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Note on the Circulation in Circuits which Cut the Streamlines in the Wake of an Aerofoil at Right-angles

By

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1. *Introduction.*—G. I. Taylor[1] in an Appendix to R. & M. 989 (1924) suggested that, in the two-dimensional flow of a real fluid, the circulations in all circuits enclosing the aerofoil and cutting the streamlines in the wake at right-angles would be very nearly the same. The present writer[2] in R. & M. 1996 (1943) gave a 'proof' that the circulations in such circuits were all equal, and Temple[3] (1943) gave a more rigorous proof of the same theorem This theorem is of fundamental importance in the calculations of the lift of aerofoils allowing for the boundary layer (*see* Preston[4] R. & M. 2725, 1949, and Spence[5], 1954) and it is re-examined in this note.

The theory is developed for convenience and simplicity, for an aerofoil with a jet issuing from the trailing edge and the effect of the wake is deduced from this. Elementary considerations, which are set out below, suggest that, in the case of an aerofoil with jet, the above theorem is not true. It would also appear that it is not quite true for an aerofoil with wake, since the same arguments can be applied. However, in this case the departure of the circulation from a constant value for circuits of the type under consideration may be expected to be small, and the effect of this on the prediction of the lift should be negligible for incidences below the stall. In the case of the aerofoil with a strong jet, the existence of circulation in circuits not enclosing the aerofoil but cutting the jet twice at right-angles may have important effects on the lift.

2. *Notation.*—

r	Radius of curvature of a streamline
q	Resultant velocity
ρ	Density
p	Pressure
	Subscripts $_1$, $_2$ refer to the upper and lower edges of the jet
	Subscript $_0$ denotes average values of quantities with subscripts $_1$ and $_2$
	Dashes refer to conditions in the jet

θ		Angle turned through by jet—measured from the downstream direction of flow at infinity
Γ		Circulation—positive in the same sense as θ
Γ_A		Circulation in circuit about aerofoil which touches trailing edge and cuts the jet or wake at right-angles
Γ_w		Circulation in a circuit which cuts the jet or wake at right-angles at the trailing edge and at infinity
Γ_∞		Circulation in circuit enclosing aerofoil and cutting the jet or wake at right-angles at infinity
H		Total pressure
χ	=	$\int \frac{dp}{\rho} + \frac{1}{2}q^2$
δ	=	$(r_2 - r_1)$. Width of jet
s		Distance measured along centre of jet
V		Velocity at infinity
L		Lift

3. *Inviscid Flow Past Aerofoil Plus Jet.*—It is assumed that the motion is steady, inviscid and incompressible so that no mixing between the jet and the outside stream takes place. The density of the jet fluid may differ from that of the surrounding fluid but is constant. The motion in the jet may be rotational though for simplification it is taken as irrotational. It is also assumed that the jet width is small compared with the radii of curvature of the streamlines*.

Consider a small section of the jet and surrounding fluid contained between two normals to the streamlines. When the normals are close together the streamlines may be taken as circles and the normals as radials $d\theta$ apart (*see* Fig. 1). Let r_1, r_2 be the radii of curvature of the jet edges, q_1, q_2 the velocities on the *outside* edges of the jet and q_1', q_2' the velocities at the *inside* edges of the jet. (Dashed symbols generally refer to conditions inside the jet.)

Then, since the motion is in circular paths, the centrifugal force on a fluid element is balanced by a pressure gradient along a radial.

Inside the jet $(r_2 < r < r_1)$

$$\frac{\rho' q'^2}{r} = \frac{\partial p'}{\partial r} \qquad (1)$$

Outside the jet $(r > r_2\,;\, r < r_1)$

$$\frac{\rho q^2}{r} = \frac{\partial p}{\partial r} \qquad (2)$$

The motion outside the jet is irrotational and hence

$$\left.\begin{aligned} qr &= K_1,\ (r < r_1) \\ qr &= K_2,\ (r > r_2) \end{aligned}\right\} \qquad (3)$$

*This assumption is not necessary to the argument and modifications to the theory can readily be made at the expense of complexity.

where K_1 and K_2 are constants, not necessarily equal. Substitute (3) in (2) and integrate to obtain

$$p + \tfrac{1}{2}q^2 = \text{const} = H \qquad (4)$$

outside the jet everywhere. If the motion inside the jet is irrotational a similar equation to (4) will apply with H replaced by H'—the total pressure in the jet.

3.1. *Steady Motion.*—Fig. 2 shows an aerofoil with jet and a circuit is drawn in any manner enclosing the aerofoil. Let the circulation in it be Γ. Then since the motion is steady

$$\frac{d\Gamma}{dt} = 0\,. \qquad (5)$$

Now Γ is equal to the sum of the strengths of all the vortices in the circuit at a given instant and hence, if (5) is to hold, the sum of the strengths of all the vortices entering and leaving the circuit in unit time must be zero. The motion outside the jet is irrotational and so there is no flux of vorticity across the part of the circuit lying in this region. Hence the net flux of vorticity across the part of the circuit cutting the jet must be zero.

Now Preston[2] in R. & M. 1996 (1943) and Temple[3] in A.R.C. 7118 (1943) have shown that the flux of vorticity across a *fixed* line joining two points A, B is

$$F_{AB} = \chi_A - \chi_B\,, \qquad (6)$$

where

$$\chi = \int \frac{dp}{\rho} + \tfrac{1}{2}q^2 \qquad (7)$$

is the Bernoulli function.

If the points A, B lie on the edges of the jet the flux of vorticity across any fixed line joining them is

$$F_{1,2} = \chi_1 - \chi_2 = 0 \qquad (8)$$

for steady motion.

But, since the motion outside the jet is irrotational

$$\chi_1 = \chi_2 \qquad (9)$$

and (8) is always satisfied.

3.2. *Continuity of Pressure.*—Assume the flow in the jet is irrotational. Then inside the jet:

$$p_2' + \tfrac{1}{2}\rho' q_2'^2 = p_1' + \tfrac{1}{2}\rho' q_1'^2 = H' \qquad (10)$$

and outside the jet

$$p_2 + \tfrac{1}{2}\rho q_2^2 = p_1 + \tfrac{1}{2}\rho q_1^2 = H. \qquad (11)$$

There can be no discontinuity of pressure across the edges of the jet, hence

$$\left.\begin{aligned} p_1 &= p_1' \\ p_2 &= p_2' \end{aligned}\right\}. \qquad (12)$$

Substitute (12) in (10)

$$p_2 + \tfrac{1}{2}\rho' q_2'^2 = p_1 + \tfrac{1}{2}\rho' q_1'^2. \qquad (13)$$

Subtract (11) from (13) to eliminate the pressure terms to obtain

$$\tfrac{1}{2}(\rho' q_2'^2 - \rho q_2^2) = \tfrac{1}{2}(\rho' q_1'^2 - \rho q_1^2),$$

or

$$\tfrac{1}{2}\rho'(q_1'^2 - q_2'^2) = \tfrac{1}{2}\rho(q_1^2 - q_2^2) = p_2 - p_1,$$

or

$$\tfrac{1}{2}\rho'(q_1' + q_2')(q_1' - q_2') = \tfrac{1}{2}\rho(q_1 + q_2)(q_1 - q_2) = p_2 - p_1. \qquad (14)$$

Since the motion in the jet is irrotational

$$q_1'r_1 = q_2'r_2 = K', \text{ say.} \qquad (15)$$

Substituting (15) in (14), then

$$\tfrac{1}{2}\rho' \frac{(r_2 + r_1)(r_2 - r_1)}{r_1^2 r_2^2} K'^2 = \tfrac{1}{2}\rho(q_1 + q_2)(q_1 - q_2) = p_2 - p_1 \qquad (16)$$

and putting

$$\left.\begin{aligned} r_2 - r_1 &= \delta \\ \frac{r_2 + r_1}{2} &= r_0 \end{aligned}\right\}, \qquad (17)$$

then, if the jet width δ is small compared with the radius,

$$r_2 \simeq r_1 \simeq r_0 . \qquad (18)$$

Define

$$q_0' = K'/r_0. \qquad (19)$$

Then equation (16) becomes

$$\frac{\rho'\delta q_0'^2}{r_0} = \rho q_0(q_1 - q_2) = p_2 - p_1 \qquad (20)$$

where

$$q_0 = \frac{q_1 + q_2}{2} . \qquad (21)$$

3.3. *Circulation in Circuits which cut the Jet twice at Right-Angles.*—The simplest elementary circuit of this type is shown in Fig. 1 where it is formed by the streamlines just outside the jet and two radials $d\theta$ apart. The circulation is

$$d\Gamma = (q_1 r_1 - q_2 r_2)\, d\theta \qquad (22)$$

and using (17) in the form

$$\left.\begin{aligned} r_1 &= r_0 - \delta/2 \\ r_2 &= r_0 + \delta/2 \end{aligned}\right\},$$

then equation (22) becomes

$$\begin{aligned} \frac{d\Gamma}{d\theta} &= q_1(r_0 - \delta/2) - q_2(r_0 + \delta/2) \\ &= r_0(q_1 - q_2) - \delta/2(q_1 + q_2) . \end{aligned} \qquad (23)$$

Using equations (20) and (21), (23) becomes

$$\frac{d\Gamma}{d\theta} = \frac{\delta(\rho' q_0'^2 - \rho q_0^2)}{\rho q_0} = \delta q_0 \left(\frac{\rho' q_0'^2}{\rho q_0^2} - 1\right). \qquad (24)$$

This result shows that in general there is the equivalent of 'bound' vorticity in the jet downstream of the aerofoil, and $d\Gamma$ only vanishes if $\rho' q_0'^2$ for the jet equals ρq_0^2 for the surrounding fluid, excepting the trivial cases of no jet ($\delta = 0$) and a straight jet ($d\theta = 0$). In general $d\Gamma$ is proportional to the product of the jet width and the excess of jet momentum over the outside stream momentum. It is inversely proportional to radius of curvature of the jet, since

$$d\theta = \frac{ds}{r_0} \qquad (25)$$

and also inversely proportional to q_0. It changes sign if $\rho' q'^2 < \rho q_0^2$ which would represent the effect of a wake. Thus in general it is not true that the circulations in all circuits cutting the streamlines in the wake at right-angles and enclosing the aerofoil are the same.

The circulation in a circuit which cuts the jet twice at right-angles and which stretches from the trailing edge to infinity (Fig. 2) is, from equation (24),

$$\Gamma_w = \int_0^{\theta_{\text{T.E.}}} \delta q_0 \left(\frac{\rho' q_0'^2}{\rho q_0^2} - 1 \right) d\theta \quad \ldots \quad (26)$$

where $\theta_{\text{T.E.}}$ is the angle which the jet at the trailing edge makes with its direction at ∞. For the lift to be finite, the jet at infinity must be parallel to the stream direction. If q_0 does not differ much from V, the velocity at ∞, so that in effect the pressure downstream of the aerofoil is very nearly the free-stream pressure, then q_0' is nearly constant and so also is δ and hence (26) becomes in these circumstances

$$\Gamma_w \simeq \delta V \left(\frac{\rho' q_0'^2}{\rho V^2} - 1 \right) \theta_{\text{T.E.}} . \quad \ldots \quad (26\text{a})$$

If Γ_A denotes the circulation in a circuit enclosing the aerofoil and cutting the jet at right-angles at the trailing edge, and if Γ_∞ denotes the circulation in a circuit enclosing the aerofoil and cutting the streamlines at ∞ at right-angles (Fig. 2) then

$$\Gamma_\infty = \Gamma_A + \Gamma_w \quad \ldots \quad (27)$$

and

$$L = \rho V \Gamma_\infty \quad \ldots \quad (28)$$

where L is the lift. This follows from the arguments advanced by G. I. Taylor in R. & M. 989. Thus prediction of the lift requires a knowledge of Γ_A and Γ_w, the latter being given by (26).

To fix ideas it is interesting to calculate the contribution (ΔC_L) to the lift coefficient due to the circulation in the wake as given by (26a). Suppose the jet issues from the trailing edge in the direction of the chord, then

$$\Delta C_L = \frac{2\Gamma_w}{Vc} = \frac{\delta}{c} \left(\frac{\rho' q_0^2}{\rho V^2} - 1 \right) \alpha .$$

Take $\delta/C = 0{\cdot}01$, $\rho' = \rho$ and $q_0'/V = 2{\cdot}0$, then $\Delta C_L = 0{\cdot}01 \times 3\alpha = 0{\cdot}03\alpha$, which when compared with ordinary lift coefficient $C_L \simeq 2\pi\alpha$ amounts to only $\frac{1}{2}$ per cent of this. On the other hand if the jet issues roughly at right-angles to the stream direction in the neighbourhood of a stagnation point so that equation (26) must be used, the value of ΔC_L may well be quite large since $(q_0'/q_0)^2$ will be very large in this neighbourhood.

3.4. *Calculation of the Circulation about the Aerofoil—Γ_A.*—The previous section shows that if the location of the jet is known relative to the aerofoil then the circulation in the jet can be found. It will be assumed for the moment that this is the case in order to find the effect of the circulation in the jet on the circulation about the aerofoil. Now the basic aerofoil can be transformed to a circle by standard methods and so a point to point correspondence can be established for the flow patterns in the planes of the aerofoil and circle. In particular the position of the jet in the plane of the circle can be found, and, at corresponding points along the jet, the circulation in an element of jet $d\Gamma$ will be the same in the two planes (*see* Figs. 2 and 3). In the plane of the circle (Fig. 3) the effect of the vortices represented by $d\Gamma$ can now be found by introducing equal image vortices of opposite sign at the inverse points inside the circle plus equal vortices of the same sign at the origin of the circle.

Consider the velocity at the point T on the circle corresponding to the trailing edge of the aerofoil (Figs. 2 and 3). For small angles of attack and for a weak jet the line joining a point in the jet to the centre of the circle must very nearly pass through T. Hence it can be assumed that the velocity at T due to the vortex $d\Gamma$ and its images is the same as that where the line joining it to the origin cuts the circle. If the vortex $d\Gamma$ is distance R_2 from the origin then the inverse point is at

$$R_1 = a^2/R_2 \qquad \ldots \qquad (29)$$

from the origin, and the velocity due to the vortices at the point T on the circle (Fig. 3) is

$$dq_i = \frac{d\Gamma}{2\pi a}\left(\frac{1}{a - R_1} - \frac{1}{a} + \frac{1}{R_2 - a}\right)$$

$$dq_i = \frac{d\Gamma}{2\pi a}\frac{2}{\left(\frac{R_2}{a} - 1\right)}\,. \qquad \ldots \qquad (30)$$

Equation (30) shows that it is the vortices near to the trailing edge which are most effective in contributing to the velocity induced by them at T.

Now if the flow is to leave the trailing-edge region smoothly, then T in the plane of the circle must be a stagnation point. Hence, in addition to the circulation Γ_0 which the aerofoil must have to satisfy this condition with no jet flow, there is an induced circulation Γ_i given from equation (30) by

$$d\Gamma_i = d\Gamma\,\frac{2}{\left(\frac{R^2}{a} - 1\right)} \qquad \ldots \qquad (31)$$

and

$$\Gamma_A = \Gamma_0 + \Gamma_i\,. \qquad \ldots \qquad (32)$$

The flow pattern can now be calculated and in particular the position and curvature of jet in the circle plane with this circulation can be found allowing for the vortices in the jet. Hence by use of conformal transformation the position and curvature of the jet in the aerofoil plane can be found. Then, if necessary, a better value for $d\Gamma$ can be deduced and the procedure can be repeated by iteration. Evidently an integral equation can be deduced which expresses the complex relation between $d\Gamma$, the jet velocity and the aerofoil shape, but such an equation would almost certainly prove intractable and it would have to be solved approximately by an iteration process. If the jet velocity is large there may be difficulties in obtaining a sufficiently good approximation to the jet shape to start the calculation. The jet, if strong, may be expected to remain nearly straight for some distance behind the trailing edge. Hence in this region $d\Gamma$ will be small and the vortices further downstream will have an increasing effect with increase of jet speed.

4. *Final Comments.*—It is clear from the calculations made in section 3.3 that in the case of an aerofoil with jet the circulation in circuits which cut the jet twice at right-angles is not zero and that the same arguments would apply to an aerofoil with a wake. The so-called ' proofs ' given by the present writer and by Temple of the ' theorem ' that the circulation is constant in circuits which enclose the aerofoil and cut the streamlines in the wake at right-angles, must contain a flaw in their arguments. In effect the first proof assumes the answer being sought: for an implicit assumption is made that, outside the wake, the increment of velocity potential on neighbouring normals to the streamlines is the same above and below the wake. This need not necessarily be the case and in fact will not be the case if there is circulation in circuits which enclose the wake and cut the streamlines in it twice at right-angles.

It has been seen how the circulation in the jet can arise, and how it affects the circulation in a large circuit in two ways. There is the direct addition of the jet or wake circulation and there is the effect of the jet circulation on the circulation round the aerofoil itself. In the case of the aerofoil plus jet, the circulation around the aerofoil will be increased by the presence of the circulation in the jet which is of the same sign as that about the aerofoil. In the case of the aerofoil with wake the circulation in the wake is of opposite sign and the circulation about the aerofoil will be reduced as will be the circulation in a large circuit. In the latter case it is expected that the effect will be very small, since the momentum loss in the wake is closely related to the drag coefficient, and this is small at incidences below the stall. Moreover, on a 'boundary-layer' basis of neglect of curvature effects, *i.e.*, neglect of pressure variation through the boundary layer and wake, the effects discussed above are negligible and the theorem of Preston and Temple is true to the order of accuracy implied by boundary-layer approximations. It is interesting to note however that the predictions of lift made by Preston[4] and by Spence[5] are slightly in excess of the measured values, a result to be expected when the circulation in the wake is neglected.

Calculation of the effect of a jet or wake will be tedious owing to the necessity of knowing the curvature of the jet or wake streamlines before the circulation in the jet or wake can be computed. However, approximations may be made to shorten the calculations, since section 3 shows that the major contributions from the jet or wake arise from the portions close to the aerofoil.

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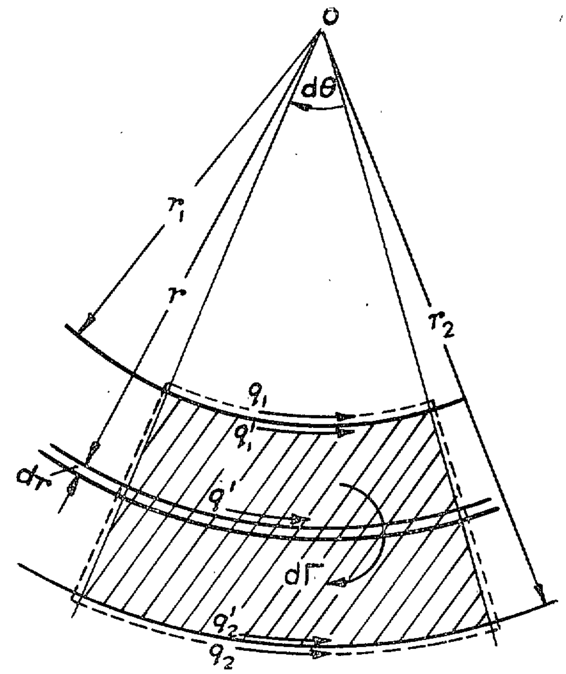

Fig. 1.

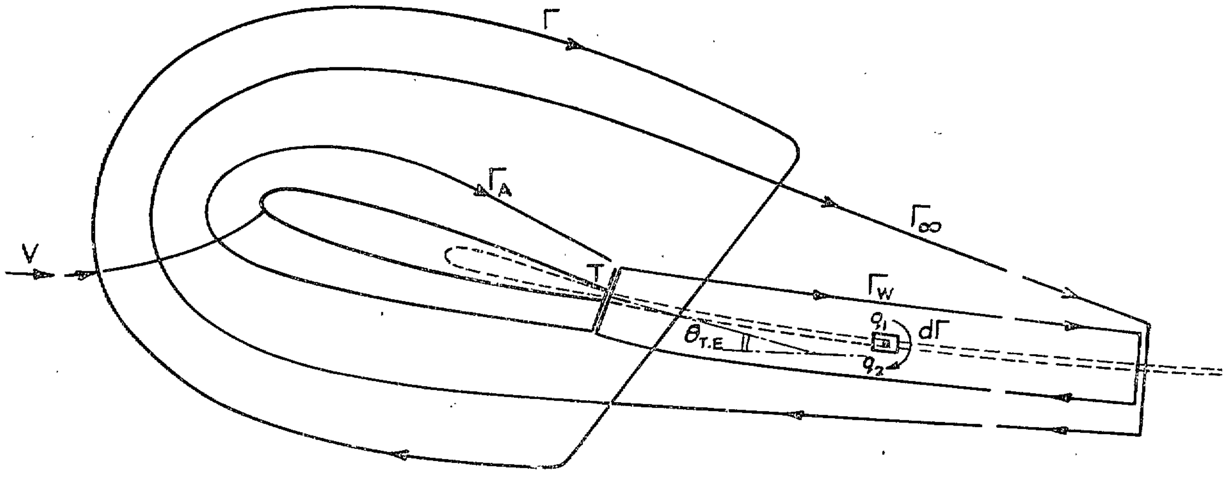

Fig. 2.

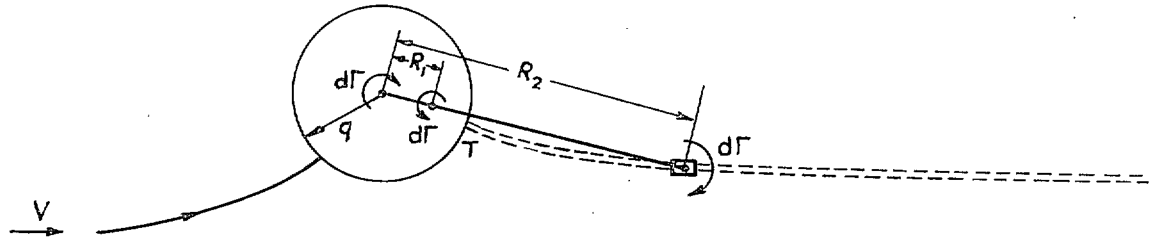

Fig. 3.

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