R. & M. No. 2940
(15,600)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

Analysis of Short-period Longitudinal Oscillations of an Aircraft—Interpretation of Flight Tests

By

S. Neumark, Techn.Sc.D., A.F.R.Ae.S.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1956
FIFTEEN SHILLINGS NET

Analysis of Short-period Longitudinal Oscillations of an Aircraft—Interpretation of Flight Tests

By

S. Neumark, Techn.Sc.D., A.F.R.Ae.S.

Communicated by the Principal Director of Scientific Research (Air),
Ministry of Supply

Reports and Memoranda No. 2940*

September, 1952

Summary.—A method is presented of analysing experimental curves obtained in flight when an aircraft is disturbed longitudinally by a suitable elevator input and performs mainly short-period oscillations. Determination of frequency, damping factor, amplitude ratios and phase angles of various oscillatory curves leads to formulae for evaluating stability derivatives. Cases of elevator fixed or oscillating, for tailed and tailless aircraft, are considered and illustrated by numerical examples.

The main results of the investigation are listed in section 6.

1. *Introduction.*—Many types of modern aircraft, especially tailless, suffer a certain loss of rotary damping at high speed, particularly in the transonic region. The effect is not always important but, in some cases, it may be so strong as to make the damping of the short-period longitudinal oscillation nearly zero or even negative, and so make the aircraft almost uncontrollable. The loss of damping depends critically on the Mach number, and a small increase of the latter in the transonic range may sometimes cause a sudden collapse of damping. The matter is being investigated theoretically and by oscillatory tests in wind tunnels, but serious difficulties are encountered. A considerable time may pass before the problem is mastered completely and the designer is able to predetermine, in a simple way, the stability derivatives for any aircraft shape, and thus design safely against the trouble. That is why the appropriate flight tests on prototypes are now of great importance, and they are frequently undertaken by aircraft firms and research establishments.

It is the object of this paper to work out the theoretical bases of such flight tests, and to find correct and convenient ways of interpreting them, *i.e.*, of determining as many aerodynamic derivatives as possible from the graphs of variables recorded by instruments during the tests. The original purpose of the tests was merely to find out whether the short-period oscillatory damping was adequate at various Mach numbers and altitudes; in particular, to determine critical conditions if the aircraft was so unfortunate as to encounter them. However, the progress of instrumentation has made it possible to obtain continuous and fairly accurate simultaneous records of several variables during appropriate oscillatory disturbances deliberately excited by

* R.A.E. Report Aero 2479, received 30th January, 1953.

the pilot. The quantities usually recorded are: angular velocity of pitch q, normal acceleration ng, and elevator deflection η, but some alternative schemes are possible. A mass of quantitative information is thus obtained which, combined with the existing theory of dynamic stability, may be used for calculating all (or almost all) aerodynamic derivatives involved. These may then be applied for predicting the behaviour of the aircraft in modified conditions (*e.g.*, predicting the damping with elevator fixed from tests with elevator oscillating, etc.). Also, the values of derivatives obtained from flight tests may be usefully compared with those estimated theoretically, or furnished by wind-tunnel tests.

The correlation of the experimental curves with the equations of aircraft dynamics requires a rather elaborate mathematical procedure. It will be shown, however, that simple final formulae can be obtained which may be directly used in practice. The indispensable simplifications introduced in the theory and the unavoidable experimental errors will cause all results to be only approximate. It would be premature to try to assess fully the order of accuracy attainable, although an attempt in this direction is made, and the value of the method proposed here will only be revealed by its application.

A longitudinally disturbed motion of an aircraft, following an arbitrary initial disturbance, consists normally of two oscillatory modes. The *short-period oscillation*, with a period no more than a few seconds (sometimes below 1 second) is usually so strongly damped as to become not recordable after two or three periods, except for the dangerous cases of inadequate damping. The properties of this oscillation depend practically on only a few most important aerodynamic derivatives (z_w, m_w, m_q, $m_{\dot{w}}$). The *phugoid oscillation* has a period of the order of one minute (up to 2 minutes or more at very high speeds), and its natural damping is always low (positive or negative), the practical importance of this damping being, however, very much less than that of the quick oscillation. The properties of the phugoid oscillation, especially its damping, depend on a great number of derivatives, and there are considerable difficulties in correlating calculated and measured characteristics. The disturbed flight consists initially of both short-period and phugoid modes but, after a comparatively short time, the former is normally damped out and the latter only persists for quite a long time. The short-period oscillation can, therefore, be only studied during the early stage of the disturbed flight. If this contains comparable amounts of the two modes, then the recorded curves of any measured quantities are complicated and not very suitable for analytical interpretation; it would be difficult to resolve the recorded motion into its two constituent modes and to isolate the short-period mode which is of the first importance. The flight tests must therefore be arranged in such a way as to produce initially mainly the short-period oscillation, while the phugoid mode should be excluded as far as possible. The usual simple manoeuvre applied for the purpose consists in pushing the stick rapidly forwards and then pulling it quickly back to its original position. The operation may, of course, be carried out in the inverse order, but the essential requirement is that the elevator is suddenly deflected one way and then quickly brought *back to its original equilibrium attitude*. It may be easily explained in general terms why only very little phugoid motion should result from such a manoeuvre. The two opposite elevator movements in quick succession produce (theoretically) two equal and opposite phugoid waves with a phase difference which is small compared to the phugoid period. Two such waves very nearly cancel each other, and so only a small residual phugoid oscillation remains*.

In spite of this, the experimental curves often present some more or less marked deviation from the ideal shape of a pure single-mode damped oscillation, and this may cause some troubles in interpreting them. If the motion consisted exclusively of the short-period oscillation, then the variation of each recorded quantity, such as q or n, would be expressed by the simple formula:

$$A e^{-\mathscr{R}(t-t_0)} \sin \mathscr{J}(t-t_0)\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

the shape of the curves being as shown in Fig. 1. Such curves have two simple properties: (*a*) the points of intersection with the horizontal axis are equidistant, the distance being half the

* The reasoning, of course, does not apply to the two rapid oscillatory waves, because the phase difference is not small compared to their short period.

period $\pi/\mathscr{J}$; (*b*) the successive peak values form a geometrical progression whose ratio depends on the damping factor $\mathscr{R}$. The curves recorded in flight tests look quite similar to those of Fig. 1, but they deviate somewhat from the theoretical shape so that neither of the above properties is reasonably satisfied. Therefore, even the simplest information required from the tests, *i.e.*, the magnitudes of frequency and damping, cannot be directly obtained with anything like the desirable accuracy. This circumstance may jeopardize the accuracy of all more elaborate conclusions to an even higher degree, and it was thought necessary to analyse the matter in a more detailed way. To this purpose, the theoretical response to an idealized elevator movement (' rectangular input ' according to Fig. 2) was calculated, and formulae for the variation of q and n, following such a movement, derived. The calculation was done first in the simplified way, *i.e.*, neglecting the changes in forward speed (putting $u = 0$), and thus eliminating the phugoid motion right from the start; and then in a more exact way, admitting u variation as well, and hence obtaining both short- and long-period oscillation. The details of the calculation are described in Appendix I, and illustrative diagrams presented in Figs. 3, 4, 5 and 6. It is seen that the response curves in both cases are quite similar, and the amount of phugoid motion introduced by an elevator input of short duration is small as expected. The theory thus confirms the intuitive expectation in broad lines. Nevertheless, the small phugoid contribution is quite sufficient to distort the response curves in a similar way to that found in experiments, *i.e.*, making it difficult to determine the frequency and damping accurately.

It appears, therefore, that the uncomfortable distortion of the experimental curves must not necessarily be attributed to instrument errors or faulty technique (though, obviously, an additional distortion due to such causes may often take place). Distorted curves should not be considered as unpleasant exceptions, but as usual occurrences, and a method is therefore required to treat them properly. The problem is quite simple if the distortion is small and, as in the given case, due to an additional oscillation of a period much greater than the duration of the recorded event. The long-period (phugoid) contribution may be safely approximated by a straight line of a small slope and, as shown in Appendix II, the parameters of such a line can be easily determined by measuring a few co-ordinates of appropriate points (upper and lower peaks or, sometimes, points of intersection with the horizontal axis), and then using simple standard formulae. It will be seen that the curves need not be re-drawn but merely referred to new, slightly inclined, axes—instead of the original horizontal axes. The proposed procedure may be termed ' filtration '. If correctly applied, it should reinstate the two properties of the oscillatory curves of the type (1.1), previously mentioned. The frequency and damping parameter may then be determined in a much more reliable way.

The response theory will not be applied at all in connection with the principal subject of this paper, as described in the main text. This is because we cannot, in general, assume the control input to be exactly known, and even if so, the input will usually be very different from the simple ' rectangular ' form of Fig. 2, and quite unsuitable for analytical representation. Our method is completely different from the elaborate American ' dynamic response ' technique reported recently in a comprehensive way by W. Milliken[17]. In this paper, only the curves obtained during a free oscillation are taken as a basis, with no specific assumptions as to the initial elevator input—except the general assumption that the amount of phugoid oscillation produced is small.

If only the period and damping of the rapid oscillation were required, it would suffice to record only one response curve, *e.g.*, q alone, or n alone. This would give some information about the stability derivatives, but not nearly enough for a complete investigation. However, in most cases, we shall have at least two such curves (*e.g.*, q and n, if the elevator is kept fixed efficaciously after the initial manoeuvre), or three of them (q, n, and η, if the elevator is free, or oscillates slightly owing to elasticity of the control circuit). In such cases, further valuable information can be obtained by determining the amplitude ratios and phase differences between particular curves. The procedure is quite simple, but it was thought not superfluous to describe it in detail, and this has been done in Appendix III.

In the main text of this paper, it will be assumed that the frequency and damping parameter of the recorded curves, and also the respective amplitude ratios and phase differences, are known, and the formulae for determining stability derivatives therefrom will be derived. It will be seen that the system of equations disposable is redundant with respect to some derivatives, and this will provide a useful check of accuracy of the experimental technique. With respect to some other derivatives the system of equations may prove to be indeterminate, and then it will be possible to calculate only some combinations of derivatives from which it will not be possible to isolate the individual derivatives. For example, the full rotary damping derivative $m_{\dot\theta}$ can be determined but not its constituent parts m_q and $m_{\dot w}$. This will have to be acknowledged as a certain imperfection of the method, but the information obtained will still be quite valuable. It may, of course, be supplemented by data furnished by alternative methods of research.

Section 2 deals with the case of elevator fixed, and sections 3 and 4 with those of elevator oscillating, on tailed and tailless aircraft, respectively. The difference between the two latter cases is merely that for a tailed aircraft the effect of elevator deflection on the total lift may normally be neglected (the derivative z_η omitted), while such a simplification does not apply to tailless. However, including this effect makes the algebra more complicated, and it was thought advisable to treat the simpler case first, whereupon the more complicated one becomes more tractable. Section 5 contains an attempt to assess the accuracy of the analysis in the case of elevator fixed. The main conclusions are summarized in section 6.

Acknowledgments are due to Z. Olenski, of A. V. Roe & Co., Ltd., to whose initiative this paper is largely due, and to A. W. Thorpe for his several valuable suggestions; also to Mrs. J. Collingbourne, W. P. Gillott and C. G. Price, who helped to work out examples, did the computation and prepared the illustrations.

2. *Case of Elevator Fixed*.—2.1. *Theory*.—Let us assume that the initial manoeuvre has been terminated by bringing the elevator back to its original equilibrium position (corresponding to level flight) and that, from this instant onwards, the elevator is kept rigidly fixed*. We may also assume that the forward speed is constant ($u = 0$), *i.e.*, that the phugoid oscillation is absent, and thus the disturbed motion will be governed by the familiar simplified system of equations (2nd order only, dimensionless, referred to a moving system of axes II, level flight):

$$(D + \tfrac{1}{2}a)\hat{w} - \hat{q} = 0 \qquad (2.1)$$

$$(\chi D + \omega)\hat{w} + (D + \nu)\hat{q} = 0\,, \qquad (2.2)$$

where the meaning of the main symbols is as follows (*see* also List of Symbols):

$$\left.\begin{aligned}
&D = d/d\tau \text{ (differential operator)}; \quad \tau = t/\hat{t} \text{ (aerodynamic time)};\\
&\hat{t} = W/g\rho SV = VC_L/2g \text{ (unit of aerodynamic time)};\\
&\hat{w} = w/V \text{ (increment of incidence)}; \quad \hat{q} = q\hat{t} \text{ (dimensionless rate of pitch)};\\
&a = \partial C_L/\partial\alpha \text{ (lift-curve slope)}\dagger; \quad \nu = -\,m_q/i_B; \quad \chi = -\,m_{\dot w}/i_B;\\
&\omega = -\frac{\mu m_w}{i_B} = \frac{\mu}{i_B}\cdot\frac{c}{2l}\,aK_m, \text{ where } K_m = -\,\partial C_m/\partial C_L \text{ (`restoring' margin)};\\
&\mu = W/g\rho Sl = V\hat{t}/l \text{ (relative density)}; \quad i_B = \frac{k_B^{\,2}}{l^2} \text{ (inertia coefficient)}.
\end{aligned}\right\} \qquad (2.3)$$

* This is not quite equivalent to keeping the stick fixed; in the latter case, the elevator may still oscillate appreciably, because of the elasticity of the control circuit. In many cases, only a special clamping device can assure a complete immobility of the elevator. If the elevator is power operated, however, it can usually be considered as fixed unless actuated by the power unit.

† The equation (2.1) is often written, as $(D - z_w)\hat{w} - \hat{q} = 0$ where, strictly, $z_w = -\,\tfrac{1}{2}(a + C_D)$. In view of our simplification through which the entire drag equation has been omitted, the term C_D may be reasonably neglected in the last expression, as small compared to a.

The notation is that of Bryant and Gates[1], with some usual modifications and additions[4, 5, 11, 13, 15]. The 'compound' derivatives ω, ν, χ are merely convenient algebraic combinations of the more familiar aerodynamic derivatives m_w, m_q, $m_{\dot{w}}$ with the factors μ and i_B, and with signs inverted so that they are positive when contributing to stability, and they are used mainly in order to simplify writing. It may be mentioned that the definition of $m_{\dot{w}}$ adopted here differs from that of Ref. 1 inasmuch as the factor μ is omitted, so that we define $m_{\dot{w}} = M_{\dot{W}}/\rho S l^2$ instead of $gM_{\dot{W}}/Wl$ (*cf.* Ref. 13, form. 5.3). This is more consistent with the definitions of all other aerodynamic derivatives (which are independent of μ and thus of the aircraft weight). It is also convenient because we now have the simple relationship:

$$m_q + m_{\dot{w}} = m_{\dot{\vartheta}} , \qquad \ldots \qquad (2.4)$$

where $m_{\dot{\vartheta}}$ represents the rotary derivative in the system of axes fixed in space, determinable directly by oscillatory tunnel tests (*cf.* Ref. 13). It should be emphasized that the derivative $m_{\dot{w}}$ has now a much wider meaning than that envisaged in Refs. 4 and 5, *i.e.*, it comprises not only the effect of downwash lag at the tail but (especially for tailless aircraft) the important effect due to the 'unsteadiness' of the oscillatory motion. It should also be mentioned that $\partial C_m/\partial C_L$ in the definition of K_m is a partial derivative 'at constant speed'; therefore, K_m does not include any effects of varying Mach number and, at high speeds, differs considerably from the static margin K_n (*cf.* Ref. 9). The term 'restoring margin' for K_m seems suitable as, when the aircraft incidence is changed from its equilibrium value, there arises a 'restoring' moment, proportional to K_m, which tends to reinstate the original incidence.

In the subsequent considerations, we shall represent all variables in complex form, which simplifies most derivations greatly. The equations (2.1) and (2.2) represent a damped oscillation of a certain *angular frequency* $\bar{J}$, with a certain *damping parameter* $\bar{R}$*. Thus, for instance, the solution for $\hat{w}$ will be:

$$\hat{w} = \hat{w}^* e^{-\bar{R}(\tau-\tau_0)} \sin \bar{J}(\tau - \tau_0) , \quad (\hat{w}^*, \tau_0\text{—constants of integration}) \qquad \ldots \qquad (2.5)$$

but it will be more convenient to write it in the complex form:

$$\hat{w} = \hat{w}^* e^{(-\bar{R}+i\bar{J})(\tau-\tau_0)} , \qquad \ldots \qquad (2.6)$$

all further deductions applying either to the real or to the imaginary part of the complex expressions. For instance, (2.5) is the imaginary part of (2.6).

Differentiating (2.6), we obtain:

$$D\hat{w} = (-\bar{R} + i\bar{J})\hat{w} , \qquad \ldots \qquad (2.7)$$

which means that the differentiation of expressions of the type (2.6) reduces to multiplying them by $(-\bar{R} + i\bar{J})$, or that the symbols D and $(-\bar{R} + i\bar{J})$ are interchangeable.

Substitution of (2.7) into (2.1) yields:

$$\hat{q} = (\tfrac{1}{2}a - \bar{R} + i\bar{J})\hat{w} . \qquad \ldots \qquad (2.8)$$

Introducing this into (2.2), and splitting real and imaginary parts, we obtain:

$$[\{\omega + \tfrac{1}{2}a\nu + \bar{R}^2 - \bar{J}^2 - (\tfrac{1}{2}a + \nu + \chi)\bar{R}\} + i\bar{J}\{(\tfrac{1}{2}a + \nu + \chi) - 2\bar{R}\}]\hat{w} = 0 . \qquad (2.9)$$

This equation must be satisfied identically, thus the real and imaginary parts in the square bracket must vanish, and hence†

$$\tfrac{1}{2}a + \nu + \chi = 2\bar{R} ; \qquad \ldots \qquad (2.10)$$

$$\omega + \tfrac{1}{2}a\nu = \bar{R}^2 + \bar{J}^2 . \qquad \ldots \qquad (2.11)$$

---

* $\bar{J}$ and $\bar{R}$ are both dimensionless; the bars are used to denote the values corresponding to the case of elevator fixed.

† The relationships (2.10) and (2.11) can, of course, be found more simply, by eliminating $\hat{w}$ and $\hat{q}$ from (2.1) and (2.2), whence:

$$D^2 + (\tfrac{1}{2}a + \nu + \varkappa)D + (\omega + \tfrac{1}{2}a\nu) = 0 ,$$

and, if we put $D = -\bar{R} \pm i\bar{J}$, the equations (2.10) and (2.11) are obvious at once. It was thought useful, however, to follow the alternative, slightly longer, derivation because an analogous method will be indispensable later on, in the case of elevator free.

The frequency $\bar{J}$ and damping parameter $\bar{R}$ can be determined from any curve recorded during an oscillation in flight, *e.g.*, from the $\bar{q}$- or $\bar{n}$-curve (for details, *see* Appendices II and III). A θ- or $\hat{w}$-curve could be used as well (if suitable instruments were available), because the frequency and damping should be exactly the same for all of them, at least theoretically. If only one curve is recorded, we have only *two equations* (2.10) and (2.11) for determining *four unknown derivatives* a, ν, χ and ω (the three last symbols representing m_q, $m_{\dot{w}}$ and m_w). It is seen that, in such a case, we obtain only the two combinations of derivatives $(\frac{1}{2}a + \nu + \chi)$ and $(\omega + \frac{1}{2}a\nu)$, but not the values of particular derivatives, unless we make use of some alternative sources of information, such as theoretical calculations, wind-tunnel tests, or steady flight tests. It is very desirable, however, to make no use of such extraneous information and try to deduce as many results as possible from the oscillatory flight tests alone, these results to be compared afterwards with any other available data. This is particularly important because the oscillatory flight tests supply the information on the values of 'unsteady' derivatives, and it is of particular interest to find out how far these deviate from the 'steady' counterparts (such as z_w and m_w obtained from static tunnel tests).

In most cases, *two curves* representing $\bar{q}$ and $\bar{n}$ are simultaneously recorded, and then, in addition to $\bar{R}$ and $\bar{J}$, two more quantities may be determined, *viz*, the amplitude ratio and phase difference of the two curves (*see* Appendix III). It seems that, with two more equations, it will be possible to calculate all four derivatives. The position is not quite so satisfactory, however, as will be seen from the following analysis.

The normal acceleration $\bar{n}g$ (in the direction of the negative z-axis, *i.e.*, of the lift in original undisturbed flight) is given by

$$\bar{n}g = Vq - \frac{dw}{dt} = \frac{V}{\hat{t}}(\hat{q} - D\hat{w}) , \qquad (2.12)$$

or, in view of (2.1):

$$\bar{n} = \frac{V}{2g\hat{t}} a\hat{w} . \qquad (2.13)$$

Dividing (2.8) by (2.13), and replacing $\hat{q}$ by $\bar{q}\hat{t}$, we obtain:

$$\frac{\bar{q}}{\bar{n}} = \frac{g}{V}\frac{\frac{1}{2}a - \bar{R} + i\bar{J}}{\frac{1}{2}a} . \qquad (2.14)$$

We assume that the amplitude ratio $\bar{q}^*/\bar{n}^*$ and the phase difference $\bar{\varphi}_{qn}$ have been obtained from the recorded curves, and so we may write:

$$\frac{\bar{q}}{\bar{n}} = \frac{\bar{q}^*}{\bar{n}^*} e^{i\bar{\varphi}_{qn}} = \frac{\bar{q}^*}{\bar{n}^*}(\cos\bar{\varphi}_{qn} + i\sin\bar{\varphi}_{qn}) . \qquad (2.15)$$

The expressions (2.14) and (2.15) must be equal, and hence:

$$\tfrac{1}{2}a - \bar{R} + i\bar{J} = \tfrac{1}{2}a \cdot \frac{V\bar{q}^*}{g\bar{n}^*}(\cos\bar{\varphi}_{qn} + i\sin\bar{\varphi}_{qn}) . \qquad (2.16)$$

Equating the real and imaginary parts in this complex equality, and introducing, for abbreviation, the symbol:

$$\bar{p} = \frac{V}{g} \cdot \frac{\bar{q}^*}{\bar{n}^*} , \qquad (2.17)$$

we obtain the equations:

$$a - 2\bar{R} = a\bar{p}\cos\bar{\varphi}_{qn} , \quad 2\bar{J} = a\bar{p}\sin\bar{\varphi}_{qn} , \qquad (2.18)$$

from which, taking into account (2.10) and (2.11), we deduce:

$$a\bar{p} = \sqrt{(4\omega - 2a\chi)} . \qquad (2.19)$$

It is seen now that each of the two new equations (2.18) contains only one unknown derivative a which can thus be determined in two different ways:

$$a = \frac{2\bar{R}}{1 - \bar{p}\cos\bar{\varphi}_{qn}}; \quad \ldots \quad (2.20a)$$

$$a = \frac{2\bar{J}}{\bar{p}\sin\bar{\varphi}_{qn}}. \quad \ldots \quad (2.20b)$$

It appears that, although we have altogether four equations (2.10, 2.11, 2.20a, 2.20b) with four unknowns, the system is redundant with respect to a, while still indeterminate with respect to the remaining derivatives ν, χ and ω. The fact that we have two formulae for a provides a useful check of accuracy of the entire procedure*.

The final result is that, when using the flight-test technique as described, we may determine a (in a twofold way), and then the sums $(\nu + \chi)$ from (2.10) and $(\omega + \frac{1}{2}a\nu)$ from (2.1). It is impossible, however, to extract the values of the individual derivatives ν, χ, ω from these sums. It may be easily shown that, if alternative or additional quantities were recorded in flight, such as, for instance, angle of pitch θ or angular acceleration $\dot{q}$, the position would still remain unaltered, and only the same combinations of derivatives could be determined. Although this indicates a certain imperfection of the method, yet the information obtained is very valuable. The combination $(\nu + \chi)$ leads immediately to the value of the important *derivative* $m_{\dot{\theta}}$ (the one directly measurable in oscillatory tunnel tests) because, in view of (2.4) and the definitions in (2.3), we have:

$$m_{\dot{\theta}} = m_q + m_{\dot{w}} = -i_B(\nu + \chi) = -i_B(2\bar{R} - \tfrac{1}{2}a). \quad \ldots \quad (2.21)$$

As to the combination $(\omega + \frac{1}{2}a\nu)$, this provides directly the *manoeuvre margin* H_m (stick fixed).

$$H_m = K_m - \frac{l}{c}\frac{m_q}{\mu} = \frac{i_B}{\mu}\cdot\frac{2l}{ca}(\omega + \tfrac{1}{2}a\nu) = \frac{i_B}{\mu}\cdot\frac{2l}{ca}(\bar{R}^2 + \bar{J}^2), \quad \ldots \quad (2.22)$$

as follows from Ref. 9, form. 114, p. 21. One might object that the value thus obtained could differ somewhat from the manoeuvre margin as defined in connection with the usual recovery manoeuvre, because in our case we have to deal with 'unsteady' (oscillatory) derivatives, while Gates' manoeuvrability theory[3] is based on the concept of a steady circling motion. However, the only derivatives involved are m_w and m_q, and these, as far as is known, are almost exactly the same in steady conditions as in an oscillation at low reduced frequency†.

We may add that some other combinations of derivatives can be derived from the above analysis, *e.g.*, from (2.19) we get:

$$\omega - \tfrac{1}{2}a\chi = \tfrac{1}{4}a^2\bar{p}^2. \quad \ldots \quad (2.23)$$

If the values of the two expressions $(\omega + \frac{1}{2}a\nu)$ and $(\omega - \frac{1}{2}a\chi)$ are known, this will give in many cases a pretty narrow range for ω, thus for the restoring margin K_m.

* It is possible to derive one more formula for a, free from the phase angle $\bar{\varphi}_{qn}$, by equating the moduli of the two parts in (2.16):

$$a = \frac{2}{\bar{p}^2 - 1}\left[\sqrt{\{\bar{p}^2\bar{R}^2 + (\bar{p}^2 - 1)\bar{J}^2\}} - \bar{R}\right], \quad \ldots \quad (2.20c)$$

but this can, of course, be also obtained by eliminating $\bar{\varphi}_{qn}$ from (2.20a and b), and really contains nothing new. We can also eliminate $\bar{p}$, and then we obtain yet another formula for a:

$$a = 2(\bar{R} + \bar{J}\cot\bar{\varphi}_{qn}). \quad \ldots \quad (2.20d)$$

Either of the four formulae may be preferred, according to which of the recorded quantities seem more reliable.

† The theoretical investigation of Ref. 14 shows that at least the two-dimensional derivatives m_w and m_q are very little affected by the varying reduced frequency, provided this is small. There are no reasons to expect that the position will be different in three dimensions, and the agreement is likely to be even closer.

More complete information may be obtained if the derivative m_q is estimated theoretically, as may often be done without great difficulties and with reasonable accuracy. Our system of equations will then become determinate, and we shall be able to find the values of $m_{\dot{w}}$ and m_w (hence also the restoring margin K_m).

No information about the static margin K_n (in the sense of its now accepted definition as given by Gates and Lyon[9]) can be obtained from oscillatory flight tests, because the definition involves the effects of varying speed, *i.e.*, first of all the varying Mach number effects. The determination of the static margin will, of course, always require flight tests at varying speeds, and does not come under the scope of this paper.

2.2. *Examples*.—The following numerical examples are not supposed to apply to any specific aircraft, but the data are chosen so as to be realistic and illustrate some typical and diverse cases. In each example, the few required design and operating data, and all relevant derivatives, are first assumed as given, and the oscillatory characteristics calculated therefrom (first stage). It is then supposed that all derivatives are unknown, while the oscillatory characteristics have been read from the curves recorded in flight, and the derivatives or their combinations are determined from our formulae, in the way it is proposed they should be in practice (second stage). The readings are occasionally supposed to deviate somewhat from the values calculated in the first stage, so as to illustrate partial effects of unavoidable errors on the accuracy of the final results. In the first example, a detailed calculating scheme is shown, in the remaining two only numerical values are listed.

The numerical data have been chosen so as to avoid rounding and ensure a very good accuracy, certainly higher than can be expected in practice. The errors assumed in the second stage are also quite small, and much greater errors can certainly be tolerated. It depends on the precision of instruments and of the entire test technique what sort of accuracy will be attainable, and this may only be ascertained experimentally. An analysis of errors is given in section 5.

Example I. Tailed Aircraft.—*First stage*.—The assumptions are as follows:

Design and operating data: $l/c = 2 \cdot 5$, $i_B = 0 \cdot 08$, $V = 644$ ft/sec, $\mu = 88$.

Derivatives: $a = 4 \cdot 25$, $m_q = -0 \cdot 282$, $m_{\dot{w}} = -0 \cdot 096$, $K_m = 0 \cdot 08$.

From these data, further derivatives are determined:

$$m_w = -\frac{ac}{2l} K_m = -0 \cdot 068; \text{ from (2.4) } m_{\dot{\theta}} = -0 \cdot 378;$$

manoeuvre margin, from (2.22) $H_m = 0 \cdot 088$.

The 'compound' derivatives now become: $\nu = 3 \cdot 525$, $\chi = 1 \cdot 2$, $\omega = 74 \cdot 8$. We then calculate the oscillatory characteristics from (2.19, 2.17, 2.10, 2.11, 2.18):

$$\bar{p} = 4, \quad \bar{q}^*/\bar{n}^* = 0 \cdot 2 \text{ radn/sec}, \quad \bar{R} = 3 \cdot 425, \quad \bar{J} = 8 \cdot 4, \quad \cos \bar{\varphi}_{qn} = -0 \cdot 1529,$$
$$\sin \bar{\varphi}_{qn} = 0 \cdot 9882, \text{ and hence } \bar{\varphi}_{qn} = 98^\circ \, 48'.$$

Second stage.—We assume that the design and operating data are known, exactly the same as in the first stage, and that the oscillatory characteristics, as obtained from the recorded curves, are:

$$\bar{R} = 3 \cdot 42, \quad \bar{J} = 8 \cdot 4, \quad \bar{q}^*/\bar{n}^* = 0 \cdot 2 \text{ radn/sec}, \quad \bar{\varphi}_{qn} = 98^\circ \, 50' \text{ (hence } \cos \bar{\varphi}_{qn} = -0 \cdot 1536, \; \sin \bar{\varphi}_{qn} = 0 \cdot 9881).$$

We calculate from (2.17, 2.20a, 2.20b, 2.10, 2.23):

$$\bar{p} = 4; \quad a = 4 \cdot 237, \text{ or } a = 4 \cdot 251, \text{ say } a = 4 \cdot 24;$$

$$\nu + \chi = 4 \cdot 72; \quad \omega + \tfrac{1}{2} a\nu = 82 \cdot 26; \quad \omega - \tfrac{1}{2} a\chi = 71 \cdot 91.$$

For a tailed aircraft, at a moderate Mach number, there is no doubt that χ is positive (ν is always positive). The two last results thus show that ω lies somewhere between 72 and 82, hence the restoring margin K_m is between $0 \cdot 077$ and $0 \cdot 088$ which is quite a narrow interval, and we may expect that the lower value will be nearer to the true one (which was $0 \cdot 08$).

The formula (2.21) now gives $m_{\dot{\vartheta}} = -0 \cdot 3776$, and the manoeuvre margin is obtained from (2.22) as $H_m = 0 \cdot 0882$.

Example II. Tailless Aircraft.—First stage.—Data assumed:

$$l/c = 1, \quad i_B = 0 \cdot 36, \quad V = 800 \text{ ft/sec}, \quad \mu = 82 \cdot 26;$$
$$a = 3, \quad m_q = -0 \cdot 36, \quad m_{\dot{w}} = -0 \cdot 18, \quad K_m = 0 \cdot 04.$$

Intermediate values calculated:

$$m_w = -0 \cdot 06, \quad m_{\dot{\vartheta}} = -0 \cdot 54, \quad H_m = 0 \cdot 0444;$$
$$\nu = 1, \quad \chi = 0 \cdot 5, \quad \omega = 13 \cdot 71.$$

Oscillatory characteristics calculated:

$$\bar{p} = 2 \cdot 4, \quad \bar{q}^*/\bar{n}^* = 0 \cdot 0966 \text{ radn/sec}, \quad \bar{R} = 1 \cdot 5, \quad \bar{J} = 3 \cdot 6, \quad \bar{\varphi}_{qn} = 90°.$$

Second stage.—The first four data assumed as above, and the oscillatory characteristics taken as

$$\bar{R} = 1 \cdot 5, \quad \bar{J} = 3 \cdot 6, \quad \bar{q}^*/\bar{n}^* = 0 \cdot 096, \quad \bar{\varphi}_{qn} = 90°.$$

The calculated results become:

$$\bar{p} = 2 \cdot 385; \quad a = 3 \cdot 00 \text{ or } a = 3 \cdot 019, \quad \text{say } a = 3;$$
$$\nu + \chi = 1 \cdot 5, \quad \omega + \tfrac{1}{2} a\nu = 15 \cdot 21, \quad \omega - \tfrac{1}{2} a\chi = 12 \cdot 80; \quad m_{\dot{\vartheta}} = -0 \cdot 54, \quad H_m = 0 \cdot 0444.$$

A rough estimate of ω and K_m is again possible. ω is certainly smaller than $15 \cdot 21$, hence $K_m < 0 \cdot 0444$. Supposing, however, that m_q has been calculated theoretically as $(-0 \cdot 432)$, *i.e.*, with as much as 20 per cent error, we obtain $\nu = 1 \cdot 2$, $\omega = 13 \cdot 41$, and $K_m = 0 \cdot 0391$, with only $2 \cdot 2$ per cent error, which is really much more accurate than required.

Example III. Tailless Aircraft.—First stage.—Data assumed:

$$l/c = 1, \quad i_B = 0 \cdot 2, \quad V = 750 \text{ ft/sec}, \quad \mu = 39 \cdot 65;$$
$$a = 4, \quad m_q = -0 \cdot 41, \quad m_{\dot{w}} = +0 \cdot 13, \quad K_m = 0 \cdot 06.$$

Intermediate values calculated:

$$m_w = -0 \cdot 12, \quad m_{\dot{\vartheta}} = -0 \cdot 28, \quad H_m = 0 \cdot 0703;$$
$$\nu = 2 \cdot 05, \quad \chi = -0 \cdot 65, \quad \omega = 23 \cdot 79.$$

Oscillatory characteristics calculated:

$$\bar{p} = 2 \cdot 5045, \quad \bar{q}^*/\bar{n}^* = 0 \cdot 1075 \text{ radn/sec}, \quad \bar{R} = 1 \cdot 7, \quad \bar{J} = 5, \quad \cos \bar{\varphi}_{qn} = 0 \cdot 0599,$$
$$\sin \bar{\varphi}_{qn} = 0 \cdot 9982, \quad \bar{\varphi}_{qn} = 86° \, 34'.$$

Second stage.—The first four data assumed as above, and the oscillatory characteristics taken as

$$\bar{R} = 1 \cdot 7, \quad \bar{J} = 5, \quad \bar{q}^*/\bar{n}^* = 0 \cdot 108, \quad \bar{\varphi}_{qn} = 86° \, 30' \text{ (hence } \cos \bar{\varphi}_{qn} = 0 \cdot 0610, \; \sin \bar{\varphi}_{qn} = 0 \cdot 9981).$$

The calculated results become:

$$\bar{p} = 2 \cdot 516, \quad a = 4 \cdot 017, \text{ or } a = 3 \cdot 982, \quad \text{say } a = 4 \cdot 00.$$
$$\nu + \chi = 1 \cdot 4, \quad \omega + \tfrac{1}{2} a\nu = 27 \cdot 89, \quad \omega - \tfrac{1}{2} a\chi = 25 \cdot 32; \quad m_{\dot{\vartheta}} = -0 \cdot 28, \quad H_m = 0 \cdot 0703.$$

In this example, $m_{\dot{w}}$ has been assumed positive (thus χ negative), but $m_{\dot{\vartheta}}$ is still negative, and the damping parameter $\bar{R}$ has a positive and quite satisfactory value. The true value of ω lies not between $27 \cdot 89$ and $25 \cdot 32$ but below the latter value, hence a direct estimate of ω would be somewhat uncertain. Supposing, however, that m_q has been calculated theoretically as $(-0 \cdot 3)$, *i.e.*, with as much as 27 per cent error, we obtain $\nu = 1 \cdot 5$, $\omega = 24 \cdot 89$, and $K_m = 0 \cdot 0628$, with only 5 per cent error.

General Remarks.—In the three examples, the phase difference $\bar{\varphi}_{qn}$ is greater than, equal to, and less than 90 deg, respectively. It is obvious from (2.18) that $\sin \bar{\varphi}_{qn}$ must be positive, hence $\bar{\varphi}_{qn}$ always lies between 0 deg and 180 deg. It will be greater than 90 deg if $2\bar{R} > a$ or, in view of (210), if

$$\nu + \chi > \tfrac{1}{2}a, \quad \text{or} \quad (-m_{\dot{\vartheta}}) > \tfrac{1}{2}ai_B \,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.24)$$

i.e., when the rotary damping is quite large.

If the worst happens and the damping becomes negative ($\bar{R} < 0$), the angle $\bar{\varphi}_{qn}$ will be much less than 90 deg. However, this angle may be 90 deg or less, with still quite adequate damping, as in our examples II and III.

3. *Case of Elevator Oscillating. Tailed Aircraft.*—3.1. *Theory.*—Let us again assume that the initial manoeuvre has been terminated by bringing the elevator back to its original equilibrium position, but then the elevator is let free or, at least, a certain measure of freedom is left to it due to elasticity of the control circuit, even while the stick is fixed*. The oscillatory motion will then differ more or less from that with the elevator rigidly fixed (as described in the previous section), and even a small periodic motion of the elevator may modify the oscillatory characteristics of the aircraft considerably. A strict theory of such a motion would involve an additional degree of freedom which would raise the order of the system of equations by two and lead to a very complicated algebra, with at least four new unknown derivatives, such as: elevator floating-moment parameter b_1, restoring-moment parameter b_2, elevator damping parameters (aerodynamic and frictional), spring constants of the control system, etc.; also some further constants, such as elevator inertia and mass unbalance. The problem is of great complexity and notoriously one of the most difficult in the theory of stability, and its existing solutions[2, 6, 7, 8] are hardly suitable for use in connection with flight tests. Moreover, it is clear already at this stage that, with one more quantity (η) to be recorded during tests, only two new independent quantities (one amplitude ratio, and one phase difference) will be available, while the number of unknown derivatives will increase by four or more, thus making the algebraic system strongly indeterminate. However, the problem may be treated in a much simpler way. It is known[2, 6, 7, 8] that there are two main effects of letting the elevator free: firstly, a new 'very rapid' oscillatory mode makes its appearance, with a much higher frequency and much stronger damping than those of the 'short-period' mode; secondly, the frequency and damping of the latter mode are considerably modified. The very rapid oscillation, owing to its heavy damping, dies out almost immediately after the initial cause of the disturbance has come to an end. Thus, in our present case, the motion of both the aircraft and the elevator consists almost exclusively of the modified short-period oscillation (with a small amount of the slowly developing phugoid oscillation, as before). The flight tests confirm this convincingly and, in the typical case, when q, n and the elevator deflection η are recorded, all three curves exhibit practically the short-period mode only, although its frequency and damping are now different from those observed with elevator fixed. This

---

* If the stick is to be let free, it may be difficult to bring it exactly to the original position and then to release it at once, and the pilot may be inclined merely to push the stick forward and then simply let it go. It is believed advisable to try to bring the stick back to its original position at least approximately, so as to avoid excessive phugoid motion. And, in any case, the stick force should have been trimmed out, by means of the trimming tab (or adjustable tailplane, or any similar device) before starting the initial manoeuvre, so that the elevator later oscillates about its original position, and these is no tendency for the aircraft to climb or descend.

suggests a simple method of analytical treatment, by using again the equations of motion (2.1) and (2.2), and merely introducing additional terms due to the varying η. The equation of motion of the elevator itself is then not introduced at all, thus avoiding the trouble of dealing with all additional derivatives. This means, of course, a somewhat defeatist attitude, shirking some difficulties temporarily, and postponing the more thorough analysis to the future. The elevator will be treated as an element of our oscillating system which, owing to its (unknown) dynamic characteristics, influences the short-period oscillation of the aircraft, and we shall try to find what new information can be obtained by investigating oscillatory curves recorded in flight. The first attempt of this kind was made by Dr. K. H. Doetsch[12] in connection with the lateral oscillations of an aircraft with rudder free; the results have been useful and interesting, and the method seems to deserve serious attention.

When dealing with conventional tailed aircraft, we may neglect the very small effect of the varying elevator angle on the total lift, and only consider its effect on the total moment. The equation (2.1) will thus remain unchanged (no $z_\eta \eta$ term, *see* section 4, equation (4.1)), and only an additional term introduced in (2.2). Our new system of equations will be:

$$(D + \tfrac{1}{2}a)\hat{w} - \hat{q} = 0 \,, \quad \ldots \quad (3.1)$$

$$(\chi D + \omega)\hat{w} + (D + \nu)\hat{q} + \delta\eta = 0 \,, \quad \ldots \quad (3.2)$$

where:

$$\delta = -\frac{\mu m_\eta}{i_B} = -\frac{\mu}{i_B} \cdot \frac{c}{2l} \cdot \frac{\partial C_m}{\partial \eta} \quad \ldots \quad (3.3)$$

is the 'compound' derivative measuring the effect of elevator deflection. The effect of $m_{\dot{\eta}}$, *i.e.*, of the change in aircraft moment due to the angular velocity of the elevator, is supposed small of the order of errors in the main terms, and therefore neglected.

We may again assume, as in the previous section (form. 2.6):

$$\hat{w} = \hat{w}^* \, \mathrm{e}^{(-R+iJ)(\tau-\tau_0)} \,, \quad \ldots \quad (3.4)$$

so that

$$D\hat{w} = (-R + iJ)\hat{w} \quad \ldots \quad (3.5)$$

and

$$\hat{q} = (\tfrac{1}{2}a - R + iJ)\hat{w} \,, \quad \ldots \quad (3.6)$$

where J and R are angular frequency and damping parameter of the oscillation with elevator free.

We now suppose that the elevator motion, as recorded in flight consists also of a single oscillatory mode, of the same frequency J and damping parameter R, and let us denote by

$$\varepsilon = \eta^*/\hat{w}^* \quad \ldots \quad (3.7)$$

the amplitude ratio of η and $\hat{w}$, and by

$$\varphi = \varphi_{\eta w} \quad \ldots \quad (3.8)$$

the phase difference by which η leads $\hat{w}$. Then we may write:

$$\eta = \varepsilon\hat{w} \, \mathrm{e}^{i\varphi} \,. \quad \ldots \quad (3.9)$$

As the first equation of motion (3.1) is the same as (2.1) before, we obtain again the formula for the normal acceleration factor n, similar to (2.13):

$$n = \frac{V}{2g\hat{t}} a\hat{w} \,, \quad \ldots \quad (3.10)$$

or, replacing $\hat{t}$ by its expression from (2.3) :

$$n = \frac{a}{C_L}\hat{w}, \quad \ldots \quad (3.11)$$

and it is seen that n is still in phase with $\hat{w}$. Dividing (3.9) by (3.11), we find:

$$\frac{\eta}{n} = \frac{\varepsilon C_L}{a} e^{i\varphi} = \frac{\eta^*}{n^*} e^{i\varphi_{\eta n}}, \quad \ldots \quad (3.12)$$

where η^*/n^* is the amplitude ratio, and $\varphi_{\eta n}$ the phase difference between η and n. It is seen that:

$$\varepsilon = \frac{a}{C_L} \cdot \frac{\eta^*}{n^*} \quad \ldots \quad (3.13)$$

and

$$\varphi = \varphi_{\eta n}, \quad \ldots \quad (3.14)$$

so that φ is the phase difference between η and $\hat{w}$, or between η and n. If curves of η and n have been recorded in flight, ε and φ can thus be determined at once. As to a, this can be found again from either of the formulae:

$$a = \frac{2R}{1 - p\cos\varphi_{qn}}, \quad \ldots \quad (3.15a)$$

$$a = \frac{2J}{p\sin\varphi_{qn}}, \quad \ldots \quad (3.15b)$$

analogous to (2.20a, b), because the derivation in section 2, depending on the first equation of motion only, still applies here, with no alteration except that φ_{qn} and

$$p = \frac{V}{g} \cdot \frac{q^*}{n^*} \quad \ldots \quad (3.16)$$

are now obtained from curves of q and n recorded in flight with elevator oscillating. Formulae analogous to (2.20c, d) will, of course, also hold true in the present case, but they are merely consequences of (3.15a, b).

We have not made use of the second equation of motion (3.2) yet. This being different from (2.2), we shall now arrive at new results different from (2.10) and (2.11). Substituting (3.5), (3.6) and (3.9) into (3.2), we obtain:

$$[\omega - \chi(R - iJ) + (\nu - R + iJ)(\tfrac{1}{2}a - R + iJ) + \delta\varepsilon\, e^{i\varphi}]\hat{w} = 0\,. \quad \ldots \quad (3.17)$$

The real and imaginary parts in the square bracket must both vanish, and hence:

$$\omega + \tfrac{1}{2}a\nu - R(\tfrac{1}{2}a + \nu + \chi) + R^2 - J^2 + \delta\varepsilon\cos\varphi = 0\,, \quad \ldots \quad (3.18)$$

$$J\{(\tfrac{1}{2}a + \nu + \chi) - 2R\} + \delta\varepsilon\sin\varphi = 0\,. \quad \ldots \quad (3.19)$$

We now obtain from (3.19):

$$2R = (\tfrac{1}{2}a + \nu + \chi) + \frac{\varepsilon\delta\sin\varphi}{J} \quad \ldots \quad (3.20)$$

and, substituting $(\frac{1}{2}a + \nu + \chi)$ from this into (3.18), and simplifying:

$$R^2 + J^2 = (\omega + \tfrac{1}{2}a\nu) + \varepsilon\delta\left(\cos\varphi + \frac{R}{J}\sin\varphi\right). \quad \ldots \quad (3.21)$$

Comparing (3.20, 21) with (2.10, 11), we see that R and J are now modified owing to the terms containing the factor $\varepsilon\delta$. The formulae should be compared for the same all-up weight, height and speed, which means the same incidence and Mach number. The frequencies $\bar{J}$ and J, and hence the reduced frequencies, will then differ, however, and this may affect the derivatives to a certain extent (but only $m_{\dot{w}}$, hence χ, may be appreciably sensitive to the variation of reduced frequency). It will be seen later that the difference between J and $\bar{J}$ is not likely to be large in most cases, and hence it will be usually permissible to assume that the derivatives a, ω, ν and χ are practically the same in the cases of elevator fixed and free. Subtracting (2.10) from (3.20) and (2.11) from (3.21), we may then write:

$$\bar{R} = R - \frac{\varepsilon\delta \sin \varphi}{2J}, \quad \ldots \quad (3.22)$$

$$\bar{R}^2 + \bar{J}^2 = R^2 + J^2 - \varepsilon\delta\left(\cos \varphi + \frac{R}{J} \sin \varphi\right), \quad \ldots \quad (3.23)$$

with an alternative formula for $\bar{J}$, obtained by eliminating $\bar{R}$:

$$\bar{J}^2 = J^2 - \varepsilon\delta \cos \varphi - \frac{\varepsilon^2\delta^2 \sin^2 \varphi}{4J^2} = \left(J + \frac{\varepsilon\delta \sin^2 \frac{\varphi}{2}}{J}\right)\left(J - \frac{\varepsilon\delta \cos^2 \frac{\varphi}{2}}{J}\right), \quad \ldots \quad (3.24)$$

and it is seen how the damping and frequency are affected by the oscillating elevator. The values of R, J and $(R^2 + J^2)$ now differ from $\bar{R}$, $\bar{J}$ and $(\bar{R}^2 + \bar{J}^2)$ by the terms containing the factor $\varepsilon\delta$. Owing to the normally large values of the elevator effect coefficient δ, these terms may often be quite large, even if the amplitude of the elevator (and hence the amplitude ratio ε) is small.

It may be convenient to introduce a new parameter:

$$k = \frac{\varepsilon\delta}{J^2}, \quad \ldots \quad (3.25)$$

and re-write the formulae (3.22 to 3.24) in the form:

$$\frac{\bar{R} - R}{J} = -\tfrac{1}{2}k \sin \varphi, \quad \ldots \quad (3.26)$$

$$\bar{R}^2 + \bar{J}^2 = R^2 + J^2 - kJ(J \cos \varphi + R \sin \varphi), \quad \ldots \quad (3.27)$$

$$\bar{J}/J = \sqrt{(1 - k \cos \varphi - \tfrac{1}{4}k^2 \sin^2 \varphi)} = \frac{k}{2}\sqrt{\left\{\left(\frac{2}{k} + 1 - \cos \varphi\right)\left(\frac{2}{k} - 1 - \cos \varphi\right)\right\}}. \quad \ldots \quad (3.28)$$

The formulae (3.22 to 3.24) or (3.26 to 3.28) may be directly used for solving the following important problem. Suppose the oscillatory flight tests have been made on an aircraft with manual controls (say a small model version of a large aircraft to be built) where it was difficult or impossible to prevent oscillations of the elevator. It may then be required to predict the behaviour of the same or analogous aircraft with power-operated controls (say of the intended full-scale type) assuming that the elevator-fixed conditions will apply. The oscillatory characteristics R, J, ε and φ having been determined by flight tests, the only additional quantity needed to calculate $\bar{R}$ and $\bar{J}$ will be δ (or m_η, or $\partial C_m/\partial\eta$, *see* (3.3)). Now, it will be usually not difficult to have the latter estimated with reasonable accuracy, either theoretically or from static tunnel tests or, better still, from previous static flight tests in similar operating conditions, and then $\bar{R}$ and $\bar{J}$ may be predicted at once. In most cases, R will be considerably smaller than J, and $\varepsilon\delta$ smaller than J^2 (hence $k < 1$), and the inspection of the formulae (3.26, 28) shows that the effect of fixing or freeing the elevator on damping will often be very significant, while the change of frequency should not normally be so large.

The formula (3.22) or (3.26) shows the importance of the phase angle φ (measured in flight as $\varphi_{\eta n}$), as regards its effect on damping. If φ is between 0 deg and 180 deg, *i.e.*, the elevator deflection leads the normal acceleration, then $\sin \varphi > 0$, and freeing the elevator increases the damping—this case will seldom occur in practice. More usually, φ will lie between 0 deg and $-$ 180 deg or, meaning exactly the same, between 360 deg and 180 deg; we shall say then that the elevator deflection lags behind the normal acceleration, and in such a case freeing the elevator decreases the damping.

The effect on frequency is somewhat more involved. It is first obvious from (3.24) or (3.28) that if $\cos\varphi > 0$, then $J > \bar{J}$. However, this may still be true when $\cos\varphi$ is small and negative, and an exact criterion is obtained by solving the inequality:

$$k\cos\varphi + \tfrac{1}{4}k^2\sin^2\varphi > 0 \,. \qquad \ldots \quad (3.29)$$

The solution is:

$$J > \bar{J} \text{ when } \cos\varphi > \frac{2}{k} - \sqrt{\left(\frac{4}{k^2} + 1\right)}, \qquad \ldots \quad (3.30)$$

and *vice versa*. This means that freeing the elevator increases the frequency if the elevator deflection leads the normal acceleration, or lags behind it, by an angle not exceeding the value:

$$\varphi_1 = \pi - \cos^{-1}\left\{\sqrt{\left(\frac{4}{k^2} + 1\right)} - \frac{2}{k}\right\}. \qquad \ldots \quad (3.31)$$

If the phase difference (positive or negative) exceeds numerically φ_1, then freeing the elevator decreases the frequency. The angle φ_1 is always between 90 deg and 180 deg, and in typical cases, when J is large and ε small, is only slightly greater than 90 deg. The intervals for φ, in which R and J are increased or decreased, respectively, by freeing the elevator, are illustrated graphically in Fig. 8.

A complete illustration of the formulae (3.26, 28) is given in Figs. 10 and 11. It may be mentioned that the formula (3.28) gives real values for $\bar{J}$ only if

$$\cos\varphi < \frac{2}{k} - 1 \,. \qquad \ldots \quad (3.32)$$

This inequality is satisfied for any φ if $k < 1$. If, however, k happens to be greater than 1, then the inequality may not be satisfied for small values of φ, and $\bar{J}$ becomes imaginary. In such a case the motion with elevator fixed would be aperiodic (consisting of two subsidences). This case is unlikely to occur in practice, and it is not proposed to discuss it in detail.

The equations (3.22, 23) can also be solved for R and J, with some little algebraic effort. Introducing an alternative auxiliary parameter*:

$$\bar{k} = \varepsilon\delta/\bar{J}^2 \,, \qquad \ldots \quad (3.33)$$

and eliminating R from (3.22, 23), we obtain a biquadratic equation for J:

$$4J^4 - 4(1 + \bar{k}\cos\varphi)\,\bar{J}^2J^2 - \bar{k}^2\bar{J}^4\sin^2\varphi = 0 \,, \qquad \ldots \quad (3.34)$$

the solution of which is:

$$J/\bar{J} = \tfrac{1}{2}\sqrt{[2\{\sqrt{(1 + 2\bar{k}\cos\varphi + \bar{k}^2)} + 1 + \bar{k}\cos\varphi\}]} \,, \qquad \ldots \quad (3.35)$$

and then, substituting this into (3.22), we get:

$$\frac{R - \bar{R}}{\bar{J}} = \frac{\bar{k}\sin\varphi}{\sqrt{[2\{\sqrt{(1 + 2\bar{k}\cos\varphi + \bar{k}^2)} + 1 + \bar{k}\cos\varphi\}]}} \,. \qquad \ldots \quad (3.36)$$

---

* The parameter $\bar{k}$ may be interpreted in an interesting way. $\bar{J}^2$ can be approximately replaced, in most cases, by ω (*cf.* examples in section 2.2). Taking into account the expression for ω and δ from (2.3) and (3.3), and for ε from (3.7), we may write:

$$\bar{k} \simeq \frac{\eta^*}{w^*}\frac{\partial C_m/\partial\eta}{\partial C_m/\partial\alpha}, \qquad \ldots \quad (3.33a)$$

and hence $\bar{k}$ represents, approximately, the ratio of ' moment increment due to the greatest elevator deflection ' to ' moment increment due to the greatest incidence change ' during the oscillation.

The two last formulae are illustrated by graphs in Figs. 12 and 13 which show again the effect of freeing the elevator on the frequency and damping of the short-period oscillation, as influenced by $\bar{k}$ (thus by the amplitude ratio ε) and by the phase angle φ. It is seen again that freeing the elevator decreases the damping if $\sin \varphi > 0$. The condition for the frequency to be increased by this operation is obtained from (3.35) in the form of the inequality:

$$\cos \varphi > \frac{2}{\bar{k}} - \sqrt{\left(\frac{4}{\bar{k}^2} + 1\right)}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.37)$$

which is exactly similar to (3.30), with k replaced by $\bar{k}$. This result is illustrated graphically in Fig. 9.

Interesting as the illustration of the solutions (3.35, 36) may be, it must be pointed out that these solutions do not, by themselves, provide means to predict the frequency and damping with elevator free from known characteristics of the oscillation with elevator fixed. This is because neither ε (thus $\bar{k}$) nor φ are known before the tests with elevator free have been made, and they cannot be predicted without solving the full system of dynamical equations, including that of the elevator, the procedure requiring the knowledge of an additional lot of troublesome derivatives and being outside the scope of this paper. The prediction in the inverse sense (from free to fixed elevator) has been shown already to be feasible and simple.

It may still be pointed out that, if flight tests with elevator both fixed and oscillating have been performed, then each of the equations (3.22, 23) may be used for determining δ (thus m_η, or $\partial C_m/\delta\eta$). The system is then redundant for δ, and hence the recorded quantities must satisfy a certain relationship. This can be obtained, by eliminating $\varepsilon\delta$ from (3.22) and (3.23), in the form:

$$2J(\bar{R} - R) \cot \varphi = \bar{J}^2 - J^2 + (\bar{R} - R)^2, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.38)$$

and may be utilized as a check of accuracy of the entire test technique.

The results of flight tests with oscillating elevator may be again interpreted in terms of the full rotary derivative $m_{\dot{\vartheta}}$ and manoeuvre margin H_m. We may still write, by analogy with (2.21, 22):

$$m_{\dot{\vartheta}}' = - i_B(2R - \tfrac{1}{2}a), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.39)$$

$$H_m' = \frac{i_B}{\mu} \cdot \frac{2l}{ca} (R^2 + J^2), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.40)$$

the dashes being used to denote 'effective' values for the case of the elevator oscillating. The meaning of the formulae (3.39, 40) is simply, that the aircraft with oscillating elevator behaves just as it would with elevator fixed, if the manoeuvre margin and the rotary derivative had been given the values resulting from these formulae. If the tests are made *with stick free*, then the value of H_m' should agree, at least approximately, with the familiar, 'manoeuvre margin stick-free' (*cf.* Ref. 9). The interpretation of $m_{\dot{\vartheta}}'$ is not quite so simple. We may expect, however, that such a value of the full rotary derivative would be obtained from oscillatory tunnel tests on the aircraft model, if its elevator were constrained to oscillate during the tests with the same frequency as the model, and with appropriate amplitude ratio and phase difference. This line does not seem very promising.

The above analysis rests upon the assumption that the quantities recorded in flight are n and η (and, in addition, q which, however, is only needed for determining lift slope a). This may not always be so, and some alternative sets of recorded quantities may perhaps be found preferable in certain conditions. This would, of course, necessitate a modified analysis, on similar lines. It is not proposed to start such an investigation now, anticipating changes in flight tests technique which may never take place. However, one case, based on recording θ and η, is examined briefly in Appendix V, because it links, in an interesting way, with an analogous flight-test technique for lateral oscillations, as described by Doetsch[12].

3.2. *Examples.*—*Example IV.*—Let us take the design and operating data from Example I, section 2.2, suppose in addition that $C_L = 0 \cdot 255$, and that the oscillatory characteristics, obtained with stick free, are:

$$R = 1 \cdot 925, \quad J = 8 \cdot 1, \quad q^*/n^* = 0 \cdot 1905; \quad \varphi_{qn} = 88°35'$$

(hence $\cos \varphi_{qn} = 0 \cdot 0247, \quad \sin \varphi_{qn} = 0 \cdot 9997), \quad \eta^*/n^* = 1 \cdot 21° = 0 \cdot 02112$ radn,

$$\varphi = \varphi_{\eta n} = -\ 106°\ 30' \text{ (hence } \cos \varphi = -\ 0 \cdot 2840, \quad \sin \varphi = -\ 0 \cdot 9588).$$

p is found from (3.16) and, as a check of compatibility, we determine again a from (3.15a, b):

$$p = 3 \cdot 81, \quad a = 4 \cdot 250 \text{ or } a = 4 \cdot 253, \text{ say } 4 \cdot 25 \text{ (as before).}$$

The amplitude ratio ε is obtained from (3.13):

$$\varepsilon = 0 \cdot 352\,.$$

Suppose, in addition, that $\partial C_m/\partial \eta$ has been determined by tunnel tests, or otherwise, as ($-\ 0 \cdot 00571$ per degree), so that

$$\partial C_m/\partial \eta = -\ 0 \cdot 327 \text{ per radian},$$

and δ may be found from (3.3):

$$\delta \simeq 72\,.$$

The formulae (3.22 to 3.24) now give:

$$\bar{R} = 3 \cdot 425, \quad \bar{R}^2 + \bar{J}^2 = 82 \cdot 29, \quad \bar{J} = 8 \cdot 4 \text{ (as before).}$$

The parameter k in this case is, from (3.28):

$$k = 0 \cdot 359\,,$$

and the formulae (3.35, 36) lead again to the values $R = 1 \cdot 925$, $J = 8 \cdot 1$.

The 'effective' rotary derivative, from (3.39):

$$m_{\dot\vartheta}' = -\ 0 \cdot 138, \text{ to be compared with } m_{\dot\vartheta} = -\ 0 \cdot 378,$$

and the manoeuvre margin stick free, from (3.40):

$$H_m' = 0 \cdot 074, \text{ to be compared with } H_m = 0 \cdot 088.$$

It is seen that, by freeing the elevator, nearly two-thirds of the rotary damping derivative $m_{\dot\vartheta}$ has been lost, and the oscillatory damping has been nearly halved. The manoeuvre margin has been somewhat reduced. The frequency has remained almost unchanged. The amplitude ratio q^*/n^* has decreased slightly, and the phase angle φ_{qn} has been reduced by about 10 deg—in obvious connection with the loss of damping.

Example V.—Assume the following design and operating data:

$$l/c = 3, \quad i_B = 0 \cdot 075, \quad V = 627 \cdot 9 \text{ ft/sec}, \quad \mu = 99, \quad C_L = 0 \cdot 23,$$

and suppose that flight tests with elevator both fixed and free have been made, the results being:

(i) with elevator fixed: $\bar{R} = 4$, $\bar{J} = 3 \cdot 6$, $q^*/n^* = 0 \cdot 08$ radn/sec, $\bar{\varphi}_{qn} = 112°37'$ (hence $\cos \bar{\varphi}_{qn} = -\ 0 \cdot 3846$, $\sin \bar{\varphi}_{qn} = 0 \cdot 9231$);

(ii) with elevator free: $R = 1$, $J = 3$, $q^*/n^* = 0 \cdot 0688$ radn/sec, $\varphi_{qn} = 63°26'$ (hence $\cos \varphi_{qn} = 0 \cdot 4472$, $\sin \varphi_{qn} = 0 \cdot 8944$), $\eta^*/n^* = 0 \cdot 65° = 0 \cdot 01134$ radn, $\varphi = \varphi_{\eta n} = -\ 125°45'$ or $+\ 234°15'$ (hence $\cos \varphi = -\ 0 \cdot 5842$, $\sin \varphi = -\ 0 \cdot 8116$, $\cot \varphi = 0 \cdot 7199$).

We calculate, from section 2:

$$\bar{p} = 1\cdot 56,\ a = 5,\ \nu + \chi = 5\cdot 5,\ \omega + 2\cdot 5\nu = 28\cdot 96,\ \omega - 2\cdot 5\chi = 15\cdot 21,$$

and hence $m_{\dot{\vartheta}} = -\ 0\cdot 4125$, manoeuvre margin stick fixed $H_m = 0\cdot 0263$.

Similarly, using the formulae of section 3, we obtain:

$$p = 1\cdot 3416,\ a = 5 \text{ (from either 3.15a or 3.15b) as before, and hence:}$$

$$m_{\dot{\vartheta}}' = +\ 0\cdot 0375, \text{ manoeuvre margin stick free } H_m' = 0\cdot 0091.$$

The effect of freeing the elevator is extremely strong in this case, although its oscillatory amplitude is not large. The effective rotary derivative $m_{\dot{\vartheta}}'$ has its sign changed to positive, and this accounts for a very striking decrease of damping. The latter is maintained positive only through the large value of a. The phase angle $\varphi_{q\eta}$ has been reduced by nearly 50 deg. The manoeuvre margin has dropped to about one-third of the stick-fixed value.

We may check that the condition (3.38) is satisfied, and hence we obtain the same value for δ from (3.22) and (3.23):

$$\delta = 90, \text{ thus (from 3.3) } \partial C_m/\partial\eta = 0\cdot 4091 \text{ per radian} = -\ 0\cdot 00714 \text{ per degree.}$$

The amplitude ratio ε is now obtained from (3.13) :

$$\varepsilon = 0\cdot 2465\ ,$$

and hence the parameter k becomes (from 3.25):

$$k = 2\cdot 465, \text{ which is a very high value.}$$

It is impossible, as usually, to extract the individual values of ω, ν, χ from the flight tests alone. Suppose, however, that the value $m_q = -\ 0\cdot 2748$ has been calculated theoretically; we then obtain:

$$m_{\dot{w}} = -\ 0\cdot 1377,\ \nu = 3\cdot 664,\ \chi = 1\cdot 836,\ \omega = 19\cdot 8, \text{ and } K_m = 0\cdot 018.$$

4. *Case of Elevator Oscillating. Tailless Aircraft.*—4.1. *Theory.*—The only difference in this case, as compared with that of tailed aircraft, is that the effect of elevator oscillation on the total lift cannot be neglected. The equations of motion will thus have to be written :

$$(D + \tfrac{1}{2}a)\hat{w} - \hat{q} - z_\eta\eta = 0\ , \qquad (4.1)$$

$$(\chi D + \omega)\hat{w} + (D + \nu)\hat{q} + \delta\eta = 0\ , \qquad (4.2)$$

there being the additional term $(-\ z_\eta\eta)$ in the first equation (*cf.* 3.1), and the derivative z_η is defined by:

$$z_\eta = -\ \tfrac{1}{2}\cdot\frac{\partial C_L}{\partial\eta}\ . \qquad (4.3)$$

The theory proceeds on exactly similar lines as before, but the algebra becomes more complicated. We have again:

$$D\hat{w} = (-\ R + iJ)\hat{w} \qquad (4.4)$$

and

$$\eta = \varepsilon\hat{w}\,e^{i\varphi}\ , \qquad (4.5)$$

but $\hat{q}$ is obtained from (4.1) in a form different from (3.6):

$$\hat{q} = \{(\tfrac{1}{2}a - \varepsilon z_\eta \cos\varphi - R) + i(J - \varepsilon z_\eta \sin\varphi)\}\hat{w}\ ,$$

or

$$\hat{q} = \{y_1 - R + i(J - y_2)\}\hat{w}\ , \qquad (4.6)$$

where auxiliary symbols:

$$y_1 = \tfrac{1}{2}a - \varepsilon z_\eta \cos\varphi\,, \qquad y_2 = \varepsilon z_\eta \sin\varphi \quad \ldots \quad (4.7)$$

have been introduced for abbreviation. The normal acceleration is still expressed by (*cf.* 2.12):

$$ng = \frac{V}{\hat{t}}(\hat{q} - D\hat{w}) \quad \ldots \quad (4.8)$$

but, in view of (4.1) and (4.5), we obtain:

$$n = \frac{V}{g\hat{t}}(y_1 - iy_2)\hat{w}\,, \quad \ldots \quad (4.9)$$

so that n and $\hat{w}$ are no longer in phase. Dividing (4.6) by (4.9) yields:

$$\frac{q}{n} = \frac{g}{V}\,\frac{y_1 - R + i(J - y_2)}{y_1 - iy_2} = \frac{q^*}{n^*}\,e^{i\varphi_{qn}}\,, \quad \ldots \quad (4.10)$$

so that, putting again (*cf.* 3.16):

$$p = \frac{V}{g}\,\frac{q^*}{n^*}\,, \quad \ldots \quad (4.11)$$

we obtain:

$$y_1 - R + i(J - y_2) = p(y_1 - iy_2)(\cos\varphi_{qn} + i\sin\varphi_{qn})\,, \quad \ldots \quad (4.12)$$

or, separating real and imaginary parts:

$$\left.\begin{aligned} y_1 - R &= p(y_1\cos\varphi_{qn} + y_2\sin\varphi_{qn})\,, \\ J - y_2 &= p(y_1\sin\varphi_{qn} - y_2\cos\varphi_{qn})\,. \end{aligned}\right\} \quad \ldots \quad (4.13)$$

Flight tests will provide the values of R, J, p and φ_{qn}, and then the equations (4.13) give the following solutions for y_1 and y_2:

$$\left.\begin{aligned} y_1 &= \frac{R(1 - p\cos\varphi_{qn}) + Jp\sin\varphi_{qn}}{1 - 2p\cos\varphi_{qn} + p^2}\,, \\ y_2 &= \frac{J(1 - p\cos\varphi_{qn}) - Rp\sin\varphi_{qn}}{1 - 2p\cos\varphi_{qn} + p^2}\,. \end{aligned}\right\} \quad \ldots \quad (4.14)$$

These are not sufficient for determining the four unknown quantities a, z_η, ε and φ in (4.7). However, flight tests also supply the amplitude ratio η^*/n^* and the phase angle $\varphi_{\eta n}$. Dividing (4.5) by (4.9), we get:

$$\frac{\eta}{n} = \varepsilon\,\frac{g\hat{t}}{V}\,\frac{e^{i\varphi}}{y_1 - iy_2} = \frac{\eta^*}{n^*}\,e^{i\varphi_{\eta n}}, \quad \ldots \quad (4.15)$$

and hence, introducing for abbreviation:

$$m = \frac{V}{g\hat{t}}\,\frac{\eta^*}{n^*} = \frac{2}{C_L}\cdot\frac{\eta^*}{n^*}\,, \quad \ldots \quad (4.16)$$

we obtain:

$$\varepsilon(\cos\varphi + i\sin\varphi) = m(y_1 - iy_2)(\cos\varphi_{\eta n} + i\sin\varphi_{\eta n})\,, \quad \ldots \quad (4.17)$$

whence:

$$\varepsilon = m\sqrt{(y_1^2 + y_2^2)} = m\sqrt{\left(\frac{R^2 + J^2}{1 - 2p\cos\varphi_{qn} + p^2}\right)} \quad \ldots \quad (4.18)$$

and

$$\cos\varphi = \frac{y_1\cos\varphi_{\eta n} + y_2\sin\varphi_{\eta n}}{\sqrt{(y_1^2 + y_2^2)}}\,, \quad \sin\varphi = \frac{y_1\sin\varphi_{\eta n} - y_2\cos\varphi_{\eta n}}{\sqrt{(y_1^2 + y_2^2)}}\,. \quad \ldots \quad (4.19)$$

It is seen that, using all three recorded curves of q, n and η, it is possible to calculate y_1 and y_2 from (4.14), then ε and φ from (4.18, 19), and finally z_η and a from (4.7) which give:

$$z_\eta = \frac{y_2}{\varepsilon \sin \varphi} = \frac{y_2}{m(y_1 \sin \varphi_{\eta n} - y_2 \cos \varphi_{\eta n})}, \quad \ldots \quad (4.20)$$

$$a = 2(y_1 + \varepsilon z_\eta \cos \varphi) = 2(y_1 + y_2 \cot \varphi) . \quad \ldots \quad (4.21)$$

It should be noticed that, if z_η is negligibly small (assumption legitimate only for most tailed aircraft), then we may put $y_2 = 0$, and in such a case the formulae (4.14) lead to both relationships (3.15a, b), while (4.18, 19) reduce to (3.13, 14), and we come back to the case considered in section 3.

It may happen that z_η is not negligible even for a tailed aircraft, especially one with a short fuselage and an all-moving tail. Such cases will be recognized by the fact that the formulae (3.15a, b) give appreciably different values for a, and then the present method should be used, instead of that given in section 3.

The second equation of motion (4.2) has not been used yet. We now substitute (4.4, 5, 6) into (4.2), and obtain:

$$[\omega - \chi(R - iJ) + (\nu - R + iJ)\{y_1 - R + i(J - y_2)\} + \varepsilon\delta\, e^{i\varphi}]\hat{w} = 0 . \quad \ldots \quad (4.22)$$

The real and imaginary parts in the square bracket must both vanish, and hence:

$$\omega + \nu y_1 - R(y_1 + \nu + \chi) + R^2 - J^2 + J y_2 + \varepsilon\delta \cos \varphi = 0 , \quad \ldots \quad (4.23)$$

$$J\{(y_1 + \nu + \chi) - 2R\} - y_2(\nu - R) + \varepsilon\delta \sin \varphi = 0 . \quad \ldots \quad (4.24)$$

We now obtain from (4.24):

$$2R = y_1 + \nu + \chi + \frac{\varepsilon\delta \sin \varphi - y_2(\nu - R)}{J} \quad \ldots \quad (4.25)$$

and, substituting $(y_1 + \nu + \chi)$ from this into (4.23), and simplifying:

$$R^2 + J^2 = \omega + \nu y_1 + \varepsilon\delta\left(\cos \varphi + \frac{R}{J} \sin \varphi\right) - y_2\left(J - R\frac{\nu - R}{J}\right). \quad \ldots \quad (4.26)$$

These formulae are analogous to (3.20, 21), and become identical when z_η is neglected. The analogy can be carried through even further. Let us introduce two new parameters δ', φ', defined by the relationships:

$$\left.\begin{aligned} \delta' \cos \varphi' &= \{\delta - z_\eta(\nu - R)\} \cos \varphi + z_\eta J \sin \varphi , \\ \delta' \sin \varphi' &= \{\delta - z_\eta(\nu - R)\} \sin \varphi - z_\eta J \cos \varphi , \end{aligned}\right\} \quad \ldots \quad (4.27)$$

which also give:

$$\delta'^2 = \delta^2 - 2z_\eta\delta(\nu - R) + z_\eta^2\{(\nu - R)^2 + J^2\} . \quad \ldots \quad (4.28)$$

Replacing then y_1 and y_2 in (4.25, 26) by their expressions from (4.7), and taking into account (2.10, 11), we obtain:

$$\bar{R} = R - \frac{\varepsilon\delta' \sin \varphi'}{2J} , \quad \ldots \quad (4.29)$$

$$\bar{R}^2 + \bar{J}^2 = R^2 + J^2 - \varepsilon\delta'\left(\cos \varphi' + \frac{R}{J} \sin \varphi'\right), \quad \ldots \quad (4.30)$$

with an alternative formula for $\bar{J}$, obtained by eliminating $\bar{R}$:

$$\bar{J}^2 = J^2 - \varepsilon\delta' \cos \varphi' - \frac{\varepsilon^2\delta'^2 \sin^2 \varphi'}{4J^2} . \quad \ldots \quad (4.31)$$

The equations (4.29 to 4.31) are exactly similar to (3.22 to 3.24), and the only difference consists in δ and φ being replaced by δ' and φ'. The entire remainder of section 3.1, including the formulae (3.25 to 3.40) and Figs. 8 to 13, apply here therefore with no modification except δ and φ being replaced by δ' and φ', and k, $\bar{k}$ being replaced by:

$$k' = \frac{\varepsilon\delta'}{J^2}, \quad \bar{k}' = \frac{\varepsilon\delta'}{\bar{J}^2}. \qquad \ldots \quad (4.32)$$

It may be mentioned that, δ being usually large and z_η small, the formulae (4.27, 28) show that the differences between δ' and δ, or φ' and φ, are small. Also, y_2 is normally much smaller than y_1, and therefore φ differs little from $\varphi_{\eta n}$. The three angles $\varphi_{\eta n}$, φ and φ' will therefore usually have very similar values.

4.2. *Example VI.*—Assume the following design and operating data:

$$l/c = 1, \quad i_B = 0\cdot3, \quad V = 849 \text{ ft/sec}, \quad \mu = 120, \quad C_L = 0\cdot225,$$

and the following oscillatory characteristics obtained with stick free:

$$R = 1, \quad J = 4, \quad q^*/n^* = 0\cdot08, \quad \varphi_{qn} = 78° \, 49'$$

(hence $\cos \varphi_{qn} = 0\cdot1939$, $\sin \varphi_{qn} = 0\cdot9810$), $\eta^*/n^* = 0\cdot716° = 0\cdot0125$ radn, $\varphi_{\eta n} = -141° \, 21'$ (hence $\cos \varphi_{\eta n} = -0\cdot7810$, $\sin \varphi_{\eta n} = -0\cdot6246$).

We determine p from (4.11):

$$p = 2\cdot1093\,.$$

Let us try to find a as for conventional tailed aircraft. The formulae (3.15a) and (3.15b) give the values $3\cdot384$ and $3\cdot866$, respectively. These values differ considerably, and clearly the method of section 4.1 must be used. We obtain from (4.14, 16, 18, 19, 20, 21):

$$y_1 = 1\cdot9148\,, \qquad y_2 = 0\cdot0637\,, \qquad m = 0\cdot1111\,, \qquad \varepsilon = 0\cdot2128\,,$$

$$\cos\varphi = -0\cdot8013\,, \quad \sin\varphi = -0\cdot5983\,, \quad \varphi = -143° \, 15'\,, \quad z_\eta = 0\cdot500\,, \quad a = 4\cdot000\,,$$

and it is seen that our final value of a differs considerably from either of the previous tentative values. Suppose now that $\partial C_m/\partial\eta$ has been determined independently as ($-0\cdot00358$ per degree) so that:

$$\partial C_m/\partial\eta = -0\cdot205 \text{ per radian, and from (3.3) } \delta = 41\,,$$

and m_q has been calculated theoretically as ($-0\cdot48$) so that:

$$\nu = 1\cdot6\,.$$

We now calculate from (4.27, 28):

$$\delta'\cos\varphi' = -31\cdot897\,, \quad \delta'\sin\varphi' = -26\cdot312\,, \quad \delta' = 41\cdot349\,,$$

$$\cos\varphi' = -0\cdot7714\,, \quad \sin\varphi' = -0\cdot6363\,, \quad \varphi' = -140° \, 29'\,,$$

and from (4.29 to 31):

$$\bar{R} = 1\cdot7\,, \qquad \bar{J} = 4\cdot722\,, \qquad \bar{R}^2 + \bar{J}^2 = 25\cdot19\,.$$

The parameters k' and $\bar{k}'$ are, from (4.32):

$$k' = 0\cdot5499\,, \qquad \bar{k}' = 0\cdot3946\,,$$

and the formulae (3.35, 36), using the values of $\bar{k}'$ and φ' lead again to $R = 1$, $J = 4$.

We obtain further, from (2.18) :

$$a\bar{p} = 9\cdot4632\,, \qquad \bar{p} = 2\cdot3658\,; \qquad \bar{q}^*/\bar{n}^* = 0\cdot0897\,,$$

$$\cos\bar{\varphi}_{qn} = 0\cdot0634\,, \qquad \sin\bar{\varphi}_{qn} = 0\cdot9980\,; \qquad \bar{\varphi}_{qn} = 86^\circ\,22'\,,$$

and it is seen that, by freeing the elevator, the amplitude ratio q^*/n^* and the phase difference φ_{qn} are both reduced, this accompanying a significant loss of damping and a slight decrease of frequency.

The formulae (2.10, 11) now give:

$$\nu + \chi = 1\cdot4\,, \qquad \chi = -\,0\cdot2\,, \qquad \omega = 21\cdot99\,,$$

and hence,

$$m_{\dot\theta} = -\,0\cdot42\,, \quad m_q = -\,0\cdot48\,, \quad m_{\dot w} = +\,0\cdot06\,, \quad K_m = 0\cdot0275\,, \quad H_m = 0\cdot0315\,.$$

The ' effective ' rotary derivative, from (3.39) becomes :

$$m_{\dot\theta}' = 0\,,$$

and the manoeuvre margin stick free, from (3.40):

$$H_m' = 0\cdot0213\,.$$

It is seen that, in this case, there is no rotary damping left at all with elevator free, and the oscillatory damping is maintained positive only through the lift derivative a.

5. *Remarks about Accuracy of the Analysis.*—The presentation of the method proposed in this report would be incomplete without an attempt to assess its accuracy. The numerical examples have been based on the assumption of almost perfect exactitude of the measured quantities and thus give an idealized picture which will never be attained in practice. All directly measured quantities will in reality be burdened with errors which will entail errors of calculated derivatives. The initial errors may be due to various causes, such as:

(*a*) incomplete fulfilment of the assumed circumstances of the flight disturbance, *e.g.*, through gusts, small unintended stick movements, friction in the elevator circuit, etc.,

(*b*) imperfection of instruments,

(*c*) individual inaccuracies in instrument readings.

The final errors in estimating the derivatives may be caused to a small degree by certain simplifying assumptions of the theory, but they will be mainly due to the effects of the initial errors in the measured quantities. They may be represented as combinations of these initial errors depending on the structure of determining formulae. All final inaccuracies may be considerably reduced by repeated tests and correct application of the theory of errors.

It is seen that we have to deal with a major problem which it would be premature to solve in its entirety at the present stage, before a considerable experience is gained in practical application of the proposed method. The main difficulty is to assess the probable magnitude of initial errors of recorded quantities which, of course, may vary considerably in different tests, and will be gradually diminished by improving the instruments and test technique. It is not intended to give here a complete study of the problem. We shall limit ourselves to examining briefly only the simplest case of oscillations with elevator fixed, as described in section 2, assuming only unrepeated tests, and introducing certain hypothetical numerical magnitudes of initial errors, based on some past experience.

Let us consider, as directly measured quantities, the frequency $\bar{J}$, damping parameter $\bar{R}$, dimensionless amplitude ratio $\bar{p}$ (*see* 2.17) and the phase angle $\bar{\varphi}_{qn}$. The initial independent errors assumed will be:

$$\Delta\bar{J}/\bar{J}\,, \qquad \Delta\bar{R}/\bar{R}\,, \qquad \Delta\bar{p}/\bar{p}\,, \text{ and } \Delta\bar{\varphi}_{qn}\,,$$

i.e., per cent errors in $\bar{J}$, $\bar{R}$ and $\bar{p}$, and absolute error in $\bar{\varphi}_{qn}$ (in degrees or radians). The main calculated quantities are:

$$a\,, \qquad m_{\dot{\theta}} = -\,i_B(\nu + \chi), \text{ and } H_m.$$

The first of them, lift slope a, is determined by either of the formulae (2.20a, b). Taking logarithms of both parts in each formula and differentiating, we obtain:

$$\frac{da}{a} = \frac{d\bar{R}}{\bar{R}} + \frac{\cos\bar{\varphi}_{qn}}{1 - \bar{p}\cos\bar{\varphi}_{qn}}\,d\bar{p} - \frac{\bar{p}\sin\bar{\varphi}_{qn}}{1 - \bar{p}\cos\bar{\varphi}_{qn}}\,d\bar{\varphi}_{qn}\,, \qquad \text{(5.1a)}$$

or

$$\frac{da}{a} = \frac{d\bar{J}}{\bar{J}} - \frac{d\bar{p}}{\bar{p}} - \cot\bar{\varphi}_{qn}\,d\bar{\varphi}_{qn}\,. \qquad \text{(5.1b)}$$

Replacing differentials by finite (but supposedly small) errors, we may re-write (5.1) as follows:

$$\frac{\Delta a}{a} = \frac{\Delta\bar{R}}{\bar{R}} + \frac{\bar{p}\cos\bar{\varphi}_{qn}}{1 - \bar{p}\cos\bar{\varphi}_{qn}}\cdot\frac{\Delta\bar{p}}{\bar{p}} - \frac{\bar{J}}{\bar{R}}\,\Delta\bar{\varphi}_{qn}\,, \qquad \text{(5.2a)}$$

or

$$\frac{\Delta a}{a} = \frac{\Delta\bar{J}}{\bar{J}} - \frac{\Delta\bar{p}}{\bar{p}} - \cot\bar{\varphi}_{qn}\cdot\Delta\bar{\varphi}_{qn}\,, \qquad \text{(5.2b)}$$

and these formulae give an estimate of percentage error in a, in terms of the initial errors, when a is calculated alternatively from (2.20a) or (2.20b). It should be noted that $\Delta\bar{\varphi}_{qn}$ must be taken in radians.

Similarly, the rotary derivative $m_{\dot{\theta}}$ is determined by the formula (2.21) which, if a is obtained from either of (2.20), becomes alternatively:

$$-\,m_{\dot{\theta}}/i_B = \bar{R}\,\frac{1 - 2\bar{p}\cos\bar{\varphi}_{qn}}{1 - \bar{p}\cos\bar{\varphi}_{qn}}\,, \qquad \text{(5.3a)}$$

or

$$-\,m_{\dot{\theta}}/i_B = 2\bar{R} - \frac{\bar{J}}{\bar{p}\sin\bar{\varphi}_{qn}}\,. \qquad \text{(5.3b)}$$

Taking logarithms again, differentiating, simplifying, and replacing differentials by finite errors, we obtain:

$$\frac{\Delta m_{\dot{\theta}}}{m_{\dot{\theta}}} = \frac{\Delta\bar{R}}{\bar{R}} - \frac{\bar{p}\cos\bar{\varphi}_{qn}}{(1 - \bar{p}\cos\bar{\varphi}_{qn})(1 - 2\bar{p}\cos\bar{\varphi}_{qn})}\cdot\frac{\Delta\bar{p}}{\bar{p}} + \frac{\bar{J}}{\bar{R}(1 - 2\bar{p}\cos\bar{\varphi}_{qn})}\cdot\Delta\bar{\varphi}_{qn} \qquad \text{(5.4a)}$$

or

$$\frac{\Delta m_{\dot{\theta}}}{m_{\dot{\theta}}} = \frac{\Delta\bar{R}}{\bar{R}} - \frac{\Delta\bar{J}}{\bar{J}} + \frac{\Delta\bar{p}}{\bar{p}} + \cot\bar{\varphi}_{qn}\cdot\Delta\bar{\varphi}_{qn}. \qquad \text{(5.4b)}$$

Finally, the manoeuvre margin H_m is determined by (2.22) or:

$$H_m = \text{const}\,(\bar{R}^2 + \bar{J}^2)\,, \qquad \text{(5.5)}$$

and hence:

$$\frac{\Delta H_m}{H_m} = \frac{2\bar{R}^2}{\bar{R}^2 + \bar{J}^2}\,\frac{\Delta\bar{R}}{\bar{R}} + \frac{2\bar{J}^2}{\bar{R}^2 + \bar{J}^2}\,\frac{\Delta\bar{J}}{\bar{J}}\,. \qquad \text{(5.6)}$$

We shall now assume two following alternative sets of initial errors, given by W. Pinsker, as reasonable upper and lower limiting values, according to the experience of the Flight Section, Aerodynamics Department, R.A.E.:

	$\Delta\bar{R}/\bar{R}$	$\Delta\bar{J}/\bar{J}$	$\Delta\bar{p}/\bar{p}$	$\Delta\bar{\varphi}_{qn}$
Large initial errors ..	$\pm 15\%$	$\pm 5\%$	$\pm 6\%$	± 10 deg $= \pm 0 \cdot 1745$ radn
Small initial errors ..	$\pm\ 2\%$	$\pm 1\%$	$\pm 2\%$	$\pm 2 \cdot 5$ deg $= \pm 0 \cdot 0436$ radn

The values of the first line seem very pessimistic; they have been thought, however, to apply when the damping is very high so that the oscillatory curves possess only two detectable peaks. The satisfactory values of the second line are the best obtainable now but only when the damping is not very strong so that several peaks are clearly seen on the curves. All sorts of intermediate values may, of course, apply. The progress of instrumentation and test technique should bring their values nearer to the present small limits, and possibly even lower.

Considering our three numerical examples of section 2.2, introducing alternatively the larger or smaller values of the initial errors as listed above, with such signs that all terms in the error formulae (5.2, 4, 6) add up (the most disadvantageous case), the following table has been obtained:

Greatest possible percentage errors of the calculated results in Examples I, II, III

Initial errors assumed	Example	$\Delta a/a$		$\Delta m_{\dot{\vartheta}}/m_{\dot{\vartheta}}$		$\Delta H_m/H_m$
		from (2.20a)	from (2.20b)	from (5.3a)	from (5.3b)	
Large	I	59·8	13·7	35·1	28·7	12·8
	II	56·9	11·0	56·9	26·0	13·0
	III	67·4	12·0	90·6	27·0	12·1
Small	I	19·3	3·7	7·1	5·7	2·3
	II	12·5	3·0	12·5	4·0	2·3
	III	15·2	5·7	21·0	7·7	2·2

The results given in the above table are instructive, and contain both an encouragement and a warning. It is seen first that, of the two alternative formulae (2.20a) and (2.20b), the latter is much more reliable and gives tolerable errors even if the initial errors are very large. If the initial errors are reasonably small, then the accuracy of all calculated results is entirely satisfactory. It must be stressed that the figures in our table have been obtained on the most pessimistic assumption that all partial errors add up, while in reality they will partly cancel each other, so that the average errors should be about half those given. It is seen, however, that an uncritical interpretation of inaccurate initial data may lead to quite unreliable results, and that no efforts should be spared to keep the initial errors as small as possible. This requires:

(i) a constant striving to improve the experimental technique and instrumentation,

(ii) efforts to improve the interpretation of the recorded curves.

The first point is beyond the scope of this paper. As to the second one, however, it is expected that the method of filtration, expounded in Appendix II, will contribute considerably to the accuracy in determination of $\bar{R}$ and $\bar{J}$ from the curves. This should apply especially in cases of heavy damping which have been considered with pessimism by experimenters using less refined methods. It seems now that, even when only three or two peaks of the curves are detectable, the method of filtration if judiciously applied (*cf.* Note 2, Appendix II) should ensure much smaller errors than heretofore. The matter may be investigated further by applying the theory of errors, but this seems somewhat premature at present.

6. *Conclusions.*—The main conclusions of the report can be summarized as follows:

(*a*) Valuable information about the main longitudinal stability derivatives may be obtained from flight tests by recording curves of rate of pitch q, normal acceleration factor n, and elevator displacement η, during the initial stage of a free disturbed flight with elevator fixed and/or free, following a rapid fore-and-aft stick movement. This set of quantities to be recorded seems most appropriate, and η should be recorded even when the elevator is intended to be fixed, if only as a check.

(*b*) The initial elevator manoeuvre required to start the disturbed flight should consist of two opposite elevator movements following each other rapidly so that, at the end of the manoeuvre, the elevator is brought back to the original equilibrium position of trimmed level flight. The resulting disturbance then consists mainly of the short-period oscillation, with only a small amount of the phugoid one.

(*c*) In order to eliminate the small distortion of the recorded oscillatory curves caused by the unavoidable phugoid intrusion, a simple filtering procedure, as explained in Appendix II, should be applied to the curves, prior to interpretation.

(*d*) The following quantities should then be read (in the manner described in Appendix III) directly from the curves: frequency J and damping factor R, common to all curves; further, amplitude ratio q^*/n^* and phase angle φ_{qn}, and (if applicable) η^*/n^* and $\varphi_{\eta n}$.

(*e*) The final step is to calculate the derivatives according to formulae given in section 2 for the case of elevator fixed, in section 3 for orthodox tailed aircraft with elevator oscillating, and in section 4 for tailless or short-tailed craft with elevator oscillating. The quantities which can be calculated are: the aircraft lift slope a, the total rotary damping derivative $m_{\dot\theta}$, and the manoeuvre margin H_m for elevator fixed or oscillating; also, for the case of section 4, the derivative z_η (aircraft lift slope due to elevator).

(*f*) Unless supplementary information from other sources is available, the technique described gives no means for isolating the partial damping derivatives m_q and $m_{\dot w}$, being the constituent parts of $m_{\dot\theta}$, nor does it furnish more than a rough estimate of the restoring margin K_m.

(*g*) From a test with elevator moving the characteristics with elevator fixed can be deduced, assuming the derivatives do not change with frequency, for both tailed and tailless aircraft—*see* sections 3 and 4. This process cannot be put in reverse.

LIST OF SYMBOLS

A	General symbol for amplitude of any oscillating quantity, *see* (I.1) and (II.1)
a	Lift-curve slope, *see* (2.3)
a_s	Speed of sound, ft/sec
B	Auxiliary constant, *see* (II.11)
B_1, C_1, D_1, E_1	Coefficients of stability quartic, *see* (I.17)
b', c', d'	Coefficients, *see* (I.8) and (I.23)
C	Free term of stability quadratic, *see* (I.2, 3)
C_D	Drag coefficient
C_L	Lift coefficient
C_m	Pitching-moment coefficient
c	Wing mean chord
D	Differential operator, *see* (2.3)
g	Gravity constant, ft/sec^2
H, h	Free terms of quadratic factors of $\Delta(D)$, *see* (I.17)
H_m	Manoeuvre margin, stick fixed, *see* (2.22)
H_m'	Manoeuvre margin, stick free, *see* (3.40)
i_B	Inertia coefficient (about y-axis), *see* (2.3)
i_C	Inertia coefficient (about z-axis), *see* (V.14)
J	Angular frequency of short-period oscillation, dimensionless, *see* (III.4)
$\mathscr{J}$	Angular frequency of short-period oscillation, in sec^{-1}, *see* (III.3)
j	Angular frequency of phugoid oscillation, dimensionless, *see* (I.19)
K	Parameter depending on height, *see* (IV.4) and Fig. 15
K_m	Restoring margin, *see* (2.3)
K_n	Static margin
$k, \bar{k}, k', \bar{k}'$	Auxiliary parameters, *see* (3.25, 33) and (4.32)
k_B	Radius of gyration of aircraft about y-axis, ft
k_N	Coefficient, *see* (II.7)
l	Representative length (tail arm or, for tailless aircraft, mean wing chord)
M	Mach number, *see* (IV.2)
$M_{\dot{w}}$	Pitching-moment derivative due to rate of change of w, dimensional, *see* text following (2.3)
m	*See* (4.16)
m_q	(Steady) rotary damping derivative in pitch, dimensionless
m_w	Pitching-moment derivative due to w, dimensionless
$m_{\dot{w}}$	Pitching-moment derivative due to rate of change of w, dimensionless, *see* (2.3) and following text

m_η	Pitching-moment derivative due to elevator displacement, dimensionless
$m_{\dot\theta}$	Full rotary damping derivative, dimensionless, *see* (2.4)
$m_{\dot\theta}'$	' Effective ' value of $m_{\dot\theta}$ for elevator oscillating, *see* (3.39)
N	Arbitrary integer, *see* (II.7)
$N(\tau)$	Shorthand symbol for function of τ representing normal acceleration factor, *see* (I.11) and (I.22)
$N_1, P_1, Q_1, R_1, S_1, T_1$	Shorthand constants, *see* (I.18)
n	Normal acceleration factor (number of ' *gs* ' recorded during a disturbance), equal to ' load factor minus one ', dimensionless, *see* (2.12) and (3.10)
$n_1, n_2, n_3 \ldots$	Consecutive peaks of n, *see* (III.5)
n_v, n_r, n_ζ	Yawing-moment derivatives, dimensionless, *see* (V.14)
P	Period of oscillation, in seconds, *see* (III.3) and Fig. 1
p	Amplitude ratio q^*/n^* made dimensionless, *see* (2.17) and (3.16)
$Q(\tau)$	Shorthand symbol for function of τ representing rate of pitch, *see* (I.10) and (I.21)
q	Rate of pitch, in radians per sec
q_1, q_2, q_3	Consecutive peaks of q, *see* (III.10) and Fig. 1
$\hat{q}$	Rate of pitch, dimensionless, *see* (2.3)
$\dot{q}$	Angular acceleration in pitch, in radians per sec^2
R	Damping factor of short-period oscillation, dimensionless, *see* (III.8)
$\mathscr{R}$	Damping factor of short-period oscillation, in sec^{-1}, *see* (III.7)
r	Damping factor of phugoid oscillation, dimensionless, *see* (I.19)
$\hat{r}$	Rate of yaw, dimensionless, *see* (V.13)
S	Gross wing area, sq ft
t	Time, secs
t_0	Value of t corresponding to the first zero value of the oscillating quantity, *see* (1.1)
$t_1, t_2, t_3 \ldots$	Values of t corresponding to consecutive peaks of x, *see* (II.20)
$t_I, t_{II}, t_{III}, \ldots$	Values of t corresponding to consecutive zeros of x, *see* (II.22)
$\hat{t}$	Unit of aerodynamic time, in seconds, *see* (2.3) and Appendix IV
u	Increment of velocity along x-axis in disturbed flight, ft/sec
$\hat{u}$ =	u/V. Increment of velocity along x-axis in disturbed flight, dimensionless
V	Velocity of aircraft in undisturbed flight, ft/sec
v	Increment of velocity along y-axis in disturbed flight, ft/sec
W	Weight of aircraft, lb

w	Increment of velocity along z-axis in disturbed flight, ft/sec
$\hat{w}$	Increment of incidence in disturbed flight, dimensionless, *see* (2.3)
x	General symbol for any recorded oscillatory quantity, *see* (II.1)
x_0	Ordinate of 'zero line', *see* (II.2)
x_1, x_2, x_3	Consecutive peak values of x, *see* (II.13)
x_u, x_w	Longitudinal force derivatives due to u or w, dimensionless, *see* (I.16)
Y	Elevator impulse, *see* (I.12)
y_1, y_2	Auxiliary symbols, *see* (4.7)
y_v	Lateral force derivative due to v, dimensionless, *see* (V.13)
z_u, z_w	Normal force derivatives due to u or w, dimebsionless, *see* footnote to equation (2.3), and (I.16)
z_η	Normal force derivative due to elevator displacement, *see* (4.1) and (4.3)
α	Wing incidence, radians
β	Angle of sideslip
α_0, β_0	Parameters of 'zero line', *see* (II.1, 2)
γ	Auxiliary constant, *see* (II.11)
$\Delta(D)$	Operational determinant (stability quadratic or quartic), *see* (I.2) and (I.17).
δ	Compound pitching-moment derivative due to elevator displacement, dimensionless, *see* (3.3)
δ_n	Compound yawing-moment derivative due to rudder displacement, dimensionless, *see* (V.14)
δ'	*See* (4.27, 28)
ε	Amplitude ratio of η and $\hat{w}$, *see* (3.7)
ζ	Angular displacement of rudder from equilibrium position
φ	$= \varphi_{\eta w}$. Phase angle by which η leads w, *see* (3.9)
φ'	*See* (4.27, 28)
φ_{qn}	Phase angle by which q leads n, *see* (2.15)
$\varphi_{\eta n}$	Phase angle by which η leads n, *see* (3.12)
$\varphi_{\eta\theta}$	Phase angle by which η leads θ, *see* (V.2)
φ_1	Value of φ corresponding to equal frequencies with elevator fixed or oscillating, *see* (3.31) and Fig. 9.
ψ	Angular displacement in yaw from equilibrium position, radians
η	Angular displacement of elevator from equilibrium position, radians
$\eta_\times$	Magnitude of instantaneous angular displacement of elevator, *see* Figs. 2a and 2b.
θ	Angular displacement in pitch from equilibrium position
ϑ	Auxiliary variable, *see* (II.3)

ϑ_0	Auxiliary constant, *see* (II.4)
$\vartheta_1, \vartheta_2, \vartheta_3$	Values of ϑ corresponding to consecutive peaks of x
$\varkappa$	$= -\mu m_u/i_B$. Compound pitching-moment derivative due to u, dimensionless, *see* (I.16)
Λ_1, Λ_2	Coefficients of response functions, *see* (I.24, 26)
μ	Relative density of aircraft, *see* (2.3)
μ_2	Alternative relative density of aircraft (for lateral disturbances)
ν	Compound (steady) rotary damping derivative, dimensionless, *see* (2.3)
ν_n	Compound rotary damping derivative (in yaw), dimensionless, *see* (V.14)
Π_1, Π_2	Coefficient of response functions, *see* (I.24, 26)
ρ	Air density, slugs/cu ft
ρ_0	Value of ρ on ground level
σ	Relative density, *see* (IV.2)
τ	Aerodynamic time, dimensionless, *see* (2.3)
τ_0	*See* (2.5)
$\tau_\times$	Time interval between two consecutive elevator displacements, *see* Fig. 2b
χ	Compound pitching-moment derivative due to rate of change of w, dimensionless, *see* (2.3)
ω	Compound pitching-moment derivative due to w, dimensionless, *see* (2.3)
ω_n	Compound yawing moment derivative due to v, dimensionless, *see* (V.14)

REFERENCES

No.	*Author*	*Title, etc.*
1	L. W. Bryant and S. B. Gates ..	Nomenclature for stability coefficients. R. & M. 1801. October, 1937.
2	R. T. Jones and D. Cohen ..	An analysis of the stability of an airplane with free controls. N.A.C.A. Report 709. 1941.
3	S. B. Gates	Proposal for an elevator manoeuvrability criterion. R. & M. 2677. June, 1942.
4	H. M. Lyon, P. M. Truscott, E. I. Auterson and J. Whatham.	A theoretical analysis of longitudinal dynamic stability in gliding flight. R. & M. 2075. August, 1942.
5	S. Neumark	The disturbed longitudinal motion of an uncontrolled aircraft and of an aircraft with automatic control. R. & M. 2078. January, 1943.
6	J. Whatham and H. M. Lyon ..	A theoretical investigation of dynamic stability with free elevators. R. & M. 1980. March, 1943.
7	W. S. Brown	The longitudinal stability of an aircraft with free elevators. R. & M. 1981. April, 1943.
8	H. Greenberg and L. Sternfield ..	A theoretical investigation of longitudinal stability of airplanes with free controls, including effect of friction in control system. N.A.C.A. Report No. 791. February, 1944. Reissued as A.R.C. 8023.
9	S. B. Gates and H. M. Lyon ..	A continuation of longitudinal stability and control analysis. Part I. General theory. R. & M. 2027. February, 1944.
10	S. Neumark	A simplified theory of the lateral oscillations of an aircraft with rudder free, including the effect of friction in the control system. R. & M. 2259. May, 1945.
11	J. Whatham and E. Priestley ..	Lontitudinal response theory by the method of Laplace transform. R.A.E. Report Aero. 2160. A.R.C. 10,375. September, 1946. (Unpublished.)
12	K. H. Doetsch	Interim report on snaking behaviour of the *Meteor III*. R.A.E. Report Aero. 2288. A.R.C. 11,947. August, 1948. (Unpublished.)
13	S. Neumark and A. W. Thorpe ..	Theoretical requirements of tunnel experiments for determining stability derivatives in oscillatory longitudinal disturbances. R. & M. 2903. June, 1950.
14	S. Neumark	Two-dimensional theory of oscillating aerofoils, with application to stability derivatives. R.A.E. Report Aero. 2449. A.R.C. 14,889. November, 1951. (Unpublished.)
15	A. W. Thorpe	An approximation to the slow mode of longitudinally disturbed motion of an aircraft in level flight. R. & M. 2907. September, 1951.
16	H. Jeffreys	Operational methods in mathematical physics. Cambridge Tracts in Mathematics and Mathematical Physics, No. 23. Cambridge University Press, 1931.
17	W. Milliken, Jr.	Dynamic stability and control research. Third Anglo-American Aeronautical Conference, Brighton, September, 1951. Proceedings published by R.Ae.Soc., London, 1951. Pp. 447 to 524.

APPENDIX I

Response to Typical Elevator Manoeuvres

The purpose of this Appendix is to calculate the response in q and n following a double elevator movement (rectangular input) as described in the Introduction and illustrated in Fig. 2b. Such an elevator movement may be considered as a combination of two consecutive instantaneous movements (with step input—*cf.* Fig. 2a), the two elevator displacements being opposite but numerically equal ($\eta_\times$) and following each other with a time interval $\tau_\times$ (in aerodynamic units). We shall start by determining the response to a single step input. The calculation will be done in two different ways: (*a*) neglecting the speed variation and thus arriving at an approximate solution with the short-period mode only, and (*b*) including the speed variation, and thus obtaining a more rigorous solution with both short-period and phugoid mode.

(*a*) *Approximate Solution* (*speed variation neglected*).—The equations of motion will be written in the form similar to (2.1, 2) and (3.1, 2):

$$\left.\begin{aligned}(D + \tfrac{1}{2}a)\hat{w} - \hat{q} &= 0\,,\\ (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} &= -\,\delta\eta_\times\,,\end{aligned}\right\} \qquad \text{(I.1)}$$

the constant term ($-\delta\eta_\times$) in the second equation expressing the effect of a sudden elevator displacement of $\eta_\times$ at $\tau = 0$. An analogous term ($z_\eta\eta_\times$) could be introduced in the first equation for greater accuracy, However, it has been decided to neglect this term because the derivative z_η is usually small, especially for conventional tailed aircraft, and the effect of this term would be often negligible and always much smaller than that of $\delta\eta_\times$.

We shall solve the system (I.1) by the method of Heaviside's operators[5, 16]. The operational determinant of the system is:

$$\Delta(D) = \begin{vmatrix} D + \tfrac{1}{2}a & -1 \\ \chi D + \omega & D + \nu \end{vmatrix} = D^2 + (\tfrac{1}{2}a + \nu + \chi)D + (\omega + \tfrac{1}{2}a\nu) = D^2 + 2RD + C\,, \qquad \text{(I.2)}$$

where R and J are the damping parameter and frequency of the oscillation with elevator fixed, as in section 2 (bars being omitted for simplicity), and

$$C = R^2 + J^2 \qquad \text{(I.3)}$$

has been introduced for abbreviation. The operational solution of (I.1) is:

$$\hat{w} = \begin{vmatrix} 0 & -1 \\ -\,\delta\eta_\times & D + \nu \end{vmatrix} : \Delta(D) = -\frac{\delta\eta_\times}{D^2 + 2RD + C}\,, \qquad \text{(I.4)}$$

$$\hat{q} = \begin{vmatrix} D + \tfrac{1}{2}a & 0 \\ \chi D + \omega & -\,\delta\eta_\times \end{vmatrix} : \Delta(D) = -\frac{\delta\eta_\times(D + \tfrac{1}{2}a)}{D^2 + 2RD + C}\,. \qquad \text{(I.5)}$$

The corresponding operational expression for the normal acceleration factor n is:

$$n = \frac{V}{g\hat{t}}(\hat{q} - D\hat{w}) = \frac{a}{C_L}\hat{w} = -\frac{a}{C_L}\cdot\frac{\delta\eta_\times}{D^2 + 2RD + C}\,. \qquad \text{(I.6)}$$

The explicit solutions in terms of aerodynamic time τ (for the case of complex stability roots, as assumed) may now be obtained by applying the interpretation formulae:

$$\frac{D(D + R)}{D^2 + 2RD + C} = e^{-R\tau}\cos J\tau\,, \quad \frac{D}{D^2 + 2RD + C} = e^{-R\tau}\frac{\sin J\tau}{J} \qquad \text{(I.7)}$$

(*see* Ref. 16, p. 112). These may be used to derive a much more general formula, suitable for calculating response for an arbitrary oscillatory system of the second order. Let us consider the algebraical identity:

$$\frac{b'D^2 + c'D + d'}{D^2 + 2RD + C} = \frac{d'}{C} + \left(b' - \frac{d'}{C}\right)\frac{D(D+R)}{D^2 + 2RD + C} + \left(c' - b'R - \frac{d'R}{C}\right)\frac{D}{D^2 + 2RD + C}, \quad \text{(I.8)}$$

which is proved without difficulty (b', c', d' are arbitrary constants). Introducing (I.7), this becomes:

$$\frac{b'D^2 + c'D + d'}{D^2 + 2RD + C} = \frac{d'}{C} + e^{-R\tau}\left[\left(b' - \frac{d'}{C}\right)\cos J\tau + \left(c' - b'R - \frac{d'R}{C}\right)\frac{\sin J\tau}{J}\right], \quad \text{(I.9)}$$

and may be applied to interpret any operational solution, such as (I.5, 6). We obtain:

$$\frac{\hat{q}}{\delta\eta_\times} = e^{-R\tau}\left[\frac{a}{2C}\cos J\tau - \left(1 - \frac{aR}{2C}\right)\frac{\sin J\tau}{J}\right] - \frac{a}{2C} = Q(\tau)\text{ , say,} \quad \text{(I.10)}$$

and

$$\frac{n}{\delta\eta_\times} = \frac{a}{C_L C}\left[e^{-R\tau}\left(\cos J\tau + \frac{R}{J}\sin J\tau\right) - 1\right] = N(\tau)\text{ , say.} \quad \text{(I.11)}$$

Considering now the case of rectangular elevator input (Fig. 2b), it will be convenient to introduce the notion of 'elevator impulse'.

$$Y = \eta_\times \tau_\times \quad \text{(I.12)}$$

which is a convenient quantitative measure of the manoeuvre (it represents the hatched area in Fig. 2b). The response formulae will be:

$$\frac{\hat{q}}{\delta Y} = \frac{Q(\tau + \tau_\times) - Q(\tau)}{\tau_\times}, \qquad \frac{n}{\delta Y} = \frac{N(\tau + \tau_\times) - N(\tau)}{\tau_\times}, \quad \text{(I.13)}$$

valid for $\tau > 0$. They may be used to calculate the response curves for arbitrary $\eta_\times$ and $\tau_\times$.

An interesting approximation can be obtained if we assume that the time interval $\tau_\times$ is small, and ultimately tends to 0, while the impulse Y remains constant. The assumption may seem absurd because it involves ever increasing values of $\eta_\times$, but we may try it to see whether the results will be similar to the solution (I.13) for small but finite $\tau_\times$. The limits are obviously:

$$\left(\frac{\hat{q}}{\delta Y}\right)_{\tau_\times \to 0} = Q'(\tau)\text{ ,} \qquad \left(\frac{n}{\delta Y}\right)_{\tau_\times \to 0} = N'(\tau)\text{ ,} \quad \text{(I.14)}$$

and the explicit formulae may be obtained either by differentiating (I.10, 11) or, more simply, by multiplying (I.5, 6) by D and interpreting by means of (I.9). We thus obtain:

$$\left.\begin{aligned}\left(\frac{\hat{q}}{\delta Y}\right)_{\tau_\times \to 0} &= e^{-R\tau}\left[-\cos J\tau + (R - \tfrac{1}{2}a)\frac{\sin J\tau}{J}\right]\\ \left(\frac{n}{\delta Y}\right)_{\tau_\times \to 0} &= -\frac{a}{C_L}e^{-R\tau}\frac{\sin J\tau}{J}\,.\end{aligned}\right\} \quad \text{(I.15)}$$

The above results have been applied in one particular case, assuming the following data:

$$C_L = 0{\cdot}4\,, \quad a = 4\,, \quad \nu = 1\,, \quad \chi = 0{\cdot}4\,, \quad \omega = 25{\cdot}89\,,$$

which give (*see* I.2):

$$R = 1{\cdot}7\,, \qquad C = 27{\cdot}89\,, \qquad J = 5\,,$$

and hence:

$$Q(\tau) = e^{-1\cdot 7\tau}(0\cdot 0717\cos 5\tau - 0\cdot 1756\sin 5\tau) - 0\cdot 0717\ ,$$

$$N(\tau) = e^{-1\cdot 7\tau}(0\cdot 3586\cos 5\tau + 0\cdot 1219\sin 5\tau) - 0\cdot 3586\ ,$$

$$\left(\frac{\hat{q}}{\delta Y}\right)_{\tau_\times \to 0} = -\ e^{-1\cdot 7\tau}(\cos 5\tau + 0\cdot 06\sin 5\tau)\ ,$$

$$\left(\frac{n}{\delta Y}\right)_{\tau_\times \to 0} = -\ 2\ e^{-1\cdot 7\tau}\sin 5\tau\ .$$

The response curves to rectangular elevator inputs are given in Fig. 3 for three values of $\tau_\times$ ($0\cdot 5$, $0\cdot 1$ and 0). They are traced only from the instant $\tau = 0$ (termination of the elevator manoeuvre) onwards, so that the initial response between the two elevator movements is omitted. It is seen that there is a considerable effect of decreasing $\tau_\times$ from $0\cdot 5$ to $0\cdot 1$. However, the differences between the curves corresponding to $\tau_\times = 0\cdot 1$ and $\tau_\times = 0$ are very small and hence the simple tentative solutions (I.14) may be considered as good enough if $\tau_\times$ is small of the order of $0\cdot 1$. It should be remembered that $\tau_\times$ represents the duration of the elevator manoeuvre in aerodynamic units, and the corresponding value in seconds is obtained by multiplying by $\hat{t}$. The magnitude of $\hat{t}$ is discussed in Appendix IV, and it is shown that it lies usually between 1 and 10 seconds, the higher values occurring at high altitudes. The value $\tau_\times = 0\cdot 1$ can thus mean anything between $0\cdot 1$ sec and 1 sec, which may often be quite a realistic value.

(*b*) *Rigorous Solution* (*speed variation included*).—The principle of calculation is exactly the same as in the previous case, but we have now to deal with the full system of differential equations, of the fourth order, for a simple elevator displacement according to Fig. 2a:

$$\left.\begin{array}{rl}(D - x_u)\hat{u} - x_w\hat{w} \qquad\qquad + \tfrac{1}{2}C_L\theta & = 0 \\ -\ z_u\hat{u} + (D - z_w)\hat{w} - \hat{q} & = 0 \\ \varkappa\hat{u} + (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} & = -\ \delta\eta_\times \\ -\ \hat{q} + D\theta & = 0\end{array}\right\} \qquad \text{(I.16)}$$

The operational determinant of this system is:

$$\Delta(D) = D^4 + B_1D^3 + C_1D^2 + D_1D + E_1 = (D^2 + 2RD + H)(D^2 + 2rD + h)\ , \qquad \text{(I.17)}$$

where:

$$\left.\begin{array}{ll}B_1 = N_1 + \nu + \chi\ , & N_1 = -\ x_u - z_w\ , \\ C_1 = P_1 + N_1\nu + Q_1\chi + \omega\ , & P_1 = x_uz_w - x_wz_u\ , \\ D_1 = P_1\nu + R_1\chi + Q_1\omega - S_1\varkappa\ , & Q_1 = -\ x_u\ , \\ E_1 = R_1\omega - T_1\varkappa\ , & R_1 = -\ \tfrac{1}{2}z_uC_L\ , \\ & S_1 = \tfrac{1}{2}C_L - x_w\ , \\ & T_1 = -\ \tfrac{1}{2}z_wC_L\ .\end{array}\right\} \qquad \text{(I.18)}$$

R, J are the damping parameter and frequency of the short-period oscillation, r, j the analogous quantities for the phugoid mode, and

$$H = R^2 + J^2\ , \qquad h = r^2 + j^2\ . \qquad \text{(I.19)}$$

The operational solutions for $\hat{w}$ and $\hat{q}$ are:

$$\frac{\hat{w}}{\delta\eta_\times} = \begin{vmatrix} D - x_u & 0 & 0 & \tfrac{1}{2}C_L \\ -\ z_u & 0 & -\ 1 & 0 \\ \varkappa & -\ 1 & D + \nu & 0 \\ 0 & 0 & -\ 1 & D\end{vmatrix} : \Delta(D) = \frac{-\ D^2 + x_uD + \tfrac{1}{2}C_Lz_u}{\Delta(D)}\ , \qquad \text{(I.20)}$$

$$\frac{\hat{q}}{\delta\eta_\times} = \begin{vmatrix} D - x_u & -x_w & 0 & \frac{1}{2}C_L \\ -z_u & D - z_w & 0 & 0 \\ \varkappa & \chi D + \omega & -1 & 0 \\ 0 & 0 & 0 & D \end{vmatrix} : \Delta(D) = -\frac{D^3 + N_1 D^2 + P_1 D}{\Delta(D)}$$

$$= Q(\tau) \text{ , say } \quad \ldots \qquad \text{(I.21)}$$

and the corresponding operational expression for n:

$$\frac{n}{\delta\eta_\times} = \frac{2}{C_L}\frac{\hat{q} - D\hat{w}}{\delta\eta_\times} = \frac{2}{C_L}\frac{z_w D^2 - (P_1 + \frac{1}{2}C_L z_u)D}{\Delta(D)} = \hat{N}(\tau) \text{ , say} . \quad \ldots \quad \ldots \quad \ldots \qquad \text{(I.22)}$$

The explicit solutions $Q(\tau)$ and $N(\tau)$ can be obtained by using the following identity, which is easily proved:

$$\frac{b'D^3 + c'D^2 + d'D}{\Delta(D)} = \Lambda_1 \frac{D(D + R)}{D^2 + 2RD + H} + \Pi_1 \frac{D}{D^2 + 2RD + H}$$

$$- \Lambda_1 \frac{D(D + r)}{D^2 + 2rD + h} + \{b' - \Pi_1 + (R - r)\Lambda_1\} \frac{D}{D^2 + 2rD + h} , \qquad \text{(I.23)}$$

where:

$$\left.\begin{aligned} \Lambda_1 &= \frac{(H - h)(2Rb' - c') - 2(R - r)(Hb' - d')}{(H - h)^2 + 4(R - r)(Rh - rH)} \\ \Pi_1 &= \frac{\{H - h - 2R(R - r)\}(Hb' - d') + \{R(H + h) - 2rH\}(2Rb' - c')}{(H - h)^2 + 4(R - r)(Rh - rH)} \end{aligned}\right\} \quad . \quad \ldots \qquad \text{(I.24)}$$

This leads to the following interpretation formula:

$$\frac{b'D^3 + C'D^2 + d'D}{\Delta(D)} = \mathrm{e}^{-R\tau}\left(\Lambda_1 \cos J\tau + \Pi_1 \frac{\sin J\tau}{J}\right)$$

$$+ \mathrm{e}^{-r\tau}\left[- \Lambda_1 \cos j\tau + \{b' - \Pi_1 + (R - r)\Lambda_1\} \frac{\sin j\tau}{j}\right] , \qquad \ldots \qquad \text{(I.25)}$$

which may now be used directly for evaluating $\hat{q}$ and n from (I.21, 22).

In this case, again, the response to a rectangular elevator input will be given by (I.13), and the approximate solution, for $\tau_\times \to 0$, by (I.14). The first derivative of (I.25) may be found, with little effort, in the following form:

$$\frac{D(b'D^3 + c'D^2 + d'D)}{\Delta(D)} = \mathrm{e}^{-R\tau}\left(\Lambda_2 \cos J\tau + \Pi_2 \frac{\sin J\tau}{J}\right)$$

$$+ \mathrm{e}^{-r\tau}\left[(b' - \Lambda_2) \cos j\tau - \left\{\frac{(Rh - rH)\Lambda_2 + h\Pi_2}{H} + rb'\right\} \frac{\sin j\tau}{j}\right] , \qquad \text{(I.26)}$$

where:

$$\left.\begin{aligned} \Lambda_2 &= \frac{2(Rh - rH)(2Rb' - c') + (H - h)(Hb' - d')}{(H - h)^2 + 4(R - r)(Rh - rH)} \\ \Pi_2 &= \frac{\{R(H + h) - 2Hr\}(Hb' - d') - \{H(H - h) + 2R(Rh - rH)\}(2Rb' - c')}{(H - h)^2 + 4(R - r)(Rh - rH)} \end{aligned}\right\} \quad . \qquad \text{(I.27)}$$

The above results have been applied in the same case as that following equation (I.15), with the following additional data:

$$C_D = 0{\cdot}02 , \qquad C_{AS} = 0{\cdot}01 , \qquad dC_D/d\alpha = 0{\cdot}2 , \qquad \varkappa = 1{\cdot}907 .$$

From these, it follows:

$$x_u = -C_D - C_{AS} = -0{\cdot}03\,, \qquad x_w = \tfrac{1}{2}\left(C_L - \frac{dC_D}{d\alpha}\right) = 0{\cdot}1\,,$$

$$z_u = -C_L = -0{\cdot}4\,, \qquad z_w = -\tfrac{1}{2}(a + C_D) = -2{\cdot}01\,.$$

Then, from (I.18):

$$N_1 = 2{\cdot}04\,,\quad P_1 = 0{\cdot}1003\,,\quad Q_1 = 0{\cdot}03\,,\quad R_1 = 0{\cdot}08\,,\quad S_1 = 0{\cdot}1\,,\quad T_1 = 0{\cdot}402;$$
$$B_1 = 3{\cdot}44\,,\quad C_1 = 28{\cdot}0423\,,\quad D_1 = 0{\cdot}7183\,;\quad E_1 = 1{\cdot}3046\,.$$

The determinantal quartic (I.17) may now be written and factorized:

$$\begin{aligned}\Delta(D) &= D^4 + 3{\cdot}44D^3 + 28{\cdot}0423D^2 + 0{\cdot}7183D + 1{\cdot}3046 \\ &= (D^2 + 3{\cdot}42D + 27{\cdot}9272)(D^2 + 0{\cdot}02D + 0{\cdot}04671)\,,\end{aligned}$$

and hence the damping factor and frequencies are obtained:

$$R = 1{\cdot}71\,,\qquad J = 5{\cdot}000\,,\qquad r = 0{\cdot}01\,,\qquad j = 0{\cdot}216\,.$$

The final solutions are:

$$\begin{aligned}
\frac{\hat{q}}{\delta\eta_\times} = Q(\tau) &= e^{-1\cdot71\tau}\,(0{\cdot}0723\cos 5\tau - 0{\cdot}1751\sin 5\tau) \\
&\quad - e^{-0\cdot01\tau}\,(0{\cdot}0723\cos 0{\cdot}216\tau + 0{\cdot}0075\sin 0{\cdot}216\tau)\,, \\
\frac{n}{\delta\eta_\times} = N(\tau) &= e^{-1\cdot71\tau}\,(0{\cdot}3607\cos 5\tau + 0{\cdot}1230\sin 5\tau) \\
&\quad - e^{-0\cdot01\tau}\,(0{\cdot}3607\cos 0{\cdot}216\tau + 0{\cdot}0097\sin 0{\cdot}216\tau)\,, \\
\frac{\hat{q}}{\delta Y} = Q'(\tau) &= -\,e^{-1\cdot71\tau}\,(0{\cdot}9991\cos 5\tau + 0{\cdot}0624\sin 5\tau) \\
&\quad - e^{-0\cdot01\tau}\,(0{\cdot}0009\cos 0{\cdot}216\tau - 0{\cdot}0157\sin 0{\cdot}216\tau)\,, \\
\frac{n}{\delta Y} = N'(\tau) &= -\,e^{-1\cdot71\tau}\,(0{\cdot}0015\cos 5\tau + 2{\cdot}0138\sin 5\tau) \\
&\quad + e^{-0\cdot01\tau}\,(0{\cdot}0015\cos 0{\cdot}216\tau + 0{\cdot}0780\sin 0{\cdot}216\tau)\,.
\end{aligned}$$

The response curves are given in Fig. 4, for the same values of $\tau_\times$ as in Fig. 3. It is seen that the curves in both figures are very similar, and hence the effect of speed variation is quite small. Comparing the final formulae with numerical coefficients in both cases, we find that the frequency and damping of the short-period mode, and also the corresponding constant factors, are nearly equal. As regards the phugoid mode in the second case, the coefficients of sine terms are all small, and those of cosine terms nearly equal to the constant terms in the first case. The two solutions, for $\tau_\times = 0$, are shown together for comparison in Fig. 5, and it is seen that they are almost identical during the first period, and some noticeable discrepancies only develop later, obviously due to the phugoid terms in the second solution. The respective contributions to the two modes are shown in Fig. 6, where the response to a rectangular input of rather long duration ($\tau_\times = 0{\cdot}5$) is illustrated on a large scale. The curves representing the phugoid oscillatory mode are seen to be very nearly straight lines, of very small ordinates, throughout the short time covered by the diagram, because this time is only a small fraction of the phugoid period ($\sim$ 29 aerodynamic units).

It may be interesting to examine also the curves representing response to a single elevator displacement (step input), *i.e.*, the curves of $Q(\tau)$ and $N(\tau)$, from both approximate and rigorous solutions. Such curves are given in Fig. 7 and, in the case of the rigorous solution, the short-period and phugoid contributions are also shown. It is seen that the resultant rigorous curves agree with the appproximate ones initially, but the agreement soon gets much worse because the phugoid contributions (especially the cosine terms) are quite large. It should be noticed that the approximate solutions tend to finite asymptotic values for $\tau \to \infty$, in agreement with Gates' manoeuvrability theory[3], and these values are nearly reached in a short time shown in Fig. 7. The rigorous curves, however, ultimately converge to 0, and it is seen that, for values of τ greater than those shown in Fig. 7, there will be a considerable amount of the phugoid oscillation left which will disappear only after a long time. It is clear that a motion produced by a single elevator displacement would not be suitable for analysing the short-period oscillation. On the contrary, a motion following a rectangular elevator input, as illustrated in Figs. 4 to 6, is eminently suitable for this purpose, especially if the small amount of the phugoid mode is eliminated by the 'filtration' procedure, as described in Appendix II.

APPENDIX II

'Filtration' of Experimental Oscillatory Curves

If an oscillatory motion consists of a single mode, of frequency $\mathscr{J}$ and damping factor $\mathscr{R}$, then its equation can be reduced to the form (1.1). The experimental curves with which we have to deal, however, are normally distorted slightly, owing to the small amount of long-period (phugoid) oscillation added to the main short-period one. It has been shown in Appendix I that the phugoid contribution may be safely approximated, in the early stage of the motion, by a straight line, of small ordinates. The recorded curve may thus be expected to satisfy, with sufficient accuracy, the equation:

$$x = A\,e^{-\mathscr{R}(t-t_0)} \sin \mathscr{J}(t - t_0) + \alpha_0 - \beta_0 \mathscr{J}(t - t_0)\,, \qquad \text{(II.1)}$$

where x denotes any recorded quantity (*e.g.*, q or n), $\mathscr{J}$ and $\mathscr{R}$ are (dimensional) frequency and damping factor, A is the amplitude, and α_0, $\beta_0\mathscr{J}$ small unknown constants. In spite of their smallness, the two additional terms in (II.1) distort the curve so as to make it difficult to determine $\mathscr{J}$ and $\mathscr{R}$ accurately. If a method is found to determine the constants α_0 and $\beta_0\mathscr{J}$, it will be possible to trace the oblique straight line:

$$x_0 = \alpha_0 - \beta_0 \mathscr{J}(t - t_0)\,, \qquad \text{(II.2)}$$

and if this 'zero line' is used instead of the original horizontal axis, then our curve will satisfy the equation (1.1).

We are going to show now that α_0 and $\beta_0\mathscr{J}$, and all other constants, can be found by merely measuring the co-ordinates of a few maxima and minima of the experimental curve. Let us introduce a more convenient variable:

$$\vartheta = \mathscr{J}(t - t_0) \qquad \text{(II.3)}$$

and an auxiliary constant:

$$\cot \vartheta_0 = \mathscr{R}/\mathscr{J}\,, \qquad \text{(II.4)}$$

and then (II.1) may be written:

$$x = A\,e^{-\vartheta \cot \vartheta_0} \sin \vartheta + \alpha_0 - \beta_0 \vartheta\,. \qquad \text{(II.5)}$$

The condition for x attaining its maximum or minimum will be:

$$A\,e^{-\vartheta \cot \vartheta_0} (\cos \vartheta - \sin \vartheta \cot \vartheta_0) - \beta_0 = 0\,. \qquad \text{(II.6)}$$

If β_0 were equal to 0, the equation (II.6) would be satisfied by any of the values $\vartheta_N = \vartheta_0 + (N - 1)\pi$, where N denotes an arbitrary integer. With β_0 different from 0 but small, the solution may be written:

$$\vartheta_N = \vartheta_0 + (N - 1)\pi - k_N \beta_0\,, \qquad \text{(II.7)}$$

where $k_N\beta_0$ is a supposedly small correction term, depending on N. We then obtain, neglecting powers of β_0 higher than the first:

$$\left.\begin{aligned} \tan \vartheta_N &= \tan(\vartheta_0 - k_N\beta_0) = \tan \vartheta_0 - k_N\beta_0 \sec^2 \vartheta_0 \\ \cos \vartheta_N &= (-1)^{N-1} \cos \vartheta_0 (1 + k_N \beta_0 \tan \vartheta_0) \\ \sin \vartheta_N &= (-1)^{N-1} \sin \vartheta_0 (1 - k_N \beta_0 \cot \vartheta_0) \end{aligned}\right\}, \qquad \text{(II.8)}$$

and, introducing (II.7, 8) into (II.6), and keeping only terms of the first order in β_0, we get:

$$k_N = \frac{(-1)^{N-1} \sin \vartheta_0}{A}\, e^{\{\vartheta_0 + \pi(N-1)\} \cot \vartheta_0}. \qquad \text{(II.9)}$$

It is seen that the term $k_N\beta_0$ is really small of the order β_0, but the coefficients k_N increase rapidly with N, so that the accuracy of our procedure may be adequate only for a few first peaks, and it deteriorates with increasing N. Let us now denote the first maximum of x by x_1, the subsequent minimum by $(-x_2)$, the next maximum by x_3, next minimum by $(-x_4)$, and so on. The values of the consecutive peaks will all be represented by the formula, correct to the first order of β_0:

$$(-1)^{N-1}x_N = (-1)^{N-1} A\, e^{-\vartheta_0 \cot \vartheta_0} e^{-\pi(N-1)\cot\vartheta_0} \sin\vartheta_0 + \alpha_0 - \beta_0\{\vartheta_0 + (N-1)\pi\}\,. \qquad \text{(II.10)}$$

Introducing for abbreviation:

$$B = A \sin\vartheta_0\, e^{-\vartheta_0 \cot\vartheta_0}\,, \qquad \gamma = e^{-\pi\cot\vartheta_0}\,, \quad \ldots \qquad \text{(II.11)}$$

we may write (II.10) more simply:

$$x_N = B\gamma^{N-1} + (-1)^{N-1}[\alpha_0 - \beta_0\{\vartheta_0 + (N-1)\pi\}]\,, \quad \ldots \qquad \text{(II.12)}$$

and hence the consecutive peak values will be:

$$\left.\begin{aligned} x_1 &= B + \alpha_0 - \beta_0\vartheta_0 \\ x_2 &= B\gamma - \alpha_0 + \beta_0(\vartheta_0 + \pi) \\ x_3 &= B\gamma^2 + \alpha_0 - \beta_0(\vartheta_0 + 2\pi) \\ x_4 &= B\gamma^3 - \alpha_0 + \beta_0(\vartheta_0 + 3\pi) \\ x_5 &= B\gamma^4 + \alpha_0 - \beta_0(\vartheta_0 + 4\pi)\,, \text{ etc.} \end{aligned}\right\} \quad \ldots \qquad \text{(II.13)}$$

We then obtain, eliminating α_0:

$$\left.\begin{aligned} x_1 + x_2 &= B(1+\gamma) + \pi\beta_0 \\ x_2 + x_3 &= B\gamma(1+\gamma) - \pi\beta_0 \\ x_3 + x_4 &= B\gamma^2(1+\gamma) + \pi\beta_0 \\ x_4 + x_5 &= B\gamma^3(1+\gamma) - \pi\beta_0\,, \text{ etc.}; \end{aligned}\right\} \quad \ldots \qquad \text{(II.14)}$$

further, eliminating β_0:

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= B(1+\gamma)^2 \\ x_2 + 2x_3 + x_4 &= B\gamma(1+\gamma)^2 \\ x_3 + 2x_4 + x_5 &= B\gamma^2(1+\gamma)^2\,, \text{ etc.,} \end{aligned}\right\} \quad \ldots \qquad \text{(II.15)}$$

and finally, eliminating B:

$$\gamma = \frac{x_2 + 2x_3 + x_4}{x_1 + 2x_2 + x_3} = \frac{x_3 + 2x_4 + x_5}{x_2 + 2x_3 + x_4}\,, \text{ etc.} \quad \ldots \qquad \text{(II.16)}$$

It is seen that γ, and hence ϑ_0, can be determined by measuring simply a few consecutive peak values of x. It may be noticed that

$$\frac{x_1 + 2x_2 + x_3}{4} = \frac{1}{2}\left(\frac{x_1 + x_2}{2} + \frac{x_2 + x_3}{2}\right)$$

is a 'second arithmetical mean' of x_1, x_2, x_3, and hence γ equals, with a good approximation, a ratio of the second mean of (x_2, x_3, x_4) to that of (x_1, x_2, x_3), or a ratio of the second mean of (x_3, x_4, x_5) to that of (x_2, x_3, x_4), etc. Once γ has been determined, we can easily find the remaining constants, *viz.*:

$$B = \frac{x_1 + 2x_2 + x_3}{(1+\gamma)^2} = \frac{x_2 + 2x_3 + x_4}{\gamma(1+\gamma)^2} = \frac{x_3 + 2x_4 + x_5}{\gamma^2(1+\gamma)^2}\,, \text{ etc.} \quad \ldots \qquad \text{(II.17)}$$

$$\beta_0 = \frac{x_1\gamma - x_2(1-\gamma) - x_3}{\pi(1+\gamma)} = \frac{-x_2\gamma + x_3(1-\gamma) + x_4}{\pi(1+\gamma)} = \frac{x_3\gamma - x_4(1-\gamma) - x_5}{\pi(1+\gamma)}\,, \text{ etc.} \qquad \text{(II.18)}$$

$$\alpha_0 = x_1 - B + \beta_0\vartheta_0 = -x_2 + B\gamma + \beta_0(\pi + \vartheta_0) = x_3 - B\gamma^2 + \beta_0(2\pi + \vartheta_0)\,, \text{ etc.} \qquad \text{(II.19)}$$

Each of the parameters γ, B, β_0 and α_0 is thus represented by a sequence of formulae, involving a few consecutive peak values, starting from the first one, or from the second one, etc. All formulae should give consistent results, but the first formula in each sequence may be expected to be most accurate. It will suffice in practice to use only the first formula in each case, more so as, usually, only few peaks can be read from the recorded curve with reasonable accuracy.

The value of $\mathscr{J}$ is still required. To find this, we write, in view of (II.7, 9) and (II.3), putting in turn $N = 1, 2, 3$, etc.:

$$\left.\begin{aligned} \vartheta_1 &= \mathscr{J}(t_1 - t_0) = \vartheta_0 - \frac{\beta_0 \sin \vartheta_0}{A} e^{\vartheta_0 \cot \vartheta_0} \\ \vartheta_2 &= \mathscr{J}(t_2 - t_0) = \vartheta_0 + \pi + \frac{\beta_0 \sin \vartheta_0}{A} e^{(\vartheta_0 + \pi) \cot \vartheta_0} \\ \vartheta_3 &= \mathscr{J}(t_3 - t_0) = \vartheta_0 + 2\pi - \frac{\beta_0 \sin \vartheta_0}{A} e^{(\vartheta_0 + 2\pi) \cot \vartheta_0}; \text{ etc.} \end{aligned}\right\}, \quad \ldots \quad \text{(II.20)}$$

and hence, subtracting the first equation from each of the following ones, and using (II.11):

$$\left.\begin{aligned} \mathscr{J}(t_2 - t_1) &= \pi + \frac{\beta_0}{B}\left(\frac{1}{\gamma} + 1\right) \sin^2 \vartheta_0 \\ \mathscr{J}(t_3 - t_1) &= 2\pi - \frac{\beta_0}{B}\left(\frac{1}{\gamma^2} - 1\right) \sin^2 \vartheta_0 \\ \mathscr{J}(t_4 - t_1) &= 3\pi + \frac{\beta_0}{B}\left(\frac{1}{\gamma^3} + 1\right) \sin^2 \vartheta_0, \text{ etc.} \end{aligned}\right\}. \quad \ldots \quad \text{(II.21)}$$

The abscissae $t_1, t_2, t_3, \ldots$ may be read directly from the diagram, and the formulae (II.21) should then give consistent values for $\mathscr{J}$; $\mathscr{R}$ will then be found from (II.4).

In such a way, we have a purely algebraic method to determine the frequency $\mathscr{J}$ and damping factor $\mathscr{R}$. It will be better, however, to trace the 'zero line' (II.2) in the diagram. To do this, we notice that the lines (II.1) and (II.2) intersect at $t = t_0$, $x = \alpha_0$. It suffices therefore to draw a horizontal line $x = \alpha_0$ and find its 'first' point of intersection with the recorded curve. The zero line will then pass through this point and have the slope $(-\beta_0\mathscr{J})$. If the recorded curve is now referred to the zero line, instead of the original horizontal axis, it should have the properties of a single-mode oscillatory curve (1.1). If several recorded curves (such as q, n, η) are to be investigated, appropriate zero lines should be traced for all of them, and then they will be ready for the procedure described in Appendix III.

Examples.—In the following examples, numerical values of all parameters of the equation (II.1) will be first assumed as given, and co-ordinates of several peaks determined exactly to the fourth decimal figure. These co-ordinates will then be used to calculate the parameters by using the formulae of this Appendix and, in such a way, the accuracy of the procedure will be established. Whenever alternative formulae are available, the first one will be used, and the consistency checked occasionally.

(1) Assume $\mathscr{R} = 1{\cdot}7$, $\mathscr{J} = 5$, $A = 1$, $\alpha_0 = -0{\cdot}06$, $\beta_0 = 0{\cdot}005$, $t_0 = 0{\cdot}2$ sec. The equation (II.1) is therefore:

$$x = e^{-1\cdot7(t-0\cdot2)} \sin 5(t - t_0) - 0{\cdot}06 - 0{\cdot}025(t - t_0)$$

(*see* Fig. 14a), and we have, from (II.4):

$$\cot \vartheta_0 = 0{\cdot}34, \text{ hence } \vartheta_0 = 1{\cdot}2431 \text{ radn}, \quad \sin \vartheta_0 = 0{\cdot}9468, \quad \cos \vartheta_0 = 0{\cdot}3219.$$

The condition (II.6) for x attaining its peak values becomes:

$$e^{-0\cdot34\vartheta} (\cos \vartheta - 0{\cdot}34 \sin \vartheta) - 0{\cdot}005 = 0\,,$$

and the exact calculated values of five first peaks are as follows:

$$t_1 = 0\cdot4472,\quad t_2 = 1\cdot0812,\quad t_3 = 1\cdot6932,\quad t_4 = 2\cdot3718,\quad t_5 = 2\cdot8702,$$
$$\vartheta_1 = 1\cdot2358,\quad \vartheta_2 = 4\cdot4060,\quad \vartheta_3 = 7\cdot4660,\quad \vartheta_4 = 10\cdot8590,\quad \vartheta_5 = 13\cdot3510,$$
$$x_1 = 0\cdot5542,\quad x_2 = 0\cdot2952,\quad x_3 = -0\cdot0242,\quad x_4 = 0\cdot1390,\quad x_5 = -0\cdot1192.$$

We calculate, from (II.16 to 19):

$$\gamma = 0\cdot3443 \text{ (alternative value } 0\cdot3489\text{), hence } \cot\vartheta_0 = 3\cdot3394,\ \vartheta_0 = 1\cdot2436,$$
$$\sin\vartheta_0 = 0\cdot9470.$$
$$B = 0\cdot6200,\quad A = 0\cdot9985,\quad \beta_0 = 0\cdot0051,\quad \alpha_0 = -0\cdot0595.$$

The formulae (II.21) then give:

$$\mathscr{J} = 5\cdot0006 \text{ (alternative value } 4\cdot9987\text{), hence } \beta_0\mathscr{J} = 0\cdot0255,$$

and (II.4) gives $\mathscr{R} = 1\cdot6972$.

The accuracy of all results is very satisfactory, and greater errors must certainly be expected from other sources.

(2) Assume $\mathscr{R} = 0\cdot8$, $\mathscr{J} = 5$, $A = 1$, $\alpha_0 = 0\cdot18$, $\beta_0 = 0\cdot02$, $t_0 = 0\cdot1$ sec. The equation (II.1) is:

$$x = e^{-0\cdot8(t-0\cdot1)} \sin 5(t - t_0) + 0\cdot18 - 0\cdot1(t - t_0)$$

(*see* Fig. 14b), and we have, from (II.4):

$$\cot\vartheta_0 = 0\cdot16,\quad \vartheta_0 = 1\cdot4121 \text{ radn},\quad \sin\vartheta_0 = 0\cdot9874,\quad \cos\vartheta_0 = 0\cdot1580.$$

The condition (II.6) for x attaining its peak values becomes:

$$e^{-0\cdot16\vartheta}(\cos\vartheta - 0\cdot16\sin\vartheta) - 0\cdot02 = 0,$$

and the exact calculated values of six first peaks are as follows:

$$t_1 = 0\cdot3775,\quad \vartheta_1 = 1\cdot3875,\quad x_1 = 0\cdot9397,$$
$$t_2 = 1\cdot0190,\quad \vartheta_2 = 4\cdot5949,\quad x_2 = 0\cdot3880,$$
$$t_3 = 1\cdot6257,\quad \vartheta_3 = 7\cdot6284,\quad x_3 = 0\cdot3151,$$
$$t_4 = 2\cdot2902,\quad \vartheta_4 = 10\cdot9511,\quad x_4 = 0\cdot2122,$$
$$t_5 = 2\cdot8596,\quad \vartheta_5 = 13\cdot7982,\quad x_5 = 0\cdot0077,$$
$$t_6 = 3\cdot5896,\quad \vartheta_6 = 17\cdot4481,\quad x_6 = 0\cdot2294.$$

We calculate, from (II.16 to 19):

$$\gamma = 0\cdot6059 \text{ (alternative values } 0\cdot6073,\ 0\cdot6116\text{), hence } \cot\vartheta_0 = 0\cdot1595,$$
$$\vartheta_0 = 1\cdot4126 \text{ (alternative values } 1\cdot4134,\ 1\cdot4155\text{)},\quad \sin\vartheta_0 = 0\cdot9875.$$
$$B = 0\cdot7875 \text{ (altern. } 0\cdot7842,\ 0\cdot7691\text{)},\ A = 0\cdot9990 \text{ (altern. } 0\cdot9978,\ 0\cdot9716\text{)},$$
$$\beta_0 = 0\cdot0201 \text{ (altern. } 0\cdot0199,\ 0\cdot0203\text{)},\quad \alpha_0 = 0\cdot1806 \text{ (altern. } 0\cdot1787,\ 0\cdot1833\text{)}.$$

The formulae (II.21) then give:

$$\mathscr{J} = 5\cdot0001 \text{ (altern. } 4\cdot9994,\ 4\cdot9990,\ 4\cdot9985\text{), hence } \beta_0\mathscr{J} = 0\cdot1005,$$

and (II.4) gives $\mathscr{R} = 0\cdot7975$.

The value of t_0 can now be found from either of (II.20):

$$t_0 = 0\cdot1000 \text{ sec (altern. } 0\cdot0999, \text{ etc.)}$$

The accuracy of all results is again very satisfactory, in spite of the increased slope of the zero line.

Note.—It is generally easy to measure accurately the peak values $x_1, x_2, x_3, \ldots$. The values of $\gamma, \vartheta_0, B, A, \beta_0$ and α_0 are expressed in terms of these peak values only, and so we may expect a good accuracy for all of them. The position is less satisfactory as regards the determination of $\mathscr{I}$ from (II.21) because these formulae involve the abscissae $t_1, t_2, t_3, \ldots$ which can be only read from the diagrams with a comparatively poor accuracy. It would perhaps be better in some cases to read instead the values of the abscissae $t_{\mathrm{I}}, t_{\mathrm{II}}, t_{\mathrm{III}}, \ldots$ of the zeros of the experimental curve, which can be determined with a better accuracy. If α_0 and β_0 are both small then, following similar lines as before, we may obtain the formulae:

$$\mathscr{I}(t_{\mathrm{I}} - t_0) = -\frac{\alpha_0}{A}, \qquad \mathscr{I}(t_{\mathrm{II}} - t_0) = \pi + \frac{\alpha_0 - \pi\beta_0}{A\gamma},$$

$$\mathscr{I}(t_{\mathrm{III}} - t_0) = 2\pi - \frac{\alpha_0 - 2\pi\beta_0}{A\gamma^2}, \text{ etc.,} \qquad \ldots \qquad \text{(II.22)}$$

and hence:

$$\left.\begin{aligned} \mathscr{I}(t_{\mathrm{II}} - t_{\mathrm{I}}) &= \pi + \frac{\alpha_0}{A} + \frac{\alpha_0 - \pi\beta_0}{A\gamma} \\ \mathscr{I}(t_{\mathrm{III}} - t_{\mathrm{I}}) &= 2\pi + \frac{\alpha_0}{A} - \frac{\alpha_0 - 2\pi\beta_0}{A\gamma^2}, \text{ etc.} \end{aligned}\right\} \qquad \ldots \qquad \text{(II.23)}$$

These formulae may be sometimes used to determine $\mathscr{I}$, instead of (II.21). However, some tentative examples have shown that the formulae (II.22) are usually less accurate than (II.20), and therefore the greater accuracy in reading the asbcissae may not necessarily lead to better values of $\mathscr{I}$. Moreover, the first determination of $\mathscr{I}$ is only needed to calculate $\beta_0\mathscr{I}$, the slope of the zero line and, obviously, this need not be very accurate. When the zero line is traced in the diagram, its points of intersection with the experimental curve should be equidistant, and the peak values should form a geometrical progression. This will give an excellent check of the entire procedure and means to determine $\mathscr{I}$ and $\mathscr{R}$ with the best accuracy, following the lines of Appendix III.

Note 2.—The entire method of filtration seems to break down when the damping parameter $\mathscr{R}$ is large compared to the frequency $\mathscr{I}$ (thus cot ϑ_0 large), and the slope $\beta_0\mathscr{I}$ comparatively large, because then the curve may present only a small number of peaks. Four peaks are needed for determining γ from (II.16). If only a lesser number is available, the problem seems indeterminate, there being only three or two of equations (II.13), while the number of unknowns is four (B, γ, α_0, β_0). However, the method may be easily modified in such cases. If *three peaks* only exist, the curve beyond the last one converges rapidly towards the zero line, and at least the slope of the latter $\beta_0\mathscr{I}$ may be determined with a good accuracy, while $\mathscr{I}$ is still found approximately from the first of (II.21), neglecting the correction of the order of β_0. In such a way, a sufficiently accurate value of β_0 is determined, and then the first two of equations (II.14) give:

$$\gamma = \frac{x_2 + x_3 + \pi\beta_0}{x_1 + x_2 - \pi\beta_0}; \qquad \ldots \qquad \text{(II.24)}$$

B is then found from (II.17) and α_0 from (II.19), and the zero line may be traced, the procedure being practically self-checking. Alternatively, the zero line may be traced backwards at sight.

If only *two peaks* are available, the curve converges so rapidly towards the zero line that the latter may be traced immediately. In some cases, a tentative zero line may be traced so as to exhibit four peaks, and then the original method applies in the usual way.

It seems that, by applying such devices, the frequency and damping can be determined for heavily damped curves with a similar accuracy as in the case of low damping.

APPENDIX III

Determination of Frequency, Damping Factor, Amplitude Ratios and Phase Angles of Experimental Curves

Experimental curves of any recorded quantities (such as q and n—Fig. 1) being available, it is required to determine the characteristic quantities as listed above. It is supposed that we have to deal with simple damped oscillatory curves of a single (short-period) mode, and therefore their equations may be written :

$$n = n^* \, e^{-\mathscr{R}(t-t_0)} \sin \mathscr{J}(t - t_0) \, , \qquad \text{(III.1)}$$

$$q = q^* \, e^{-\mathscr{R}(t-t_0)} \sin \{\mathscr{J}(t - t_0) + \varphi_{qn}\} \, . \qquad \text{(III.2)}$$

The quantities required are $\mathscr{J}$, $\mathscr{R}$, q^*/n^* and φ_{qn}.

The period P can be read directly from the diagram, as the horizontal distance between any two alternate zeros (or between alternate peaks, but this will be usually less accurate). The *frequency* is then:

$$\mathscr{J} = 2\pi/P \, , \text{ (in sec}^{-1}) \qquad \text{(III.3)}$$

and the dimensionless frequency:

$$J = \mathscr{J}\hat{t} = 2\pi\hat{t}/P \, . \qquad \text{(III.4)}$$

To determine the *damping factor* $\mathscr{R}$, we notice that the consecutive peaks of the curve (III.1) are (with signs inverted for the minima):

$$n_1 = n^* \, e^{-\mathscr{R}(t_1-t_0)} \mathscr{J}/\sqrt{(\mathscr{R}^2 + \mathscr{J}^2)} \, , \text{ corresponding to } t_1 = t_0 + \frac{1}{\mathscr{J}} \tan^{-1}\frac{\mathscr{J}}{\mathscr{R}} \, ,$$

$$n_2 = n^* \, e^{-\mathscr{R}(t_2-t_0)} \mathscr{J}/\sqrt{(\mathscr{R}^2 + \mathscr{J}^2)} \, , \text{ corresponding to } t_2 = t_1 + \pi/\mathscr{J}, \text{ etc.,} \qquad \text{(III.5)}$$

hence:

$$n_1/n_2 = e^{\mathscr{R}(t_2-t_1)} = e^{\pi\mathscr{R}/\mathscr{J}} \, , \qquad \text{(III.6)}$$

and thus:

$$\mathscr{R} = \frac{\mathscr{J}}{\pi} \ln \frac{n_1}{n_2} \text{ (in sec}^{-1}) \, . \qquad \text{(III.7)}$$

The dimensionless damping factor becomes:

$$R = \mathscr{R}\hat{t} = \frac{2\hat{t}}{P} \ln \frac{n_1}{n_2} \, . \qquad \text{(III.8)}$$

The ratio n_1/n_2 may be replaced by n_2/n_3, etc., or similarly by q_1/q_2, q_2/q_3, etc., and the results should be consistent.

To find the *phase angle* φ_{qn} (by which q leads n), it suffices to measure the time interval ΔP between two corresponding zeros (or peaks) of the two curves, as shown in Fig. 1. We have then:

$$\varphi_{qn} = 2\pi \frac{\Delta P}{P} \text{ (in radians), or } = 360 \frac{\Delta P}{P} \text{ (in degrees).} \qquad \text{(III.9)}$$

The *amplitude ratio* q^*/n^* differs, of course, generally from the ratio of peaks q_1/n_1—unless $\varphi_{qn} = 0$. The peak value q_1 is, by analogy with (III.5):

$$q_1 = q^* \, e^{-\mathscr{R}(t_1'-t_0)} \mathscr{J}/\sqrt{(\mathscr{R}^2 + \mathscr{J}^2)} \, , \qquad \text{(III.10)}$$

where

$$t_1' = t_1 - \frac{\varphi_{qn}}{\mathscr{J}} = t_1 - \Delta P \, . \qquad \text{(III.11)}$$

We have therefore:

$$\frac{q_1}{n_1} = \frac{q^*}{n^*} \mathrm{e}^{\mathscr{R}(t_1 - t_1')} = \frac{q^*}{n^*} \mathrm{e}^{\mathscr{R}\varphi_{qn}/\mathscr{I}}, \quad \ldots \quad \text{(III.12)}$$

and hence:

$$\frac{q^*}{n^*} = \frac{q_1}{n_1} \mathrm{e}^{-\mathscr{R}\varphi_{qn}/\mathscr{I}} = \frac{q_1}{n_1}\left(\frac{n_1}{n_2}\right)^{-\varphi_{qn}/\pi},$$

or finally:

$$\frac{q^*}{n^*} = \frac{q_1}{n_1}\left(\frac{n_1}{n_2}\right)^{-2\Delta P/P}. \quad \ldots \quad \text{(III.13)}$$

As expounded in Appendices I and II, the experimental curves will normally be distorted by a small amount of the phugoid mode, and they have to be ' filtered '. The zero line (Figs. 14a and 14b) is then determined, and this is to be used instead of the horizontal axis, for reading off the period and the peak values. The latter should then, strictly speaking, be measured at the points where tangents are not horizontal but parallel to the zero line, but this refinement is usually negligible.

A usual graphical procedure is illustrated in Figs. 15 and 16. In the first one, logarithms of the peak values n_1, n_2, . . . and q_1, q_2 . . . are plotted against the corresponding time abscissae t_1, t_2, . . . and t_1', t_2', . . . (best done on graph paper with a logarithmic vertical scale), and the nearest straight lines drawn through each series of points. The two lines should be parallel, and their slope determines the damping factor $\mathscr{R}$ because, from (III.6) we have*:

$$\frac{\log n_1 - \log n_2}{t_2 - t_1} = \frac{\mathscr{R}}{2 \cdot 3026}. \quad \ldots \quad \text{(III.14)}$$

The vertical distance between the two lines represents the amplitude ratio in logarithmic scale because, from (III.12), we have:

$$\log (n^*/q^*) = \log n_1 - \left[\log q_1 - \frac{\mathscr{R}}{2 \cdot 3026}(t_1 - t_1')\right]. \quad \ldots \quad \text{(III.15)}$$

The period P and phase time interval ΔP can also be read directly from Fig. 15, thus using the time abscissae of the peaks. A better accuracy, however, is obtained by using zeros, and this may be done graphically as shown in Fig. 16. The indices I, II, III, . . . of the consecutive zeros are plotted against the corresponding time abscissae t_{I}, t_{II}, t_{III} for n, and similarly against t_{I}', t_{II}', t_{III}', for q, and the nearest straight lines drawn through each series of points. The two lines should again be parallel; their slope determines P, and their horizontal distance ΔP.

---

* On the logarithmic scale, decimal logarithms (log) are used, instead of natural logarithms (ln), and this explains the factor $2 \cdot 3026 = \ln 10$ in (III.14) and (III.15).

APPENDIX IV

Remarks about the Unit of Aerodynamic Time

The unit of aerodynamic time $\hat{t}$ is defined by:

$$\hat{t} = \frac{W}{\rho g S V}, \quad \ldots \quad \text{(IV.1)}$$

but a somewhat more convenient formula may be proposed. We have:

$$\rho = \rho_0 \sigma \text{ , and } V = a_s M , \quad \ldots \quad \text{(IV.2)}$$

and hence:

$$\hat{t} = K \frac{W/S}{M}, \quad \ldots \quad \text{(IV.3)}$$

where:

$$K = \frac{1}{g \rho_0 \sigma a_s} = \frac{13 \cdot 072}{\sigma a_s}. \quad \ldots \quad \text{(IV.4)}$$

It is seen that $\hat{t}$ is proportional to the wing loading W/S and inversely proportional to the Mach number M, and the coefficient K depends on the height only. The values of K, calculated for the Standard Atmosphere, are given in Table 1, and the corresponding curve shown in Fig. 17. The Table 2 gives some illustrative values of $\hat{t}$ corresponding to $M = 0 \cdot 8$, for several heights and wing loadings, and it will be quite easy to obtain values for lower or higher Mach Numbers. It is seen that $\hat{t}$ may vary within surprisingly wide limits, but will seldom fall below 1 second. It increases very strongly with height, and unusually large values of the order of 20 seconds or even more may be encountered in the foreseeable future. For the present, $\hat{t}$ seldom exceeds 10 seconds.

APPENDIX V

Alternative Analysis with Attitude Recorded Instead of Normal Acceleration. Analogy with the Case of Lateral Oscillations

Let us suppose that, during flight tests with oscillating elevator, the quantities recorded are η and θ. The latter is the angular displacement in pitch and can be registered by taking photographs of a distant fixed object continuously. This technique is not usual, but may sometimes be found convenient.

We shall consider equations (3.1, 2), thus neglecting z_η. The relationship (3.6) still holds, being merely a consequence of (3.1), and hence we obtain :—

$$\theta = D^{-1}\hat{q} = \frac{\frac{1}{2}a - R + iJ}{-R + iJ}\,\hat{w} = \left(1 - \frac{\frac{1}{2}a}{R - iJ}\right)\hat{w}\,. \quad \ldots \quad \ldots \quad (\text{V.1})$$

The amplitude ratio η^*/θ^* and the phase angle $\varphi_{\eta\theta}$ having been determined from flight tests, we have :

$$\frac{\eta}{\theta} = \frac{\eta^*}{\theta^*}\,\mathrm{e}^{i\varphi_{\eta\theta}}\,, \quad \ldots \quad (\text{V.2})$$

and hence:

$$\eta = \frac{\eta^*}{\theta^*}\,\mathrm{e}^{i\varphi_{\eta\theta}}\left(1 - \frac{\frac{1}{2}a}{R - iJ}\right)\hat{w}\,. \quad \ldots \quad (\text{V.3})$$

Substituting (3.6) and (V.3) into (3.2), we obtain:

$$\left[\omega - \chi(R - iJ) + (\nu - R + iJ)(\tfrac{1}{2}a - R + iJ) + \delta\,\frac{\eta^*}{\theta^*}\,\mathrm{e}^{i\varphi_{\eta\theta}}\left(1 - \frac{\frac{1}{2}a}{R - iJ}\right)\right]\hat{w} = 0\,. \quad (\text{V.4})$$

The real and imaginary parts in the square brackets must both vanish, and hence :

$$\omega + \tfrac{1}{2}a\nu - R(\tfrac{1}{2}a + \nu + \chi) + R^2 - J^2 + \delta\,\frac{\eta^*}{\theta^*}\left(\cos\varphi_{\eta\theta} + \tfrac{1}{2}a\,\frac{J\sin\varphi_{\eta\theta} - R\cos\varphi_{\eta\theta}}{R^2 + J^2}\right) = 0\,, \quad (\text{V.5})$$

$$J(\tfrac{1}{2}a + \nu + \chi - 2R) + \delta\,\frac{\eta^*}{\theta^*}\left(\sin\varphi_{\eta\theta} - \tfrac{1}{2}a\,\frac{J\cos\varphi_{\eta\theta} + R\sin\varphi_{\eta\theta}}{R^2 + J^2}\right) = 0\,. \quad \ldots \quad (\text{V.6})$$

We now obtain from (V.6):

$$2R = (\tfrac{1}{2}a + \nu + \chi) + \frac{\delta}{J}\,\frac{\eta^*}{\theta^*}\left(\sin\varphi_{\eta\theta} - \tfrac{1}{2}a\,\frac{J\cos\varphi_{\eta\theta} + R\sin\varphi_{\eta\theta}}{R^2 + J^2}\right) \quad \ldots \quad (\text{V.7})$$

and, substituting $(\frac{1}{2}a + \nu + \chi)$ from this into (V.5):

$$R^2 + J^2 = (\omega + \tfrac{1}{2}a\nu) + \delta\,\frac{\eta^*}{\theta^*}\left\{\left(1 - \frac{aR}{R^2 + J^2}\right)\left(\cos\varphi_{\eta\theta} + \frac{R}{J}\sin\varphi_{\eta\theta}\right) + \frac{a}{2J}\sin\varphi_{\eta\theta}\right\}. \quad \ldots \quad (\text{V.8})$$

Comparing (V.7, 8) with (2.10, 11), we may write:

$$\bar{R} = R - \frac{\delta}{2J}\,\frac{\eta^*}{\theta^*}\left(\sin\varphi_{\eta\theta} - \tfrac{1}{2}a\,\frac{J\cos\varphi_{\eta\theta} + R\sin\varphi_{\eta\theta}}{R^2 + J^2}\right), \quad \ldots \quad (\text{V.9})$$

$$\bar{R}^2 + \bar{J}^2 = R^2 + J^2 - \delta\,\frac{\eta^*}{\theta^*}\left\{\left(1 - \frac{aR}{R^2 + J^2}\right)\left(\cos\varphi_{\eta\theta} + \frac{R}{J}\sin\varphi_{\eta\theta}\right) + \frac{a}{2J}\sin\varphi_{\eta\theta}\right\}, \quad \ldots \quad (\text{V.10})$$

with an alternative formula for $\bar{J}$, obtained by eliminating $\bar{R}$:

$$\bar{J}^2 = J^2 - \delta \frac{\eta^*}{\theta^*}\left(\cos \varphi_{\eta\theta} + \tfrac{1}{2}a \frac{J \sin \varphi_{\eta\theta} - R \cos \varphi_{\eta\theta}}{R^2 + J^2}\right)$$

$$- \left\{ \frac{\delta}{2J} \frac{\eta^*}{\theta^*}\left(\sin \varphi_{\eta\theta} - \tfrac{1}{2}a \frac{J \cos \varphi_{\eta\theta} + R \sin \varphi_{\eta\theta}}{R^2 + J^2}\right)\right\}^2 . \quad \text{(V.11)}$$

The formulae (V.9 to V.11) are rather similar to (3.22 to 3.24) but considerably more complicated. They may be directly used for finding the frequency and damping with elevator fixed from those with elevator oscillating, but the value of a is then required. This cannot be determined from θ- and η-curves only, so one more quantity must be recorded, and this can hardly be anything but n. Recording q would not help because the relationship between θ and q:

$$\hat{q} = (-R + iJ)\theta \quad \text{(V.12)}$$

involves only the parameters R and J, but not a.

It is seen therefore that our alternative analysis does not seem very promising. It is interesting, however, because of the similar technique proposed by Doetsch[12] for investigating lateral oscillations. We may expect a complete analogy between the problems of pitching and yawing oscillations, with elevator (or rudder) fixed and oscillating, provided speed variation is neglected in the former, rolling in the latter case. The equations of lateral oscillation with no rolling are[10]:

$$(D - y_v)\beta + \hat{r} = 0 ,$$

$$- \omega_n \beta + (D + \nu_n)\hat{r} + \delta_n \zeta = 0 , \quad \text{(V.13)}$$

where:

$$\omega_n = \mu_2 \frac{n_v}{i_C} , \qquad \nu_n = - \frac{n_r}{i_C} , \qquad \delta_n = - \mu_2 \frac{n_\zeta}{i_C} , \quad \text{(V.14)}$$

and an analysis, exactly similar to that expounded before, leads to the formula:

$$\bar{R} = R - \frac{\delta_n}{2J} \frac{\zeta^*}{\psi^*}\left(\sin \varphi_{\zeta\psi} + y_v \frac{J \cos \varphi_{\zeta\psi} + R \sin \varphi_{\zeta\psi}}{R^2 + J^2}\right) . \quad \text{(V.15)}$$

and to another formula similar to (V.11). The symbols R and J now refer to the lateral oscillation. If the entire second term in brackets were neglected, we would have a simpler approximate relationship:

$$\bar{R} = R - \frac{\delta_n}{2J} \frac{\zeta^*}{\psi^*} \sin \varphi_{\zeta\psi} , \quad \text{(V.16)}$$

which is exactly identical (apart from notation) with Doetsch's formula (Ref. 12, page 6). Although no derivation is given in Ref. 12, it is clear that the terms containing y_v have been neglected by Doetsch. Our formula (V.15) thus represents an improvement on his formula. It must be mentioned that the correction term in (V.15) may often be of little significance, because y_v is usually small ($-0{\cdot}2$ to $-0{\cdot}4$). In some cases, however, the correction term may be far from negligible and, if $\sin \varphi_{\zeta\psi}$ and J happen to be small and y_v comparatively large, the correction term may become greater than the main term $\sin \varphi_{\zeta\psi}$. As to the case of the longitudinal oscillations, the terms with the factor a should never be neglected in (V.9 to V.11) because a is usually quite large (of the order of 4). For instance, the second term in brackets in (V.9) attains its maximum value when $\tan \varphi_{\eta\theta} = R/J$, and then it becomes $a/2\sqrt{(R^2 + J^2)}$, while the first term $\sin \varphi_{\eta\theta}$ is then equal to $R/\sqrt{(R^2 + J^2)}$, and the two terms are of the same order of magnitude.

In the case of longitudinal oscillations, a relationship may be easily derived between the amplitude ratios ε, η^*/θ^* and the phase angles φ and $\varphi_{\eta\theta}$. Making use of (3.9) and (V.1, 2), we obtain:

$$\left.\begin{aligned} \frac{\eta^*}{\theta^*} &= \varepsilon\sqrt{\left\{\frac{R^2+J^2}{(\frac{1}{2}a-R)^2+J^2}\right\}} \\ \cos\varphi_{\eta\theta} &= \frac{(R^2-\frac{1}{2}aR+J^2)\cos\varphi-\frac{1}{2}aJ\sin\varphi}{\sqrt{[(R^2+J^2)\{(\frac{1}{2}a-R)^2+J^2\}]}} \\ \sin\varphi_{\eta\theta} &= \frac{(R^2-\frac{1}{2}aR+J^2)\sin\varphi+\frac{1}{2}aJ\cos\varphi}{\sqrt{[(R^2+J^2)\{(\frac{1}{2}a-R)^2+J^2\}]}} \end{aligned}\right\} \quad \ldots \ldots \quad \text{(V.17)}$$

Taking, *e.g.*, the numerical data from Example V, we obtain :

$$\eta^*/\theta^* = 0{\cdot}2324, \quad \cos\varphi_{\eta\theta} = 0{\cdot}1608, \quad \sin\varphi_{\eta\theta} = -\,0{\cdot}9870, \quad \varphi_{\eta\theta} = -\,80^\circ\,45',$$

and the formulae (V.9, 11) give $\bar{R} = 4$, $\bar{J} = 3{\cdot}6$, as before.

TABLE 1

Coefficient K (for calculating unit of aerodynamic time $\hat{t}$, see Appendix IV) for Varying Altitude—Illustrated in Fig. 17

Altitude thousands feet	K	Altitude thousands feet	K
0	0·01170	35	0·04335
1	0·01209	36	0·04525
2	0·01250	37	0·04747
3	0·01293	38	0·04981
4	0·01336	39	0·05226
5	0·01382	40	0·05483
6	0·01429	41	0·05754
7	0·01479	42	0·06037
8	0·01531	43	0·06334
9	0·01585	44	0·06646
10	0·01642	45	0·06975
11	0·01701	46	0·07315
12	0·01762	47	0·07677
13	0·01826	48	0·08057
14	0·01894	49	0·08451
15	0·01964	50	0·08867
16	0·02036	51	0·09320
17	0·02113	52	0·09764
18	0·02193	53	0·1025
19	0·02279	54	0·1076
20	0·02366	55	0·1128
21	0·02457	56	0·1187
22	0·02553	57	0·1245
23	0·02653	58	0·1306
24	0·02758	59	0·1370
25	0·02868	60	0·1435
26	0·02987		
27	0·03106	65	0·1830
28	0·03235	70	0·2332
29	0·03371	75	0·2961
30	0·03512	80	0·3772
31	0·03658	85	0·4806
32	0·03813	90	0·6083
33	0·03980	95	0·7761
34	0·04151	100	1·023

TABLE 2

Illustrative Values of the Unit of Aerodynamic Time $\hat{t}$ (see Appendix IV)

Mach Number $M = 0{\cdot}8$

Wing loading lb/sq ft \ Height thousands feet	0	20	40	60	80	100
40	0·585	1·183	2·741	7·175	18·86	51·15
80	1·170	2·366	5·483	14·350	37·72	
120	1·755	3·549	8·224	21·525		
160	2·340	4·732	10·966			
200	2·925	5·915				
240	3·510					

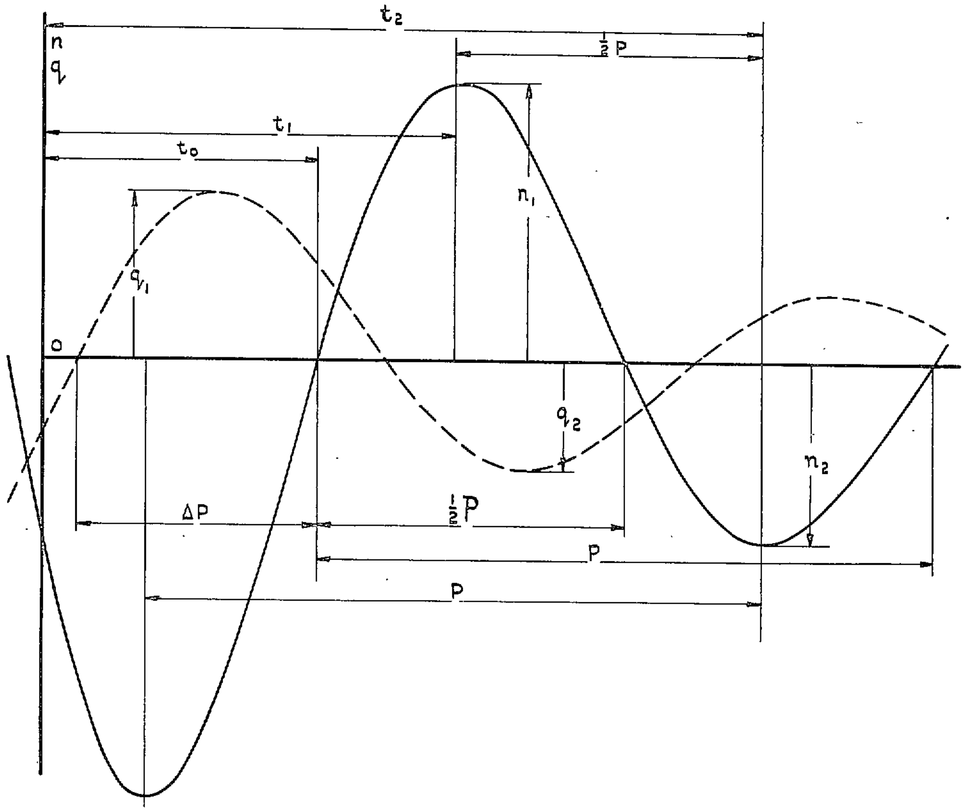


FIG. 1. Typical curves of a damped oscillation.

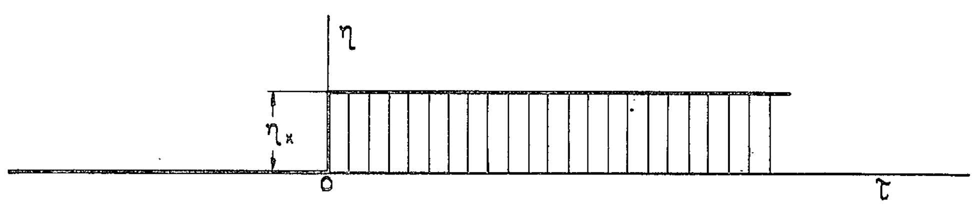


FIG. 2a. Single instantaneous elevator movement. (Step input.)

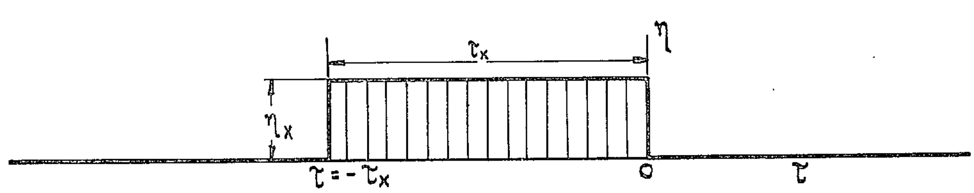


FIG. 2b. Double elevator movement. (Rectangular input.)

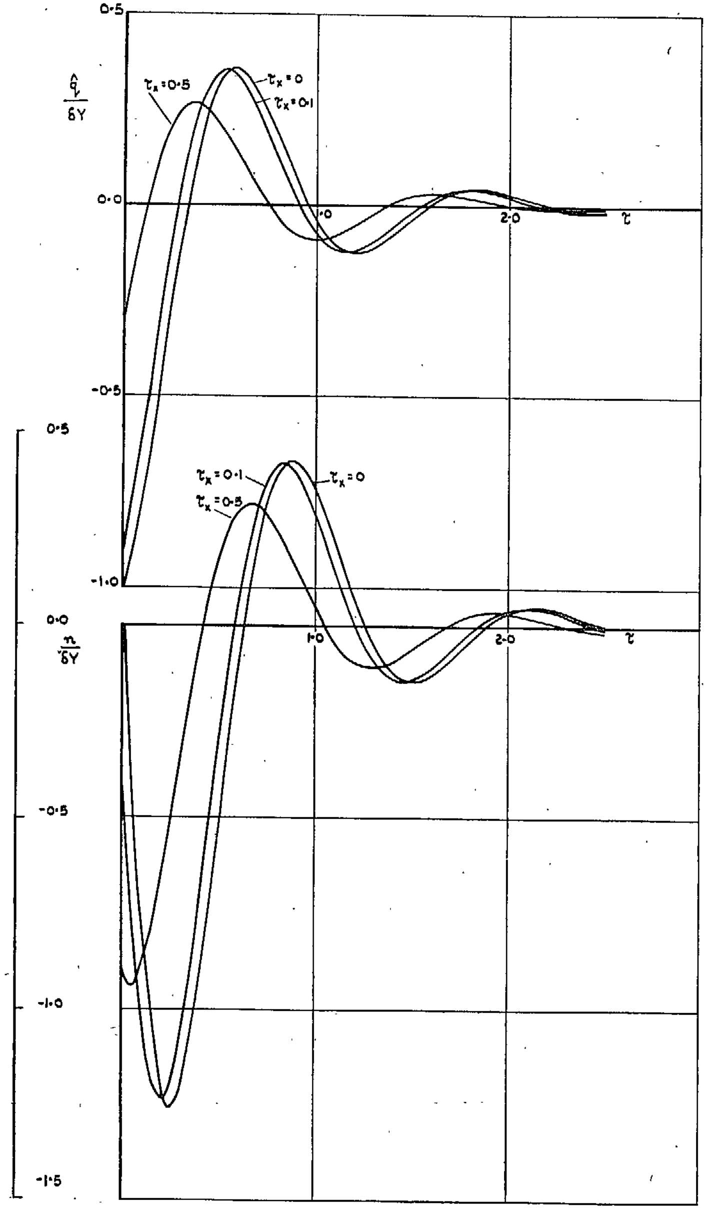


FIG. 3. Response in $\hat{q}$ and n to rectangular elevator inputs of the same impulse and different duration (neglecting effects of speed variation).

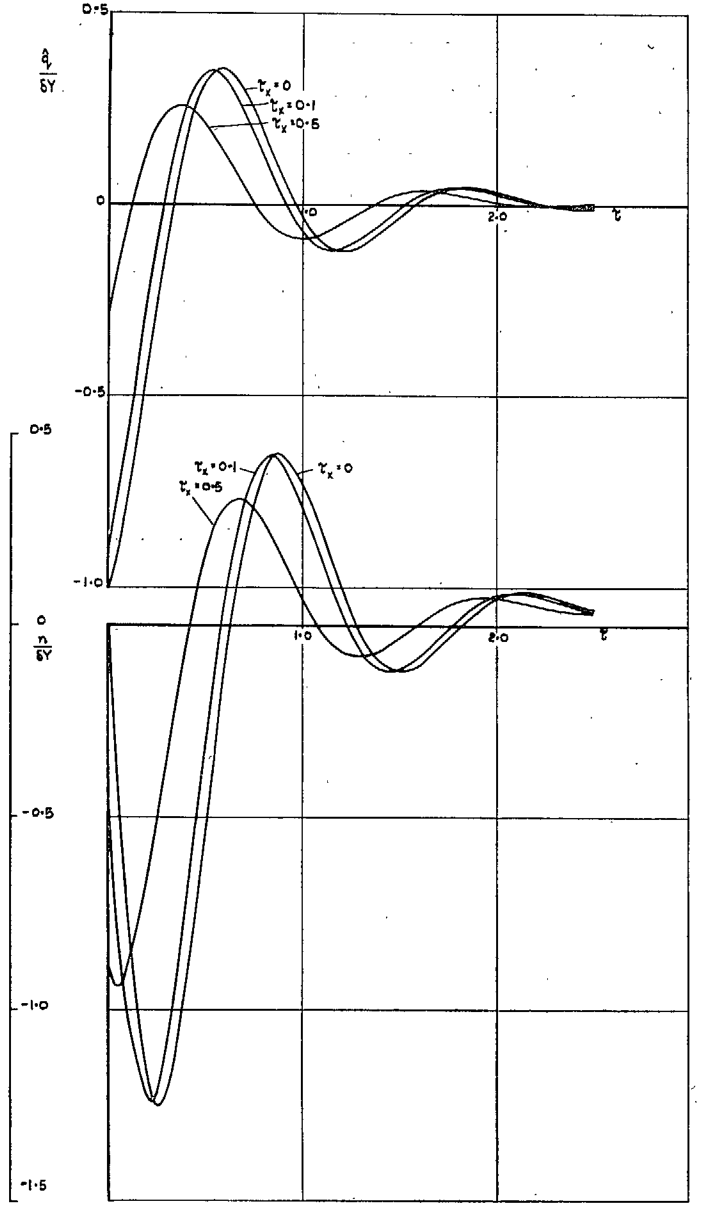


FIG. 4. Response in $\hat{q}$ and n to rectangular elevator inputs of the same impulse and different duration (including effects of speed variation).

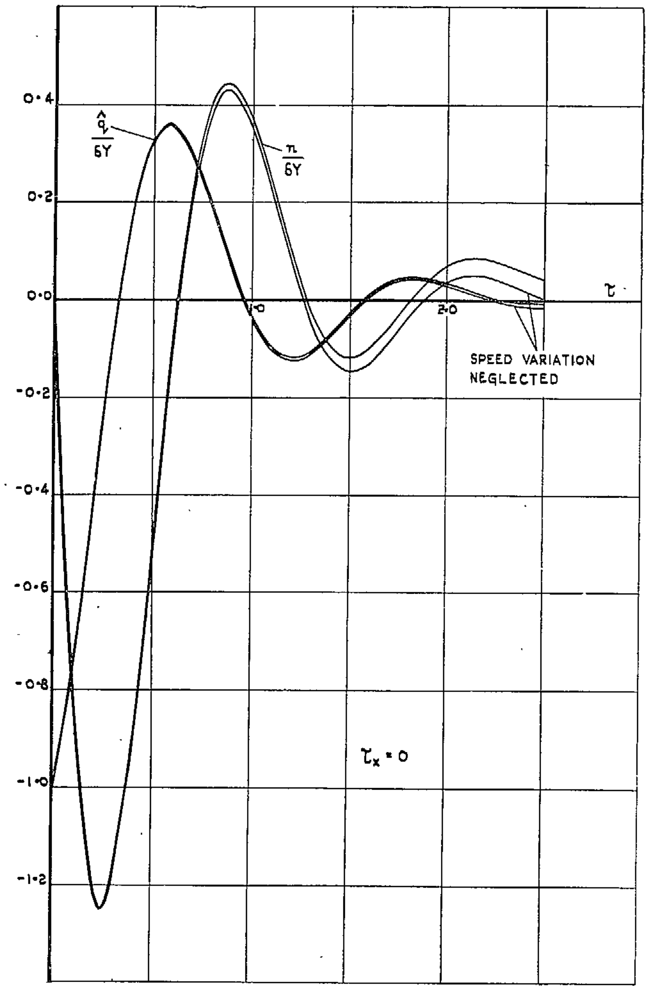


FIG. 5. Comparison of response in $\hat{q}$ and n to a rectangular input of very short duration neglecting or including the effects of speed variation.

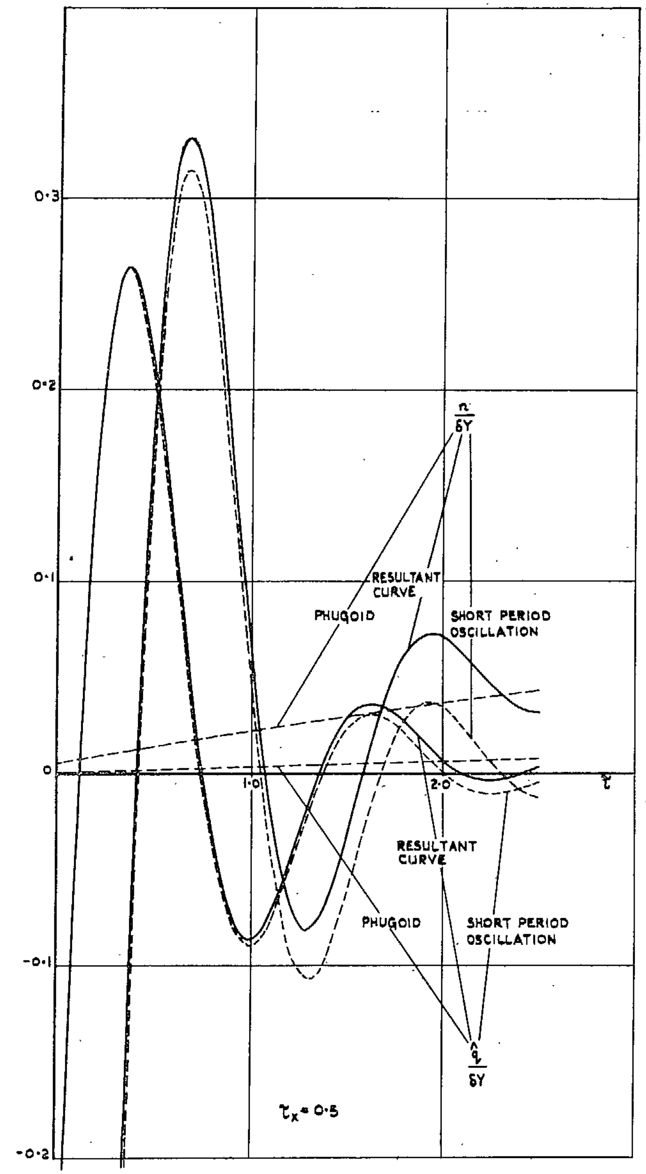


FIG. 6. Contribution of short-period and phugoid modes to response curves following a rectangular elevator input of rather long duration.

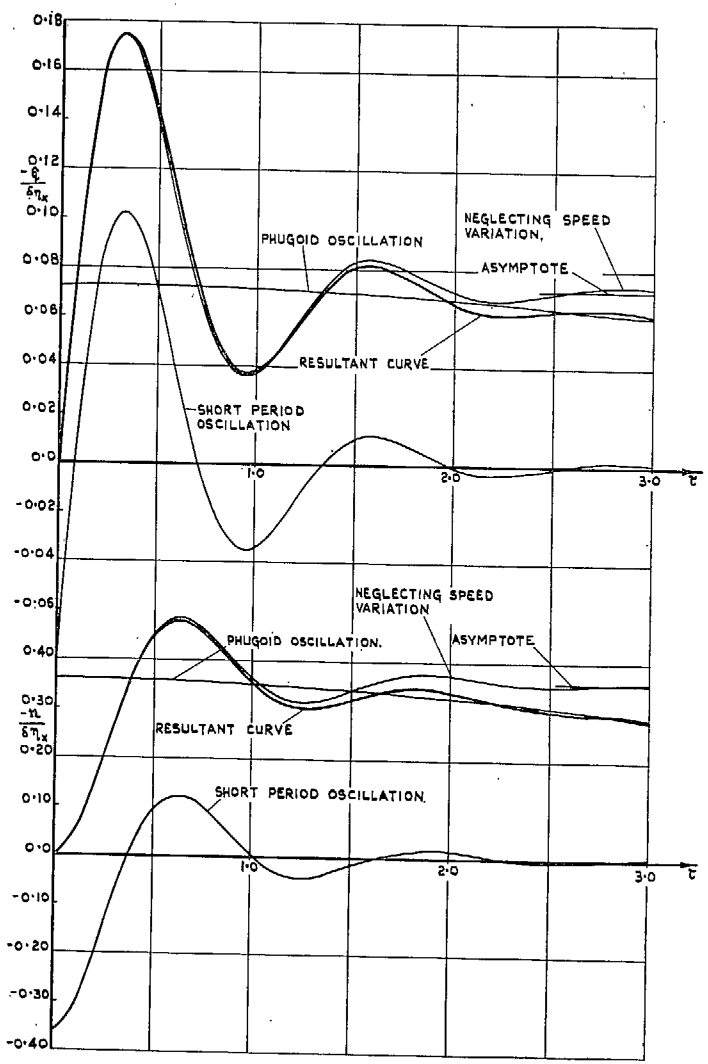


FIG. 7. Response in $\hat{q}$ and n to a single instantaneous elevator movement (step input), neglecting or including effect of speed variation.

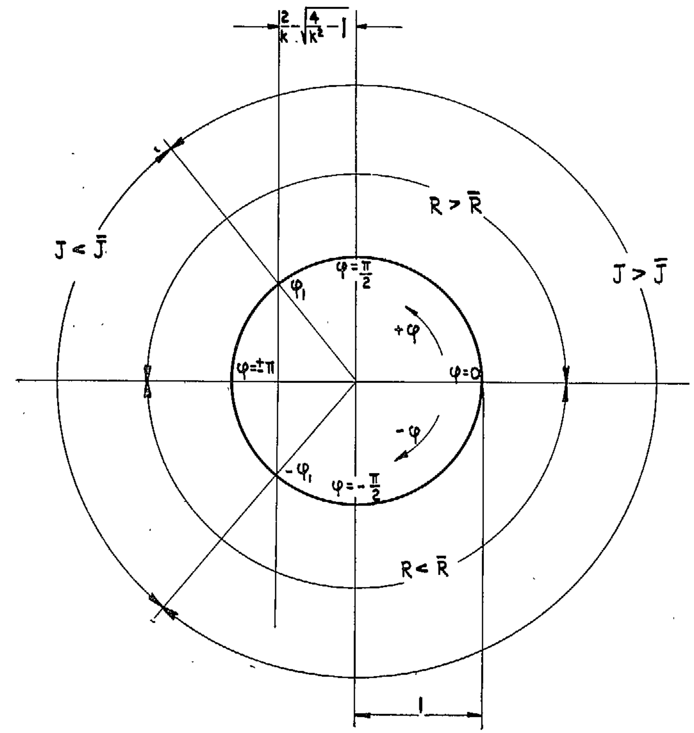


FIG. 8. Regions of phase angle φ in which damping or frequency is increased or decreased by freeing the elevator.

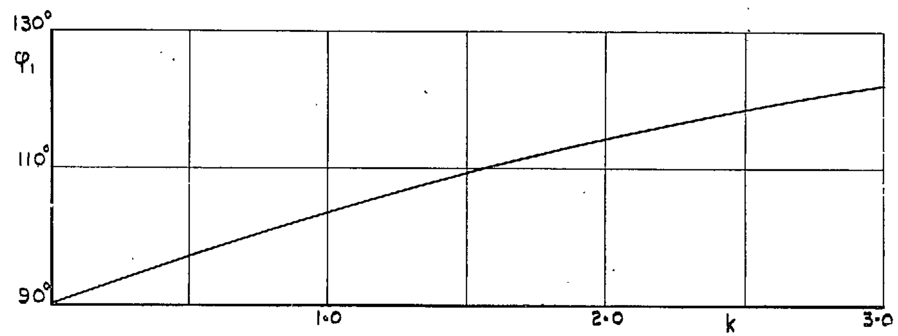


FIG. 9. Phase angle φ_1 corresponding to equal frequencies with elevator fixed or oscillating, against parameter k (or $\bar{k}$).

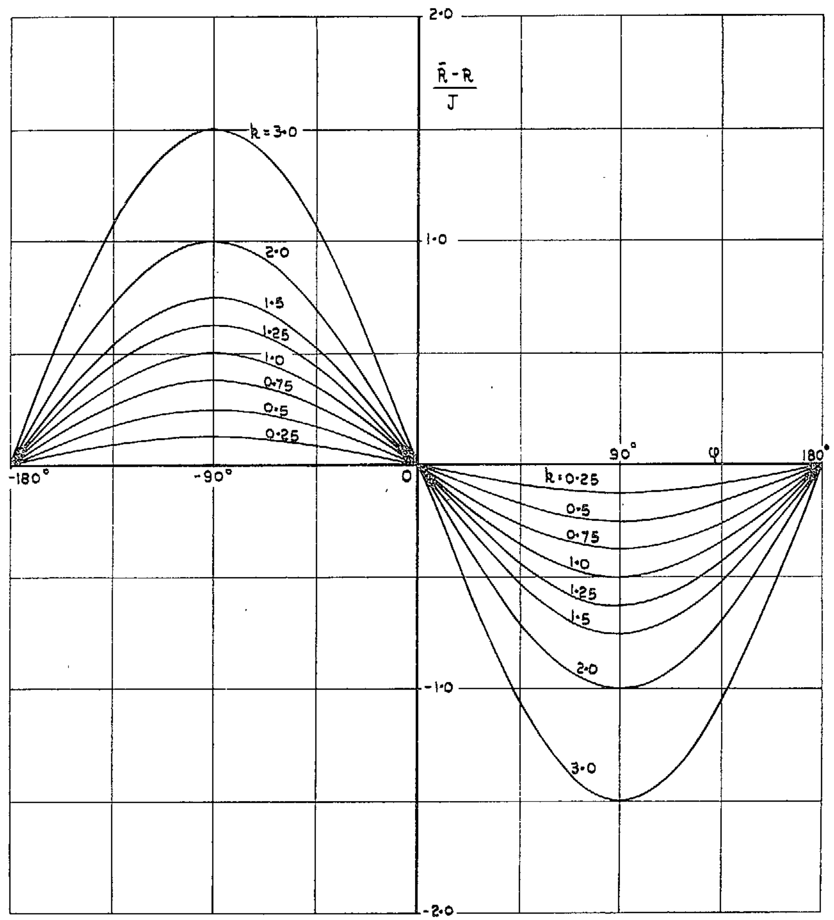


FIG. 10. Effect on damping of fixing the elevator, in terms of parameters k and φ.

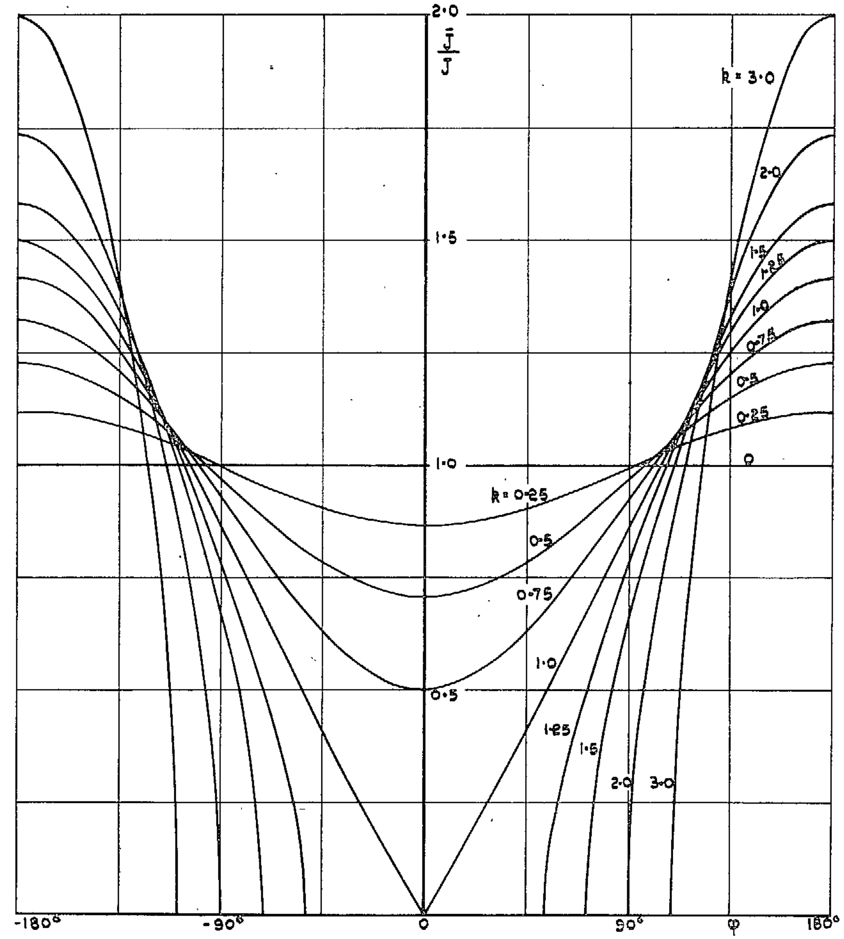


FIG. 11. Effect on frequency of fixing the elevator, in terms of parameters k and φ.

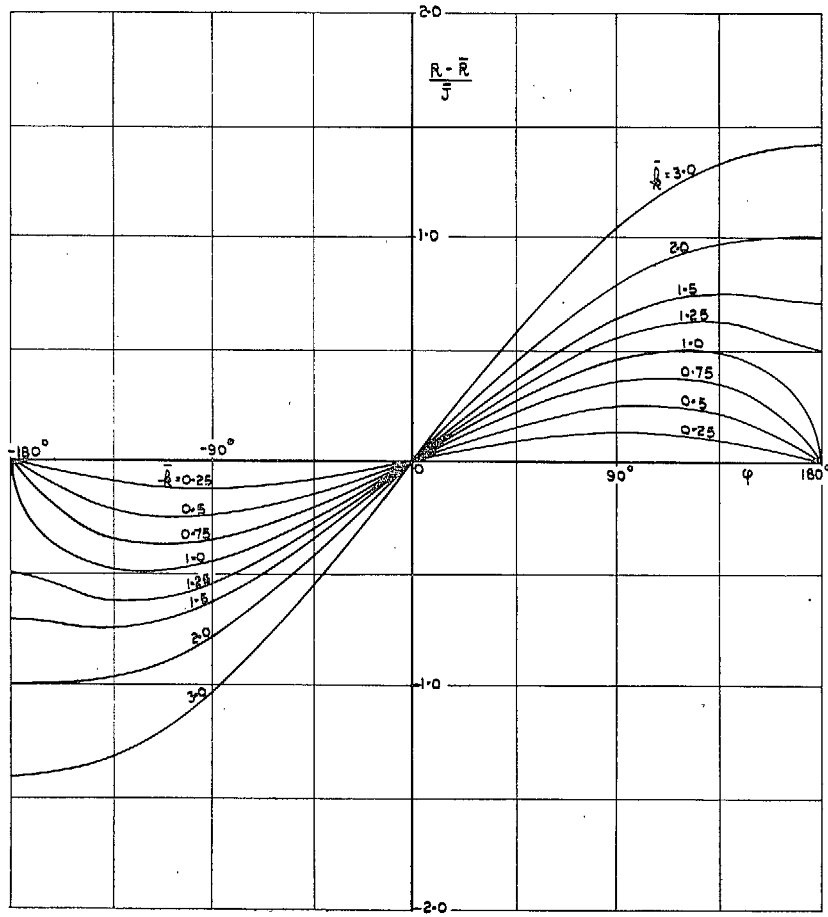


FIG. 12. Effect on damping of freeing the elevator, in terms of the parameters $\bar{k}$ and φ.

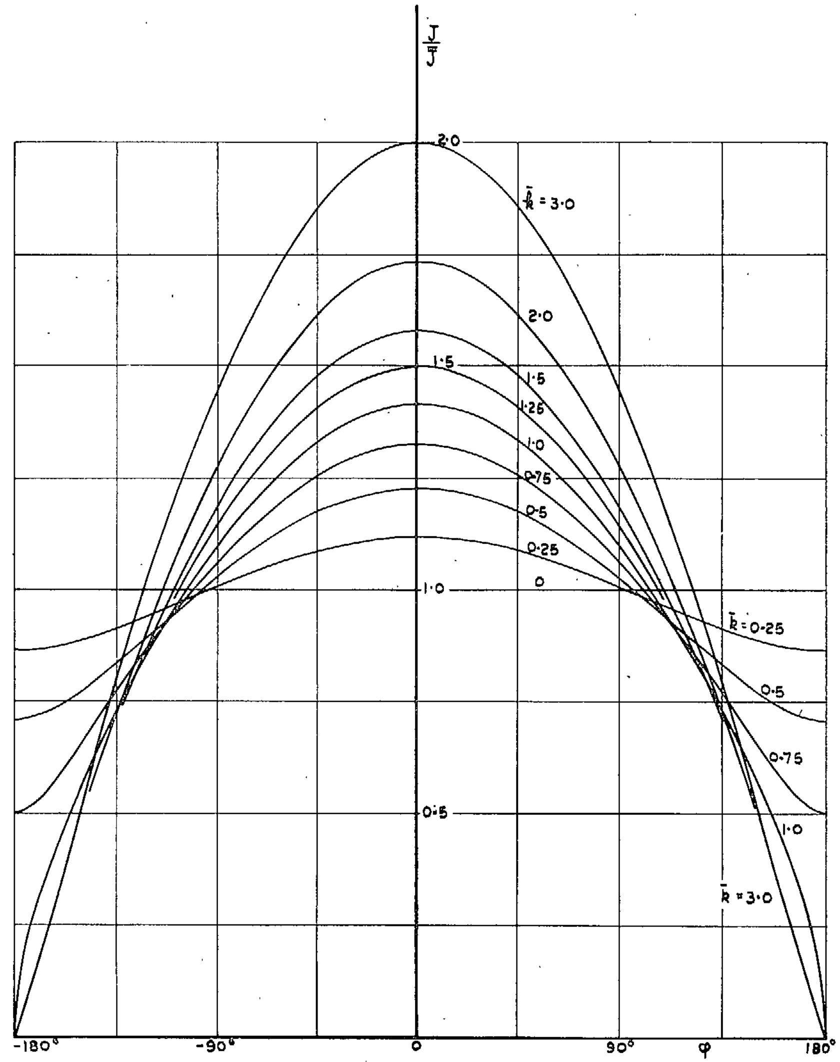
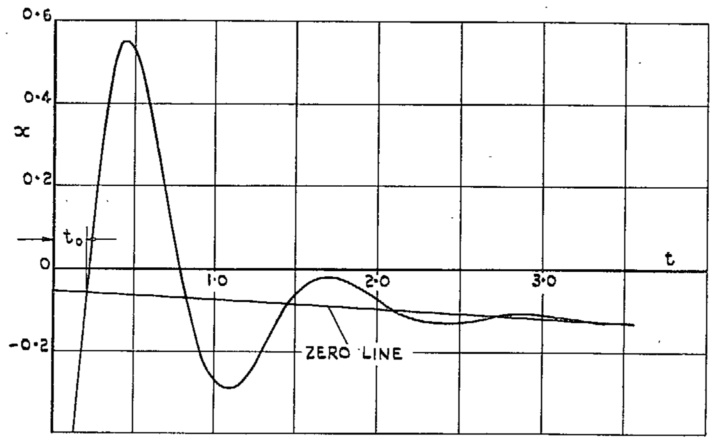
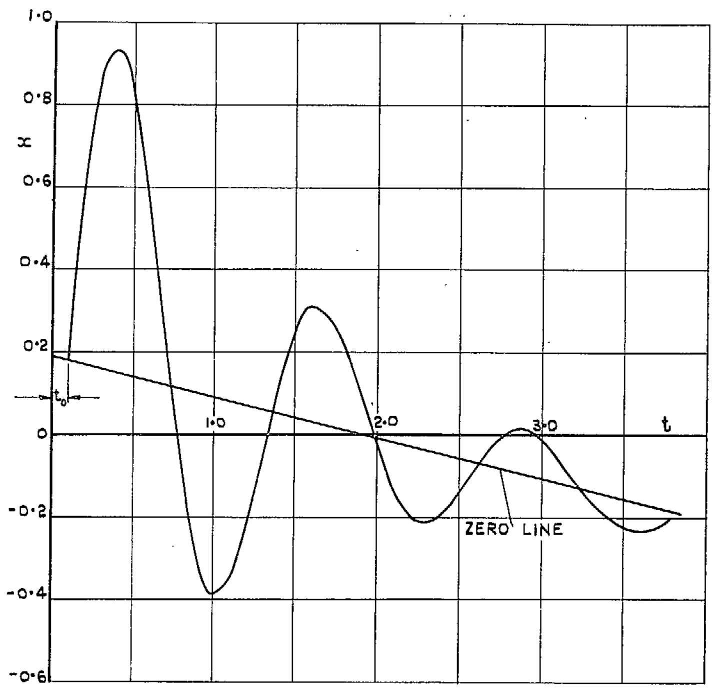


FIG. 13. Effect on frequency of freeing the elevator, in terms of the parameters $\bar{k}$ and φ.



(a). $\mathcal{R}$ = 1·7; k = 5; A = 1; α_0 = – 0·06; β_0 = 0·005; t_0 = 0·2 SEC.



(b). $\mathcal{R}$ = 0·8; k = 5; A = 1; α_0 = 0·18; β_0 = 0·02; t_0 = 0·1 SEC.

FIGS. 14a and 14b. Examples of ' filtration ' of experimental curves.

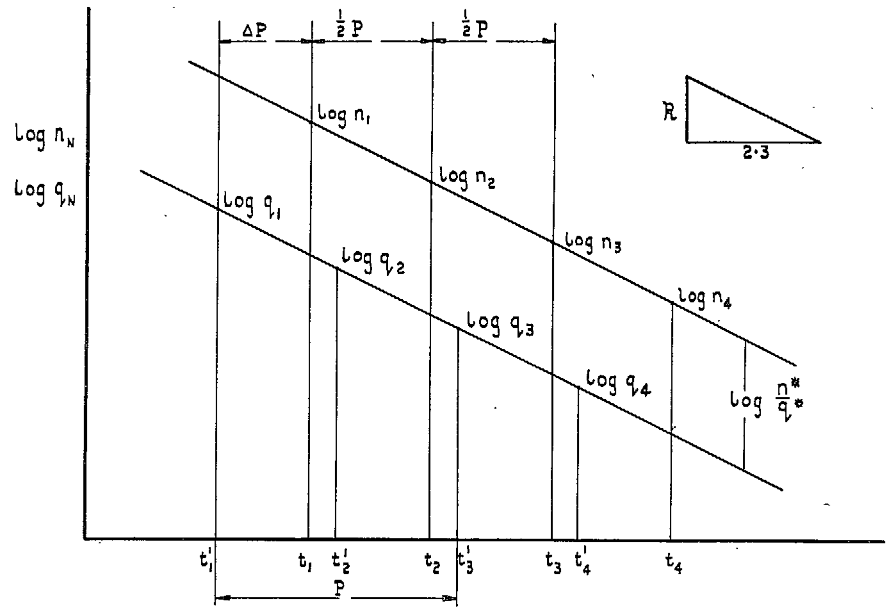


FIG. 15. Graphical determination of damping factor $\mathcal{R}$ and amplitude ratio n^*/q^*.

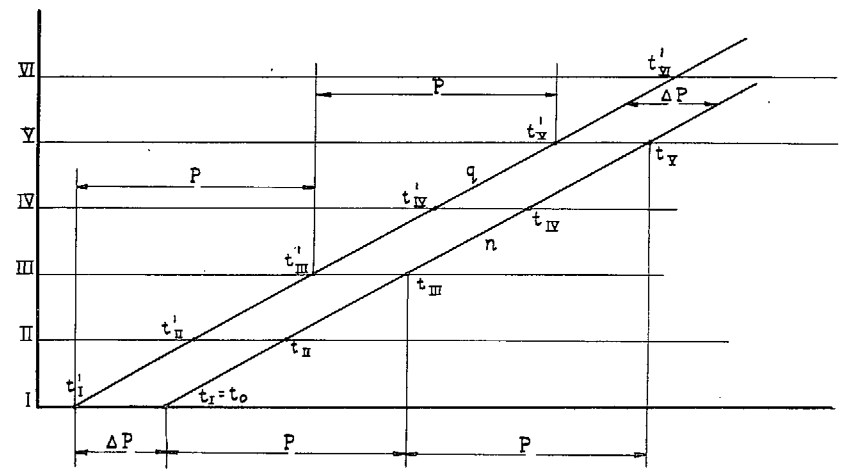


FIG. 16. Graphical determination of period P and phase-time interval ΔP.

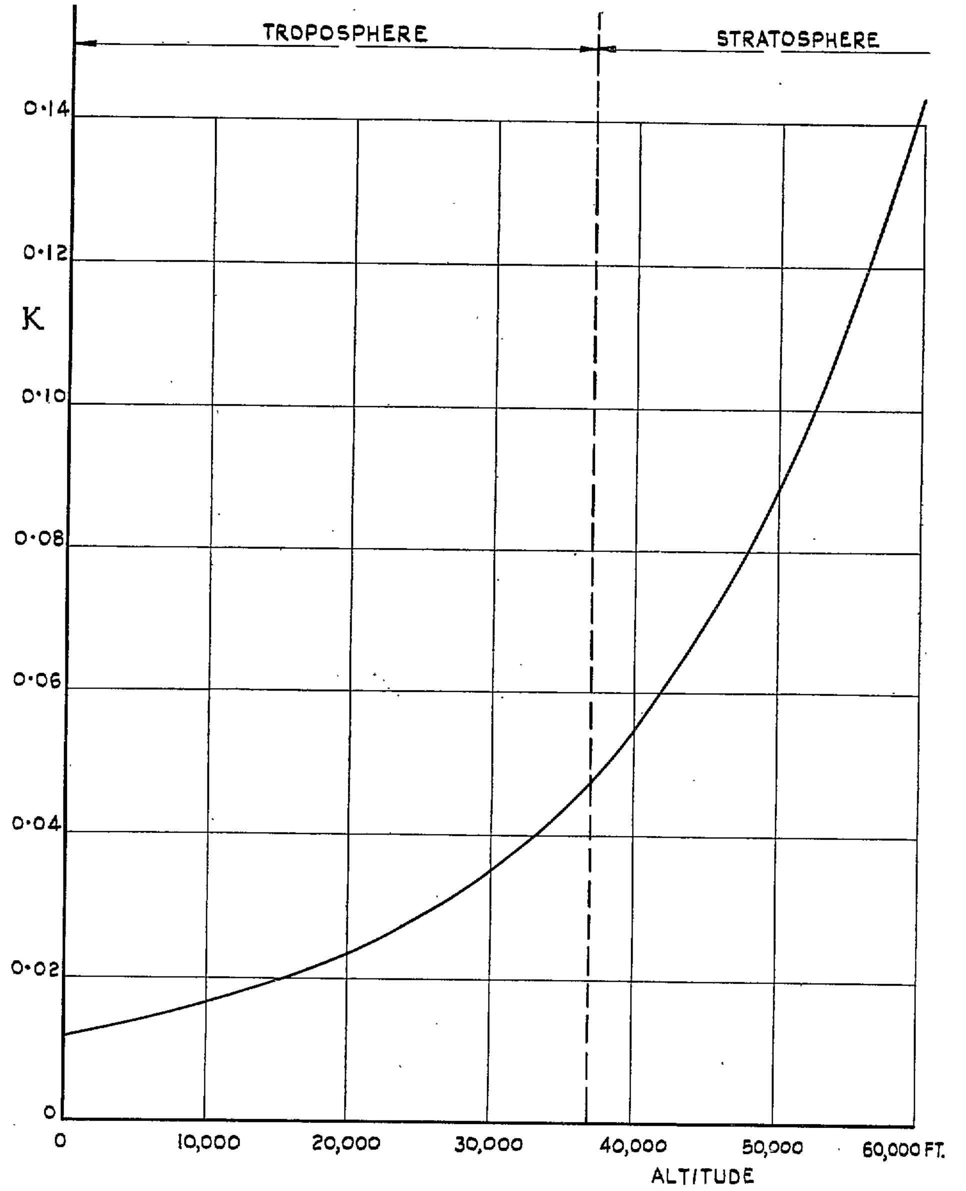


FIG. 17. Coefficient K for varying altitude.

(1960) Wt. 18/9296 K.7 7/56 Hw. PRINTED IN GREAT BRITAIN

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1938 Vol. I. Aerodynamics General, Performance, Airscrews. 50s. (51s. 8d.)
Vol. II. Stability and Control, Flutter, Structures, Seaplanes. Wind Tunnels, Materials. 30s. (31s. 8d.)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50s. (51s. 8d.)
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63s. (64s. 8d.)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50s. (51s. 8d.)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control, Structures. 63s. (64s. 8d.)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75s. (76s. 8d.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47s. 6d. (49s. 2d.)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80s. (81s. 8d.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90s. (91s. 11d.)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84s. (86s. 9d.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84s. (86s. 9d.)

Annual Reports of the Aeronautical Research Council—

1933–34	1s. 6d. (1s. 8½d.)	1937	2s. (2s. 2½d.)
1934–35	1s. 6d. (1s. 8½d.)	1938	1s. 6d. (1s. 8½d.)
April 1, 1935 to Dec. 31, 1936	4s. (4s. 5½d.)	1939–48	3s. (3s. 3½d.)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - R. & M. No. 2600 2s. 6d. (2s. 7½d.)

Author Index to all Reports and Memoranda of the Aeronautical Research Council—

1909–January, 1954. R. & M. No. 2570 15s. (15s. 5½d.)

Indexes to the Technical Reports of the Aeronautical Research Council—

December 1, 1936 — June 30, 1939	R. & M. No. 1850	1s. 3d. (1s. 4½d.)
July 1, 1939 — June 30, 1945	R. & M. No. 1950	1s. (1s. 1½d.)
July 1, 1945 — June 30, 1946	R. & M. No. 2050	1s. (1s. 1½d.)
July 1, 1946 — December 31, 1946	R. & M. No. 2150	1s. 3d. (1s. 4½d.)
January 1, 1947 — June 30, 1947	R. & M. No. 2250	1s. 3d. (1s. 4½d.)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2251–2349	R. & M. No. 2350	1s. 9d. (1s. 10½d.)
Between Nos. 2351–2449	R. & M. No. 2450	2s. (2s. 1½d.)
Between Nos. 2451–2549	R. & M. No. 2550	2s. 6d. (2s. 7½d.)
Between Nos. 2551–2649	R. & M. No. 2650	2s. 6d. (2s. 7½d.)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1 (Post Orders: P.O. Box 569, London S.E.1); 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol, 1; 80 Chichester Street, Belfast, *or through any bookseller*

S.O. Code No. 23-2940

R. & M. No. 29