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The Torsional Rigidity of Solid Cylinders of Double-wedge Section

By

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COMMUNICATED BY THE PRINCIPAL DIRECTOR OF SCIENTIFIC RESEARCH (AIR), MINISTRY OF SUPPLY

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Summary.—The torsional rigidity of solid cylinders of double-wedge section is considered theoretically. Minimum energy methods are used to determine close upper and lower limits to the rigidity. The results are presented in graphical form.

1. *Introduction.*—In this report the torsional rigidity of solid cylinders of double-wedge section is considered theoretically. A lower limit for the rigidity has been obtained in a manner similar to that used by Duncan[1]; a parabolic variation of the stress function across the thickness is assumed and the Ritz[2] method is then used in conjunction with a variational technique to determine the rigidity. An upper limit has been obtained from the static analogue of Kelvin's theorem[3]; a linear variation of the warping function across the thickness is assumed and a variational technique then used to determine the rigidity.†

2. *List of symbols* (*see* Fig. 1).

Structure properties

C		Torsional rigidity
G		Shear modulus
t		Maximum thickness of section
c		Chord of section
λ		Fraction of chord at which maximum thickness occurs
m	$=$	t/c ratio

Non-dimensional parameters

$$m_1 = \frac{m}{2\lambda}$$

$$m_2 = \frac{m}{2(1-\lambda)}$$

$$r_1 = -4m_1 + \sqrt{(10 + 6m_1^2)}$$

$$r_2 = -4m_2 + \sqrt{(10 + 6m_2^2)}$$

$$p_1 = -m_1 + \sqrt{(3 + m_1^2)}$$

$$p_2 = -m_2 + \sqrt{(3 + m_2^2)}$$

* R.A.E. Report Structures 163, received 31st May, 1954.

† Since this paper was completed the author's attention was drawn to a similar paper by J. H. Argyris and S. Kelsey in *Aircraft Engineering*, December 1954.

$$\alpha = \frac{\dfrac{r_1 m_1^2}{1 - m_1^2} + \dfrac{r_2 m_2^2}{1 - m_2^2} - 5(m_1 + m_2)}{r_1 + r_2 + 5(m_1 + m_2)}$$

$$\beta = \frac{m_2(1 + 3m_1^2)}{(1 - m_1^2)^2} + \frac{m_1(1 + 3m_2^2)}{(1 - m_2^2)^2}$$

$$B_1 = \frac{(m_1 + m_2)\{m_1 - m_2 + p_1(1 - m_1 m_2)\}}{(p_1 + p_2)(1 - m_1^2)(1 - m_2^2)}$$

$$B_2 = \frac{(m_1 + m_2)\{m_2 - m_1 + p_1(1 - m_1 m_2)\}}{(p_1 + p_2)(1 - m_1^2)(1 - m_2^2)} .$$

3. *Lower and Upper Limits for the Torsional Rigidity.*—A lower limit for the rigidity has been found in Appendix I on the assumption that the stress function varies parabolically across the thickness; the rigidity is then determined by the Ritz method and a variational technique. An upper limit for the rigidity has been found in Appendix II on the assumption that the warping function varies linearly across the thickness; the rigidity is then determined from the static analogue of Kelvin's theorem and a variational technique. It follows that the torsional rigidity satisfies the inequality:

$$C_{\text{lower}} < C < C_{\text{upper}} \quad \ldots \quad (1)$$

where

$$C_{\text{lower}} = \frac{Gct^3}{12}\left[\left(\frac{\lambda}{1 - m_1^2}\right)\left\{1 + \frac{\alpha - (1 + \alpha)m_1^2}{1 + \dfrac{r_1}{8m_1}}\right\} + \left(\frac{1 - \lambda}{1 - m_2^2}\right)\left\{1 + \frac{\alpha - (1 + \alpha)m_2^2}{1 + \dfrac{r_2}{8m_2}}\right\}\right] \quad (2)$$

and

$$C_{\text{upper}} = \frac{Gct^3}{12}\left(\frac{1}{m_1 + m_2}\right)\left[\beta + 4m_1 m_2(p_1 B_1^2 + p_2 B_2^2) - 8m_1 m_2\left(\frac{B_1}{1 - m_1^2} + \frac{B_2}{1 - m_2^2}\right)\right]. \quad (3)$$

These limits have been plotted in Fig. 2 for various values of λ up to $t/c = 0 \cdot 3$. It will be seen that over the range considered the limits are close; the maximum error that can arise by taking the mean of the two limits is less than $1 \cdot 6$ per cent.

Equations (2) and (3) may be simplified for the special cases in which the section becomes a diamond or a triangle.

3.1. *Special Case : Diamond Section* ($\lambda = 0 \cdot 5$).—For a diamond section equations (2) and (3) reduce to

$$C_{\text{lower}} = \frac{Gct^3}{12}\left[\frac{2 - 9m^2(1 + m^2) + 4m^3\sqrt{(10 + 6m^2)}}{(2 + m^2)(1 - m^2)^2}\right] \quad \ldots \quad (4)$$

and

$$C_{\text{upper}} = \frac{Gct^3}{12}\left[\frac{1 - 5m^2 - 4m^4 + 4m^3\sqrt{(3 + m^2)}}{(1 - m^2)^2}\right]. \quad \ldots \quad (5)$$

3.2. *Special Case : Triangular Section* ($\lambda = 0$ *or* 1).—For a triangular section equations (2) and (3) reduce to

$$C_{\text{lower}} = \frac{Gct^3}{12}\left[\frac{4\{20 + 11m^2 - 4m\sqrt{(40 + 6m^2)}\}}{5(4 - m^2)^2}\right] \quad \ldots \quad (6)$$

and

$$C_{\text{upper}} = \frac{Gct^3}{12}\left[\frac{4\{12 + 5m^2 - 4m\sqrt{(12 + m^2)}\}}{3(4 - m^2)^2}\right] \ldots \quad (7)$$

4. *Discussion of Results.*—It will be seen from Fig. 2 that when the maximum thickness is near the mid-chord (*i.e.*, $\lambda = 0 \cdot 5$) the torsional rigidity is practically independent of λ, which is to be expected from considerations of symmetry. For a cylinder for which $t/c < 0 \cdot 05$ and $0 \cdot 2 < \lambda < 0 \cdot 8$ the torsional rigidity is approximately $Gct^3/12$ which, for materials in which $\nu = \frac{1}{4}$, is $1 \cdot 5$ times the flexural rigidity.

For a given t/c ratio the lower and upper limits are closest when the section is a diamond and are furthest apart when the section is a triangle. If $t/c = 1$ and $\lambda = 0 \cdot 5$ (corresponding to the limiting case of a square) and lower and upper limits are each in error by $3 \cdot 6$ per cent, and if $t/c = 2/\sqrt{3}$ and $\lambda = 0$ (corresponding to the limiting case of an equilateral triangle) the lower limit is correct and the upper limit in error by $12 \cdot 6$ per cent.

5. *Conclusions.*—The torsional rigidity of solid cylinders of double-wedge section has been considered theoretically. Minimum energy methods have been used to determine close upper and lower limits to the rigidity. The variation of the torsional rigidity with the t/c ratio and with the position at which the maximum thickness occurs has been investigated and the results presented in graphical form.

REFERENCES

No.	*Author*	*Title, etc.*
1	W. J. Duncan	*Phil. Mag.* Vol. 16. 1933.
2	W. Ritz	*Jour. Reine Angew. Math.* Vol. 135. 1908.
3	D. Williams	The use of the principle of minimum potential energy in problems of static equilibrium. R. & M. 1827. January, 1938.
4	S. Timoshenko	*Theory of Elasticity.* McGraw-Hill Book Co. 1934.

Additional Symbols used only in Appendices (*see* Fig. 1)

Ox, Oy; O_1x_1, O_1y_1; O_2x_2, O_2y_2	Cartesian co-ordinates
$\bar{x}$, $\bar{y}$	Co-ordinates of centre of twist
ϕ	Torsion stress function
w	Warping stress function
c_1	$= \lambda c$
c_2	$= (1 - \lambda)c$
K, H	Surface integrals
f_1, g_1	Functions of x_1
f_2, g_2	Functions of x_2

APPENDIX I

Calculation of Lower Limit

In the Ritz method a form for the stress function ϕ is chosen that vanishes on the boundary of the section and which may contain a number of arbitrary parameters. For unit twist per unit length the closest approximation to the stress function is that for which the surface integral

$$K = \iint_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 - 4G\phi\right] dA \qquad (8)$$

is a minimum. When ϕ satisfies this condition we have

$$C_{\text{lower}} = 2\iint_A \phi \, dA \qquad (9)$$

The Ritz method will now be used in conjunction with a variational technique in a manner similar to that used by Duncan[1]. The double-wedge section and the position of the origin and axes are shown in Fig. 1. In considering the region O_1BB′ it is convenient to have the origin at O_1, and similarly at O_2 for the region O_2BB′. A parabolic variation of the stress function across the thickness of the section is assumed, so that in the region O_1BB′

$$\phi = (m_1^2 x_1^2 - y_1^2)Gf_1 \qquad (10)$$

and in the region O_2BB′

$$\phi = (m_2^2 x_2^2 - y_2^2)Gf_2 \qquad (11)$$

In the above equations f_1 and f_2 are functions of x_1 and x_2 and they will be chosen to make the surface integral K a minimum.

Substituting equations (10) and (11) in equation (8) and integrating with respect to y across the thickness gives K as the sum of two integrals of x_1, f_1, f_1' and x_2, f_2, f_2'. Variations δf_1 in f_1 and δf_2 in f_2 will give rise to a variation δK, and for K to be a minimum δK must vanish, whence

$$\begin{aligned}\delta K = {} & \frac{16}{15} m_1^3 \int_0^{c_1} x_1^3 \{5(1 - m_1^2)f_1 - 10m_1^2 x_1 f_1' - 2m_1^2 x_1^2 f_1'' - 5\} \, \delta f_1 \, dx_1 \\ & + \frac{16}{15} m_2^3 \int_0^{c_2} x_2^3 \{5(1 - m_2^2)f_2 - 10m_2^2 x_2 f_2' - 2m_2^2 x_2^2 f_2'' - 5\} \, \delta f_2 \, dx_2 \\ & + \frac{t^4}{15} [\{5m_1 f_1(c_1) + t f_1'(c_1)\} \, \delta f_1(c_1) + \{5m_2 f_2(c_2) + t f_2'(c_2)\} \, \delta f_2(c_2)] = 0 \, . \end{aligned} \qquad (12)$$

The variations δf_1 and δf_2 are quite arbitrary provided there is continuity at BB′. *i.e.*,

$$\left.\begin{aligned} f_1(c_1) &= f_2(c_2) \\ \delta f_1(c_1) &= \delta f_2(c_2) \end{aligned}\right\} \qquad (13)$$

and the expressions under the integral signs in equation (12) must therefore vanish. Similarly the expression in square brackets in equation (12) must vanish subject to condition (13). The solution of these equations is:

$$\left.\begin{aligned} f_1 &= \frac{1}{1 - m_1^2} + \left(\alpha - \frac{m_1^2}{1 - m_1^2}\right)\left(\frac{x_1}{c_1}\right)^{r_1/2m_1} \\ f_2 &= \frac{1}{1 - m_2^2} + \left(\alpha - \frac{m_2^2}{1 - m_2^2}\right)\left(\frac{x_2}{c_2}\right)^{r_2/2m_2} \end{aligned}\right\} \qquad (14)$$

Substitution of equation (14) in equations (9), (10) and (11) and integrating gives

$$C_{\text{lower}} = \frac{Gct^3}{12}\left[\left(\frac{\lambda}{1-m_1^2}\right)\left\{1 + \frac{\alpha - (1+\alpha)m_1^2}{1 + \dfrac{r_1}{8m_1}}\right\} + \left(\frac{1-\lambda}{1-m_2^2}\right)\left\{1 + \frac{\alpha - (1+\alpha)m_2^2}{1 + \dfrac{r_2}{8m_2}}\right\}\right]. \quad (15)$$

APPENDIX II

Calculation of Upper Limit

The method for obtaining an upper limit is based on the static analogue of Kelvin's theorem:—'The strain energy of a structure corresponding to a given deformation is less than if the freedom had been limited by the introduction of constraints'. The given deformation is assumed to be a unit twist per unit length and the internal constraints are those necessary to impose a chosen warping w of the cross-section. The position of the centre of twist is arbitrary since it may be altered by a rigid body movement[4], but if it is chosen to be at the point $(\bar{x},\bar{y})$ the strain energy per unit length of cylinder[4] is proportional to

$$H = \iint_A \left[\left(\frac{\partial w}{\partial x} - y + \bar{y}\right)^2 + \left(\frac{\partial w}{\partial y} + x - \bar{x}\right)^2\right] dA \quad \ldots \quad (16)$$

and the closest approximation to the warping function is that for which H is a minimum. When H satisfies this condition we have

$$C_{\text{upper}} = GH \quad \ldots \quad (17)$$

The steps in the analysis are similar to those used in calculating the lower limit. It is convenient to let the section twist about the centre C, but in considering the region O_1BB$'$ it is convenient to have the origin at O_1, and similarly at O_2 for the region O_2BB$'$. A linear variation of the warping function across the thickness of the section is assumed, so that in the region O_1BB$'$

$$w = y_1 g_1 \quad \ldots \quad (18)$$

and in the region O_2BB$'$

$$w = y_2 g_2 \quad \ldots \quad (19)$$

In the above equations g_1 and g_2 are functions of x_1 and x_2 and they will be chosen to make the surface integral H a minimum.

With the origins at O_1 and O_2 for the two parts of the double-wedge, equation (16) becomes

$$\begin{aligned} H &= \int_0^{c_1}\int_{-m_1x_1}^{m_1x_1} [y_1^2(g_1' - 1)^2 + (g_1 + x_1 - c_1)^2]\, dx_1\, dy_1 \\ &\quad + \int_0^{c_2}\int_{-m_2x_2}^{m_2x_2} [y_2^2(g_2' - 1)^2 + (g_2 + x_2 - c_2)^2]\, dx_2\, dy_2 \\ &= \frac{2m_1}{3}\int_0^{c_1} [m_1^2x_1^3(g_1' - 1)^2 + 3x_1(g_1 + x_1 - c_1)^2]\, dx_1 \\ &\quad + \frac{2m_2}{3}\int_0^{c_2} [m_2^2x_2^3(g_2' - 1)^2 + 3x_2(g_2 + x_2 - c_2)^2]\, dx_2 \quad \ldots \quad (20) \end{aligned}$$

on integrating with respect to y_1 and y_2.

Variations δg_1 in g_1 and δg_2 in g_2 will give rise to a variation δH, and for H to be a minimum δH must vanish, whence

$$\delta H = \frac{4m_1}{3}\int_0^{c_1}[3x_1^2 - 3x_1c_1 + 3x_1g_1 - m_1^2x_1^2\{3(g_1' - 1) + x_1g_1''\}]\,\delta g_1\,dx_1$$

$$+ \frac{4m_2}{3}\int_0^{c_2}[3x_2^2 - 3x_2c_2 + 3x_2g_2 - m_2^2x_2^2\{3(g_2' - 1) + x_2g_2''\}]\,\delta g_2\,dx_2$$

$$+ \frac{t^3}{6}[\{g_1'(c_1) - 1\}\,\delta g_1(c_1) + \{g_2'(c_2) - 1\}\,\delta g_2(c_2)]\,. \qquad (21)$$

The variations δg_1 and δg_2 are quite arbitrary, apart from continuity at BB′, so that each of the expressions in square brackets under the integral signs in equation (21) vanish. The last expression in square brackets in equation (21) will vanish provided there is continuity at BB′, *i.e.*, provided

$$\left.\begin{aligned} g_1(c_1) &= -\,g_2(c_2) \\ \delta g_1(c_1) &= -\,\delta g_2(c_2) \end{aligned}\right\} \qquad (22)$$

The minus signs in equation (22) are because of the reversed directions of y_1 and y_2.

The solution of these equations is

$$\frac{g_1}{c_1} = 1 - \left(\frac{1 + m_1^2}{1 - m_1^2}\right)\frac{x_1}{c_1} + 2m_1B_1\left(\frac{x_1}{c_1}\right)^{p_1/m_1} \qquad (23)$$

and

$$-\frac{g_2}{c_2} = 1 - \left(\frac{1 + m_2^2}{1 - m_2^2}\right)\frac{x_2}{c_2} + 2m_2B_2\left(\frac{x_2}{c_2}\right)^{p_2/m_2} \qquad (24)$$

where

$$B_1 = \frac{(m_1 + m_2)\{m_1 - m_2 + p_2(1 - m_1m_2)\}}{(p_1 + p_2)(1 - m_1^2)(1 - m_2^2)}$$

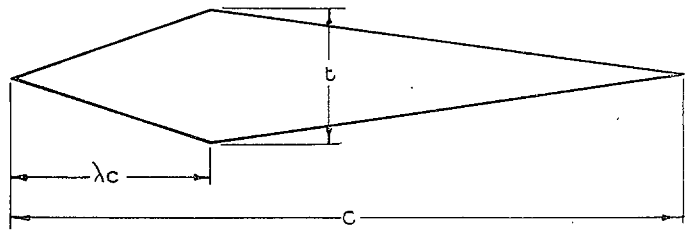
and

$$B_2 = \frac{(m_1 + m_2)\{m_2 - m_1 + p_1(1 - m_1m_2)\}}{(p_1 + p_2)(1 - m_1^2)(1 - m_2^2)}\,.$$

Substitution of equations (23) and (24) in equations (16) and (17) and integrating gives

$$C_{\text{upper}} = \frac{Gct^3}{12}\left(\frac{1}{m_1 + m_2}\right)\Bigg[\frac{m_2(1 + 3m_1^2)}{(1 - m_1^2)^2} + \frac{m_1(1 + 3m_2^2)}{(1 - m_2^2)^2}$$

$$+ 4m_1m_2(p_1B_1^2 + p_2B_2^2) - 8m_1m_2\left(\frac{B_1}{1 - m_1^2} + \frac{B_2}{1 - m_2^2}\right)\Bigg]\,. \qquad (25)$$



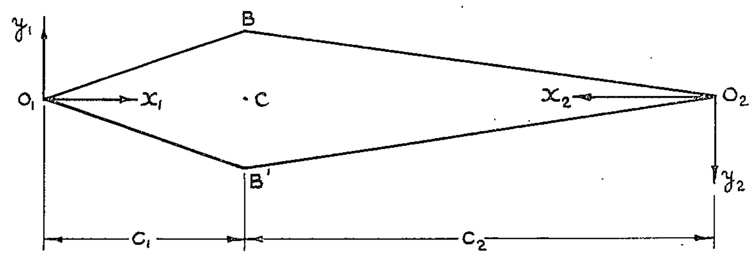


FIG. 1. Figure showing notation.

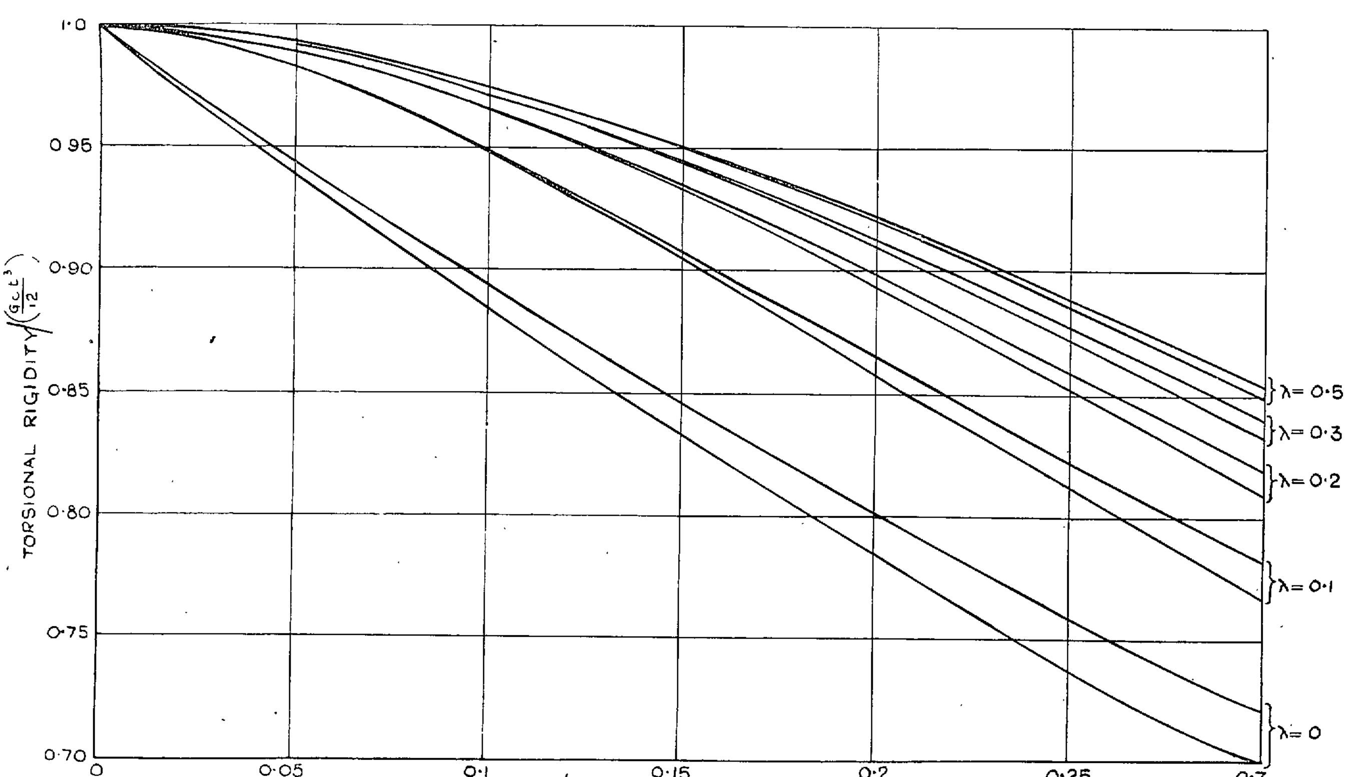


FIG. 2. The torsional rigidity of solid cylinders of double-wedge section.

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