ldots \quad (34)$$

But a similar argument starting from equation (20*a*) shows also that

$$\mathbf{r} = \mathbf{b}'\mathbf{f}\bar{\mathbf{b}}\mathbf{R} . \quad \ldots \quad (31a)$$

Therefore, the flexibility $\mathbf{F}$ of the structure in the given m directions is:

$$\mathbf{F} = \bar{\mathbf{b}}'\mathbf{f}\mathbf{b} = \mathbf{b}'\mathbf{f}\mathbf{b} = \mathbf{b}'\mathbf{f}\bar{\mathbf{b}} . \quad \ldots \quad (35)$$

The last relation follows, of course, also from the reciprocal theorem of Maxwell-Betti (symmetry of the flexibility matrix). Note finally the interesting dual relationships:

$$\mathbf{S} = \mathbf{b}\mathbf{R} , \qquad \bar{\mathbf{S}} = \bar{\mathbf{b}}\mathbf{R} \quad \ldots \quad (8)$$

and

$$\mathbf{r} = \bar{\mathbf{b}}'\mathbf{v} = \mathbf{b}'\mathbf{v} = \mathbf{b}'\bar{\mathbf{v}} . \quad \ldots \quad (30b)$$

7. *The Calculation of the Redundancies* $\mathbf{X}$ *and the True Stress Matrix* $\mathbf{S}$.—We consider now once more an n times redundant structure under a system of m loads R_j. We select the basic system, in which the redundancies $\mathbf{X} = \mathbf{O}$, and apply to it the loads $\mathbf{R}$. The conditions of compatibility in the original structure demand that the generalised relative displacements v_{Xi} at the n ' cut ' redundancies are zero if we impose also the correct magnitudes of the redundancies $\mathbf{X}$ on the basic system. Thus, in matrix form:

$$\mathbf{v}_X = \{v_{X1}\, v_{X2} \ldots\ldots v_{Xi} \ldots\ldots v_{Xn}\} = \mathbf{O} . \quad \ldots \quad (36)$$

Applying now the unit-load method in the form (30) and noting that in the present case

$$\mathbf{r} \equiv \mathbf{v}_X \text{ and } \bar{\mathbf{b}} \equiv \mathbf{b}_1 \quad \ldots \quad (37)$$

we find immediately, using equation (33):

$$\mathbf{v}_X = \mathbf{b}_1'\mathbf{v} = \mathbf{b}_1'\mathbf{f}\mathbf{b}\mathbf{R} = \mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R} + \mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X} = \mathbf{O} \ldots \quad \ldots \quad (38)$$

Thus:

$$\mathbf{X} = -\mathbf{D}^{-1}\mathbf{D}_0\mathbf{R}\,, \qquad (39)$$

where

$$\mathbf{D} = \mathbf{b}_1'\mathbf{f}\mathbf{b}_1 \text{ and } \mathbf{D}_0 = \mathbf{b}_1'\mathbf{f}\mathbf{b}_0\,. \qquad (40)$$

The solution of the n equations (38) in the unknowns $\mathbf{X}$ is here obtained formally by inversion of the matrix of the coefficients of the unknowns. Naturally, we may apply other methods of solution. In practice, the most appropriate technique will depend on the number of unknowns and the capacity of the store of the digital computer (*see* Refs. 1, 2 and 3). Thus, the Mercury computer of Ferranti with a store of 16,000 numbers or orders can solve directly say 110 linear equations by inversion of the matrix of the coefficients. With a medium-sized computer like the Pegasus of Ferranti we can invert directly matrices of order say 32×32; for larger matrices we should have to use on this computer the method of partitioning or other suitable techniques.

Substituting now equation (39) into equation (6) we find the true stresses

$$\mathbf{S} = [\mathbf{b}_0 - \mathbf{b}_1[\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0]\,\mathbf{R} = \mathbf{b}\mathbf{R}\,. \qquad (41)$$

Hence:

$$\mathbf{b} = \mathbf{b}_0 - \mathbf{b}_1[\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0 = \mathbf{b}_0 - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{D}_0 \qquad (42)$$

which solves the problem of the stress distribution completely.

To derive the flexibility matrix for the loads $\mathbf{R}$ we use equation (35) and note that here we can put $\bar{\mathbf{b}} = \mathbf{b}_0$. Thus, $\mathbf{F} = \mathbf{b}_0'\mathbf{f}\mathbf{b}$, which, using equation (42), becomes:

$$\mathbf{F} = \mathbf{F}_0 - \mathbf{D}_0'\mathbf{D}^{-1}\mathbf{D}_0 \qquad (43)$$

where

$$\mathbf{F}_0 = \mathbf{b}_0'\mathbf{f}\mathbf{b}_0\,, \qquad (44)$$

is the flexibility of the basic system for the directions of the loads $\mathbf{R}$.

Equations (41) to (44) show that for the complete structural analysis of any redundant structure under a given set of loads $\mathbf{R}$ we need only three basic matrices: $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{f}$. All three can be compiled very easily once we follow the procedure suggested in sections 4 and 5.

The flexibility $\mathbf{F}$ of the actual system may also be considered as the condensed matrix of the flexibility of the basic system for the directions of both $\mathbf{R}$ and $\mathbf{X}$. Thus, the deflections in the actual structure can be written :

$$\begin{bmatrix} \mathbf{F}_0 & \mathbf{D}_0' \\ \mathbf{D}_0 & \mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{R} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \mathbf{o} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{O} \end{bmatrix}\mathbf{R}$$

from which we deduce immediately equation (43). This derivation of the flexibility matrix (43) may be used to solve a slightly more general problem. Assume that we know the flexibility $\mathbf{F}$ of a structure for m of its points and require the flexibility $\mathbf{F}_1$ for k points $(k < m)$ only when the remaining $l = m - k$ points are fixed in space. Again we write $\mathbf{F}$ in the obvious partitioned form:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{kk} & \mathbf{F}_{lk}' \\ \mathbf{F}_{lk} & \mathbf{F}_{ll} \end{bmatrix}$$

and find easily:

$$\mathbf{F}_1 = \mathbf{F}_{kk} - \mathbf{F}_{lk}'\mathbf{F}_{ll}^{-1}\mathbf{F}_{lk}\,. \qquad (43a)$$

As an example of the application of equation (43*a*) consider a wing of which some of the spars transfer only shear at the root, their flanges being unattached. The flexibility of this wing is known for a set $m = k + l$ points and directions including those of the unattached flanges at the root. If a number l of the latter flanges is now fixed at the root we can derive the new flexibility $\mathbf{F}_1$ immediately from equation (43*a*).

The analysis of this paragraph may easily be generalised for a statically indeterminate basic system (*see* Ref. 1).

8. *The Redundancies and Stress Distribution for Thermal or Other Initial Strains* $\mathbf{H}$.—Assume that an n times redundant structure is subjected to initial (*e.g.*, thermal) strains η. In the basic (statically determinate) structure these can develop freely and their magnitude is defined conveniently by the (generalised) strain matrix:

$$\mathbf{H} = \{\mathbf{H}_a\ \mathbf{H}_b \ldots \mathbf{H}_g \ldots \mathbf{H}_s\}, \qquad (45)$$

where $\mathbf{H}_g$ is the generalised initial strain of element g and may be determined easily from equation (13). We have found, for example, on page 14, the matrix $\mathbf{H}$ for a flange subject to a linearly varying initial direct strain. For a shear field g under an initial shear strain η_g we confirm immediately the trivial result $H_g = \Phi_g \eta_g$.

Denoting the unknown redundancies due to $\mathbf{H}$ by $\mathbf{X}_H$ the true strains $\mathbf{v}$ of the system are obviously:

$$\mathbf{v} = \mathbf{fS} + \mathbf{H} = \mathbf{fb}_1\mathbf{X}_H + \mathbf{H}. \qquad (46)$$

Hence the compatibility equation (38) becomes here:

$$\mathbf{v}_X = \mathbf{b}_1'\mathbf{v} = \mathbf{b}_1'\mathbf{fb}_1\mathbf{X}_H + \mathbf{b}_1'\mathbf{H} = \mathbf{O} \qquad (47)$$

or

$$\mathbf{X}_H = -\mathbf{D}^{-1}\mathbf{b}_1'\mathbf{H}. \qquad (48)$$

Using equation (48) in equation (10) we obtain the stresses $\mathbf{S}_H$ due to the initial strain matrix $\mathbf{H}$:

$$\mathbf{S}_H = \mathbf{b}_1\mathbf{X}_H = -\mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_1'\mathbf{H}. \qquad (49)$$

The thermal distortion (deflection) $\mathbf{r}_H$ of the structure in the directions of the m loads R_j of section 6 are found from equation (30) as:

$$\begin{aligned}\mathbf{r}_H &= \bar{\mathbf{b}}'\mathbf{v} = \bar{\mathbf{b}}'[\mathbf{fb}_1\mathbf{X}_H + \mathbf{H}] \\ &= \bar{\mathbf{b}}'[\mathbf{I}_s - \mathbf{fb}_1\mathbf{D}^{-1}\mathbf{b}_1']\,\mathbf{H}, \qquad (50)\end{aligned}$$

where $\mathbf{I}_s$ is the unit matrix of order s. We derive an interesting and surprising result by substituting $\mathbf{b}_0$ for $\bar{\mathbf{b}}$ in the last equation. In fact,

$$\begin{aligned}\mathbf{r}_H &= [\mathbf{b}_0' - \mathbf{b}_0'\mathbf{fb}_1\mathbf{D}^{-1}\mathbf{b}_1']\,\mathbf{H} \\ &= [\mathbf{b}_0' - \mathbf{D}_0'\mathbf{D}^{-1}\mathbf{b}_1']\,\mathbf{H} \\ &= [\mathbf{b}_0 - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{D}_0]'\,\mathbf{H} = \mathbf{b}'\mathbf{H}, \qquad (51)\end{aligned}$$

which is, of course, a direct consequence of equation (30*a*). Thus, to determine the thermal distortion of a structure at a given set of points and directions due to imposed initial strains (*e.g.*, thermal strains) we need not solve the redundant problem for these strains if we know the true stress matrix $\mathbf{b}$ corresponding to unit loads in the prescribed directions.

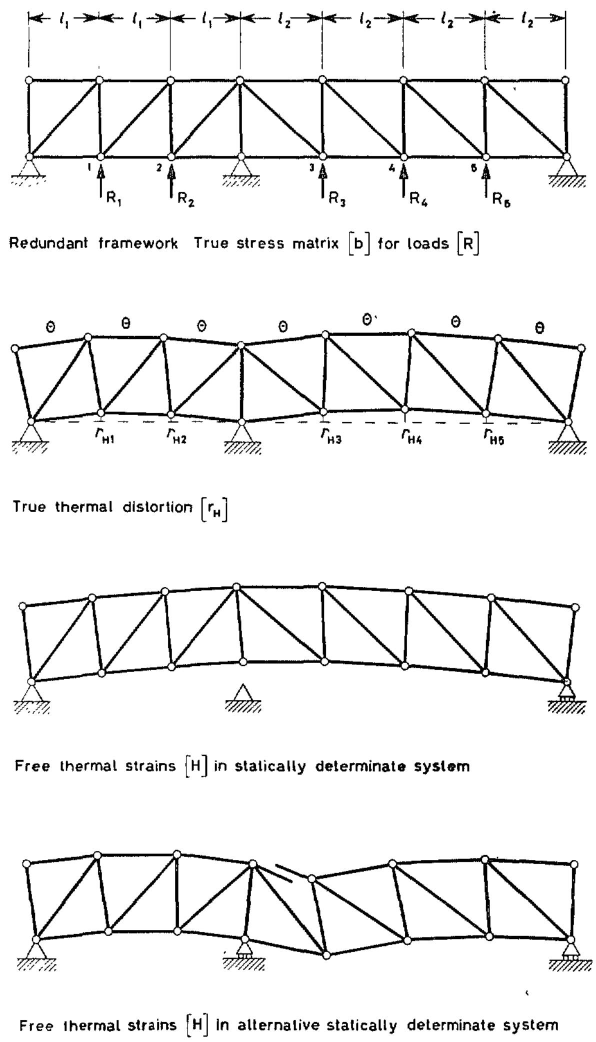

FIG. 9. Thermal distortion of a redundant, pin-jointed framework.

To illustrate the application of equation (51), consider the pin-jointed framework of Fig. 9 whose upper flange is subjected to a uniform temperature rise Θ. It is required to find the thermal deflections r_{Hj} at points 1 to 5. Assume that we know from previous calculations the true stress matrix $\mathbf{b}$ corresponding to loads R_1 to R_5. Then, noting that the initial strain matrix in the present case is simply given by (*see* equation (25*b*)):

$$\mathbf{H} = \{ l_1\alpha\Theta,\ l_1\alpha\Theta,\ l_1\alpha\Theta,\ l_2\alpha\Theta,\ l_2\alpha\Theta,\ l_2\alpha\Theta,\ l_2\alpha\Theta \},$$

the deflection matrix $\mathbf{r}_H$ follows as:

$$\mathbf{r}_H = \{r_{H1}\ r_{H2}\ r_{H3}\ r_{H4}\ r_{H5}\} = \mathbf{b}'\{\mathbf{H}\ \mathbf{O}\} = \mathbf{b}_u'\mathbf{H},$$

where $\mathbf{b}_u$ is the submatrix of $\mathbf{b}$ corresponding to the upper flange only.

PART II

The Structure with Cut-Outs or Modified Elements

9. *The New Approach to the Problem of Cut-Outs.*—The force method developed above is naturally valid for structures with any kind of cut-outs stiffened or unstiffened by closed frames as long as the overall geometry and idealisation conforms with the initial assumptions. Nevertheless it must be admitted that cut-outs require special attention in this approach, both as far as the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices are concerned. In fact, in the selection of the basic system the existence of a cut-out will, in general, enforce a more complicated choice than in the corresponding continuous structure without cut-outs. Moreover, in the region of a cut-out we shall have to use special non-standard redundant force systems. A further drawback arising from cut-outs is that the checking of the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices is in such cases not so straightforward since the uniformity of the patterns of their elements, characteristic of these matrices in continuous structures, is lost. All these points are considered in some detail in Ref. 1, where the appropriate procedure is described for each type of wing cut-out.

To avoid these complications in the presence of cut-outs it is worthwhile to apply an artifice first developed in Ref. 1, which avoids all the above-mentioned special considerations. Moreover, it gives us the ideal method of finding the redistribution of stresses due to the subsequent introduction of cut-outs in our system without having to repeat all the computations *ab initio*.

The method is as follows. To preserve the pattern of the matrices and equations disturbed by missing shear panels or flanges we fill in the cut-outs by introducing fictitious shear panels or flanges with arbitrary thicknesses or cross-sectional areas. Naturally, it is usually preferable for computational reasons to select for these dimensions those of the surrounding structure. To obtain, nevertheless, the same flange loads and shear flows in our continuous structure as in the original system, initial strains are imposed on the additional elements of such magnitude that their total stresses due to both loads and initial strains become zero. The effect of the fictitious elements is thus nullified whilst the uniform pattern of our equations is retained.

Let the column matrix of the unknown (generalised) initial strains, *imposed on the additional elements only*, be $\mathbf{H}$.

In the new continuous structure we determine the flexibility matrix $\mathbf{f}$ and the matrices $\mathbf{b}_0$ and $\mathbf{b}_1$. For the subsequent developments we require $\mathbf{b}_1$ also in the partitioned form:

$$\mathbf{b}_1 = \begin{bmatrix} \mathbf{b}_{1g} \\ \mathbf{b}_{1h} \end{bmatrix}, \quad \ldots \quad (52)$$

where the suffixes $_g$ and $_h$ refer to the stresses or forces in the elements of the original structure and the fictitious new elements respectively.

Denoting the column matrix of the redundancies of the continuous structure by $\mathbf{X}$ and writing the initial strain matrix imposed on this system in the partitioned form

$$\begin{bmatrix} \mathbf{O} \\ \mathbf{H} \end{bmatrix} \quad \ldots \quad (53)$$

we can find the unknown $\mathbf{X}$ from equations (38) and (47) which become here:

$$\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X} + \mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R} + \mathbf{b}_1'\begin{bmatrix} \mathbf{O} \\ \mathbf{H} \end{bmatrix} = \mathbf{O} . \quad \ldots \quad (54)$$

Hence, using equation (52):

$$\mathbf{X} = -\mathbf{D}^{-1}\mathbf{D}_0\mathbf{R} - \mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}, \qquad (55)$$

where $\mathbf{D}$ and $\mathbf{D}_0$ are given by equation (40) and are found, of course, in the continuous structure.

The stress matrix $\mathbf{S}$ follows as:

$$\mathbf{S} = [\mathbf{b}_0 - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{D}_0]\,\mathbf{R} - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}. \qquad (56)$$

The expression in the square bracket is the matrix $\mathbf{b}$ of equation (42) which we write in the partitioned form:

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_g \\ \mathbf{b}_h \end{bmatrix}, \qquad (57)$$

where the suffices $_g$ and $_h$ have the same meaning as before.

To find now the column matrix $\mathbf{H}$ we put the stresses in the additional elements to zero. Thus, the matrix $\mathbf{S}$ must become:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_g \\ \mathbf{O} \end{bmatrix}, \qquad (58)$$

where $\mathbf{S}_g$ are the true stresses (forces) in the original structure with cut-outs.

Applying equations (52), (57), (58) in equation (56) we find:

$$\begin{bmatrix} \mathbf{S}_g \\ \mathbf{O} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_g \\ \mathbf{b}_h \end{bmatrix} \mathbf{R} - \begin{bmatrix} \mathbf{b}_{1g} \\ \mathbf{b}_{1h} \end{bmatrix} \mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}.$$

Hence:

$$\mathbf{O} = \mathbf{b}_h\mathbf{R} - \mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}$$

or

$$\mathbf{H} = [\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1}\mathbf{b}_h\mathbf{R}. \qquad (59)$$

The true stresses in our actual cut structure are then:

$$\mathbf{S}_g = [\mathbf{b}_g - \mathbf{b}_{1g}\mathbf{D}^{-1}\mathbf{b}_{1h}'[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1}\mathbf{b}_h]\,\mathbf{R}, \qquad (60)$$

which solves our problem. The important and unique characteristic of the present method is that it yields the stresses in the cut structure solely in terms of the stresses already calculated in the fictitious continuous structure. It should be noted that the order of the square matrix to be inverted*

$$[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1}$$

is equal to the number of *linearly independent* stresses or stress resultants to be nullified†. Thus, if we remove one shear panel the order is one and the matrix is a mere scalar number. If we eliminate one flange between two adjoining nodal points the order is two, etc. The amount of work in any practical calculation is surprisingly small as is illustrated in the examples of Part III.

---

* The inversion of $\mathbf{D}$ will have been performed previously in finding $\mathbf{b}$.

† See example in authors' paper: "Structural analysis by the matrix force method with applications to aircraft wings", *Wissenschaftliche Gesellschaft für Luftfahrt*, Jahrbuch 1956.

The operations leading to equation (60) are easily programmed on the digital computer (*see* Hunt[2]). To check the computations it is useful to include in the final stress matrix the condition of zero stress in the fictitious elements. This is achieved by writing equation (60) in the form:

$$\mathbf{S}_c = \begin{bmatrix} \mathbf{S}_g \\ \mathbf{O} \end{bmatrix} = \mathbf{b}_c \mathbf{R} \, , \qquad \qquad (61)$$

where

$$\mathbf{b}_c = \mathbf{b} - \mathbf{b}_1 \mathbf{D}^{-1} \mathbf{b}_{1h}' [\mathbf{b}_{1h} \mathbf{D}^{-1} \mathbf{b}_{1h}']^{-1} \mathbf{b}_h \, . \qquad \qquad (62)$$

The suffix $_c$ indicates the stresses in the cut structure.

As mentioned already the method is ideally suited for finding the alteration to the stresses in a structure through a subsequent introduction of cut-outs such as access doors, which usually seem to materialise at a late stage of design. But even if all cut-outs are known initially, the new approach will easily be seen to be preferable to the standard method when analysing wings and fuselages. Thus, in fuselage stressing, bomb bays, doors and window openings should present no difficulties when formula (62) is used. Naturally the degree of redundancy is increased by the ' filling in ' of the cut-outs but this is of no importance for the automatic computations envisaged here.

10. *Illustration of the Validity of Equation* (62).—Consider a structure with n redundancies under any system of loads. Analysing this system by the method of Part I we obtain the complete **b** matrix. Assume now that we eliminate all redundancies of the original structure by the technique of the previous paragraph. In this case equation (62) should reduce to:

$$\mathbf{b}_c = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{O} \end{bmatrix} ,$$

i.e., to the stress distribution in the basic system since the latter is by assumption identical to the cut system.

The proof is straightforward. A simple consideration shows that the matrix $\mathbf{b}_{1h}$ reduces now, for a certain sequence of the redundancies, to the unit matrix $\mathbf{I}_n$ of the nth order; *see*, for example, the framework of Fig. 1 where $\mathbf{b}_{1h} = \mathbf{I}_3$ can be checked directly on equation (12). Equation (62) reduces hence to:

$$\mathbf{b}_c = \mathbf{b} - \mathbf{b}_1 \mathbf{D}^{-1} \mathbf{I}' [\mathbf{I} \mathbf{D}^{-1} \mathbf{I}']^{-1} \mathbf{b}_h$$

$$\mathbf{b}_c = \mathbf{b} - \mathbf{b}_1 \mathbf{b}_h \, .$$

Also

$$\mathbf{S}_c = \mathbf{b}_c \mathbf{R} = \mathbf{b} \mathbf{R} - \mathbf{b}_1 \mathbf{b}_h \mathbf{R} \, .$$

However, in the present case $\mathbf{b}_h \mathbf{R} = \mathbf{X}$, and thus using also equation (6):

$$\mathbf{S}_c = \mathbf{b} \mathbf{R} - \mathbf{b}_1 \mathbf{X} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{O} \end{bmatrix} \mathbf{R} \, .$$

q.e.d.

(68577) C * 2

11. *The Flexibility of the Cut Structure.*—Following equation (35) the flexibility of the cut structure is:

$$\mathbf{F}_c = \bar{\mathbf{b}}_c'\mathbf{f}\mathbf{b}_c = \mathbf{b}_c'\mathbf{f}\mathbf{b}_c \quad \ldots \quad (63)$$

If the cuts do not affect the basic system chosen for the calculation of the continuous structure then, but only then, may we write in place of equation (63)

$$\mathbf{F}_c = \mathbf{b}_0'\mathbf{f}\mathbf{b}_c \quad \ldots \quad (64)$$

Note that $\mathbf{f}$ includes the finite flexibilities of the fictitious elements but the result is still correct since the corresponding rows in $\mathbf{b}_c$ are zero. If $\mathbf{b}_c$ were defined only for the elements of the original cut structure* (*i.e.*, we exclude the zero rows), then the flexibility could be expressed as:

$$\mathbf{F}_c = \bar{\mathbf{b}}_c'\mathbf{f}_c\mathbf{b}_c = \mathbf{b}_c'\mathbf{f}_c\mathbf{b}_c , \quad \ldots \quad (63a)$$

where $\mathbf{f}_c$ is the flexibility matrix of the unassembled elements of the true cut structure. However, the form (63) appears preferable since the matrix $\mathbf{f}$ would anyhow have to be stored in the computer for the calculation of the $\mathbf{D}$ and $\mathbf{D}_0$ matrices.

An alternative approach to the flexibility $\mathbf{F}_c$ is of particular interest since it relates it to the flexibility $\mathbf{F}$ of the continuous structure. Following equations (4) and (51) the deflections of the cut structure may be written in the two alternative forms:

$$\mathbf{r}_c = \mathbf{F}_c\mathbf{R} = \mathbf{F}\mathbf{R} + \mathbf{b}'\begin{bmatrix}\mathbf{O}\\ \mathbf{H}\end{bmatrix} \quad \ldots \quad (64)$$

or using equations (57) and (59):

$$\mathbf{F}_c = \mathbf{F} + \mathbf{b}_h'[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1}\mathbf{b}_h . \quad \ldots \quad (65)$$

Note the symmetry and simplicity of the formula.

12. *A Generalisation of the Previous Method. The Structure with Modified Elements.*—The previous developments on the method of cut-outs suggest that it should be possible to devise a similar procedure when elements of the structure are modified subsequent to the completion of the structural analysis. Thus, we seek now a device to find the stress distribution in the altered system solely in terms of the stress distribution in the original system. Naturally, we can always find the stress distribution of the modified structure by an *ab initio* analysis, but this will involve, in general, considerably more lengthy calculations.

The method is as follows:

Assume that the $\mathbf{b}$ matrix for the original structure has been determined. As in section 9, equations (57) and (52), we partition $\mathbf{b}$ and $\mathbf{b}_1$ in the form:

$$\mathbf{b} = \begin{bmatrix}\mathbf{b}_g\\ \mathbf{b}_h\end{bmatrix} \text{ and } \mathbf{b}_1 = \begin{bmatrix}\mathbf{b}_{1g}\\ \mathbf{b}_{1h}\end{bmatrix},$$

where the suffices $_g$ and $_h$ refer now to elements to remain unaltered and to be modified respectively. We denote, furthermore, the flexibilities of the unassembled elements in the original and modified state by the respective symbols $\mathbf{f}_h$ and $\mathbf{f}_{hm}$. We impose now on the h elements of the original structure such initial strains $\mathbf{H}$ (again a column matrix) that their total strains are identical to those in the modified elements of the new structure. The stresses in the original system are then also those of our altered system subject to the same loads but no initial strains and this solves our problem completely.

---

* $\mathbf{b}_c$ is then identical to $\mathbf{b}_g$ of equation (57).

The stress matrix in both systems is given by:

$$\mathbf{bR} - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H} \ ,$$

see also equation (56). Hence the strains in the h elements are, in the original structure:

$$\mathbf{f}_h[\mathbf{b}_h\mathbf{R} - \mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}] + \mathbf{H}$$

and in the new structure:

$$\mathbf{f}_{hm}[\mathbf{b}_h\mathbf{R} - \mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}] \ .$$

Equality of the two expressions yields:

$$\mathbf{H} = [\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' + [\mathbf{f}_{hm} - \mathbf{f}_h]^{-1}]^{-1}\mathbf{b}_h \ . \qquad \text{(66)}$$

Hence the stresses $\mathbf{S}_m$ in the modified system are given by:

$$\mathbf{S}_m = \mathbf{b}_m\mathbf{R} \ , \qquad \text{(67)}$$

where

$$\mathbf{b}_m = \mathbf{b} - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' + \Delta\,\mathbf{f}_h^{-1}]^{-1}\mathbf{b}_h \qquad \text{(68)}$$

and

$$\Delta\,\mathbf{f}_h = \mathbf{f}_{hm} - \mathbf{f}_h \ . \qquad \text{(69)}$$

Note that the inversion of the matrix in the square bracket is only of the order of $\mathbf{f}_h$ as was also the case of the cut-out. The modification of the elements may, naturally, involve either a reinforcement or lightening; in the first case $\Delta\,\mathbf{f}_h < \mathbf{O}$ and in the second $\Delta\,\mathbf{f}_h > \mathbf{O}$. Formula (68) may, of course, also be applied when the stresses arise from initial strains $\mathbf{H}$ instead of loads $\mathbf{R}$. In fact, in such a case we have only to substitute:

$$\mathbf{b}_1\mathbf{X}_H = \mathbf{S}_H \text{ for } \mathbf{bR} \text{ and } \mathbf{b}_{1h}\mathbf{X}_H = \mathbf{S}_{Hh} \text{ for } \mathbf{b}_h\mathbf{R}$$

to find the stress $\mathbf{S}_{Hm}$ in the modified structure as :

$$\mathbf{S}_{Hm} = \mathbf{S}_H - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' + \Delta\,\mathbf{f}_h^{-1}]^{-1}\mathbf{S}_{Hh} \ . \qquad \text{(68a)}$$

Limiting cases.

(*a*) Elimination of the h elements. Then $\mathbf{f}_{hm} \to \infty$, and $\mathbf{b}_m$ reduces to the expression of equation (62)

$$\mathbf{b}_m = \mathbf{b}_c = \mathbf{b} - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1}\mathbf{b}_h \ . \qquad \text{(62)}$$

(*b*)* Rigidification of the h elements. Then $\mathbf{f}_{hm} \to \mathbf{O}$, and $\mathbf{b}_m$ reduces to:

$$\mathbf{b}_m = \mathbf{b} - \mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' - \mathbf{f}_h^{-1}]^{-1}\mathbf{b}_h \ . \qquad \text{(70)}$$

$\mathbf{f}_h^{-1}$ is, of course, $\mathbf{k}_h$, the original stiffness of the unassembled h elements.

A further special modification of the h elements deserves mention. In many instances the alteration of each of the h elements will be geometrically similar to the original elements. Then:

$$\mathbf{f}_{hm} = \left[\frac{1}{\beta}\right]\mathbf{f}_h \ ,$$

* The dual theorem in the displacement method solves the cut-out problem, see Ref. 7.

where $[1/\beta]$ is a mere diagonal matrix. Equation (68) becomes now:

$$\mathbf{b}_m = \mathbf{b} - \mathbf{b}_1 \mathbf{D}^{-1} \mathbf{b}_{1h}' \left[\mathbf{b}_{1h} \mathbf{D}^{-1} \mathbf{b}_{1h}' - \left[\frac{\beta}{\beta - 1} \right] \mathbf{k}_h \right]^{-1} \mathbf{b}_h . \qquad (71)$$

13. *The Flexibility of the Modified Structure.*—The flexibility $\mathbf{F}_m$ of the modified structure is:

$$\mathbf{F}_m = \bar{\mathbf{b}}_m' \mathbf{f}_m \mathbf{b}_m = \mathbf{b}_m' \mathbf{f}_m \mathbf{b}_m , \qquad (72)$$

where $\mathbf{f}_m$ is the flexibility of the unassembled elements in the modified structure:

$$\mathbf{f}_m = \begin{bmatrix} \mathbf{f}_g & \mathbf{O} \\ \mathbf{O} & \mathbf{f}_{hm} \end{bmatrix} . \qquad (73)$$

We may also write for $\mathbf{F}_m$:

$$\mathbf{F}_m = \mathbf{b}_0' \mathbf{f}_m \mathbf{b}_m \qquad (74)$$

but this form is inapplicable in the limiting case of modifications consisting of cut-outs affecting the basic system $\mathbf{b}_0$ (*see* also section 10).

As in section 10 it is important from the practical point of view to relate $\mathbf{F}_m$ to the flexibility $\mathbf{F}$ of the original structure. Following the same argument as in the case of cut-outs, we have for the deflections $\mathbf{r}_m$ of the modified system:

$$\mathbf{r}_m = \mathbf{F}_m \mathbf{R} = \mathbf{F}\mathbf{R} + \mathbf{b}' \begin{bmatrix} \mathbf{O} \\ \mathbf{H} \end{bmatrix} \qquad (69a)$$

or using equation (66):

$$\mathbf{F}_m = \mathbf{F} + \mathbf{b}_h' [\mathbf{b}_{1h} \mathbf{D}^{-1} \mathbf{b}_{1h}' + \Delta \mathbf{f}_h^{-1}]^{-1} \mathbf{b}_h . \qquad (75)$$

PART III

Simple Applications of the Theory

14. *The Four-Flange Tube under Transverse Loads.*—We illustrate now the theory developed in Parts I and II on a simple example. To this effect we consider the singly-symmetrical four-flange tube shown in Fig. 10 under the set of transverse loads:

$$\mathbf{R} = \{R_1 \, R_2 \, \ldots . \, R_6 \, \ldots . \, R_{12}\} . \qquad (1)$$

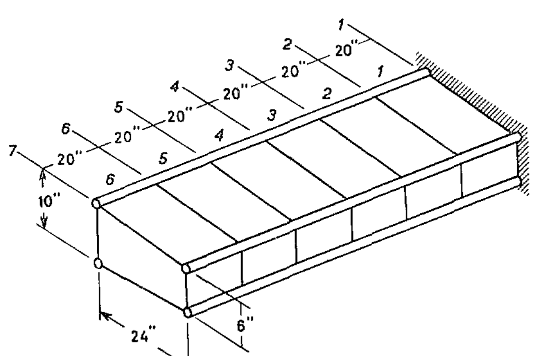

FIG. 10. Four-flange tube.

TABLE 2

Geometric and elastic data for tube

Rib	1	2	3	4	5	6	7
Bay	1	2	3	4	5	6	
A_a (in.^{2})	0·85	0·85	0·85	0·55	0·45	0·35	0·25
A_b (in.^{2})	1·20	0·80	0·40	0·40	0·40	0·40	0·40
t_{wa} (in.)	0·048	0·048	0·048	0·036	0·036	0·036	
t_{wb} (in.)	0·080	0·080	0·064	0·064	0·036	0·036	
t_s (in.)	0·048	0·048	0·048	0·048	0·048	0·048	
t_r (in.)	0·08	0·036	0·036	0·036	0·036	0·036	0·036
G_r (10^6 lb/in.^{2})	3·85	2·50	2·50	2·50	2·50	2·50	2·50

$E = 10 \times 10^6$ lb/in.^{2}

$G = 3{\cdot}85 \times 10^6$ lb/in.^{2} except for ribs

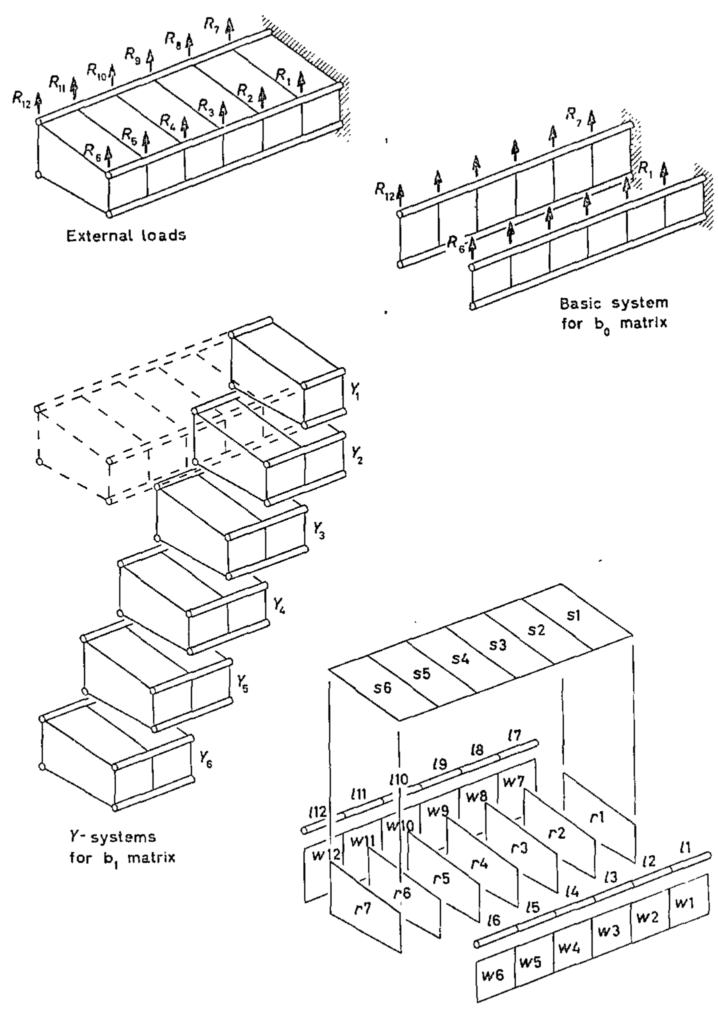

FIG. 11. Structural elements of four-flange tube, basic system and redundancies.

At first we assume that the structure is continuous, *i.e.*, we exclude any kind of cut-outs. The selected elements of the system and their numbering are depicted in Fig. 11, and the dimensions and elastic properties are given in Table 2. Only the outline of the calculations is presented here since this case is investigated in greater detail in Ref. 1. The complete programming of the problem has been developed by Hunt[2].

As basic system we select the two spars acting independently as beams and it is obvious that top and bottom covers and ribs are unloaded in such a system. For the redundancies we choose the six Y-systems of Fig. 4 applied at the rib stations 1 to 6 and denote the matrix of the redundancies:

$$\mathbf{Y} = \{Y_1 \, Y_2 \, Y_3 \, Y_4 \, Y_5 \, Y_6\} \, . \qquad \ldots \qquad (76)$$

The elements of the structure affected by each of the Y-systems are shown in Fig. 11. The rib flanges are loaded neither in the basic system nor by the redundancies and need not be considered at all.

To calculate the stress distribution under any system of loads $\mathbf{R}$ and the corresponding flexibility $\mathbf{F}$ we require initially only the very simple matrices $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{f}$. However, the computational work and the programming is considerably simplified by suitable partitioning of the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices; the reader may consult Ref. 1 and in particular Hunt[2] for more details of the procedure. Suffice it here to state that we write $\mathbf{b}_0$ in the form:

$$\mathbf{b}_0 = \overset{1 \, \ldots \, 12}{\begin{bmatrix} \mathbf{b}_{0la} \\ \mathbf{b}_{0lb} \\ \mathbf{b}_{0s} \\ \mathbf{b}_{0wa} \\ \mathbf{b}_{0wb} \\ \mathbf{b}_{0r} \end{bmatrix}} = \overset{1 \, \ldots \, 12}{\begin{bmatrix} \mathbf{b}_{0la} \\ \mathbf{b}_{0lb} \\ \mathbf{0} \\ \mathbf{b}_{0wa} \\ \mathbf{b}_{0wb} \\ \mathbf{0} \end{bmatrix}} \qquad \ldots \qquad (77)$$

where the numbers above the columns refer to the loads 1 to 12. Subscripts $_l$, $_s$, $_w$, $_r$ are used to denote longitudinal flanges, covers, spar webs and ribs respectively; subscripts $_a$ and $_b$ refer to front and rear spar respectively. Note that by virtue of the choice of the basic system $\mathbf{b}_{0s}$ and $\mathbf{b}_{0r}$ are zero matrices. In compiling $\mathbf{b}_0$ we take advantage of the symmetry of the structure by writing terms for the top surface only. The rows follow in each of the submatrices the numbering of the elements which, in each case, is from the root to the tip. The total number of rows in $\mathbf{b}_0$ is thus:

$$2 \times 6 + 2 \times 6 + 1 \times 6 + 1 \times 6 + 1 \times 6 + 1 \times 7 = 49 \, .$$

The choice of the basic system ensures very simple submatrices which can, in fact, be written down by inspection. This may be seen from Table 3, where $\mathbf{b}_{0la}$ and $\mathbf{b}_{0wa}$ of dimensions 12×12 and 6×12 respectively are given. On the other hand if we had selected the E.T.B. and Bredt-Batho stresses of the tube as statically equivalent stress system, the calculation of $\mathbf{b}_0$ would, in comparison, have been quite complicated. This illustrates clearly our comments on page 8 on the best choice of the basic system.

The $\mathbf{b}_1$ matrix of 49 rows is similarly partitioned to give:

$$\mathbf{b}_1 = \overset{1 \, \ldots \, 6}{\begin{bmatrix} \mathbf{b}_{1la} \\ \mathbf{b}_{1lb} \\ \mathbf{b}_{1s} \\ \mathbf{b}_{1wa} \\ \mathbf{b}_{1wb} \\ \mathbf{b}_{1r} \end{bmatrix}} \qquad \ldots \qquad (78)$$

where the numbers above the columns refer to the six redundancies and the suffices have the same meaning as in the case of $\mathbf{b}_0$. The elements of the submatrices are obtained immediately from the information on Fig. 4. The simplicity of these matrices may be seen on $\mathbf{b}_{1la}$ and $\mathbf{b}_{1wa}$ shown in Table 3.

The corresponding partitioned form of the flexibility matrix $\mathbf{f}$ of the unassembled elements is:

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_{la} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{f}_{lb} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{f}_{s} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{f}_{wa} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{f}_{wb} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{f}_{r} \end{bmatrix} \quad \ldots \quad (79)$$

where $\mathbf{f}_{la}$, $\mathbf{f}_{lb}$ are calculated from equation (22) and $\mathbf{f}_s$ to $\mathbf{f}_r$ from equation 17. The submatrix $\mathbf{f}_{lb}$ is shown in Table 4; the factor 2 is introduced to take care of the lower flange since $\mathbf{b}_0$ and $\mathbf{b}_1$ contain only the loads for the top flange.

We cannot emphasise sufficiently the great simplicity of the derivation of the three basic matrices $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{f}$. Although our example is naturally trivial the method is basically the same in the case of much more complex aircraft structures, *e.g.*, delta wings, once we can ignore the effect of cut-outs, have selected the simplest possible basic system and have tabulated all information for the $\mathbf{b}_1$ and $\mathbf{f}$ matrices.

The analysis of section 6 requires the matrix multiplications $\mathbf{b}_1'\mathbf{f}\mathbf{b}_0$, $\mathbf{b}_1'\mathbf{f}\mathbf{b}_1$ and $\mathbf{b}_0'\mathbf{f}\mathbf{b}_0$. Using the above partitioned form of the basic matrices we find the simple result:

$$\left.\begin{aligned} \mathbf{D}_0 &= \mathbf{b}_1'\mathbf{f}\mathbf{b}_0 = [\mathbf{b}_1'\mathbf{f}\mathbf{b}_0]_{la} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_0]_{lb} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_0]_{wa} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_0]_{wb} \\ \mathbf{D} &= \mathbf{b}_1'\mathbf{f}\mathbf{b}_1 = [\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]_{la} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]_{lb} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]_{s} + \\ &\qquad + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]_{wa} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]_{wb} + [\mathbf{b}_1'\mathbf{f}\mathbf{b}_1]_{r} \\ \mathbf{F}_0 &= \mathbf{b}_0'\mathbf{f}\mathbf{b}_0 = [\mathbf{b}_0'\mathbf{f}\mathbf{b}_0]_{la} + [\mathbf{b}_0'\mathbf{f}\mathbf{b}_0]_{lb} + [\mathbf{b}_0'\mathbf{f}\mathbf{b}_0]_{wa} + [\mathbf{b}_0'\mathbf{f}\mathbf{b}_0]_{wb} \end{aligned}\right\} . \quad (80)$$

These matrix multiplications and additions are very easily programmed for the digital computer where the form of $\mathbf{D}$, $\mathbf{D}_0$, $\mathbf{F}$ in (80) is most useful, especially in the case of large structures, since it allows an efficient use of the computer store (*see* Hunt[2,3]). Naturally it is also advantageous when we use a mere desk machine. If the number of external forces exceeds say 50 and that of the redundancies say 32, it will be necessary to partition the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices also by columns (the numbers given refer to the Ferranti Pegasus). The partitioning of the $\mathbf{b}_1$ matrix by columns is actually closely related to the idea of a statically indeterminate basic system (*see* section 1 and Ref. 1. p. 82. March, 1955).

Having $\mathbf{D}$ we find the inverted matrix $\mathbf{D}^{-1}$ and the product $\mathbf{D}^{-1}\mathbf{D}_0$ most conveniently and speedily on the digital computer especially if the latter operates, as do the Ferranti machines, with a matrix interpretive scheme (*see* Hunt[2]). In our present example $\mathbf{D}$ is of order 6×6 and the inversion requires on the Pegasus only 14 sec, whilst approximately 6 hours are necessary on the desk machine when using the Jordan technique. If we are interested only in the stress distribution for a particular load group $\mathbf{R}$, we can form on the digital computer $\mathbf{D}_0\mathbf{R}$ (which is a column matrix) and then find $\mathbf{D}^{-1}[\mathbf{D}_0\mathbf{R}]$ without first obtaining $\mathbf{D}^{-1}$. This shortens further the computing time. For example, on the Pegasus the time of 14 sec for $\mathbf{D}^{-1}$ is reduced to 9 sec when calculating the column matrix $\mathbf{D}^{-1}[\mathbf{D}_0\mathbf{R}]$. Naturally, these times increase rapidly with the order of the $\mathbf{D}$ matrix, but they are not more than 17 min 52 sec and 7 min 19 sec respectively when this order is 32×32. Again the quoted times apply to the Pegasus on which we would have to use partitioning when $\mathbf{D}$ is of higher order (*see* also statement at end of previous paragraph).

The $\mathbf{D}^{-1}$ and $\mathbf{D}$ matrices for the present example are given in Table 4. It is now very simple to find from equations (42) and (43) the final $\mathbf{b}$ matrix and the exact 12×12 flexibility $\mathbf{F}$. Table 5 shows, in particular, the submatrices $\mathbf{b}_{la}$ and $\mathbf{b}_{xa}$ whilst $\mathbf{F}$ is found on Table 6. Finally, we present the flange loads and shear flows in the webs for a single load $R_3 = 1{,}000$ lb. in Fig. 12.

15. *Thermal Stresses.*—We determine next the stresses and distortion of the same structure due to a non-uniform temperature distribution. For this purpose we assume that the upper flange of spar 'a' has the temperature distribution shown in Fig. 15. The variation is taken to be linear between nodal points and hence is defined by the values at the nodal points. The column matrix $[\eta]$ of the initial strains at the beginning and end of each element of the top flange 'a' is:

$$[\eta] = \Big\{ \overset{l1}{\{\alpha\Theta_1\ \alpha\Theta_2\}}\ \overset{l2}{\{\alpha\Theta_2\ \alpha\Theta_3\}}\ \overset{l3}{\{\alpha\Theta_3\ \alpha\Theta_4\}}\ \overset{l4}{\{\alpha\Theta_4\ \alpha\Theta_5\}}\ \overset{l5}{\{\alpha\Theta_5\ \alpha\Theta_6\}}\ \overset{l6}{\{\alpha\Theta_6\ \alpha\Theta_7\}} \Big\} \quad \ldots \quad (81)$$

Following equations (46) and (48) of section 7 we require the generalised strains $\mathbf{H}$ corresponding to the linearly varying flange loads. These may be found from equation (24) which in the present case may be written:

$$\mathbf{H} = \mathbf{l}[\eta] \quad \ldots \quad (82)$$

where

$$\mathbf{l} = \begin{bmatrix} \mathbf{l}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{l}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{l}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{l}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{l}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{l}_6 \end{bmatrix}, \quad \ldots \quad (83)$$

in which, since l is the same in each bay, all $\mathbf{l}_i$'s are identical matrices given by equation (25). The matrix $\mathbf{l}$ is written out in full in Table 7.

The redundancies $\mathbf{Y}_\theta$ due to temperature may now be derived from equation (48):

$$\mathbf{Y}_\theta = -\mathbf{D}^{-1}\mathbf{b}_1'\{\mathbf{H}\ \mathbf{O}\} = -\mathbf{D}^{-1}\mathbf{b}_{1la}'\mathbf{H} \,. \quad \ldots \quad (84)$$

This last relation follows since the initial strains are only applied to the upper flange of spar 'a'. Using $\mathbf{b}_{1la}$ from Table 3 and taking $\alpha = 23 \times 10^{-6}$, we find (in lb in.):

$$\mathbf{Y}_\theta = \{-41{,}330\ -36{,}590\ -27{,}770\ -18{,}000\ -13{,}980\ -11{,}230\} \,. \quad (85)$$

The thermal stresses can now be determined from:

$$\mathbf{S}_\theta = \mathbf{b}_1\mathbf{Y}_\theta \quad \ldots \quad (86)$$

In particular, Fig. 15 shows the flange loads and web shear flows.

Of some interest is finally the thermal distortion of the structure. Thus, using the simple relation equation (51), the deflections $r_{\theta j}$ at the stations 1 to 12 are:

$$\mathbf{r}_\theta = \{r_{01}\ r_{02} \ldots r_{06} \ldots r_{012}\} = \mathbf{b}_{la}'\mathbf{H} \,, \quad \ldots \quad (87)$$

since the free thermal strains exist only in the upper flange 'a'. $\mathbf{b}_{la}$ is the true stress matrix corresponding to the unit loads R_1 to R_{12} and is given in Table 5. We find (in in.):

$$\mathbf{r}_\theta = \{ \overset{1}{-0\cdot019}\ \ \overset{2}{-0\cdot081}\ \ \overset{3}{-0\cdot191}\ \ \overset{4}{-0\cdot327}\ \ \overset{5}{-0\cdot492}\ \ \overset{6}{-0\cdot693}$$

$$\overset{7}{-0\cdot016}\ \ \overset{8}{-0\cdot071}\ \ \overset{9}{-0\cdot171}\ \ \overset{10}{-0\cdot308}\ \ \overset{11}{-0\cdot474}\ \ \overset{12}{-0\cdot661}\} \,. \quad \ldots \quad (88)$$

It will be seen that the complete determination of the thermal stresses and distortion is surprisingly simple.

16. *Effect of Cut-Outs.*—In this paragraph we give a number of applications of the method developed in sections 8 and 10 for cut-outs.

Three different cases of cut-outs are investigated (*see* Fig. 11 for notation):

(1) Elimination of web $w3$, *i.e.*, web of spar ' a ' in the third bay

(2) Elimination of two fields, web $w3$ and cover $s2$, and an additional cut of the flanges of spar ' a ' at the root

(3) Elimination of flange $l3$.

Two particular loading cases are considered:

(*a*) Single load $R_3 = 1{,}000$ lb

(*b*) Thermal loading of Fig. 15, but only for cut-out cases (1) and (2).

Inspection of formulae (62), (65) and (68*a*) shows that we require the submatrices $\mathbf{b}_{1h}$ and $\mathbf{b}_h$. These are easily extracted from the available matrices $\mathbf{b}_1$ and $\mathbf{b}$. Then we have to form:

$$\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}'$$

and invert it to find:

$$[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1} . \qquad (89)$$

The order of this matrix is 1×1, 3×3 and 2×2 for cut-outs (1), (2) and (3) respectively. All these data are collected in Table 8. By substitution of these matrices into equations (62) and (68*a*) we obtain the stress distributions for the cut-outs and loading cases considered. The results are given in Figs. 12, 13, 14 for the load $R_3 = 1{,}000$ lb and Figs. 15 and 16 for the thermal loading. For the single cut-out (1) we consider also the flexibility $\mathbf{F}_c$ of the cut-out structure. Its derivation is particularly simple since in this case:

$$[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1} = \frac{192^2}{3{\cdot}801 \times 10^6} , \qquad (90)$$

i.e., a scalar. Thus:

$$\mathbf{F}_c = \mathbf{F} + \Delta\mathbf{F} = \mathbf{F} + \frac{192^2}{3{\cdot}801 \times 10^6}\,\mathbf{b}_h'\mathbf{b}_h , \qquad (91)$$

where the elements of the 12×12 matrix $\mathbf{b}_h'\mathbf{b}_h$ are obtained merely by single multiplications. The incremental change $\Delta\mathbf{F}$ of the flexibility due to the cut-out is given in Table 6.

17. *Modified Structure.*—The last example to be analysed is one illustrating the applications of the method in section 11. The case investigated is that of the tube of section 13 but with the cross-sectional areas of flanges $l1$ and $l2$ firstly halved and secondly doubled.

The stress distribution in the two modified structures may now be derived from the original stress distribution using equation (71). To do this we have to find:

$$\left[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' + \frac{\beta}{\beta - 1}\,\mathbf{f}_h^{-1}\right]^{-1} \qquad (92)$$

which, in the present case, is a matrix of order 4×4. The factor β is $\frac{1}{2}$ and 2 when the cross-sectional areas are halved and doubled respectively, and Table 9 gives the corresponding two matrices (92). For a single load of $R_3 = 1{,}000$ lb the flange loads and web shear flows in the two modified structures are shown in Fig. 17 and can be compared with the stress distribution in the original structure.

In conclusion we draw attention to the extreme simplicity of the methods on cut-outs and modifications given in Ref. 1 and this paper. The reader may consult in this connection also Hunt[2,3].

TABLE 3

Basic Matrices

$\mathbf{b}_{0la} = \frac{20}{6}$

1	2	3	4	5	6	7	8	9	10	11	12	Element
−1	−2	−3	−4	−5	−6	0	0	0	0	0	0	1
0	−1	−2	−3	−4	−5	0	0	0	0	0	0	
0	−1	−2	−3	−4	−5	0	0	0	0	0	0	2
0	0	−1	−2	−3	−4	0	0	0	0	0	0	
0	0	−1	−2	−3	−4	0	0	0	0	0	0	3
0	0	0	−1	−2	−3	0	0	0	0	0	0	
0	0	0	−1	−2	−3	0	0	0	0	0	0	4
0	0	0	0	−1	−2	0	0	0	0	0	0	
0	0	0	0	−1	−2	0	0	0	0	0	0	5
0	0	0	0	0	−1	0	0	0	0	0	0	
0	0	0	0	0	−1	0	0	0	0	0	0	6
0	0	0	0	0	0	0	0	0	0	0	0	

$\mathbf{b}_{0wa} = \frac{1}{6}$

1	2	3	4	5	6	7	8	9	10	11	12	Element
1	1	1	1	1	1	0	0	0	0	0	0	1
0	1	1	1	1	1	0	0	0	0	0	0	2
0	0	1	1	1	1	0	0	0	0	0	0	3
0	0	0	1	1	1	0	0	0	0	0	0	4
0	0	0	0	1	1	0	0	0	0	0	0	5
0	0	0	0	0	1	0	0	0	0	0	0	6

$\mathbf{b}_{1la} = \frac{1}{6}$

1	2	3	4	5	6	Element
1	0	0	0	0	0	1
0	1	0	0	0	0	
0	1	0	0	0	0	2
0	0	1	0	0	0	
0	0	1	0	0	0	3
0	0	0	1	0	0	
0	0	0	1	0	0	4
0	0	0	0	1	0	
0	0	0	0	1	0	5
0	0	0	0	0	1	
0	0	0	0	0	1	6
0	0	0	0	0	0	

$\mathbf{b}_{1wa} = \frac{10}{6 \times 320}$

1	2	3	4	5	6	Element
−1	1	0	0	0	0	1
0	−1	1	0	0	0	2
0	0	−1	1	0	0	3
0	0	0	−1	1	0	4
0	0	0	0	−1	1	5
0	0	0	0	0	−1	6

Numbers against rows denote elements.

TABLE 4

Basic Matrices—continued

$$\mathbf{f}_b = \frac{2}{3 \times 10^6}$$

$l7$		$l8$		$l9$		$l10$		$l11$		$l12$	
2·00	1·00	0	0	0	0	0	0	0	0	0	0
1·00	2·00	0	0	0	0	0	0	0	0	0	0
0	0	3·33	1·67	0	0	0	0	0	0	0	0
0	0	1·67	3·33	0	0	0	0	0	0	0	0
0	0	0	0	5·00	2·50	0	0	0	0	0	0
0	0	0	0	2·50	5·00	0	0	0	0	0	0
0	0	0	0	0	0	5·00	2·50	0	0	0	0
0	0	0	0	0	0	2·50	5·00	0	0	0	0
0	0	0	0	0	0	0	0	5·00	2·50	0	0
0	0	0	0	0	0	0	0	2·50	5·00	0	0
0	0	0	0	0	0	0	0	0	0	5·00	2·50
0	0	0	0	0	0	0	0	0	0	2·50	5·00

$$\mathbf{D} = 10^{-8} \times$$

Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
16·800	− 8·334	2·083	0	0	0
− 8·334	40·126	−11·448	2·083	0	0
2·083	−11·448	44·133	−10·712	2·083	0
0	2·083	−10·712	47·892	−10·974	2·083
0	0	2·083	−10·974	52·081	−10·270
0	0	0	2·083	−10·270	57·240

$$\mathbf{D}^{-1} = 10^{6} \times$$

6·6384	1·3916	0·0347	− 0·0558	− 0·0132	− 0·0003
1·3916	2·9840	0·7156	0·0249	− 0·0244	− 0·0053
0·0347	0·7156	2·5828	0·5494	0·0088	− 0·0184
− 0·0558	− 0·0249	0·5494	2·3167	0·4661	− 0·0007
− 0·0132	− 0·0244	0·0088	0·4661	2·0885	0·3578
− 0·0003	− 0·0053	− 0·0184	− 0·0007	0·3578	1·8112

TABLE 5

True Stress Distribution : Typical Matrices

$\mathbf{b}_{ta} =$

	1	2	3	4	5	6	7	8	9	10	11	12	
	−1·447	−2·098	−2·726	−3·385	−4·071	−4·749	−0·433	−1·100	−1·797	−2·479	−3·156	−3·832	1
	0·398	−0·841	−1·658	−2·434	−3·268	−4·088	−0·096	−0·681	−1·482	−2·311	−3·134	−3·954	2
	0·011	0·430	−0·971	−1·966	−2·974	−3·994	−0·004	−0·164	−0·950	−1·964	−2·997	−4·024	3
	−0·016	−0·017	0·323	−1·080	−2·134	−3·182	0·004	−0·006	−0·213	−1·082	−2·158	−3·239	4
	−0·004	−0·021	−0·030	0·246	−1·066	−1·992	0·001	0·006	−0·006	−0·207	−1·038	−2·002	5
	−0·000	−0·003	−0·017	−0·038	0·266	−0·945	0·000	0·001	0·007	0·002	−0·126	−0·824	6
	0	0	0	0	0	0	0	0	0	0	0	0	7

Numbers opposite rows refer to rib stations. Duplicated rows for the two elements meeting there are merged into one.
Numbers above columns refer to loads R_1 to R_{12}.

$\mathbf{b}_{wa} = 10^{-3} \times$

120·15	101·78	95·87	92·22	87·62	83·13	10·52	13·08	9·82	5·24	0·69	− 3·81	1
− 12·11	102·21	83·99	77·11	71·69	65·44	2·88	16·15	16·63	10·86	4·28	− 2·18	2
− 0·83	− 13·97	102·93	90·20	88·74	87·88	0·22	4·95	23·05	27·55	26·20	24·52	3
0·38	− 0·14	− 11·03	103·94	95·87	89·35	− 0·08	0·36	6·47	27·35	35·02	38·66	4
0·12	0·57	0·40	− 8·87	104·11	99·70	− 0·03	− 0·14	0·38	6·54	28·49	36·82	5
0·00	0·10	0·53	1·18	− 8·30	95·20	− 0·00	− 0·04	− 0·21	− 0·07	3·93	25·74	6

Numbers opposite rows refer to web elements.

TABLE 6

Flexibility of Continuous Tube and Increment Due to Cut-Out Web w3

$\mathbf{F} = 10^{-6} \times$

1	2	3	4	5	6	7	8	9	10	11	12
21·54	27·62	35·67	41·42	48·36	55·32	4·43	11·27	18·42	25·43	32·38	39·32
27·62	73·68	103·4	132·9	164·0	194·8	10·82	36·50	67·23	98·25	129·0	159·7
35·67	103·4	192·4	265·9	342·3	419·1	17·30	65·22	136·1	212·8	289·4	365·9
41·42	132·9	265·9	428·3	576·0	722·2	23·71	94·30	210·9	355·1	504·3	653·1
48·36	164·0	342·3	576·0	843·6	1090·3	30·05	122·9	285·1	501·9	747·7	997·3
55·32	194·8	419·1	722·2	1090·3	1510·6	36·42	151·6	359·3	649·3	996·9	1373·0
4·43	10·82	17·30	23·71	30·05	36·42	10·81	17·21	23·53	29·88	36·25	42·61
11·27	36·50	65·22	94·30	122·9	151·6	17·21	49·40	78·73	107·4	136·1	164·9
18·42	67·23	136·1	210·9	285·1	359·3	23·53	78·73	157·7	232·1	306·2	380·4
25·43	98·25	212·8	355·1	501·9	649·3	29·88	107·4	232·1	383·8	530·6	677·5
32·38	129·0	289·4	504·3	747·7	996·9	36·25	136·1	306·2	530·6	788·1	1036·9
39·32	159·7	365·9	653·1	997·3	1373·0	42·61	164·9	380·4	677·5	1036·9	1424·7

$\Delta\mathbf{F} = 10^{-6} \times$

0·01	0·11	−0·83	−0·73	−0·71	−0·71	0·00	−0·04	−0·19	−0·22	−0·21	−0·20
0·11	1·89	−13·95	−12·22	−12·02	−11·91	−0·03	−0·67	−3·12	−3·73	−3·55	−3·32
−0·83	−13·95	102·8	90·04	88·59	87·73	0·22	4·94	23·01	27·50	26·15	24·48
−0·73	−12·22	90·04	78·91	77·63	76·88	0·19	4·33	20·16	24·10	22·92	21·45
−0·71	−12·02	88·59	77·63	76·37	75·63	0·19	4·26	19·84	23·71	22·55	21·10
−0·71	−11·91	87·73	76·88	75·63	74·90	0·19	4·22	19·66	23·48	22·33	20·90
0·00	−0·03	0·22	0·19	0·19	0·19	0·00	0·01	0·05	0·06	0·06	0·05
−0·04	−0·67	4·94	4·33	4·26	4·22	0·01	0·24	1·11	1·32	1·26	1·18
−0·19	−3·12	23·01	20·16	19·84	19·66	0·05	1·11	5·15	6·16	5·86	5·48
−0·22	−3·73	27·50	24·10	23·71	23·48	0·06	1·32	6·16	7·36	7·00	6·55
−0·21	−3·55	26·15	22·92	22·55	22·33	0·06	1·26	5·86	7·00	6·66	6·23
−0·20	−3·32	24·48	21·45	21·10	20·90	0·05	1·18	5·48	6·55	6·23	5·83

TABLE 7

$$\mathbf{l} = \frac{10}{3}\left[\begin{array}{cc|cc|cc|cc|cc|cc}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{array}\right]$$

TABLE 8

1. Cut-out web $w3$

$$\mathbf{b}_{1h} = \frac{1}{192}\begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' = 103\cdot 11 \qquad \left[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}'\right]^{-1} = 0\cdot 009698$$

2. Cut-out web $w3$, covers $s2$ and flange $l1$ at root

$$\mathbf{b}_{1h} = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{192} & \frac{1}{192} & 0 & 0 \\ 0 & \frac{1}{320} & \frac{1}{320} & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' = 10^3 \times \begin{bmatrix} 184\cdot 40 & 0\cdot 07856 & -0\cdot 7067 \\ 0\cdot 07856 & 0\cdot 10310 & 0\cdot 02185 \\ -0\cdot 7067 & 0\cdot 02185 & 0\cdot 04039 \end{bmatrix}$$

$$[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1} = 10^{-3} \times \begin{bmatrix} 0\cdot 005893 & -0\cdot 02979 & 0\cdot 1192 \\ -0\cdot 02979 & 11\cdot 1063 & -6\cdot 5305 \\ 0\cdot 1192 & -6\cdot 5305 & 30\cdot 380 \end{bmatrix}$$

3. Cut-out flange element $l3$

$$\mathbf{b}_{1h} = \frac{1}{6}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' = 10^4 \times \begin{bmatrix} 7\cdot 1744 & 1\cdot 5261 \\ 1\cdot 5261 & 6\cdot 4353 \end{bmatrix} \qquad [\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}']^{-1} = 10^{-4} \times \begin{bmatrix} 0\cdot 1468 & -0\cdot 0348 \\ -0\cdot 0348 & 0\cdot 1636 \end{bmatrix}$$

TABLE 9

Flange Elements l1, l2 Modified

$$\mathbf{b}_{1h} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' = 10^6 \times \begin{bmatrix} 0\cdot18440 & 0\cdot03865 & 0\cdot03865 & 0\cdot00096 \\ 0\cdot03865 & 0\cdot08289 & 0\cdot08289 & 0\cdot01987 \\ 0\cdot03865 & 0\cdot08289 & 0\cdot08289 & 0\cdot01987 \\ 0\cdot00096 & 0\cdot01987 & 0\cdot01987 & 0\cdot07174 \end{bmatrix}$$

$$\mathbf{f}_h^{-1} = 10^6 \times \begin{bmatrix} 0\cdot650 & -0\cdot325 & 0 & 0 \\ -0\cdot325 & 0\cdot650 & 0 & 0 \\ 0 & 0 & 0\cdot650 & -0\cdot325 \\ 0 & 0 & -0\cdot325 & 0\cdot650 \end{bmatrix}$$

1. $\beta = 1/2$. (Flange areas $l1$, $l2$ halved), $\beta/(\beta - 1) = -1$

$$[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' + \mathbf{f}_h^{-1}]^{-1} = 10^{-6} \times \begin{bmatrix} 1\cdot4025 & 0\cdot5705 & -0\cdot1769 & -0\cdot0924 \\ 0\cdot5705 & 1\cdot6237 & -0\cdot2823 & -0\cdot1648 \\ -0\cdot1769 & -0\cdot2823 & 1\cdot7100 & 0\cdot7309 \\ -0\cdot0924 & -0\cdot1648 & 0\cdot7309 & 1\cdot6992 \end{bmatrix}$$

2. $\beta = 2$. (Flange areas $l1$, $l2$ doubled), $\beta/(\beta - 1) = 2$

$$[\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}' - 2\mathbf{f}_h^{-1}]^{-1} = 10^{-6} \times \begin{bmatrix} -1\cdot3971 & -0\cdot8025 & -0\cdot1525 & -0\cdot0973 \\ -0\cdot8024 & -1\cdot2898 & -0\cdot1788 & -0\cdot1190 \\ -0\cdot1525 & -0\cdot1788 & -1\cdot2007 & -0\cdot6578 \\ -0\cdot0973 & -0\cdot1190 & -0\cdot6578 & -1\cdot1749 \end{bmatrix}$$

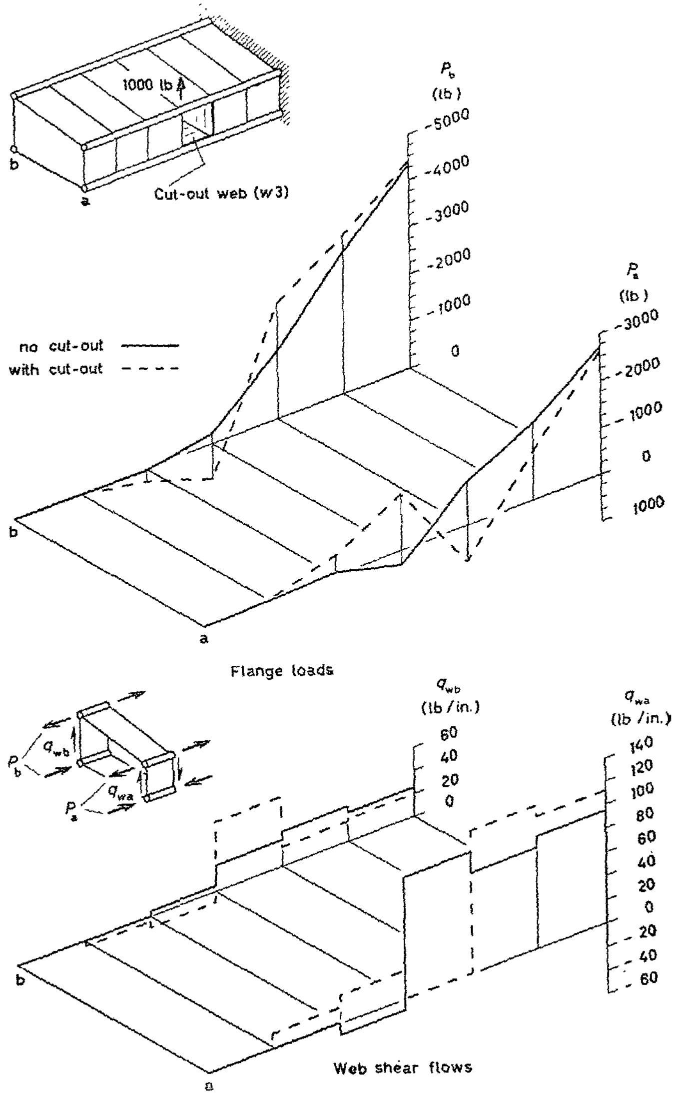

FIG. 12. Stress distribution in four-flange tube due to a single force, $R_3 = 1{,}000$ lb, with and without cut-out web $w3$.

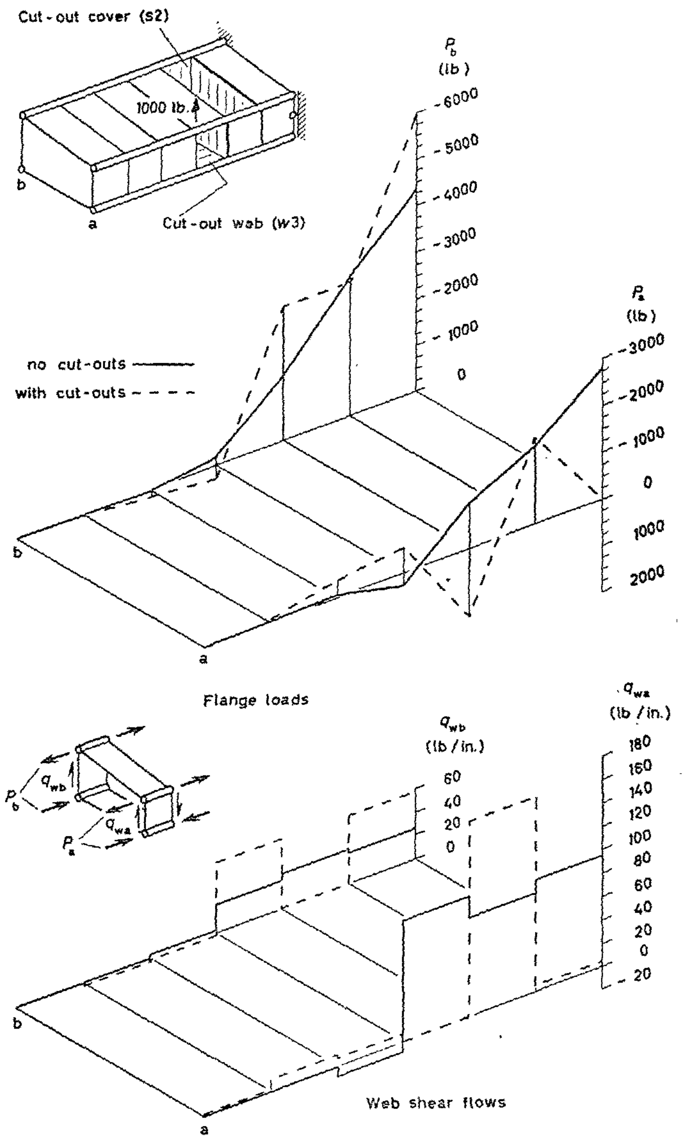

FIG. 13. Stress distribution in four-flange tube due to a single force, $R_3 = 1{,}000$ lb with and without cut-out web $w3$, cover $s2$ and flange $l1$ at root.

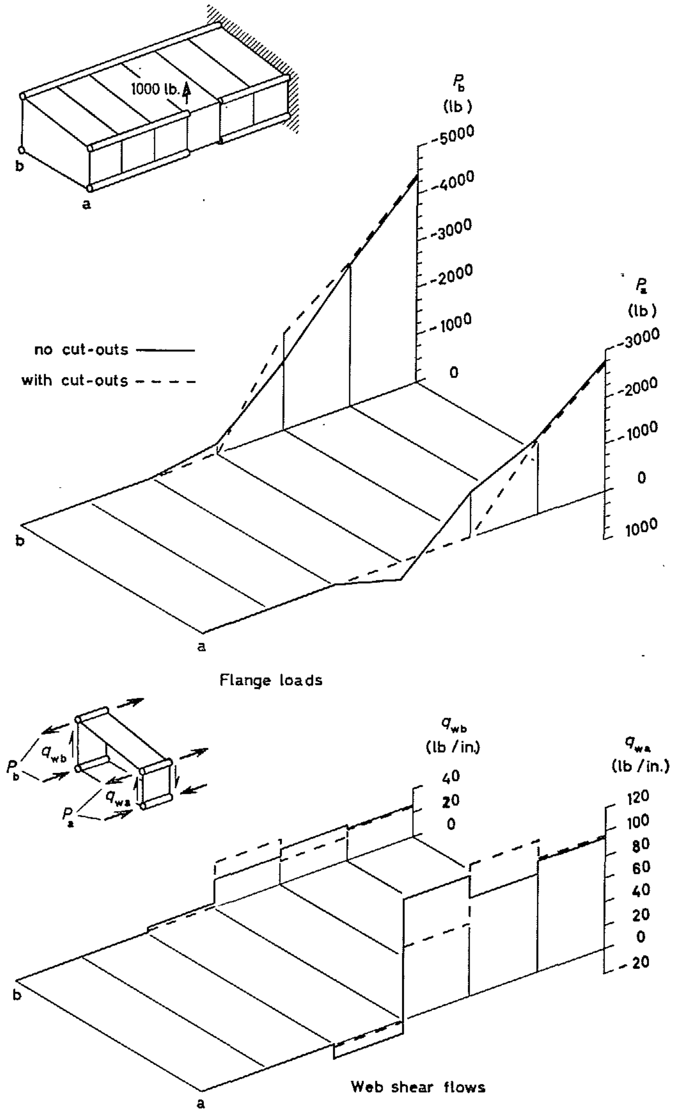

FIG. 14. Stress distribution in four-flange tube due to a single force, $R_3 = 1{,}000$ lb, with and without cut-out flange elements $l3$.

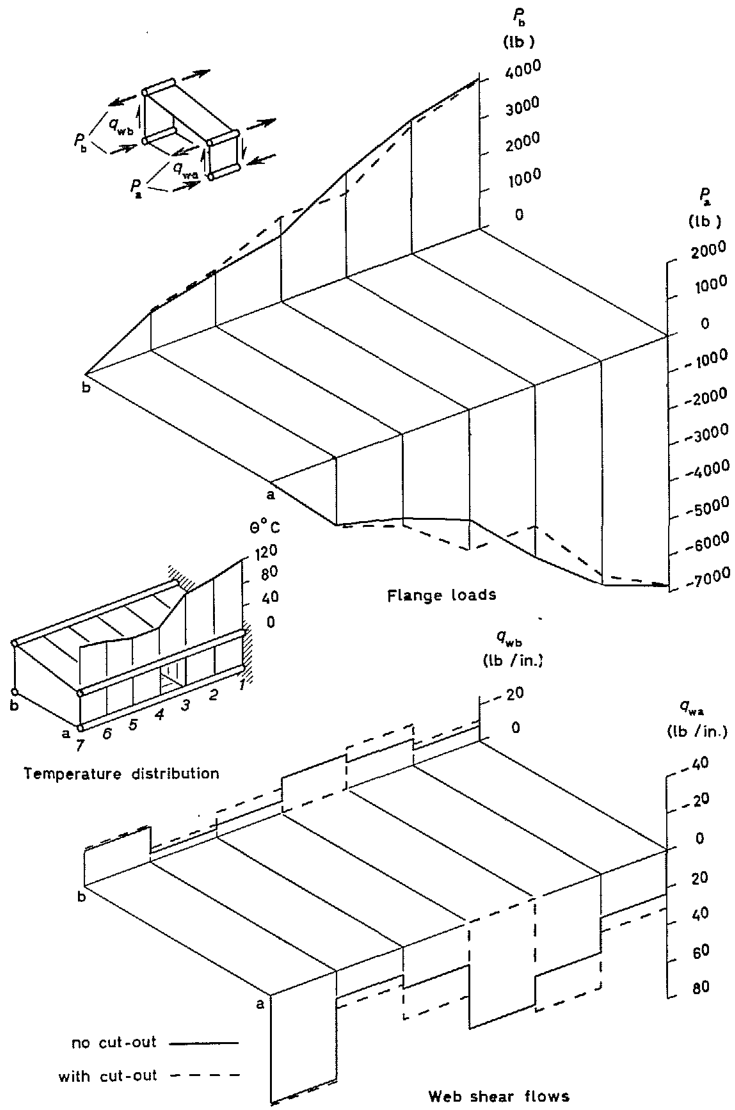

FIG. 15. Stress distribution in four-flange tube due to thermal loading with and without cut-out web $w3$.

(68577) Wt. 20/9036 K.7 10/57 Hw.

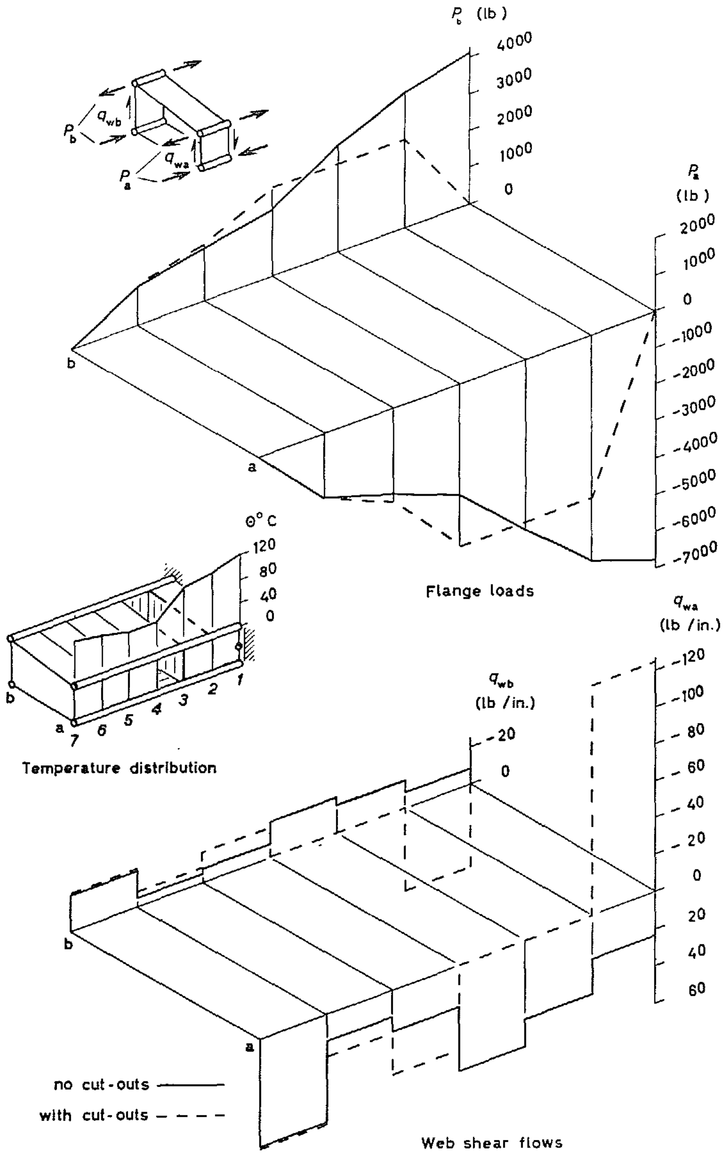

Fig. 16. Stress distribution in four-flange tube due to thermal loading with and without cut-out web $w3$, cover $s2$ and flange $l1$ at root.

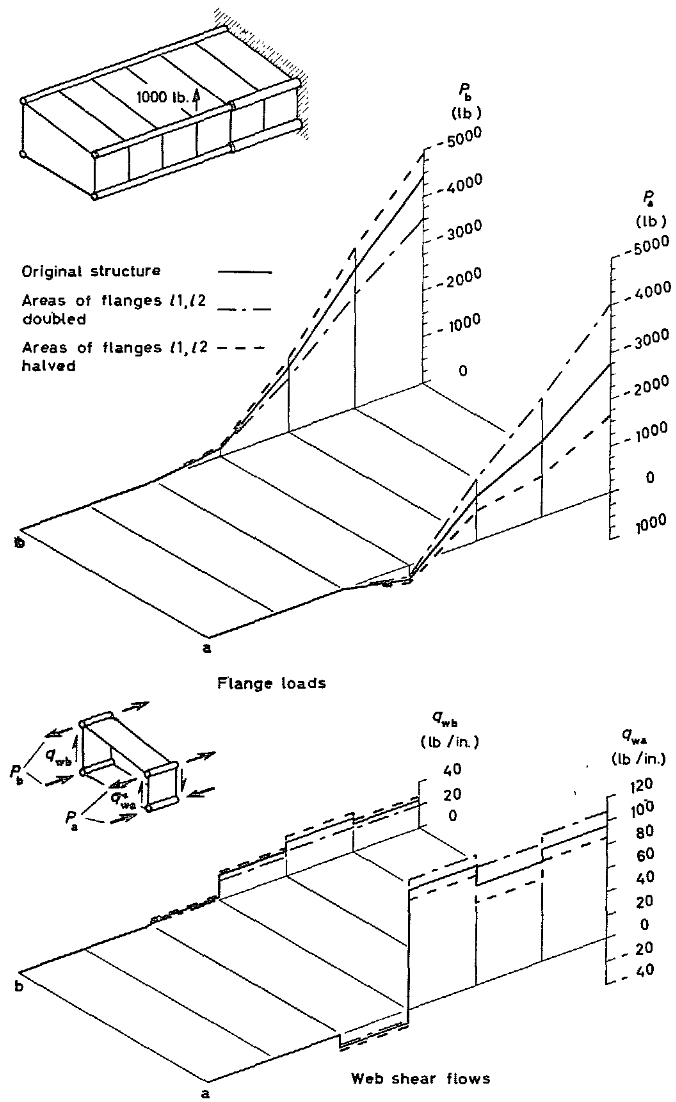

Fig. 17. Stress distribution in four-flange tube due to a single force $R_3 = 1{,}000$ lb, with areas of flange elements $l1$, $l2$, doubled and halved.

PRINTED IN GREAT BRITAIN

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50*s.* (51*s.* 9*d.*).
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63*s.* (64*s.* 9*d.*)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control Structures, and a miscellaneous section. 50*s.* (51*s.* 9*d.*)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control Structures. 63*s.* (64*s.* 9*d.*)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75*s.* (76*s.* 9*d.*)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47*s.* 6*d.* (49*s.* 3*d.*)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80*s.* (81*s.* 9*d.*)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90*s.* (92*s.* 6*d.*)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84*s.* (86*s.* 3*d.*)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84*s.* (86*s.* 3*d.*)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130*s.* (132*s.* 6*d.*)
Vol. II. Aircraft, Airscrews, Controls. 130*s.* (132*s.* 6*d.*)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130*s.* (132*s.* 3*d.*)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130*s.* (132*s.* 3*d.*)

Annual Reports of the Aeronautical Research Council—

1937 2*s.* (2*s.* 2*d.*) 1938 1*s.* 6*d.* (1*s.* 8*d.*) 1939–48 3*s.* (3*s.* 3*d.*)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - R. & M. 2600 2*s.* 6*d.* (2*s.* 8*d.*)

Author Index to all Reports and Memoranda of the Aeronautical Research Council—

1909—January, 1954 R. & M. No. 2570 15*s.* (15*s.* 6*d.*)

Indexes to the Technical Reports of the Aeronautical Research Council—

December 1, 1936—June 30, 1939	R. & M. No. 1850	1*s.* 3*d.* (1*s.* 5*d.*)
July 1, 1939—June 30, 1945	R. & M. No. 1950	1*s.* (1*s.* 2*d.*)
July 1, 1945—June 30, 1946	R. & M. No. 2050	1*s.* (1*s.* 2*d.*)
July 1, 1946—December 31, 1946	R. & M. No. 2150	1*s.* 3*d.* (1*s.* 5*d.*)
January 1, 1947—June 30, 1947	R. & M. No. 2250	1*s.* 3*d.* (1*s.* 5*d.*)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2251–2349	R. & M. No. 2350	1*s.* 9*d.* (1*s.* 11*d.*)
Between Nos. 2351–2449	R. & M. No. 2450	2*s.* (2*s.* 2*d.*)
Between Nos. 2451–2549	R. & M. No. 2550	2*s.* 6*d.* (2*s.* 8*d.*)
Between Nos. 2551–2649	R. & M. No. 2650	2*s.* 6*d.* (2*s.* 8*d.*)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London, W.C.2; 423 Oxford Street, London, W.1; 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol 1; 80 Chichester Street, Belfast, *or through any bookseller.*

S.O. Code No. 23-3034

R. & M. No. 3034