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By

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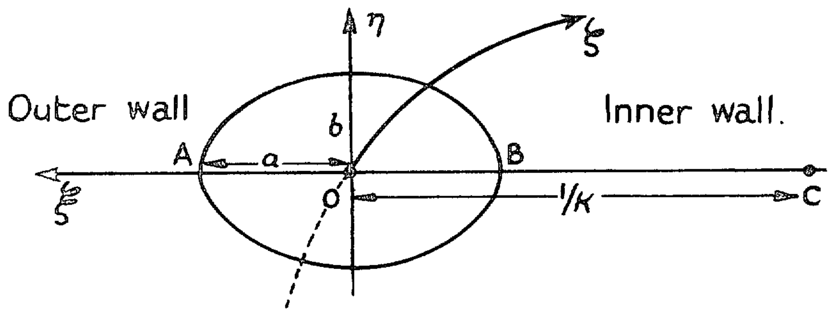
Summary.—The Navier-Stokes equations for the flow of a viscous incompressible fluid through curved pipes of different sections are solved in power series of the curvature of the pipe. The solution is given as far as the first power of the curvature for the case of an elliptic section and a discussion given of the effect of the aspect ratio of the pipe on the intensity of the secondary flow. It is shown that the axial velocity is modified by two curvature terms of opposite effect. For values of the aspect ratio near unity the first of these predominates and the resultant effect is an increase of velocity in the outer half of the bend and a decrease in the inner: for large values of the aspect ratio the second term is numerically much greater and there is a resultant decrease in axial velocity in the outer half of the bend and an increase in the inner half. The solution is also given to the first power of the curvature for the case of a square section. This shows that the intensity of the secondary flow in a pipe of square section is greater than that in a pipe of circular section. Finally the solution is given as far as the second power of the curvature for the case of flow through a curved pipe of circular section when suction proportional to the curvature is applied at the walls. The result shows that with the particular distribution of suction considered the diminution in flux through a curved pipe may be almost entirely eliminated.

1. *Introduction.*—When a fluid flows through a curved pipe of any cross-section it is observed that a secondary flow occurs in planes perpendicular to the curved central axis of the pipe. The theoretical explanation of this secondary flow was first given by Thompson[1]. There must be a pressure gradient across the pipe to balance the centrifugal force on the fluid due to its curved trajectory, the pressure being greatest at the outer wall of the pipe and least at the inner wall. The fluid near the top and bottom walls of the pipe is moving more slowly than that near the central plane due to viscosity and therefore requires a smaller pressure gradient to balance its reduced centrifugal force. Consequently a secondary flow occurs in which the fluid near the top and bottom walls of the pipe moves inwards towards the centre of curvature of the central axis and the fluid near the central plane moves outwards. This in turn modifies the axial velocity. The faster-moving fluid near the central plane pushes the fluid in the boundary layer at the outer wall to the top and bottom walls and then inwards along the top and bottom walls (where it is retarded due to its proximity to these walls) towards the inner wall. Faster-moving fluid is therefore constantly transported to the outer wall and retarded fluid is carried to the inner wall. The accumulation of the retarded layer at the inner wall results in a diminution of flux through the pipe.

Experimental investigations of the secondary flow were made by Eustace[2], White[3] and Taylor[4] and the first theoretical analysis given by Dean[5] for the case of an incompressible fluid in steady motion through a pipe of circular cross-section whose axis is bent to the form of a circular arc of several revolutions. Dean expanded the velocities and pressure as a power series of the curvature of the axis of the pipe and gave the solution as far as the fourth power of this parameter, the analysis being confined to a region of the pipe sufficiently far downstream for derivatives of the velocities along the pipe to vanish.

The first part of this paper extends the same method to deal with pipes of elliptic section and a discussion is given of the effect of aspect ratio of the pipe on the intensity of the secondary flow. The second part applies the method to pipes of rectangular section. An approximate solution for the particular case of a square section is given explicitly and shows that the flow is more intense than in a pipe of circular section. The last part investigates the effect of a particular distribution of suction around the walls of a circular pipe on the flux through it. It is shown that with a suitable choice of suction the diminution in flux may be almost entirely overcome. Both the first and last parts contain Dean's results as special cases.

2. *The Secondary Flow in a Pipe of Elliptic Section.*—2.1. *The Stream Function.*—



The following right-handed, orthogonal system of axes is employed: $O\xi$ along the major axis of a section; $O\eta$ along the minor axis and $O\zeta$ along the curved central axis. Let the velocity components along these axes be (u, v, w); C the centre of curvature of the axis $O\zeta$; $OC = 1/\varkappa$ where $\varkappa$ is the curvature of $O\zeta$; $2a$, $2b$ the major and minor axes of a section. Let $b/a = \lambda$, this we shall call the aspect ratio of the pipe.

The flow is determined by the Navier-Stokes[6] equations of motion

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times \operatorname{curl} \boldsymbol{v} = -\operatorname{grad}\left(\tfrac{1}{2}\boldsymbol{v}^2 + p/\rho\right) - \nu \operatorname{curl}\operatorname{curl} \boldsymbol{v} \quad \ldots \quad (1)$$

and the equation of continuity

$$\operatorname{div} \boldsymbol{v} = 0 \quad \ldots \quad (2)$$

together with the boundary condition that $\boldsymbol{v} = 0$ at the walls.

Making the non-dimensional substitutions $x = \xi/a$, $y = \eta/\lambda a$, $z = \zeta/a$; $u = \nu\bar{u}/a$, $v = \nu\bar{v}/a$, $w = \nu\bar{w}/a$, $p = \rho\nu^2/a^2 \,.\, \bar{p}$ and putting $\partial u/\partial z = \partial v/\partial z = \partial w/\partial z = 0$ we obtain from equations (1) and (2)

$$\left.\begin{aligned}
&\bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\bar{v}}{\lambda}\frac{\partial \bar{u}}{\partial y} - \frac{\varkappa a \bar{w}^2}{1 + \varkappa a x} = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{\lambda^2}\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{1}{\lambda}\frac{\partial^2 \bar{v}}{\partial x\,\partial y} \\
&\bar{u}\frac{\partial \bar{v}}{\partial x} + \frac{\bar{v}}{\lambda}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\lambda}\frac{\partial \bar{p}}{\partial y} + \frac{\partial^2 \bar{v}}{\partial x^2} - \frac{1}{\lambda}\frac{\partial^2 \bar{u}}{\partial x\,\partial y} + \frac{\varkappa a}{1 + \varkappa a x}\left(\frac{\partial \bar{v}}{\partial x} - \frac{1}{\lambda}\frac{\partial \bar{u}}{\partial y}\right) \\
&\bar{u}\frac{\partial \bar{w}}{\partial x} + \frac{\bar{v}}{\lambda}\frac{\partial \bar{w}}{\partial y} + \frac{\varkappa a \bar{u}\bar{w}}{1 + \varkappa a x} = -\frac{1}{1 + \varkappa a x}\frac{\partial \bar{p}}{\partial z} + \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\varkappa a}{1 + \varkappa a x}\frac{\partial \bar{w}}{\partial x} - \frac{\varkappa^2 a^2 \bar{w}}{(1 + \varkappa a x)^2} \\
&\frac{\partial}{\partial x}\left\{(1 + \varkappa a x)\bar{u}\right\} + \frac{1}{\lambda}\frac{\partial}{\partial y}\left\{(1 + \varkappa a x)\bar{v}\right\} = 0\,.
\end{aligned}\right\} \quad \ldots \quad (3)$$

The boundary condition is that $\bar{u} = \bar{v} = \bar{w} = 0$ for $x^2 + y^2 = 1$.

Differentiating the third equation in (3) with respect to z we have $\partial^2\bar{p}/\partial z^2 = 0$; it follows from the first and second equations that $\partial\bar{p}/\partial x$ and $\partial\bar{p}/\partial y$ are independent of z. $\partial\bar{p}/\partial z$ is therefore constant. To solve equations (3) to the first power of $\varkappa$ we put

$$\left.\begin{aligned} \bar{u} &= u_0 + \varkappa a R^2 u_1 & \bar{v} &= v_0 + \varkappa a R^2 v_1 \\ \bar{p} &= p_0 + \varkappa a R^2 p_1 & \bar{w} &= \tfrac{1}{2}R(w_0 + \varkappa a R^2 w_1 + \varkappa a w_2) \end{aligned}\right\} \quad \ldots \quad (4)$$

where $$R = -\frac{\lambda^2}{1+\lambda^2}\frac{\partial\bar{p}}{\partial z}.$$

The terms independent of $\varkappa$ yield the equations of motion for flow through a straight elliptic pipe, *viz.*,

$$u_0 = v_0 = 0; \qquad \frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial y} = 0$$

and $$\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 w_0}{\partial y^2} = -2(1 + 1/\lambda^2)$$

with the solution $$w_0 = 1 - x^2 - y^2. \quad \ldots \quad (5)$$

Equating coefficients of $\varkappa$ we obtain

$$\left.\begin{aligned} -\frac{w_0^2}{4} &= -\frac{\partial p_1}{\partial x} + \frac{1}{\lambda^2}\frac{\partial^2 u_1}{\partial y^2} - \frac{1}{\lambda}\frac{\partial^2 v_1}{\partial x\,\partial y} \\ 0 &= -\frac{1}{\lambda}\frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial x^2} - \frac{1}{\lambda}\frac{\partial^2 u_1}{\partial x\,\partial y} \\ u_1\frac{\partial w_0}{\partial x} + \frac{v_1}{\lambda}\frac{\partial w_0}{\partial y} &= \frac{\partial^2 w_1}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 w_1}{\partial y^2} \\ 0 &= -2\left(1 + \frac{1}{\lambda^2}\right)x + \frac{\partial^2 w_2}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 w_2}{\partial y^2} + \frac{\partial w_0}{\partial x} \\ \frac{\partial u_1}{\partial x} + \frac{1}{\lambda}\frac{\partial v_1}{\partial y} &= 0 \end{aligned}\right\} \quad \ldots \quad (6)$$

The last equation may be satisfied identically by a stream function ψ defined by the equations

$$\left.\begin{aligned} u_1 &= \frac{1}{\lambda^2}\frac{\partial\psi}{\partial y} \\ v_1 &= \frac{-1}{\lambda}\frac{\partial\psi}{\partial x} \end{aligned}\right\} \quad \ldots \quad (7)$$

Eliminating p_1 between the first and second equations of (6) we obtain

$$\frac{\partial^4\psi}{\partial x^4} + \frac{2}{\lambda^2}\frac{\partial^4\psi}{\partial x^2\,\partial y^2} + \frac{1}{\lambda^4}\frac{\partial^4\psi}{\partial y^4} = (1 - x^2 - y^2)y \quad \ldots \quad (8)$$

with the boundary conditions $\partial\psi/\partial x = \partial\psi/\partial y = 0$ for $x^2 + y^2 = 1$. The solution is

$$\psi = (1 - x^2 - y^2)^2(B_1 + B_2x^2 + B_3y^2)y \quad \ldots \quad (9)$$

where
$$B_1 = \lambda^4(375 + 820\lambda^2 + 1{,}114\lambda^4 + 212\lambda^6 + 39\lambda^8)/360(5 + 2\lambda^2 + \lambda^4)G(\lambda)$$
$$B_2 = -\lambda^4(75 + 2\lambda^2 + 3\lambda^4)/360G(\lambda)$$
$$B_3 = -\lambda^4(15 + 26\lambda^2 + 39\lambda^4)/360G(\lambda)$$

and $$G(\lambda) = 35 + 84\lambda^2 + 114\lambda^4 + 20\lambda^6 + 3\lambda^8. \quad \ldots \quad (10)$$

2.2. *Nature of the Secondary Flow.*—It appears from equations (4) that $\varkappa a R^2$ is the dynamical parameter of the secondary flow. If W_0 be the velocity along the central axis for the flow through a straight elliptic pipe, from equation (5) $W_0 = \nu R/2a$, i.e., $R = 2aW_0/\nu$. R may therefore be interpreted as the Reynolds number for steady flow through the unbent pipe. From equations (7) and (9)

$$\left.\begin{aligned} u &= \frac{\varkappa \nu R^2}{\lambda^2}(1 - x^2 - y^2)[(1 - x^2 - y^2)(B_1 + B_2x^2 + 3B_3y^2) \\ &\quad - 4y^2(B_1 + B_2x^2 + B_3y^2)] \\ v &= \frac{2\varkappa \nu R^2}{\lambda^2}(1 - x^2 - y^2)[2(B_1 + B_2x^2 + B_3y^2) - B_2(1 - x^2 - y^2)]xy \end{aligned}\right\} . \qquad (11)$$

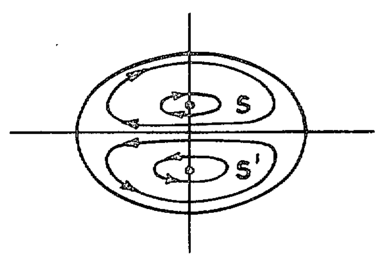
v vanishes along both axes: therefore a fluid particle originally in the upper (or lower) half of the pipe will remain in it during the subsequent motion. Also

$$[u]_{(0,0)} = \frac{\varkappa \nu R^2}{\lambda^2} \cdot B_1 > 0$$

and

$$[u]_{(0,1-\delta)} = -\frac{8\varkappa \nu}{\lambda^2} R^2(B_1 + B_3)\delta < 0 \text{ for } \delta \text{ small and positive.}$$

The fluid near the top and bottom walls is therefore moving inwards whilst that in the central plane is moving outwards. There must exist two points, say S and S′, on the minor axis at which u vanishes. Since v vanishes everywhere along the minor axis these must be stagnation points of the secondary flow. The following diagram shows the essential features of the flow consisting of two opposed vortex motions centred about the stagnation points S and S′.



An observation of some interest may be made on the pressure variation across the major axis. Substitution of u_1 and v_1 in the first two equations of (6) shows that $p_1 - p_1'$ (where p_1' is the value of p for $x = 0$) is odd in x and even in y. Putting $y = 0$ and $x = \pm 1$ in turn we have

$$(p_1)_A - p_1' = -[(p_1)_B - p_1']$$

i.e.,

$$p_1' = \frac{(p_1)_A + (p_1)_B}{2};$$

the pressure along the minor axis is equal to the arithmetic mean of the pressures at A and B. This result is verified experimentally by Richter[7] for turbulent flow and assumed for laminar flow by Keulegan and Beij[8].

2.3. *Effect of the Aspect Ratio on the Intensity of the Flow.*—It appears from equations (10) and (11) that for small λ, $u = O(\lambda^2)$ and $v = O(\lambda^3)$. For pipes of small aspect ratio the secondary flow is therefore greatly reduced. For large λ, $u = O(1/\lambda^2)$ and $v = O(1/\lambda)$; the intensity is therefore diminished for large aspect ratios. In the case of $\lambda = \infty$, that is for the flow along the channel between two infinite concentric cylinders, the secondary flow vanishes altogether. This is readily explained on physical grounds: the top and bottom walls, which retard the fluid near them and thus induce the secondary flow, are now absent. The following table shows the variation of $(\bar{u})_{y=0}$ across the central plane for various values of the aspect ratio in the case when $\varkappa a = 0 \cdot 2$ and $R = 100$. The results are shown graphically in Fig. 1.

TABLE 1

x	-1	$-2/3$	$-1/3$	0	$1/3$	$2/3$	1
$\bar{u}_{\lambda=1/2}$	0	0·6	1·9	2·6	1·9	0·6	0
$\bar{u}_{\lambda=1}$	0	1·9	5·3	6·9	5·3	1·9	0
$\bar{u}_{\lambda=2}$	0	2·4	6·3	8·1	6·3	2·4	0
$\bar{u}_{\lambda=3}$	0	1·7	4·3	5·5	4·3	1·7	0
$\bar{u}_{\lambda=4}$	0	1·1	2·9	3·7	2·9	1·1	0
$\bar{u}_{\lambda=\infty}$	0	0	0	0	0	0	0

It is natural to take as a measure of the intensity of the secondary flow the total vorticity Ω in either the upper or lower half of the pipe (these are equal and opposite).

$$\Omega = \int_{-a}^{a}\int_{0}^{b}\left(\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) d\xi \, d\eta$$

$$= \frac{4\varkappa a\nu R^2}{525} \frac{\lambda^2(125 + 310\lambda^2 + 428\lambda^4 + 82\lambda^6 + 15\lambda^8)}{(5 + 2\lambda^2 + \lambda^4)G(\lambda)}$$

If Ω_1 is the value of Ω for $\lambda = 1$, *i.e.*, Ω_1 is the total vorticity in the upper half of a pipe of circular section

$$\frac{\Omega}{\Omega_1} = \frac{128\lambda^6(125 + 310\lambda^2 + 428\lambda^4 + 82\lambda^6 + 15\lambda^8)}{15(5 + 2\lambda^2 + \lambda^4)(1 + \lambda^2)^2(35 + 84\lambda^2 + 114\lambda^4 + 20\lambda^6 + 3\lambda^8)}$$

and

$$\Omega_1 = \frac{\varkappa a\nu}{1120}\left(\frac{\partial \bar{p}}{\partial z}\right)^2 . \qquad (12)$$

The variation of Ω/Ω_1 with λ is shown graphically in Fig. 2 from which it appears that Ω attains a maximum value of $3 \cdot 1\Omega_1$ when $\lambda = 2 \cdot 2$. The diminution of flux through a curved pipe is due to the accumulation of a retarded layer of fluid at the inner wall and this in turn is due to the secondary flow; it therefore appears probable that the diminution is greatest for values of λ in the neighbourhood of 2. For $\lambda = 6$ the flow is of the same strength as for $\lambda = 1$; further increase in λ diminishes the intensity of the flow asymptotically to zero.

2.4. *The Axial Velocity.*—Substituting for w_0 in the fourth equation of (6) we have

$$\frac{\partial^2 w_2}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 w_2}{\partial y^2} = 2\left(2 + \frac{1}{\lambda^2}\right)x$$

the solution of which is

$$w_2 = -\frac{1 + 2\lambda^2}{1 + 3\lambda^2}(1 - x^2 - y^2)x . \qquad (13)$$

The addition of this term in the axial velocity represents an increase of velocity in the inside of the bend and a decrease of velocity in the outside of the bend. The solution of w_1, from the third equation in (6), is a polynomial of degree nine in x and y together, odd in x and even in y.

Since w_2 is also odd in x and even in y, it follows that the diminution in flux is zero to the first order of the curvature of the pipe. This polynomial represents an increase of velocity in the outside of the bend and a decrease in the inside of the bend. The effect of curvature of the pipe on the axial velocity is therefore represented by two terms w_1 and w_2 of opposite sign. Of these, w_1 represents the effect generally associated with the flow through a curved pipe and w_2 the effect generally associated with the flow along a curved channel. In fact, both effects are always present together. For values of λ near unity the w_1 term predominates; for large λ w_1 is of order $1/\lambda^2$ but w_2 tends to the limiting value $-\frac{2}{3}x(1-x^2)$. The following table gives the variation in $(\bar{w})_{y=0}$ across the major axis when $R = 100$, $\varkappa a = 0 \cdot 2$.

TABLE 2

x	-1	$-2/3$	$-1/3$	0	1/3	2/3	1
$\bar{w}_0$	0	27·8	44·5	50	44·5	27·8	0
$\bar{w}_{\lambda=1}$	0	21·7	36·0	50	53·0	33·9	0
$\bar{w}_{\lambda=2}$	0	15·4	30·6	50	58·4	40·2	0
$\bar{w}_{\lambda=3}$	0	19·3	34·5	50	54·5	36·3	0
$\bar{w}_{\lambda=4}$	0	22·9	38·3	50	50·7	32·7	0
$\bar{w}_{\lambda=\infty}$	0	30·3	46·5	50	42·5	25·3	0

These results are shown graphically in Fig. 3. As λ increases the point of maximum velocity shifts from the outside of the bend over to the inside of the bend. The transition from one type of flow to the other may be delayed by increasing R since w_1 is preceded by an extra factor R^2.

2.5. *The Special Case $\lambda = 1$; the Circular Section.*—Dean's results for a pipe of circular section are readily obtained by putting $\lambda = 1$ and transforming to polar co-ordinates by means of the substitutions

$$\left.\begin{aligned} x &= r\cos\theta & u_r &= \frac{\partial\psi}{r\,\partial\theta} \\ y &= r\sin\theta & u_\theta &= -\frac{\partial\psi}{\partial r} \end{aligned}\right\} \qquad (14)$$

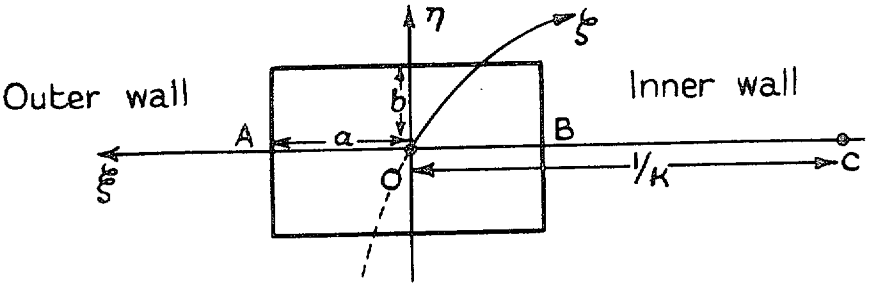
where u_r, u_θ refer to velocities in the radial and transverse directions respectively. Then $R_1 = -\frac{1}{2}\partial\bar{p}/\partial z$; from equations (10), $B_1 = \frac{1}{288}$, $B_2 = B_3 = -\frac{1}{1152}$ and from equation (9)

$$\psi = \frac{1}{1152}(1-r^2)^2(4-r^2)r\sin\theta\,. \qquad (15)$$

Therefore

$$\left.\begin{aligned} u_r &= \frac{\varkappa\nu R_1^2}{1152}(1-r^2)^2(4-r^2)\cos\theta \\ u_\theta &= -\frac{\varkappa\nu R_1^2}{1152}(1-r^2)(7r^4-23r^2+4)\sin\theta \end{aligned}\right\} \qquad (16)$$

3. *The Secondary Flow in a Pipe of Rectangular Section.*—3.1. *The Stream Function.*—



Using the same notation as in the previous section, equations (1) to (3) may be applied identically whilst the boundary conditions are now

$$\bar{u} = \bar{v} = \bar{w} = 0 \qquad \text{for} \qquad x = \pm 1 \text{ (all } y)$$
$$y = \pm 1 \text{ (all } x)$$

We put

$$\bar{u} = u_0 + \varkappa a R_R^2 u_1 \qquad \bar{v} = v_0 + \varkappa a R_R^2 v_1$$
$$\bar{p} = p_0 + \varkappa a R_R^2 p_1 \qquad \bar{w} = R_R(w_0 + \varkappa a R_R^2 w_1 + \varkappa a w_2)$$

where $$R_R = -\frac{16\lambda^2}{\pi^3}\operatorname{sech}\frac{\pi}{2\lambda}\frac{\partial \bar{p}}{\partial z}. \qquad (17)$$

The terms independent of $\varkappa$ yield the equations of motion for flow through a straight rectangular pipe, *viz.*,

$$u_0 = v_0 = 0; \qquad \frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial y} = 0$$

and $$\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 w_0}{\partial y^2} = -\frac{\pi^3}{16\lambda^2}\cosh\frac{\pi}{2\lambda}$$

with the solution[9]

$$w_0 = \sum_{n=0}^{\infty} P_n\left[\cosh(2n+1)\frac{\pi}{2\lambda} - \cosh(2n+1)\frac{\pi x}{2\lambda}\right]\cos(2n+1)\frac{\pi y}{2}$$

where $$P_n = \frac{(-1)^n}{(2n+1)^3}\cosh\frac{\pi}{2\lambda}\operatorname{sech}(2n+1)\frac{\pi}{2\lambda}. \qquad (18)$$

This series for w_0 is rapidly convergent and we approximate to w_0 by retaining only the first term, *viz.*,

$$w_0 = \left(\cosh\frac{\pi}{2\lambda} - \cosh\frac{\pi x}{2\lambda}\right)\cos\frac{\pi y}{2}. \qquad (19)$$

Equating coefficients of $\varkappa$ we obtain

$$\left.\begin{aligned} -w_0^2 &= -\frac{\partial p_1}{\partial x} + \frac{1}{\lambda^2}\frac{\partial^2 u_1}{\partial y^2} - \frac{1}{\lambda}\frac{\partial^2 v_1}{dx\,dy} \\ 0 &= -\frac{1}{\lambda}\frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial x^2} - \frac{1}{\lambda}\frac{\partial^2 u_1}{\partial x\,\partial y} \\ \frac{\partial u_1}{\partial x} &+ \frac{1}{\lambda}\frac{\partial v_1}{\partial y} = 0 \end{aligned}\right\} \qquad (20)$$

The last equation may be satisfied identically by a stream function ψ defined by the equations

$$\left.\begin{aligned} u_1 &= \frac{\pi}{2\lambda^2}\frac{\partial \psi}{\partial y} \\ v_1 &= -\frac{\pi}{2\lambda}\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad \ldots \quad (21)$$

Eliminating p_1 between the first and second equations we obtain

$$\frac{\partial^4 \psi}{\partial x^4} + \frac{2}{\lambda^2}\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{1}{\lambda^4}\frac{\partial^4 \psi}{\partial y^4} = \left(\cosh\frac{\pi}{2\lambda} - \cosh\frac{\pi x}{2\lambda}\right)^2 \sin \pi y \quad \ldots \quad (22)$$

whence

$$\psi = \sum_{n=1}^{\infty}\left(A_n \cosh\frac{n\pi x}{\lambda} + B_n x \sinh\frac{n\pi x}{\lambda}\right)\sin n\pi y$$
$$+ \sum_{n=1}^{\infty}(C_n \sinh n\lambda\pi y + D_n y \cosh n\lambda\pi y)\cos n\pi x$$
$$+ \frac{\lambda^2}{144\pi^4}\left[9\pi^2 x^2 \cosh\frac{\pi x}{\lambda} - 512\lambda^2 \cosh\frac{\pi}{2\lambda}\cosh\frac{\pi x}{2\lambda} + 144\lambda^2\right.$$
$$\left. + 72\lambda^2 \cosh\frac{\pi}{\lambda}\right]\sin \pi y . \quad \ldots \quad (23)$$

Insertion of the boundary conditions leads to the following equations

$$\frac{n\pi}{\lambda}A_n \sinh\frac{n\pi}{\lambda} + B_n\left(\sinh\frac{n\pi}{\lambda} + \frac{n\pi}{\lambda}\cosh\frac{n\pi}{\lambda}\right)$$
$$+ \frac{\delta_{1n}\lambda}{144\pi^3}\left[18\pi\lambda \cosh\frac{\pi}{\lambda} + (9\pi^2 - 128\lambda^2)\sinh\frac{\pi}{\lambda}\right] = 0 \quad \ldots \quad \text{(i)}$$

$$C_n \sinh n\lambda\pi + D_n \cosh n\lambda\pi = 0 \quad \ldots \quad \text{(ii)}$$

$$\sum_{n=1}^{\infty}\left\{A_n \cosh\frac{n\pi}{\lambda} + B_n \sinh\frac{n\pi}{\lambda} + \frac{\delta_{1n}\lambda^2}{144n\pi^3}\left[(9\pi^2 - 184\lambda^2)\cosh\frac{\pi}{\lambda} - 112\lambda^2\right]\right\}\cos n\pi y$$
$$+ \sum_{n=1}^{\infty}(-1)^n\left\{C_n\lambda \cosh n\lambda\pi y + D_n\left(\frac{1}{n\pi}\cosh n\lambda\pi y + \lambda y \sinh n\lambda\pi y\right)\right\} = 0 \quad \ldots \quad \text{(iii)}$$

$$\sum_{n=1}^{\infty}(-1)^n n\pi\left\{A_n \cosh\frac{n\pi x}{\lambda} + B_n x \sinh\frac{n\pi x}{\lambda}\right\}$$
$$+ \sum_{n=1}^{\infty}\left\{C_n n\lambda\pi \cosh n\lambda\pi + D_n(\cosh n\lambda\pi + n\lambda\pi \sinh n\lambda\pi)\right\}\cos n\pi x$$
$$- \frac{\lambda^2}{144\pi^3}\left[9\pi^2 x^2 \cosh\frac{\pi x}{\lambda} - 512\lambda^2 \cosh\frac{\pi}{2\lambda}\cosh\frac{\pi x}{2\lambda} + 144\lambda^2 + 72\lambda^2 \cosh\frac{\pi}{\lambda}\right] = 0 \quad \ldots \quad \text{(iv)}$$

where δ_{1p} $= 1, \quad p = 1$
$= 0, \quad p \neq 1 .$

Solving equations (i) to (iv) we find

$$\left(1+\frac{\lambda}{2p\pi}\sinh\frac{2p\pi}{\lambda}\right)B_p=\frac{4\lambda p}{\pi^2}\sinh\frac{p\pi}{\lambda}\sum_{n=1}^{\infty}(-1)^{n+p}\frac{\sinh n\lambda\pi}{(p^2+\lambda^2n^2)^2}D_n$$
$$-\frac{\lambda^3\delta_{1p}}{144\pi^4}\left[28\lambda\sinh\frac{2\pi}{\lambda}+112\sinh\frac{\pi}{\lambda}+9\pi\left(1+\cosh\frac{2\pi}{\lambda}\right)\right]\quad\ldots\quad(24)$$

and

$$\left(\frac{\sinh 2p\lambda\pi}{2p\lambda\pi}-1\right)D_p=\frac{4\lambda}{p\pi^2}\sinh p\lambda\pi\sum_{n=1}^{\infty}(-1)^{n+p}\frac{n^3}{(n^2+\lambda^2p^2)^2}\sinh\frac{n\pi}{\lambda}B_n$$
$$+(-1)^p\frac{\sinh p\lambda\pi}{p\lambda\pi}\left[\frac{\lambda^5}{36\pi^4}\cdot\frac{73+101\lambda^2p^2+64\lambda^4p^4}{(1+\lambda^2p^2)^3}\sinh\frac{\pi}{\lambda}\right.$$
$$\left.-\frac{64\lambda^5}{9\pi^4}\,\frac{\sinh\frac{\pi}{\lambda}}{1+4\lambda^2p^2}-\frac{\lambda^4\cosh\frac{\pi}{\lambda}}{2\pi^3(1+\lambda^2p^2)^2}\right].\quad\ldots\quad(25)$$

3.2. *The Special Case $\lambda = 1$; the Square Section.*—When $\lambda = 1$ we have the case of a square section. Solving equations (24) and (25) and retaining only A_1, B_1, C_1 and D_1 as a first approximation we find

$$\left.\begin{aligned}&A_1=0\cdot03337 \qquad C_1=-0\cdot001864\\&B_1=-0\cdot02621 \qquad D_1=0\cdot001858\\&R_S=(R_R)_{\lambda=1}=-\frac{16}{\pi^3}\operatorname{sech}\frac{\pi}{2}\frac{\partial\bar{p}}{\partial z}\end{aligned}\right\}\quad\ldots\quad(26)$$

Let U_s be the velocity at the centre of a pipe of square section; then

$$U_s=\varkappa\nu R_S^{\,2}\frac{\pi}{2}\left(\frac{\partial\psi}{\partial y}\right)_{(0,\,0)}$$
$$=0\cdot05062\varkappa\nu R_S^{\,2}.\quad\ldots\quad(27)$$

If U_c be the corresponding velocity at the centre of a pipe of circular section we have from equation (16)

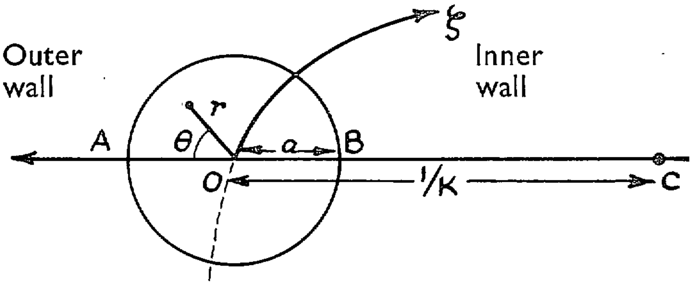
$$U_c=\frac{\varkappa\nu R_1^{\,2}}{288}.$$

Therefore

$$\frac{U_s}{U_c}=14\cdot58\left(\frac{R_S}{R_1}\right)^2$$
$$=2\cdot47.\quad\ldots\quad(28)$$

The secondary flow is therefore more intense in a pipe of square section than in a pipe of circular section.

4. *The Effect of Suction on the Flux through a Curved Pipe of Circular Section.*—4.1. *First Approximation to the Axial Velocity.*—



We take as co-ordinate system polar co-ordinates in the plane of a section and $O\zeta$ along the central axis: the normal suction velocity at the wall is taken to be $\varkappa aU\cos\theta$, that is, proportional to the curvature of the central axis. Making the non-dimensional substitutions $x = r/a$, $z = \zeta/a$; $u = \nu\bar{u}/a$, $v = \nu\bar{v}/a$, $w = \nu\bar{w}/a$, $p = \rho\nu^2/a^2 . \bar{p}$ and putting

$$\partial u/\partial z = \partial v/\partial z = \partial w/\partial z = 0$$

we obtain from equations (1) and (2)

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\bar{v}}{x}\frac{\partial \bar{u}}{\partial \theta} - \frac{\bar{v}^2}{x^2} - \frac{\varkappa a \cos\theta \bar{w}^2}{1 + \varkappa a x \cos\theta} = -\frac{\partial \bar{p}}{\partial x} - \frac{1}{x}\frac{\partial^2 \bar{v}}{\partial x \partial \theta} - \frac{1}{x^2}\frac{\partial v}{\partial \theta} + \frac{1}{x^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{\varkappa a \sin\theta}{1 + \varkappa a x \cos\theta}\left(\frac{\partial \bar{v}}{\partial x} + \frac{\bar{v}}{x} - \frac{1}{x}\frac{\partial \bar{u}}{\partial \theta}\right)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \frac{\bar{v}}{x}\frac{\partial \bar{v}}{\partial \theta} + \frac{\bar{u}\bar{v}}{x} + \frac{\varkappa a \sin\theta \bar{w}^2}{1 + \varkappa a x \cos\theta} = -\frac{\partial \bar{p}}{x \partial \theta} + \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{1}{x}\frac{\partial \bar{v}}{\partial x} - \frac{\bar{v}}{x^2} - \frac{1}{x}\frac{\partial^2 \bar{u}}{\partial x \partial \theta} + \frac{1}{x^2}\frac{\partial \bar{u}}{\partial \theta} + \frac{\varkappa a \cos\theta}{1 + \varkappa a x \cos\theta}\left(\frac{\partial \bar{v}}{\partial x} + \frac{\bar{v}}{x} - \frac{1}{x}\frac{\partial \bar{u}}{\partial \theta}\right)$$

$$\bar{u}\frac{\partial \bar{w}}{\partial x} + \frac{\bar{v}}{x}\frac{\partial \bar{w}}{\partial \theta} + \frac{\varkappa a \bar{w}}{1 + \varkappa a x \cos\theta}(\bar{u}\cos\theta - \bar{v}\sin\theta) = -\frac{1}{1 + \varkappa a x \cos\theta}\frac{\partial \bar{p}}{\partial z} + \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{1}{x}\frac{\partial \bar{w}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \bar{w}}{\partial \theta^2} + \frac{\varkappa a}{1 + \varkappa a x \cos\theta}\left(\frac{\partial \bar{w}}{\partial x}\cos\theta - \frac{\partial \bar{w}}{\partial \theta}\sin\theta\right) - \frac{\varkappa^2 a^2 \bar{w}}{(1 + \varkappa a x \cos\theta)^2}$$

$$\frac{\partial}{\partial x}\left\{(1 + \varkappa a x \cos\theta) x \bar{u}\right\} + \frac{\partial}{\partial \theta}\left\{(1 + \varkappa a x \cos\theta)\bar{v}\right\} = 0. \quad \ldots \quad (29)$$

The boundary conditions are that $\nu/a . \bar{u} = \varkappa a U \cos\theta$, $\bar{v} = \bar{w} = 0$ for $x = 1$. We put

$$\begin{aligned} \bar{u} &= u_0 + \varkappa a R_1^2 u_1 + \varkappa^2 a^2 R_1^4 u_2 & \bar{v} &= v_0 + \varkappa a R_1^2 v_1 + \varkappa^2 a^2 R_1^4 v_2 \\ \bar{p} &= p_0 + \varkappa a R_1^2 p_1 + \varkappa^2 a^2 R_1^4 p_2 & \bar{w} &= \frac{R_1}{2}(w_0 + \varkappa a R_1^2 w_1 + \varkappa^2 a^2 R_1^4 w_2) \end{aligned} \quad (30)$$

where $R_1 = -\frac{1}{2}\frac{\partial p}{\partial z}$;

We have approximated to $\bar{w}$ by neglecting a term $\varkappa a w_1'$ in comparison with $\varkappa a R_1^2 w_1$; this has been shown valid for an aspect ratio of unity in section 2.4. The terms independent of $\varkappa$ yield the equations for the flow through a straight circular cylinder with the suction velocity everywhere zero, *viz.*,

$$u_0 = v_0 = 0; \qquad \frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial \theta} = 0$$

and

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{x}\frac{\partial w_0}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w_0}{\partial \theta^2} = -4$$

with the axisymmetric solution

$$w_0 = 1 - x^2. \quad \ldots \quad (31)$$

Equating coefficients of $\varkappa$ we find

$$\left.\begin{aligned} -\frac{w_0^2}{4}\cos\theta &= -\frac{\partial p_1}{\partial x} - \frac{1}{x}\frac{\partial^2 v_1}{\partial x \partial \theta} - \frac{1}{x^2}\frac{\partial v_1}{\partial \theta} + \frac{1}{x^2}\frac{\partial^2 u_1}{\partial \theta^2} \\ \frac{w_0^2}{4}\sin\theta &= -\frac{\partial p_1}{x \partial \theta} + \frac{\partial^2 v_1}{\partial x^2} + \frac{1}{x}\frac{\partial v_1}{\partial x} - \frac{v_1}{x^2} - \frac{1}{x}\frac{\partial^2 u_1}{\partial x \partial \theta} + \frac{1}{x^2}\frac{\partial u_1}{\partial \theta} \\ u_1\frac{\partial w_0}{\partial x} &= \frac{\partial^2 w_1}{\partial x^2} + \frac{1}{x}\frac{\partial w_1}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w_1}{\partial \theta^2} \\ \frac{\partial}{\partial x}(x u_1) &+ \frac{\partial v_1}{\partial \theta} = 0 \end{aligned}\right\} \quad \ldots \quad (32)$$

The last equation may be satisfied identically by a function $f(x)$ defined by the equations

$$\left.\begin{aligned} u_1 &= f(x)\cos\theta \\ v_1 &= -\frac{d}{dx}[xf(x)]\sin\theta \end{aligned}\right\} \quad \ldots \quad (33)$$

Eliminating p_1 between the first and second equations of (32) we have, putting $df/dx = h(x)$,

$$x^3\frac{d^3h}{dx^3} + 6x^2\frac{d^2h}{dx^2} + 3x\frac{dh}{dx} - 3h = x^3 - x^5 \quad \ldots \quad (34)$$

with the solution $\quad h(x) = A^1x + \dfrac{B}{x} + \dfrac{C^1}{x^3} + \dfrac{1}{192}(4x^3 - x^5)$

whence $\quad f(x) = Ax^2 + B\log x + \dfrac{C}{x^2} + D + \dfrac{1}{1152}(6x^4 - x^6)\,. \quad \ldots \quad (35)$

The boundary conditions are that $v_1 = 0$, $u_1 = aU/\nu R_1^2.\cos\theta$ for $x = 1$; i.e., $d(xf)/dx = 0$, $f(x) = aU/\nu R_1^2$ for $x = 1$. Since u_1 and v_1 are both finite at $x = 0$, $B = C = 0$. Put $\gamma = 192aU/\nu R_1^2$, then $A = -\frac{1}{1152}(9 + 3\gamma)$, $D = \frac{1}{1152}(4 + 9\gamma)$, whence

$$\left.\begin{aligned} u_1 &= \tfrac{1}{1152}[(4 + 9\gamma) - (9 + 3\gamma)x^2 + 6x^4 - x^6]\cos\theta \\ v_1 &= -\tfrac{1}{1152}[(4 + 9\gamma) - 9(3 + \gamma)x^2 + 30x^4 - 7x^6]\sin\theta \end{aligned}\right\} \quad \ldots \quad (36)$$

Putting $w_1 = g(x)\cos\theta$ in the third of equations (32)

$$\begin{aligned} x^2\frac{d^2g}{dx^2} + x\frac{dg}{dx} - g &= -2x^3f(x) \\ &= \tfrac{1}{576}[x^9 - 6x^7 + (9 + 3\gamma)x^5 - (4 + 9\gamma)x^3]\,. \end{aligned} \quad \ldots \quad (37)$$

The boundary condition is that $g(x) = 0$ for $x = 1$. Solving we find

$$g(x)\ \tfrac{1}{46{,}080}[(19 + 80\gamma) - (21 + 10\gamma)x^2 + 9x^4 - x^6](1 - x^2)x\,. \quad \ldots \quad (38)$$

4.2. *The Second Approximation to the Axial Velocity.*—Since w_1 is of the form $g(x)\cos\theta$, it follows that this term makes no contribution to the flux through the pipe; the diminution in flux (which was deduced on physical grounds in the Introduction) is therefore zero to the first power of the curvature. Equating coefficients of $\varkappa^2$ in equations (32) we find that the equation of continuity may be satisfied by a function $f_1(x)$ defined by the equations

$$\begin{aligned} u_2 &= f_1(x)\cos 2\theta \\ v_2 &= -\frac{1}{2}\frac{d}{dx}[xf_1(x)]\sin 2\theta\,. \end{aligned}$$

and $\quad u_1\dfrac{\partial w_1}{\partial x} + u_2\dfrac{\partial w_0}{\partial x} + \dfrac{v_1}{x}\dfrac{\partial w_1}{\partial \theta} = \dfrac{\partial^2 w_2}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial w_2}{\partial x} + \dfrac{1}{x^2}\dfrac{\partial^2 w_2}{\partial \theta^2}$

i.e., $\quad \dfrac{\partial^2 w_2}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial w_2}{\partial x} + \dfrac{1}{x^2}\dfrac{\partial^2 w_2}{\partial \theta^2} = f\dfrac{dg}{dx}\cos^2\theta + \dfrac{g}{x}\dfrac{d}{dx}(xf)\sin^2\theta - 2xf_1\cos 2\theta$

i.e., $\quad x\dfrac{\partial^2 w_2}{\partial x^2} + \dfrac{\partial w_2}{\partial x} + \dfrac{1}{x}\dfrac{\partial^2 w_2}{\partial \theta^2} = \dfrac{1}{2}\left[xf\dfrac{dg}{dx} + x\dfrac{df}{dx}g + fg\right]$

$$+\frac{1}{2}\cos 2\theta\left[xf\frac{dg}{dx} - x\frac{df}{dx}g - fg - 4x^2f_1\right] \quad \ldots \quad (39)$$

from which it follows that w_2 is of the form $m(x) + n(x)\cos 2\theta$. The integral of the second term over a cross-section is zero, consequently to find the diminution of flux it will suffice to determine $m(x)$. Equating terms independent of θ in equation (39)

$$x\frac{d^2m}{dx^2}+\frac{dm}{dx}=\frac{1}{2}\left[xf\frac{dg}{dx}+x\frac{df}{dx}g+fg\right]$$

i.e.,

$$\frac{d}{dx}\left(x\frac{dm}{dx}\right)=\frac{1}{2}\frac{d}{dx}(xfg)$$

whence

$$\frac{dm}{dx}=\frac{1}{2}fg \quad \ldots \quad (40)$$

the constant of integration being zero to preserve a finite value of dm/dx at $x=0$. The boundary condition on m is that $m(x)=0$ for $x=1$.

Therefore

$$\begin{aligned} m(x) &= \frac{1}{2}\int_1^x f(t)g(t)\,dt \\ &= \frac{1}{\Delta}\int_1^x \Big\{ 76+491\gamma+720\gamma^2)t-(331+1497\gamma+1050\gamma^2)t^3 \\ &\quad +(594+1720\gamma+360\gamma^2)t^5-(569+890\gamma+30\gamma^2)t^7 \\ &\quad +(314+189\gamma)t^9-(99+13\gamma)t^{11}+16t^{13}-t^{15}\Big\}\,dt \quad \ldots \quad (41) \end{aligned}$$

where

$$\Delta = 80\times(1152)^2.$$

Let F_s, F_c be the fluxes through the straight and curved pipes respectively; then

$$\begin{aligned} F_c &= 2\pi a^2\int_0^1\left\{\frac{\nu R_1}{2a}(1-x^2)x+\frac{\nu}{2a}\cdot\varkappa^2a^2R_1^5xm(x)\right\}dx \\ &= \frac{\pi}{4}a\nu R_1+\varkappa^2a^3\nu R_1^5\pi\int_0^1 xm(x)\,dx \end{aligned}$$

and

$$F_s=\frac{\pi}{4}a\nu R_1 .$$

Therefore

$$\begin{aligned} \frac{F_c}{F_s} &= 1+\frac{4}{\Delta}\varkappa^2a^2R_1^4\int_0^1 xm(x)\,dx \\ &= 1-\left(\frac{\varkappa aR_1^2}{1152}\right)^2\left[0\cdot03058+0\cdot3518\gamma+1\cdot175\gamma^2\right]. \quad \ldots \quad (42) \end{aligned}$$

4.3. *Variation of Flux with Suction.*—Dean's[10] value for the flux it obtained by putting $\gamma=0$, that is the suction is everywhere zero. Then

$$\frac{F_c}{F_s}=1-0\cdot03058\left(\frac{\varkappa aR_1^2}{1152}\right)^2. \quad \ldots \quad (43)$$

Put $y(\gamma)=0\cdot03058+0\cdot3518\gamma+1\cdot175\gamma^2$. This expression has a minimum value of $0\cdot00424$ when $\gamma=-0\cdot1497$.

Therefore

$$\left(\frac{F_c}{F_s}\right)_{\max}=1-0\cdot00424\left(\frac{\varkappa aR_1^2}{1152}\right)^2$$

and is attained when $\gamma=-0\cdot1497$, *i.e.*, when $U=-0\cdot0007796\nu R_1^2/a$. It appears from equation (43) that the series for F_c/F_s converges provided $\varkappa aR_1^2<1152$. This condition is satisfied for $\varkappa a=0\cdot1$ and $R_1=100$. In this case

$$\frac{F_c}{F_s}=1-0\cdot7535y(\gamma)\,. \quad \ldots \quad (44)$$

The following table shows the variation in F_c/F_s with γ; these results are shown graphically in Fig. 4.

TABLE 3

γ	$-0\cdot4$	$-0\cdot2$	$-0\cdot1497$	0	$0\cdot2$	$0\cdot4$
F_c/F_s	$0\cdot94133$	$0\cdot99456$	$0\cdot99681$	$0\cdot97696$	$0\cdot88853$	$0\cdot72927$
U	$-20\cdot833\nu/a$	$-10\cdot417\nu/a$	$-7\cdot797\nu/a$	0	$10\cdot417\nu/a$	$20\cdot833\nu/a$

These results may easily be interpreted physically. The diminution in flux is brought about by the accumulation of a retarded layer of fluid in the inner part of the bend. When γ is positive (and hence U) the retarded fluid is forced towards the outer part of the bend; the region of slowly-moving fluid is extended and the flux further diminished. When γ is negative (and hence U) the slowly-moving fluid is removed by suction over the inner wall and replaced by faster-moving fluid injected over the outer wall. The particular distribution of suction we have considered is capable of almost entirely overcoming the diminution of flux due to curvature: for the optimum value of $U = - 7\cdot796\nu/a$, the flux in the example considered attains a maximum value equal to $99\cdot7$ per cent of its value in the flow through a straight pipe.

5. *Conclusions.*—The Navier-Stokes equations for the flow of a viscous incompressible fluid through a curved pipe of elliptic section have been solved to the first power of the curvature. It has been shown that the dynamical parameter of the motion is, in fact, $\varkappa aR^2$ where R is the Reynolds number for steady flow in a straight pipe of the same section and that the secondary flow consists of two opposed vortex motions in the top and bottom halves of the pipe, the direction of flow in the central plane being away from the centre of curvature. This flow diminishes for both high and low values of the aspect ratio of the pipe and in the case when the ratio is infinitely large (that is, when the motion takes place in the channel between two infinite concentric cylinders) the secondary flow vanishes altogether. This suggests that the secondary flow in a bend may be reduced by the introduction of a number of guide vanes following the curvature of the bend; the pipe is then divided into a number of channels of larger aspect ratio in each of which the intensity of the flow will be decreased. To the degree of approximation considered the axial velocity is modified by two curvature terms. The first of these, involving the square of the Reynolds number, represents an increase of velocity in the outer half of the bend and a decrease in the inner half; that is, the effect generally associated with flow in a curved pipe. The second term, independent of the Reynolds number, represents the reverse effect; that is, the effect generally associated with flow through a curved channel. For values of the aspect ratio not far removed from unity the first of these terms is the larger; as the aspect ratio increases the second term eventually predominates. Neither of these terms gives any contribution to the flux through the pipe so that the flux remains unaltered to the first power of the curvature. For the special case when the aspect ratio is unity we have the solution for the flow through a curved pipe of circular section. The same method of solution has also been applied to a curved pipe of rectangular section. The results are given explicitly for the special case of a square section and it is shown that the secondary flow in a square section is stronger than that in a circular section.

The effect of the increased resistance of a curved pipe is to cause a diminution in the flux through it. The equations for flow through a curved pipe of circular section have therefore been solved as far as the second power of the curvature to find the effect of a distribution of suction proportional to the curvature in reducing this diminution. The results obtained show that with the distribution of suction considered the diminution may be almost entirely overcome.

LIST OF SYMBOLS

$\varkappa$		Curvature of pipe
a, b		Semi-major and semi-minor axes of pipe
λ	=	b/a
ξ, η, ζ		Dimensional Cartesian co-ordinates
x, y, z		Non-dimensional Cartesian co-ordinates
u, v, w		Velocity components referred to these axes
ν		Coefficient of kinematic viscosity
R, R_1, R_R, R_S		Reynolds numbers for steady flow in straight pipes of elliptic, circular, rectangular and square sections
Ω		Total vorticity in the upper half of curved pipe
U_c, U_s		Secondary flow velocity at the centre of pipes of circular and square sections
$\varkappa a U \cos \theta$		The suction velocity at the wall of a pipe of circular section
F_c, F_s		Fluxes through curved and straight pipes of circular section

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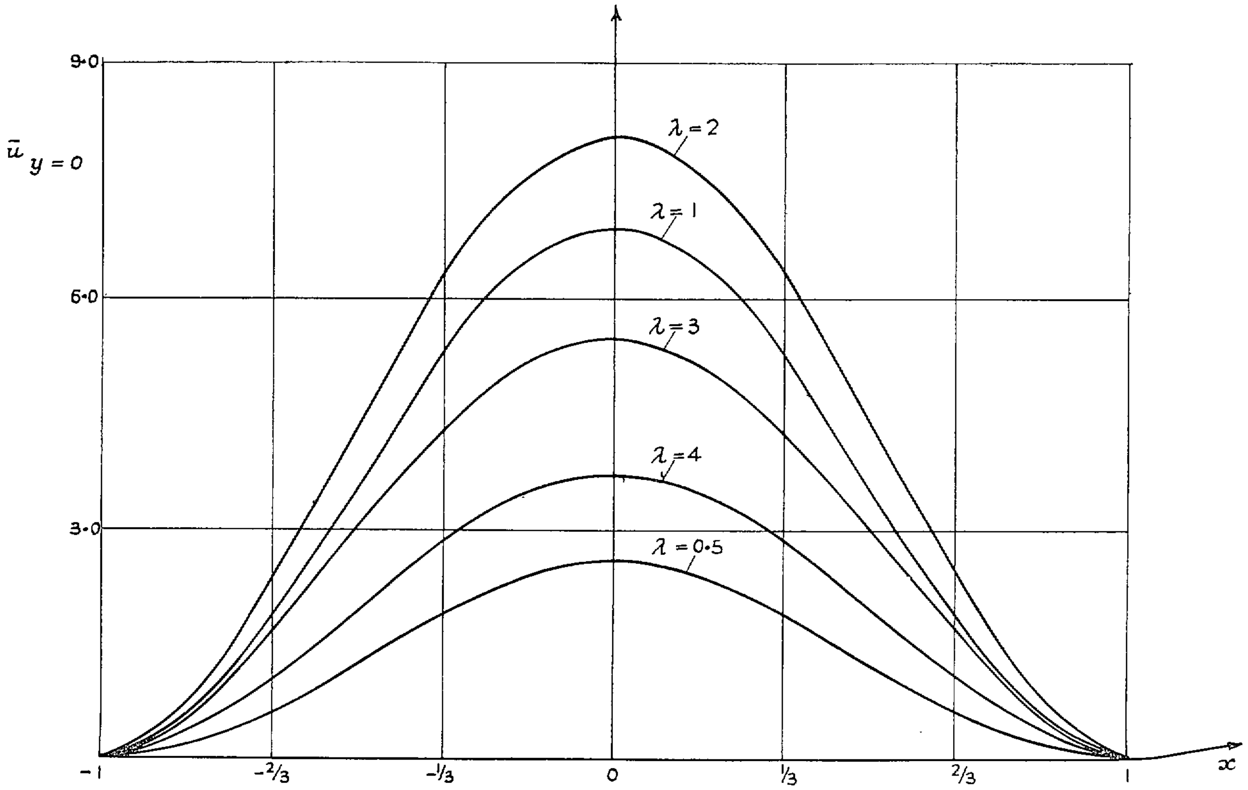
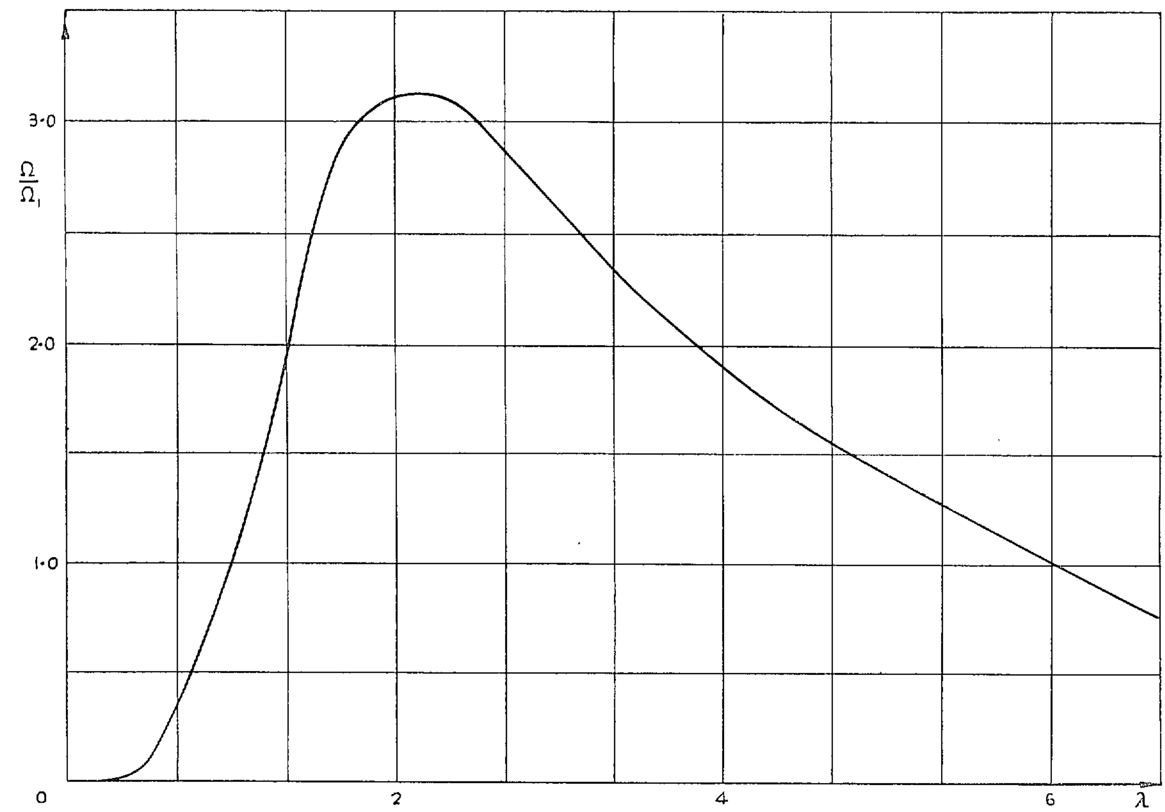


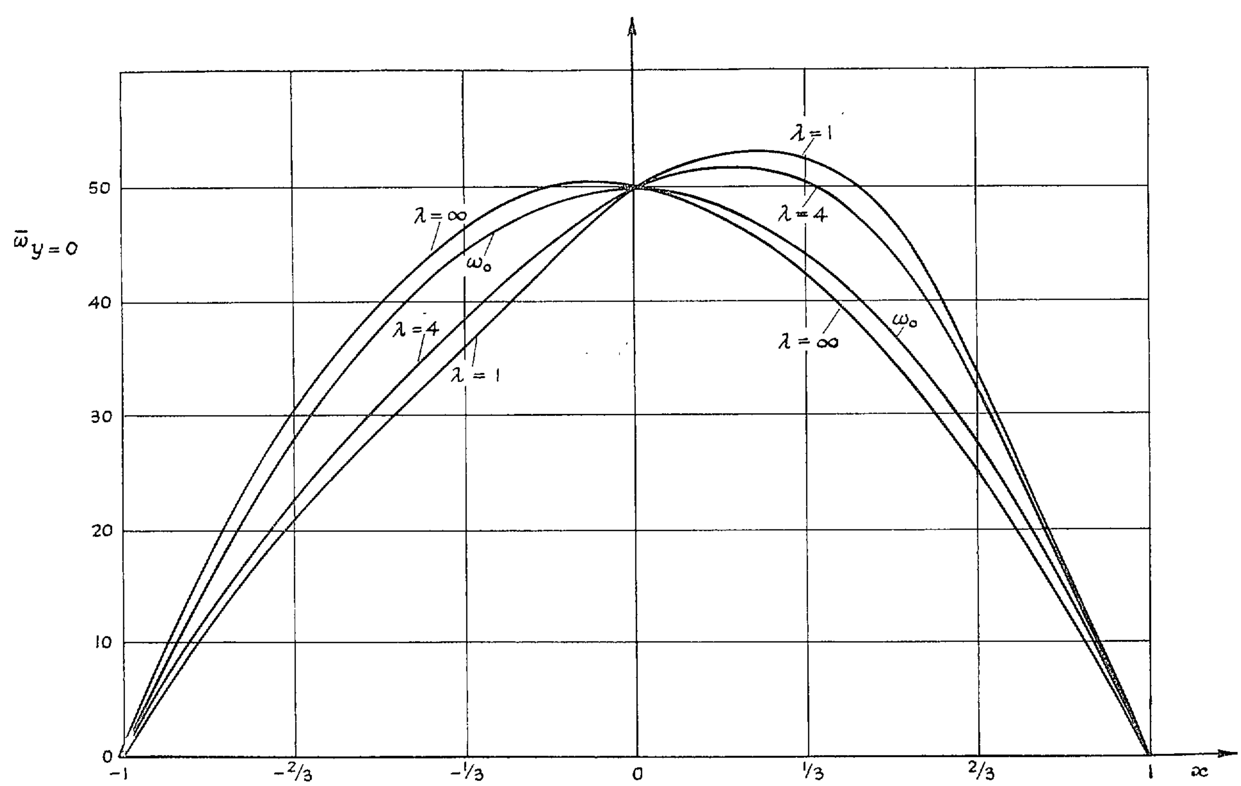
9·0
6·0
3·0
$\bar{u}_{y=0}$
$\lambda=2$
$\lambda=1$
$\lambda=3$
$\lambda=4$
$\lambda=0{\cdot}5$
−1
−2/3
−1/3
0
1/3
2/3
1
x

FIG. 1.



3·0
$\frac{\Omega}{\Omega_1}$
2·0
1·0
0
2
4
6
λ

FIG. 2.



$\bar{\omega}_{y=0}$
50
40
30
20
10
0
λ = ∞
ω_0
λ = 4
λ = 1
λ = 1
λ = 4
λ = ∞
ω_0
-1
-2/3
-1/3
0
1/3
2/3
1
x

FIG. 3.

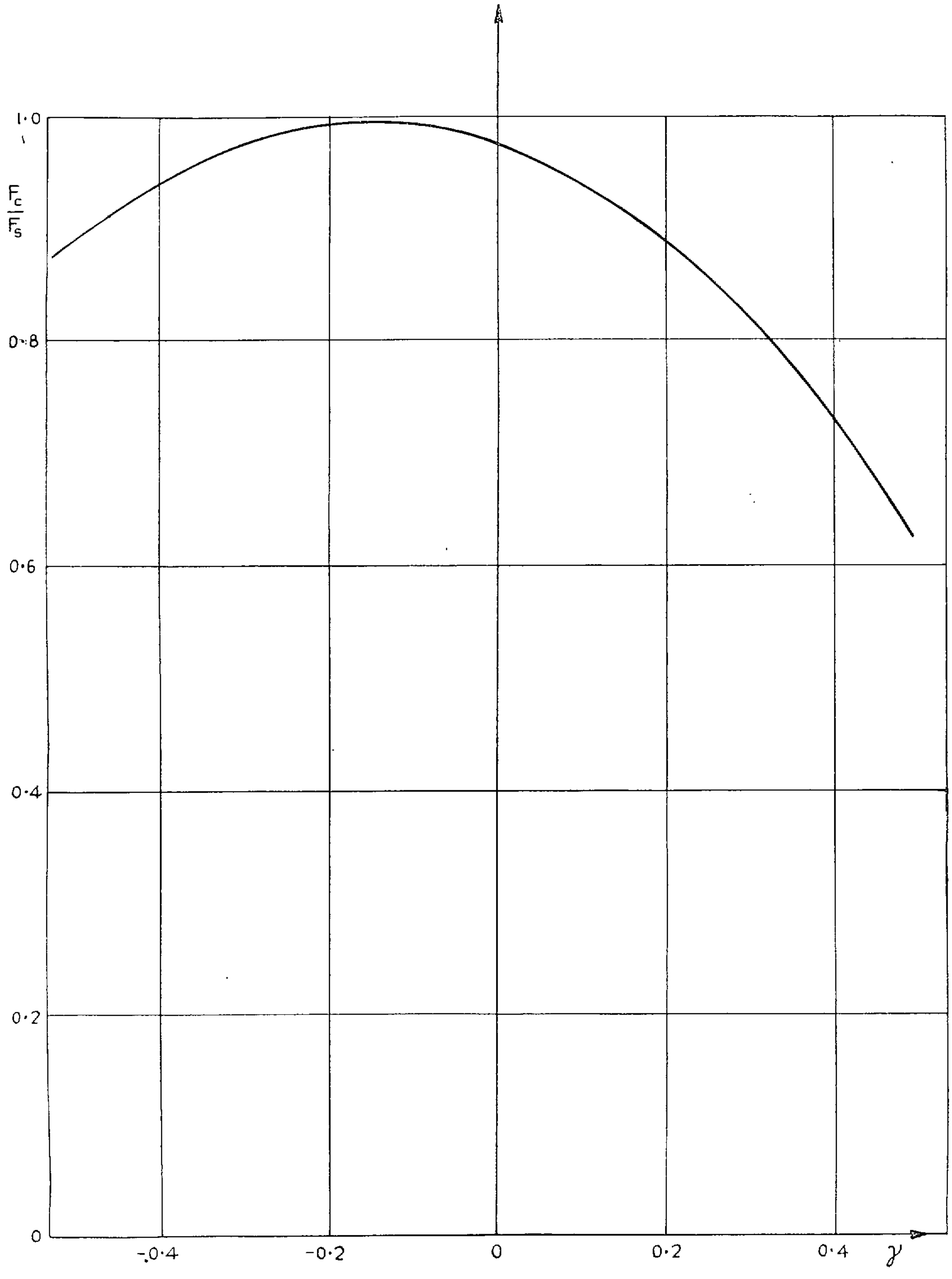


Fig. 4.

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