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By

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Summary.—The determination of the stick-free flutter characteristics of a control system when the inertia of the stick is allowed for is considered. A method of solution is proposed which corresponds to impedance matching between circuit and control surface in the flutter condition. The method is applied, by way of illustration, to two typical cases, an elevator system and a servo-tab system, and the effect of variations in stick inertia and circuit stiffness demonstrated. Conclusions drawn from these two cases are listed separately, but it is concluded generally that stick-free flutter can occur in the absence of stick-fixed flutter, and that the stick-free flutter characteristics may be quite different from those for the circuit-cut condition.

1. *Introduction.*—In considering the phenomenon of control-surface flutter a distinction is generally drawn in respect of the type of motion, whether it is consistent with normal operation of the control or not. In cases where the motion is consistent with normal operation (antisymmetric aileron, antisymmetric rudder, symmetric elevator) the control column, being connected through the control circuit to the control surface, will tend to take part in the flutter motion unless it is prevented by the pilot or other external agent. Where the motion is inconsistent with normal operation (symmetric aileron, symmetric rudder, antisymmetric elevator) the control column is, by hypothesis, stationary; the effect of the control-column motion on the flutter does not then arise.

Confining attention therefore to cases where the motion is consistent with normal operation, the control-column motion will depend upon the type and degree of external restraint. The restraint which may be applied by the pilot is difficult to predict and to represent mathematically in terms convenient for analysis. It has therefore been the practice to simplify matters by considering two extreme conditions, one in which the control column is fixed relative to the aircraft, and one in which the control column is entirely free of any external restraint. These two conditions have been termed the 'stick (or pedal)-fixed' condition and the 'stick (or pedal)-free' condition. It is important to emphasise that in the latter condition the control column is free only from external restraint; it is of course attached to the control circuit and is in that respect still constrained. Alternatively, one may regard the control column as constraining, by its inertia, the control circuit and hence the control surface.

For convenience, the term 'stick' is used throughout the remainder of this report to refer generally to the control column appropriate to aileron, elevator, or rudder.

Most investigations in the past have simplified the stick-free condition by ignoring the stick and circuit inertia, which means that the operating lever at the control surface is then assumed to be completely unconstrained. The same condition would be given by assuming the circuit disconnected from the operating lever. For the sake of preciseness, this condition is here termed the 'circuit-cut' condition. Although the circuit-cut condition may considerably misrepresent the stick motion which occurs in the authentic stick-free condition the important question is to what extent are the flutter characteristics of the system as a whole affected. So far as is known this question has not in a general way been answered, although there have been particular

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investigations in which the stick inertia was allowed for*. It is the purpose of the present report to consider this question generally ; to establish clearly how the stick affects the flutter characteristics in the stick-free condition and how its effects may be taken into account ; and to show in typical cases how the flutter characteristics in the stick-free condition compare with those in the stick-fixed condition and in the circuit-cut condition.

From dynamical considerations it can be shown that, ignoring any damping or friction in the circuit, the combined effect of the free stick and circuit in the critical flutter condition, when the system is oscillating in simple harmonic motion, is to apply an elastic restraint to the control surface which varies with the frequency. This elastic restraint is also a function of the inertia of stick and circuit and of the circuit stiffness. A general method of solution is given in which two curves are drawn, the ' flutter ' curve representing critical flutter conditions for the control surface, and a ' circuit ' curve representing the dynamic conditions for the circuit and stick. Intersections of the flutter and circuit curves then represent critical flutter conditions for the complete system in the stick-free condition. The process in fact corresponds to impedance matching between circuit and control surface. (The principle of impedance–or its reciprocal, admittance–matching as applied to mechanical vibration problems was first used by Carter[5] in connection with engine-airscrew systems, and is now well established.) An advantage of the method is that the effect of variations in stick inertia or circuit stiffness can be quickly assessed, as these affect only the circuit curve. The method is equally applicable to cases of control-surface–tab flutter.

Application of the method is illustrated in two numerical examples, one a hypothetical elevator case, the other a specific case of an aileron with aerodynamic servo-tab. The effect of variations in stick inertia and circuit stiffness is demonstrated, as is also the effect of fitting an additional spring to the stick. General conclusions reached are that stick-free flutter can occur in the absence of stick-fixed flutter, and that the stick-free flutter characteristics may be quite different from those for the circuit-cut condition. More detailed conclusions derived from the numerical examples are given in section 6.

2. *Dynamics of the Control System.*—In any general motion of the system, aerodynamic loads are excluded from the balance of forces acting directly on the stick. Ignoring damping or friction in the control system, the only forces which act on the stick are then structural elastic and inertia forces. The response of the control system to an imposed sinusoidal motion from the control surface, such as occurs in the critical flutter condition, is then determined by purely dynamical considerations. For simplicity, inertia of the circuit between stick and control surface is in the first place ignored, but its effect is considered separately later.

If θ is the angular stick movement and β the angular movement of the operating lever at the control surface, β being measured in the direction it would have if the control circuit were rigid, then the circuit stretch is $(\theta - \beta/\phi)$ in terms of stick movement, where ϕ is the circuit gearing β/θ with rigid circuit. The equation of motion for the stick is then

$$I_\theta \ddot{\theta} + K_\theta(\theta - \beta/\phi) = 0 \quad \ldots \quad (1)$$

where I_θ is the stick inertia about its hinge, and K_θ is the circuit stiffness as given by the moment applied to the stick to produce unit stick movement θ with the operating lever held fixed.

If the system is oscillating in simple harmonic motion, as in the critical flutter condition, β and θ may be written

$$\beta = \bar{\beta} \sin \omega t, \ \theta = \bar{\theta} \sin \omega t \quad \ldots \quad (2)$$

ω being the circular frequency of the oscillation.

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* Some general consideration was given by Frazer and Duncan to the problem of stick-free flutter in the case of the *Puss Moth* rudder[4], where the effect of varying stick inertia was investigated in the authentic stick-free condition.

Since writing this report the author has become aware of a paper by K. Leiss (Germany, 1942) which considers the problem in a general and quite comprehensive manner. An English translation of Leiss's paper is given in A.R.C. 11,583, ' Effect of the Control Circuit on Flutter,' June, 1948.

Substituting (2) in equation (1) then gives the ratio between stick and operating lever movements as

$$\theta/\beta = \frac{1}{\phi}\Big/\frac{\omega_\theta^2}{\omega_\theta^2 - \omega^2} \quad \ldots \quad (3)$$

where ω_θ [$= \sqrt{(K_\theta/I_\theta)}$] is the natural frequency of the stick with the operating lever held fixed.

If K_β represents the circuit stiffness as given by the moment applied to the operating lever to produce unit movement β with the stick held fixed, then the restoring moment applied to the operating lever by the circuit is $K_\beta(\beta - \phi\theta)$. The effect of the stick and circuit in the oscillating condition is thus to apply an effective elastic restraint to the operating lever given by

$$\bar{K}_\beta = K_\beta(\beta - \phi\theta)/\beta = K_\beta(1 - \phi\theta/\beta). \quad \ldots \quad (4)$$

Substituting for θ/β from (3) in equation (4), the effective restraint is finally obtained as

$$\bar{K}_\beta = K_\beta \frac{\omega^2}{\omega^2 - \omega_\theta^2}. \quad \ldots \quad (5)$$

The effective restraint $\bar{K}_\beta$ is thus a function of the circuit stiffness, the frequency w, and the stick and circuit inertias. Plotting $\bar{K}_\beta$ against ω gives what may be called the 'circuit' curve, representing the dynamic conditions of the circuit and stick as the variation of effective elastic restraint with frequency.

A typical circuit curve is illustrated in Fig. 1 (full curve). The curve consists of two branches, an upper frequency branch involving positive values of the effective restraint $\bar{K}_\beta$, and a lower frequency branch involving negative values of $\bar{K}_\beta$. As the value of ω approaches that of ω_θ the value of $\bar{K}_\beta$ approaches infinity, negative on the lower frequency branch, positive on the upper frequency branch. At zero frequency the value of $\bar{K}_\beta$ is zero (on the lower frequency branch) : at very high frequencies (on the upper frequency branch) the value of $\bar{K}_\beta$ approaches that of K_β.

From (3) it is seen that θ/β is positive if $\omega < \omega_\theta$ and negative if $\omega > \omega_\theta$. The lower frequency branch of the circuit curve is thus characterised by an in-phase motion of the stick and operating lever, and the upper frequency branch by an out-of-phase motion of the stick and operating lever. The two branches may thus be referred to as the 'in-phase' and 'out-of-phase' branches respectively.

The circuit curve is merely a particular representation of the familar forced oscillation phenomenon. For a given sinusoidal motion of the operating lever the stick responds in a manner prescribed by the elastic-inertia characteristics of the system, resulting in the application of an effective elastic restraint to the operating lever. The motion of the operating lever would then be unaffected if the control system were replaced by a simple spring earthed at one end to the main aircraft structure, the stiffness of the spring being equal to the effective elastic restraint $\bar{K}_\beta$. This representation applies equally to cases in which the operating lever is fixed or is not fixed to the main control surface, and is thus applicable to spring and servo-tab systems as well as to direct-control systems.

In the case of spring and particularly servo-tab systems it is sometimes found desirable to fit a spring to the stick (the spring being earthed to the aircraft structure), in order to provide additional 'feel' or to modify the stick-force characteristics from a control point of view. The effect of such a spring on the circuit curve is now briefly examined.

If h_θ represents the spring stiffness as given by the moment applied to the stick to produce unit stick movement θ against the action of the spring alone, then the effect of the spring is to add a term $h_\theta\theta$ to the left-hand side of equation (1). Following through the same process as before, it can then be shown that equation (5) for the circuit curve becomes

$$\bar{K}_\beta = K_\beta \frac{\omega^2 - \omega_{00}^2}{\omega^2 - \omega_0^2} \quad \ldots \quad (6)$$

where $\omega_0 = \sqrt{\left(\frac{K_\theta + h_\theta}{I_\theta}\right)}$ = natural frequency of stick with operating lever fixed and spring present.

$\omega_{00} = \sqrt{\left(\frac{h_\theta}{I_\theta}\right)}$ = value of ω_θ with K_θ zero.

The effect of the stick spring is thus to produce some rather marked changes in the circuit curve, as illustrated by the dotted curve in Fig. 1. The value of ω_θ is increased, and hence the value of the frequency at which $\bar{K}_\beta$ becomes infinite. At zero frequency (on the in-phase branch) the value of $\bar{K}_\beta$ is now positive and equal to $K_\beta . \omega_{00}^2/\omega_0^2 = K_\beta/(1 + K_\theta/h_\theta)$. The in-phase branch now gives positive values of $\bar{K}_\beta$ in the frequency range

$$0 < \omega < \omega_{00}.$$

The effect on the out-of-phase branch is less, and in particular the asymptotic value of $\bar{K}_\beta$ (as $\omega \to \infty$) is still K_β.

The effect of the circuit inertia may be considered at this stage. It is assumed that there are no concentrated masses in the circuit large enough to introduce additional degrees of freedom of appreciable amplitude ; so that the motion of the circuit remains effectively a combination of two types of motion, one as given by a load at the stick with the operating lever held fixed, and the other as given by a load at the operating lever with the stick held fixed. To represent the effect of the circuit inertia, it is necessary to define two additional gear ratios.

Gearing g_1 is the linear movement of a point in the circuit corresponding to unit movement β with the stick held fixed. Gearing g_2 is the linear movement of a point in the circuit corresponding to unit movement θ with the operating lever held fixed. The total linear movement of a point in the circuit is then $(g_1\beta + g_2\theta)$. Gearings g_1 and g_2 have each the dimensions of length and are expressed in this form for convenience.

It is shown in Appendix III that, for a direct-control system in the critical flutter condition, the effect of the circuit inertia is to add contributions to the direct inertias of the stick and control surface and to provide an inertia coupling between stick and control-surface movements. This is in addition to the elastic coupling already present between stick and control-surface movements. The net result, for the case without a stick spring, is to add the contribution to the direct inertia of the control surface and to provide an effective elastic restraint applied to the operating lever given by

$$\bar{K}_\beta = K_\beta \frac{\omega^2}{\omega^2 - \omega_\theta^2}\left[1 + 2\frac{I_c}{I_\theta'} + \left(\frac{\omega I_c}{\omega_\theta I_\theta'}\right)^2\right] \quad \ldots \quad (7)$$

where now $\omega_\theta[= \sqrt{(K_\theta/I_\theta')}]$ is again the natural frequency of the stick with operating lever fixed, but including circuit inertia effects,

$$\text{and } I_c = \phi \, \Sigma m g_1 g_2 \quad \ldots \quad (8)$$

$$I_\theta' = I_\theta + \sum_c m g_2^2 \quad \ldots \quad (9)$$

m being an element of mass, and the symbol c denoting summation throughout the circuit.

The effect of circuit inertia on the circuit curve is thus given by a comparison of (5) and (7). Normally I_c will be small compared with I_θ', and the last term in the square brackets in (7) will be small except at high frequencies. Since the gearings g_1, g_2 are essentially the same sign at any point, it follows from (8) that I_c must be positive. I_θ' is also essentially positive. At the same time the contribution of the circuit inertia to I_θ' reduces the value of ω_θ. The effect on the circuit curve is thus to reduce the value of ω which separates the in-phase and out-of-phase branches, and to increase numerically the negative values of $\bar{K}_\beta$ on the in-phase branch. The effect on the out-of-phase branch will be less at moderate frequencies, although as ω approaches infinity $\bar{K}_\beta$ also approaches infinity instead of the finite value K_β. In the practical range of frequencies the quantitative changes will normally be fairly small, and the essential character of the circuit curve is not basically affected by inclusion of the circuit inertia.

Equations (5), (6) and (7) for the circuit curve are applicable to aileron, elevator, or rudder circuits. The only distinction to be drawn is in the definition of the circuit stiffness. In the case of K_θ there is no difficulty : it represents the stiffness in respect of stick movement with either say the elevator lever fixed or both aileron levers fixed. In the case of an elevator K_β similarly represents the stiffness in respect of elevator lever movement with the stick fixed, and in this case $K_\theta = \phi^2 K_\beta$, ϕ being the circuit gearing. In the case of an aileron or twin-rudder system, however, K_β as defined represents the stiffness in respect of the lever movement of one of the surfaces when the stick is fixed and both surfaces are appropriately loaded. To be more specific, if each aileron lever is loaded with a moment M_β, nose-up in one case and nose-down in the other, and with the stick fixed each aileron moves through an angle β due to the elasticity of the circuit, then the stiffness is $K_\beta = M_\beta/\beta$. In this case, $K_\theta = 2\phi^2 K_\beta$. It is, however, only in this relationship between K_θ and K_β that any difference arises between the different types of circuit. Equations (5), (6) and (7) for the circuit curves are not affected, provided of course that K_β and $\bar{K}_\beta$ are defined in the same way.

3. *Limiting Forms of the Stick-free Condition.*—As mentioned in section 1, the two conditions most commonly considered in connection with control-surface flutter are the stick-fixed condition and the circuit-cut condition. In terms of the analysis of section 2, these two conditions are defined respectively by $\bar{K}_\beta = K_\beta$ (stick-fixed) and $\bar{K}_\beta = 0$ (circuit-cut). It is interesting to examine under what circumstances the stick-free condition of section 2 approximates to these limiting conditions.

Considering equation (7) for the standard case without a stick spring, and assuming I_c/I_θ' small, it is seen that $\bar{K}_\beta \simeq K_\beta$ if ω is large (though not infinite), or if ω_θ is small (though not zero) by K_θ being small. For a given K_β the latter condition could arise only with a small ϕ. Also $\bar{K}_\beta \rightarrow K_\beta$ when $\omega_\theta \rightarrow 0$ by $I_\theta' \rightarrow \infty$.

The stick-free condition therefore approximates to the stick-fixed condition if the frequency of the flutter is very high, the stick inertia is very large, or the circuit gearing is very low.

Considering equation (7) again, it is seen that $\bar{K}_\beta \rightarrow 0$ when $\omega \rightarrow 0$ (as already mentioned in section 2), or when $\omega_\theta \rightarrow \infty$ by $K_\theta \rightarrow \infty$. For a finite K_β the latter condition could arise only by $\phi \rightarrow \infty$. If $\omega_\theta \rightarrow \infty$ by $I_\theta' \rightarrow 0$, then $\bar{K}_\beta$ approaches a finite value which tends to zero if $I_c \rightarrow 0$. Finally, if $K_\beta \rightarrow 0$ and therefore $K_\theta \rightarrow 0$ (unless $\phi \rightarrow \infty$), then $\bar{K}_\beta$ approaches a finite value which again tends to zero if $I_c \rightarrow 0$.

The stick-free condition therefore approximates to the circuit-cut condition if the frequency of the flutter is very low, the circuit gearing is very high ; or, providing the circuit inertia is small, if the stick inertia or circuit stiffness is small. As a special case, it should also be noted from equation (6) that the circuit-cut condition is exactly reproduced where a stick spring is fitted if $\omega = \omega_{\theta\,0}$, assuming the circuit inertia negligible.

From equation (5) it can further be seen that if ω is small compared with ω_θ, and ignoring circuit inertia, then

$$\bar{K}_\beta \simeq - K_\beta \frac{\omega^2}{\omega_\theta^2} = - \frac{I_\theta}{\phi^2} \omega^2$$

for the elevator case say, where $K_\theta = \phi^2 K_\beta$

At low frequencies, therefore, or when the circuit stiffness is large or the stick inertia small, the effect of the circuit in the stick-free condition is simply to transmit the inertia of the stick as if the stick were rigidly connected to the operating lever. This approximation has been used in some investigations as a closer representation of the stick-free condition than that given by the circuit-cut condition.

4. *Flutter of the Complete System.*—In the critical flutter condition the complete system is oscillating in simple harmonic motion. As shown in section 2, the effect of the stick and circuit in the stick-free condition is then to supply to the operating lever of the control surface an elastic restraint whose value is a function of the frequency of the flutter. From a theoretical point of view the unknowns are the air speed V and the frequency ω ; and the problem is how, with a given system, to solve the flutter equations for these unknowns.

4.1. *The Flutter Equations.*—If the analysis is made, as is usual, in terms of a semi-rigid representation, the equations of motion representing the critical flutter condition will be equal in number to the number of degrees of freedom chosen. Each equation then represents the balance of forces in respect of the co-ordinate appropriate to a particular degree of freedom.

Taking a typical ternary case involving say wing flexure, wing torsion, and aileron rotation, the equations of motion can be expressed in the following non-dimensional form.

$$\left.\begin{aligned} (\delta_{11} + e_{11}y)q_1 + \delta_{12}q_2 + \delta_{13}q_3 &= 0 \\ \delta_{21}q_1 + (\delta_{22} + e_{22}y)q_2 + \delta_{23}q_3 &= 0 \\ \delta_{31}q_1 + \delta_{32}q_3 + (\delta_{33} + e_{33}y)q_3 &= 0 \end{aligned}\right\} \quad \ldots \quad (10)$$

where
$$\delta_{rs} = - a_{rs}\nu^2 + ib_{rs}\nu + c_{rs}. \quad \ldots \quad (10a)$$

q_1, q_2, q_3 are the co-ordinates appropriate to the degrees of freedom representing wing flexure, wing torsion, and aileron rotation respectively. The a_{rs} are the structural-plus-aerodynamic inertia coefficients, the b_{rs} the aerodynamic damping coefficients, and the c_{rs} the aerodynamic stiffness coefficients. ν is the frequency parameter $\omega c/V$, ω being the circular frequency and c the wing chord at a particular station. The $e_{rs}y$ are the structural stiffness coefficients, y being the speed parameter $1/V^2$. The case illustrated by equations (10) involves no structural cross-stiffnesses, *i.e.*, $e_{rs} = 0$ for $r \neq s$.

If equations (10) in the order given are appropriate to the co-ordinates q_1, q_2, q_3 respectively, then e_{11} and e_{22} (representing the wing structural stiffnesses) are known, and e_{33} (representing the aileron constraint) is of the form $k_3\bar{K}_\beta$, where k_3 is known and $\bar{K}_\beta$ is the elastic restraint applied to the operating lever from the circuit.

Eliminating q_1, q_2, q_3 from equations (10) gives the single determinantal equation

$$\begin{vmatrix} \delta_{11} + e_{11}y, & \delta_{12}, & \delta_{13} \\ \delta_{21}, & \delta_{22} + e_{22}y, & \delta_{32} \\ \delta_{31}, & \delta_{32}, & \delta_{33} + k_3\bar{K}_\beta y \end{vmatrix} = 0 \quad \ldots \quad (11)$$

where e_{33} has now been written in the form $k_3\bar{K}_\beta$.

Since from (10a) the δ_{rs} coefficients are complex, equation (11) when expanded will also be complex. Equating the real and imaginary parts separately to zero then gives two real equations, which in theory are sufficient for the solution of the two unknowns V and ω.

The functional form of the equations is however so complicated that a direct solution for V and ω is out of the question. The b_{rs} and c_{rs} coefficients are functions of the frequency parameter ν, and for the stick-free condition $\bar{K}_\beta$ is a function of the frequency ω as given by equations (5), (6), or (7) of section 2. There is also the additional complication that in some cases structural cross-stiffness coefficients $e_{rs}y$ $(r \neq s)$ may exist.

4.2. *Suggested Method of Solution.*—An indirect method of solution is possible which has the advantage of simplicity and a certain physical clarity. The flutter equation (11) can be solved in terms of V, ω, and $\bar{K}_\beta$ as variables, ignoring the dependance of $\bar{K}_\beta$ upon ω as determined by the circuit conditions of section 2. The technique of such flutter solutions is fairly standard, but they are discussed in Appendix I to this report. The important aspect of the flutter solutions is that they should cover a range of positive *and negative* values of $\bar{K}_\beta$.

The flutter solutions are presented as curves, termed the 'flutter' curves, of $\omega \sim \bar{K}_\beta$ and $V \sim \bar{K}_\beta$. A given value of $\bar{K}_\beta$, together with the associated values of ω and V as given by the flutter curves, then represents a particular solution of the flutter equation (11). Superimposing the circuit curve $\omega \sim \bar{K}_\beta$ as given by equations (5), (6) or (7) on the flutter curve $\omega \sim \bar{K}_\beta$, intersections of the two curves then represent critical flutter solutions for the complete system in the stick-free condition. Values of the frequency ω and speed V appropriate to these solutions are read directly from the flutter curves.

A particular advantage of the method is that the effect of changes in circuit stiffness or stick inertia can be quickly assessed, as these affect only the circuit curve. It will be appreciated also that the method is equally applicable to spring or servo-tab systems.

The system considered may be regarded as a special case of a system possessing a real circuit impedance. Defining circuit impedance as the moment M applied from the circuit to the operating lever per unit lever movement β, the impedance may in general be complex, involving real and imaginary parts either or both of which may be a function of frequency. Complex impedances are possessed by circuits in which there is damping of either the velocity or hysteresis type. Powered control circuits will in general possess complex impedances of a rather complicated nature[1,2]. The essential characteristic of a circuit with a complex impedance is that the restoring moment applied to the operating lever is at any instant made up of two components which are respectively in phase and 90 deg out of phase with the operating lever.

In the present case the circuit impedance contains only a real component (in phase with the operating lever) whose value is $M/\beta = \bar{K}_\beta$. In terms of circuit impedance, therefore, the method of solution suggested involves the matching of the actual circuit impedance (as given by the circuit curve) with the impedance required to satisfy the flutter condition (as given by the flutter curves).

It should be noted that the method relies on the fact that the effect of the free stick is to change, from a constant to a variable with frequency, only the value of $\bar{K}_\beta$. This is true only if there are no masses in the circuit or attached to the stick which introduce appreciable cross coupling terms with any structural mode present in the calculations. In the case of an elevator system, for instance, in which the degrees of freedom might be elevator rotation and fuselage vertical bending (in addition of course to stick movement), any mass attached directly to the top of the stick moves sensibly in a direction perpendicular to the vertical fuselage movement at that point, and no coupling is therefore involved. A bob-weight attached to the stick on a horizontal arm (as is sometimes fitted for g-restriction purposes) will however introduce an inertia coupling with the fuselage mode, and (as shown in Appendix II) the effect of this inertia coupling on the flutter equations is not confined to $\bar{K}_\beta$. In such a case the method given here cannot be used and the analysis would have to be based directly on the flutter equations for the system as a whole.

5. *Numerical Examples.*—The application of the method given in section 4 is well illustrated by the two following examples, one of a hypothetical elevator system, the other of a specific aileron servo-tab system. At the same time the examples serve to demonstrate the effect of stiffness and inertia changes in the control system on the stick-free flutter characteristics, as well as the relation which the stick-free condition bears to the stick-fixed and circuit-cut conditions. Circuit inertia effects have been ignored in both examples.

5.1. *Hypothetical Elevator System.*—An idealised elevator system representative of a fairly large aircraft was considered in which the degrees of freedom (apart from stick movement) were rotation of the elevator about its hinge-line and normal translation of the tailplane, the latter representing fuselage vertical bending. Tailplane and elevator were taken as torsionally rigid. Basic data assumed was as follows :—

Mean chord of tailplane and elevator, 10 ft

Elevator chord 30 per cent of total chord

Moment of inertia of elevator about hinge-line, 13·85 slugs ft^2

C.G. of elevator at 0·1 elevator chord behind hinge-line

No aerodynamic balance on elevator

Natural frequency of tailplane (with no elevator rotation), 10 c.p.s.

Features which might at first sight appear rather unrepresentative are the backward position of the elevator c.g. and the absence of aerodynamic balance on an elevator of this size with direct control. No aerodynamic balance was taken for convenience, and the elevator c.g. was then located so as to give flutter characteristics reasonably well suited to the present purpose of demonstration.

Flutter solutions for the binary tailplane–elevator system are presented appropriately as flutter curves in Fig. 2 (full curves), $\omega/2\pi \sim \bar{K}_\beta$ in the upper diagram, $V \sim \bar{K}_\beta$ in the lower diagram. The curves extend over both positive and negative regions of the $\bar{K}_\beta$ axis.

That part of the $V \sim \bar{K}_\beta$ curve (lower diagram) over the positive region of the $\bar{K}_\beta$ axis is typical, and may be familiar from a study of other similar investigations that have been made in the past. At a given value of $\bar{K}_\beta$ there are a lower critical speed and an upper critical speed, the region between the two representing instability. Above a certain critical value of $\bar{K}_\beta$ ($5\cdot5 \times 10^4$ lb ft per radn) there is however no critical flutter condition at any speed and the system is stable at all speeds. This critical value of $\bar{K}_\beta$ corresponds to a natural frequency of the elevator, rotating about its hinge against the stiffness $\bar{K}_\beta$, of 10·0 c.p.s., *i.e.*, 1·0 times the tailplane natural frequency. At a slightly lower value of $\bar{K}_\beta$ the lower critical speed is a minimum, at a natural elevator frequency of 9·06 c.p.s. or roughly 0·91 times the tailplane natural frequency.

Points A, B, C, D denote roughly corresponding points on the two flutter curves, A on the upper curve corresponding to A on the lower curve, and so on. It is thus seen that the frequency appropriate to flutter with positive values of $\bar{K}_\beta$ varies little (between 9·8 and 10·9 c.p.s.), whereas with negative values of $\bar{K}_\beta$ there is a considerably greater frequency variation.

For the stick and circuit, four cases were considered, in each of which the circuit gearing ϕ was kept the same and there is no stick spring.

Case 1A $K_\beta = 4 \times 10^4$ lb ft per radn, giving an elevator natural frequency (stick-fixed) of 53·7 radn per sec (ω_β).

$$\omega_\theta = \omega_\beta = 53\cdot7 \text{ radn per sec.}$$

Case 1B $K_\beta = 4 \times 10^4$ lb ft per radn $\omega_\theta = 2\omega_\beta = 107\cdot4$ radn per sec. This corresponds to the stick inertia of Case 1A being divided by 4.

Case 2A $K_\beta = 8 \times 10^4$ lb ft per radn giving $\omega_\beta = 76 \cdot 0$ radn per sec. $\omega_\theta = \omega_\beta = 76 \cdot 0$ radn per sec. This corresponds to the circuit stiffness of Case 1A being doubled.

Case 2B $K_\beta = 8 \times 10^4$ lb ft per radn $\omega_\theta = 2\omega_\beta = 152$ radn per sec. This corresponds to the circuit stiffness of Case 1B being doubled, or to the stick inertia of Case 2A being divided by 4.

In these four cases, going from 1 to 2 represents the effect of doubling the circuit stiffness, going from A to B represents the effect of reducing the stick inertia by 75 per cent. The appropriate circuit curves from equation (5) are shown dotted in Fig. 2, superimposed on the flutter curve in the upper diagram. Intersections with the flutter curve are marked with the values of the flutter speed as obtained from the $V \sim \bar{K}_\beta$ curve in the lower diagram.

5.2. *Aileron Servo-Tab System.*—Flutter calculations on an experimental servo-tab system fitted to the aileron of a medium sized aircraft had been made in connection with another investigation. The results were found to be convenient for the present purpose and are plotted as flutter curves $\omega/2\pi \sim \bar{K}_\beta$ and $V \sim \bar{K}_\beta$ in the upper and lower diagrams of Fig. 3. The system has a follow-up ratio of $N = 1$, and $\bar{K}_\beta$ again represents the elastic restraint applied by the circuit to the operating lever, which in this case is not of course directly connected to the aileron. The system investigated involved four degrees of freedom : the wing fundamental mode, aileron rotation, tab rotation, and aircraft roll.

The flutter curves of Fig. 3 possess certain interesting features. There is a closed loop between B and C (points A, B, C, D again denote roughly corresponding points on the two curves) ; and at B on the lower diagram, for instance, two critical flutter conditions co-exist at the same speed but at slightly different frequencies. In the lower frequency range the curves lie wholly in the region of positive $\bar{K}_\beta$, and for very small positive values of $\bar{K}_\beta$ they give a critical flutter condition at low speed and frequency, In the calculations no account was taken of any structural damping in the wing, but it is very probable that the effect of such damping would be to contract the unstable regions of the flutter curves in such a way as to isolate the closed loop and remove the low-speed flutter conditions at low positive $\bar{K}_\beta$. The flutter curves might then be as pictured in Fig. 4.

For the stick and circuit, three cases were considered, in each of which the circuit gearing ϕ was again kept the same.

Case 1 $K_\beta = 150$ lb ft per radn
$\omega_0/2\pi = 2$ c.p.s. $\omega_{\theta 0} = 0$.

Case 2 $K_\beta = 75$ lb ft per radn
$\omega_0/2\pi = 2$ c.p.s. $\omega_{\theta 0} = 0$.
This represents Case 1 with the circuit stiffness and stick inertia both halved.

Case 3 $K_\beta = 150$ lb ft per radn
$\omega_0/2\pi = \sqrt{6}$ c.p.s. $\omega_{\theta 0}/2\pi = \sqrt{2}$ c.p.s.
This represents Case 1 with the addition of a stick spring.

The appropriate circuit curves from equation (6) are shown dotted in Fig. 3, superimposed on the flutter curve in the upper diagram. Intersections with the flutter curve are again marked with the values of the flutter speed as obtained from the $V \sim \bar{K}_\beta$ curve in the lower diagram. For the intersection of the in-phase (lower frequency) branch of the case 3 circuit curve with the flutter curve, the value of the speed cannot be obtained very accurately from the $V \sim \bar{K}_\beta$ curve directly because of the vertical run of the latter curve at this point. The speed can however be obtained from a subsidiary plot $\omega \sim V$ of the flutter solutions, for the value of the frequency appropriate to the intersection.

5.3. *Discussion of the Results.*—Taking the elevator system first (Fig. 2) the solutions for the stick-fixed and circuit-cut conditions may first be noted. The stick-fixed condition is given by the intersection of the flutter curve in the upper diagram of Fig. 2 with the vertical line $\bar{K}_\beta = K_\beta$. In the present case, in which circuit inertia effects have been ignored, the vertical line $\bar{K}_\beta = K_\beta$ is also the vertical asymptote to the out-of-phase branch of the circuit curve for the stick-free condition with circuit stiffness K_β. The circuit-cut condition is similarly given by the intersection of the flutter curve with the vertical exis $\bar{K}_\beta = 0$.

Values of the critical speed V and frequency ω for the stick-fixed and circuit-cut conditions, and for the various stick-free conditions considered, are given in the following table.

TABLE 1

Solutions for the Hypothetical Elevator System

Condition	V ft/sec	$\omega/2\pi$ c.p.s.	Type of stick-free flutter
Stick-fixed Case 1A, 1B	140	10·1	
Case 2A, 2B	—	—	
Circuit-cut	430	9·8	
Stick-free Case 1A	1070	7·7	in-phase
Case 1B	530	9·6	in-phase
Case 2A	840	8·9	in-phase
Case 2B	520	9·65	in-phase

No stick-fixed flutter occurs in Cases 2A, 2B because for these cases the vertical line $\bar{K}_\beta = K_\beta$ misses the flutter curve completely. At the same time, the out-of-phase branches of the stick-free circuit curves for these cases (which do not appear on Fig. 2) also miss the flutter curve because they lie to the right of the asymptote $\bar{K}_\beta = K_\beta$. With circuit inertia effects included (*see* section 2), the out-of-phase branches of the stick-free circuit curves miss the flutter curve by a still greater margin. It is evident, therefore, that with the flutter curve of Fig. 2 there can be no out-of-phase stick-free flutter if there is no stick-fixed flutter.

Cases 1A and 1B, for which stick-fixed flutter does occur, still involve no out-of-phase stick-free flutter. In case 1B the out-of-phase branch of the circuit curve lies above a horizontal asymptote at $\omega = \omega_\theta = 107 \cdot 4$ radn per sec (17·1 c.p.s.) and is therefore well clear of the flutter curve. In case 1A the out-of-phase branch of the circuit curve (which appears in Fig. 2), though nearer to the flutter curve than in any of the other cases, is still clear of it. Even when stick-fixed flutter does occur, therefore, there may still be no out-of-phase stick-free flutter.

In-phase stick-free flutter does however occur in all four cases. From a comparison of the speeds and frequencies for the four cases, it appears that increasing the circuit stiffness reduces the flutter speed and increases the frequency, moving nearer to the circuit cut condition : the effect is much less however with the small stick inertia. Reducing the stick inertia has a similar effect, again less with the higher circuit stiffness. The stick-free flutter speeds are in all cases higher (and the frequencies lower) than for the circuit-cut condition, which in turn has a higher flutter speed and lower frequency than any stick-fixed condition for which flutter occurs. It is to be noted that if ω_θ is very large and K_β not too large (*i.e.*, high circuit gearing or low stick inertia) the in-phase stick-free condition approaches the circuit-cut condition.

From the general form of the flutter and circuit curves of Fig. 2, it is evident that the effects noted above operate only so long as the circuit curve cuts the flutter curve in the region above the point C. If the intersection occurs below the point C, as will happen with low values of

ω_θ (low circuit stiffness or high stick inertia), the effects on flutter speed will be reversed. In particular, it is to be noted that very low values of ω_θ may give in-phase flutter speeds lower than the flutter speed for the circuit-cut condition : it is very unlikely however that ω_θ would in practice ever be as low as this.

The servo-tab system (Fig. 3) shows some interesting differences compared with the elevator system. Stick-fixed flutter again occurs for the case with the lowest circuit stiffness (Case 2), but for this system the stick fixed flutter speed remains fairly constant as the circuit stiffness is reduced. As the circuit stiffness approaches zero, however, the stick-fixed flutter speed rapidly decreases, and for the circuit-cut condition the flutter speed is actually zero. For the more realistic system of Fig. 4 (wing structural damping included) the position would however be somewhat different. The stick-fixed condition would then be flutter free in the range between the two branches of the flutter curve, and the flutter speed for the circuit-cut condition would be roughly the same as for any stick-fixed condition in which flutter occurred.

Returning to Fig. 3, it is seen that the conclusions derived for the elevator system relevant to out-of-phase stick-free flutter apply also to the servo-tab system. For Cases 1 and 3 there is no stick-fixed flutter and (inevitably) no out-of-phase stick-free flutter. Case 2 however illustrates a case for which there is stick-fixed flutter *and* out-of-phase stick-free flutter : the speeds and frequencies are almost identical for the two conditions. In-phase stick-free flutter occurs in all three cases, in Cases 1 and 2 at zero speed, and in Case 3 (addition of a stick spring) at a higher though still comparatively low speed. With the flutter curve of Fig. 3 it is evident that in-phase flutter could only be prevented with a stiff stick spring and a large value of ω_θ, so that the in-phase branch of the circuit curve cleared the flutter curve to the right of point C. Even then flutter might still occur by intersection with the flutter curve in the region beyond point A, though the flutter speed would then be comparatively high.

It is however worth noting the possibilities of in-phase stick-free flutter with the more realistic flutter curve of Fig. 4 (wing structural damping included). Although in-phase flutter might still occur in all three circuit cases, the flutter speeds would be generally higher and it is likely that Case 3 with the stick spring would still have the highest flutter speed. It would be unwise to draw any detailed conclusions based on the form of the flutter curves of Fig. 4, which are largely hypothetical, but certain possibilities can be envisaged. If the flutter curve $\omega \sim \bar{K}_\beta$ crosses the axis of $\bar{K}_\beta$, then in-phase flutter can be prevented only by a fairly stiff spring and a large value of ω_θ. If on the other hand the flutter curve does not cross the axis of $\bar{K}_\beta$, in-phase flutter could be prevented without a stick spring by keeping the value of ω_θ low enough (low circuit stiffness or high stick inertia). Addition of a stick spring could in that case produce in-phase flutter.

It is interesting to note, incidentally, that the addition of a stick spring to the elevator system of Fig, 2 would reduce the in-phase flutter speed, assuming the intersections to occur in the practical range of the flutter curve above point C. The case is of little practical interest, of course, as a stick spring is rarely used with a direct-control system.

From these two examples it is evident that the stick-free flutter characteristics are acutely dependent upon the form of the flutter curves, and this point should be emphasised. The examples considered, though typical, are not all-embracing, and the conclusions drawn from them cannot be taken as universally applicable. The examples do however demonstrate the usefulness of the method suggested ; and they serve to illustrate the importance of considering the stick-free flutter characteristics, as distinct from those of the simplified circuit-cut condition.

6. *Conclusions*.—The problem of predicting the stick-free flutter characteristics of a control system is considered and a method of solution proposed which corresponds to impedance matching between circuit and control surface in the flutter condition. The method is applied, by way of illustration, to two typical cases, an elevator system and a servo-tab system. Conclusions drawn from these two cases are listed below, but it should be emphasised that some of these

conclusions may not apply universally. Stick-free flutter is termed ' in-phase ' and 'out-of-phase ' when the stick is respectively in-phase and out-of-phase with the control surface end of the circuit.

(a) If there is no stick-fixed flutter there will be no out-of-phase stick-free flutter.

(b) A high value of ω_θ (high circuit gearing or low stick inertia) favours the elimination of out-of-phase stick-free flutter when stick-fixed flutter is present.

(c) In-phase stick-free flutter may occur even if stick-fixed flutter is absent.

(d) With a high circuit gearing or low stick inertia the in-phase stick-free condition approaches the circuit-cut condition. Otherwise, the two conditions may be quite different. For the elevator system considered in this report, in-phase stick-free flutter speeds would normally be higher than for the circuit-cut condition.

(e) Addition of a stick-spring to a spring or servo-tab system may improve or worsen the stick-free flutter characteristics, depending upon the form of the flutter curves.

(f) The stick-free flutter characteristics are acutely dependent upon the form of the flutter curves. The effect of structural damping on the flutter curves may be important in some cases.

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LIST OF SYMBOLS

a	Length of bob-weight arm
a_{rs}	Non-dimensional structural-plus-aerodynamic inertia coefficients
A_{rs}	Structural inertia coefficients
b_{rs}	Non-dimensional aerodynamic damping coefficients.
c_{rs}	Non-dimensional aerodynamic stiffness coefficients
$e_{rs}y$	Non-dimensional structural stiffness coefficients
E_{rs}	Structural stiffness coefficients
g_1, g_2	Gearings defining the movement of a point in the circuit (*see* section 2)
h_0	Stick spring stiffness
I_1, I_2	Real and imaginary parts of the circuit impedance (*see* Appendix I)
I_c	Inertia coupling introduced by the circuit (*see* equation (8) and Appendix III)
I_0	Stick inertia
I_θ'	Effective stick inertia, including circuit (*see* equation (9) and Appendix III)
k_n	Constant (*see* Appendix I)
K_θ	Circuit stiffness in terms of stick movement

K_β	Circuit stiffness in terms of operating lever movement
$\bar{K}_\beta$	Effective value of K_β in the stick-free condition
K_s	Subsidiary spring stiffness[3]
m	General element of mass
M_b	Bob-weight mass
N	Tab follow-up ratio
N_1	Tab eccentricity ratio[3]
q	Generalised co-ordinate appropriate to the aircraft mode
$\bar{q}$	Amplitude of q
q_1, q_2, q_3	Generalised co-ordinates
Q_q	Generalised (aerodynamic) force appropriate to co-ordinate q
V	Air speed
x	Distance aft of control-surface hinge
y	Speed parameter $1/V^2$
z	Function defining vertical displacement in the aircraft mode
z_b	Value of z at the stick
β	Operating lever movement
$\bar{\beta}$	Amplitude of β
δ_{rs}	Non-dimensional coefficients occurring in the flutter equations (*see* section 4.1)
θ	Stick movement
$\bar{\theta}$	Amplitude of θ
ν	Frequency parameter appropriate to ω
$\sum_a$	Summation over the aircraft
$\sum_c$	Summation over the circuit
$\sum_e$	Summation over the elevator
ϕ	Circuit gearing β/θ with rigid circuit
ω	Circular flutter frequency
ω_β	Natural frequency of control surface with stick fixed (direct-control system) } in angular measure
ω_θ	Natural frequency of stick with operating lever fixed } in angular measure
$\omega_{\theta 0}$	Natural frequency of stick on stick spring alone } in angular measure

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APPENDIX I

Technique of the Flutter Solutions

The flutter equation may be written generally as

$$| \delta + ey | = 0 \quad \ldots \quad (A.1)$$

where the order of the determinant is equal to the number of degrees of freedom chosen (excluding stick movement). The complex δ coefficients represent the structural inertia and aerodynamic forces,

$$\delta_{rs} = - a_{rs}\nu^2 + ib_{rs}\nu + c_{rs} \quad \ldots \quad (A.2)$$

and the real ey coefficients the structural elastic forces, y being the speed parameter $1/V^2$. The a_{rs} are known, and for a given frequency parameter ν the b_{rs} and c_{rs} are also known.

The problem is to solve the flutter equation (A.1) for the relationship between the speed V, the frequency ω, and the elastic restraint $\bar{K}_\beta$ applied from the circuit.

1. *Direct-Control Systems.*—In a direct-control system the circuit is attached directly to the control surface and there is normally no elastic coupling between the control surface and any other degree of freedom. If the determinant in (A.1) is of order n say, this means that $e_{rn} = e_{ns} = 0$ $(r, s \neq n)$. The elastic restraint $\bar{K}_\beta$ occurs only in e_{nn}, which is of the form $e_{nn} = k_n\bar{K}_\beta$, k_n being known. The remaining coefficients e_{rs} $(r, s < n)$ are known.

For the simplest case of all, a binary system, the flutter equation (A.1) becomes

$$\begin{vmatrix} \delta_{11} + e_{11}y, & \delta_{12} \\ \delta_{21}, & \delta_{22} + e_{22}y \end{vmatrix} = 0 \quad \ldots \quad (A.3)$$

where $e_{22} = k_2\bar{K}_\beta$. The flutter solution is in this case quite straightforward. For a given frequency parameter the determinant of (A.3) is expanded with $e_{11}y$ and $e_{22}y$ as variables. Equating real and imaginary parts separately to zero then gives two real equations from which the values of $e_{11}y$, $e_{22}y$ are quickly determined, involving the solution of a quadratic only. For the known value of e_{11} the values of y and e_{22}, and hence of V, ω, and $\bar{K}_\beta$, are given. Repeating for other frequency parameters gives the required relationship between V, ω, and $\bar{K}_\beta$.

For ternary and higher cases the solution is less straightforward. In the case of a ternary system with no structural cross-stiffnesses, for instance, the flutter equation (A.1) becomes

$$\begin{vmatrix} \delta_{11} + e_{11}y, & \delta_{12}, & \delta_{13} \\ \delta_{21}, & \delta_{22} + e_{22}y, & \delta_{23} \\ \delta_{31}, & \delta_{32}, & \delta_{33} + e_{33}y \end{vmatrix} = 0 \quad \ldots \quad (A.4)$$

where $e_{33} = k_3\bar{K}_\beta$. The binary type of solution described above is obviously not applicable to this ternary case, and a different procedure must be followed.

An obvious method is to solve equation (A.4) for V and ω in the normal manner for a range of arbitrary values of $\bar{K}_\beta$. For a given value of $\bar{K}_\beta$ the three e coefficients are known. Assuming a value of the frequency parameter ν for the calculation of the b_{rs} and c_{rs} coefficients in (A.2)) equation (A.4) can be expanded as a polynomial in ν. Equating real and imaginary parts separately to zero then gives the two real equations

$$-p_0\nu^6 + p_2\nu^4 - p_4\nu^2 + p_6 = 0 \quad \ldots \quad (A.5)$$

$$p_1\nu^4 - p_3\nu^2 + p_5 = 0 \quad \ldots \quad (A.6)$$

where the p coefficients are in general functions of $e_{11}y$, $e_{22}y$, $e_{33}y$. Equations (A.5) and (A.6) are then solved indirectly for V and ω in the following manner. For a given speed V the p coefficients are determined. Solving (A.6) as a quadratic for ν^2, the resulting values of ν^2 are substituted in the left-hand side of (A.5). Repeating for a range of speeds, the speed V and frequency parameter ν (and hence the frequency ω) are found for which the left-hand side of (A.5) becomes zero. The frequency parameter should agree reasonably well with the value assumed for the calculation of the b_{rs} and c_{rs} coefficients : otherwise an iteration is necessary. The whole process is then repeated for other values of $\bar{K}_\beta$, and the variation between V, ω, and $\bar{K}_\beta$ finally obtained. This method can also be applied to a quaternary system, for which the equations corresponding to (A.5) and (A.6) will be respectively a quartic and cubic in ν^2: but by simple algebraic manipulation these equations are transformed into a cubic and quadratic, and the solution obtained as in the ternary case. The method is also applicable where structural cross-stiffnesses exist ; these merely add further terms to the expressions for the p coefficients, but do not otherwise complicate the solution.

There is an alternative method of solution for ternary and higher cases which has the advantage of simplicity of form and a certain generality, As explained in the main text (section 4.2), the elastic restraint $\bar{K}_\beta$ represents a circuit impedance that is wholly real, *i.e.*, in phase with the operating lever. In general, the impedance of a circuit is complex, containing a component that is 90 deg out-of-phase with the operating lever. Such an impedance could be written as $(I_1 + iI_2)$, where I_1 and I_2 are respectively the real and imaginary components. Flutter solutions for the ternary and higher cases are conveniently obtained in terms of a general complex circuit impedance, followed by reduction to the special case considered here, *viz.*, $I_1 = \bar{K}_\beta$, $I_2 = 0$.

Considering again the ternary system represented by equation (A.4), and replacing $\bar{K}_\beta$ in e_{33} by a general complex impedance $(I_1 + iI_2)$, the flutter equation becomes

$$\begin{vmatrix} \delta_{11} + e_{11}y, & \delta_{12}, & \delta_{13} \\ \delta_{21}, & \delta_{22} + e_{22}y, & \delta_{23} \\ \delta_{31}, & \delta_{32}, & \delta_{33} + k_3(I_1 + iI_2)y \end{vmatrix} = 0. \quad \ldots \quad (A.7)$$

For a given frequency parameter ν and a given speed V (and hence a given y) the determinant in (A.7) can be readily expanded with I_1, I_2 as variables. Equating real and imaginary parts separately to zero then gives two linear simultaneous equations in I_1 and I_2, from which their values are quickly obtained. Repeating for a range of speeds but the same frequency parameter (which leaves the δ coefficients unchanged) a curve of the variation $I_2 \sim I_1$ can then be drawn, each point of the curve corresponding to a particular speed and frequency. Repeating the process for a range of frequency parameters then gives a family of curves, each for a particular frequency parameter, and along each of which the speed and frequency vary and are known. Such a family of curves represents a series of flutter curves appropriate to the system with a complex circuit impedance: they could, as suggested elsewhere[1,2], be used to provide solutions for the case with a powered control system. Reduction to the present case ($I_1 = \bar{K}_\beta$, $I_2 = 0$) is given by the intersections of the flutter curves with the axis $I_2 = 0$, each intersection providing a particular set of values of V, ω, and I_1 ($= \bar{K}_\beta$) in the $V \sim \omega \sim \bar{K}_\beta$ relationship. For the present purpose the generalised flutter curves $I_2 \sim I_1$ need be calculated only in the region of $I_2 = 0$. The method is equally applicable where structural cross-stiffnesses exist; these merely add to the real parts of the δ coefficients and do not complicate the solution in any way.

2. *Spring and Servo-Tab Systems.*—Spring and servo-tab systems require special consideration because of the nature of the structural elastic coefficients in such systems. The spring-tab system (of which the servo-tab is a special case) involves in general three springs in the control-surface–tab linkage: a main spring between the operating lever and the control surface, a subsidiary spring between the operating lever and the tab, and a spring representing the circuit stiffness applied to the operating lever (i.e., $\bar{K}_\beta$). In a servo-tab system the main spring is omitted.

In a spring-tab system the effect of the various springs is to supply direct and cross elastic coefficients in the control-surface and tab degrees of freedom, assuming the latter to be represented directly by rotation of the control surface relative to the main lifting surface (wing, tailplane, or fin) and rotation of the tab relative to the control surface. For a typical ternary system (say wing-aileron-tab) the flutter equation (A.1) would be

$$\begin{vmatrix} \delta_{11} + e_{11}y, & \delta_{12}, & \delta_{13} \\ \delta_{21}, & \delta_{22} + e_{22}y, & \delta_{23} + e_{23}y \\ \delta_{31}, & \delta_{32} + e_{32}y, & \delta_{33} + e_{33}y \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.8)$$

where the last two rows of the determinant are appropriate to the control surface and tab degrees of freedom respectively. In the general case the elastic coefficients e_{22}, e_{23} ($= e_{32}$), e_{33} are each functions of the stiffnesses of the three springs in the control linkage. In particular, the circuit stiffness $\bar{K}_\beta$ occurs in all four coefficients: the case is therefore quite different from that of a direct-control system already considered.

In the case of servo-tab systems, and of spring-tab systems in which the stiffness of the main spring is small compared with that of the subsidiary spring and with the circuit stiffness $\bar{K}_\beta$, the general form of the elastic coefficients reduces to a simpler form which leads to a convenient flutter solution. With the main spring stiffness zero (or effectively so), the matrix of the elastic coefficients in the control-surface and tab degrees of freedom is of the form

$$\begin{bmatrix} e_{22}y & e_{23}y \\ e_{32}y & e_{33}y \end{bmatrix} = \begin{bmatrix} N^2 e_{33}y & -Ne_{33}y \\ -Ne_{33}y & e_{33}y \end{bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.9)$$

where N is the follow-up ratio of the tab system. The coefficient e_{33} is then of the form

$$e_{33} = k_3 \frac{\bar{K}_\beta K_s}{\bar{K}_\beta / N_1^2 + K_s} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (A.10)$$

where k_3 is known, K_s is the stiffness of the subsidiary spring, and N_1 is the eccentricity ratio[3]. With K_s very large, (A.10) reduces to $e_{33} = k_3\bar{K}_\beta$.

For a given frequency parameter (A.8), with (A.9) substituted, can be expanded in terms of $e_{11}y$ and $e_{33}y$ as variables. Because the determinant of the matrix in (A.9) is zero, this expansion will contain no terms in $(e_{33}y)^2$. Equating the real and imaginary parts of the expansion separately to zero then gives two equations from which the values of $e_{11}y$ and $e_{33}y$ are determined, involving quadratic solutions only. Knowing e_{11} then gives y (and hence the speed V and frequency ω), e_{33}, and from (A.10) the value of $\bar{K}_\beta$. Repeating for a range of frequency parameters finally gives the complete $V \sim \omega \sim \bar{K}_\beta$ relationship.

It will be appreciated that, because of the simplification represented by (A.9), the solution for the ternary system (A.8) is in form the same as that for a binary direct-control system already described. For quaternary cases of the servo-tab system the solutions will likewise be of the same form as for ternary cases of the direct control system : either by expansion in terms of ν^2 for a range of arbitrary values of $\bar{K}_\beta$, or in terms of a complex circuit impedance.

For a spring-tab system in which the stiffness of the subsidiary spring is comparable with that of the main spring, the relationship (A.9) is no longer true and the solutions based upon it cannot be used. Solution is still possible however by expansion in terms of ν^2 for a range of arbitrary values of $\bar{K}_\beta$.

APPENDIX II

Stick-free Flutter with a Bob-weight Fitted to the Stick

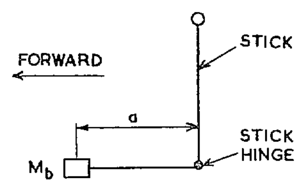
The case considered here is one where a concentrated weight is attached to the stick on a horizontal arm, as is sometimes fitted for g-restriction purposes. To demonstrate the effect of such a weight in the flutter condition, a simple binary direct-control system is chosen—say fuselage bending and elevator rotation, in addition of course to stick movement—and a routine flutter analysis applied to the complete ternary system.

The degrees of freedom for such a case will then be :

(a) Fuselage bending, involving a vertical displacement zq at any point of the aircraft, positive downward. z is a non-dimensional function of the point location, defining the mode ; q is the generalised co-ordinate and defines the displacement at the reference point, where $z = 1$.

(b) Elevator rotation β about hinge (positive downward)

(c) Stick rotation θ (positive forward)



If M_b is the mass of the bob-weight, and z_b the value of z at the stick hinge, the total kinetic energy (including stick and bob-weight) is

$$T = \tfrac{1}{2}I_\theta\dot{\theta}^2 + \tfrac{1}{2}M_b(z_b\dot{q} + a\dot{\theta})^2 + \sum_a \tfrac{1}{2}m(z\dot{q})^2 + \sum_e \tfrac{1}{2}m(z\dot{q} + x\dot{\beta})^2 \quad \ldots \quad \ldots \text{(A.11)}$$

where m is a general element of mass and x is the distance aft of the elevator hinge. Suffixes a and e attached to the summation signs denote summation over the aircraft (excluding elevator) and the elevator respectively. Circuit inertia effects are here ignored.

The potential energy is given by

$$V = \tfrac{1}{2}K_\theta(\theta - \beta/\phi)^2 + \tfrac{1}{2}E_{11}q^2 \quad \ldots \quad \ldots \text{(A.12)}$$

where E_{11} is the stiffness coefficient appropriate to the aircraft mode (a). It is assumed that there is no elastic coupling between this mode and either elevator rotation or stick movement.

Symbols I_θ, K_θ, and ϕ denote stick inertia, circuit stiffness, and circuit gearing, as in the main text.

The Lagrangian equations of motion for the three co-ordinates q, β, θ, *viz.*,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} + \frac{\partial V}{\partial q} = Q_q, \text{ etc.,}$$

are then, respectively

$$A_{11}\ddot{q} + A_{12}\ddot{\beta} + A_{13}\ddot{\theta} + E_{11}q = Q_q \quad \text{.. (A.13)}$$

$$A_{21}\ddot{q} + A_{22}\ddot{\beta} + E_{22}\beta + E_{23}\theta = Q_\beta \quad \text{.. (A.14)}$$

$$A_{31}\ddot{q} + A_{33}\ddot{\theta} + E_{32}\beta + E_{33}\theta = 0 \quad \text{.. (A.15)}$$

where $A_{11} = \sum_{ae} mz^2$ (including bob-weight)

$$A_{12} = A_{21} = \sum_{e} mxz$$

$$A_{13} = A_{31} = M_b a z_b$$

$$A_{22} = \sum_{e} mx^2 . A_{33} = I\theta + M_b a^2$$

$$E_{22} = \frac{K_\theta}{\phi 2} = K_\beta . E_{33} = K_\theta$$

$$E_{23} = E_{32} = -\frac{K_\theta}{\phi} = -\phi K_\beta .$$

Q_q, Q_β are the generalised (aerodynamic) forces appropriate to the co-ordinates q, β. In the flutter condition, with frequency ω and $q = \bar{q}\,e^{i\omega t}$, etc., Q_q, Q_β will be linear complex functions of q, β but are independent of θ. Equations (A.13), (A.14) and (A.15) are then reducible to a non-dimensional form similar to that of equations (10). The generalised force Q_θ in (A.15) is zero because there is obviously no aerodynamic force in respect of θ alone.

Coefficients A_{13}, A_{31} represent the inertia coupling between the bob-weight and the fuselage mode. With these zero, equations (A.13), (A.14), and (A.15) can be reduced to the form

$$A_{11}\ddot{q} + A_{12}\ddot{\beta} + E_{11}q = Q_q \quad \text{.. (A.16)}$$

$$A_{21}\ddot{q} + A_{22}\ddot{\beta} + \bar{E}_{22}\beta = Q_\beta \quad \text{.. (A.17)}$$

$$A_{33}\ddot{\theta} + E_{32}\beta + E_{33}\theta = 0 \quad \text{.. (A.18)}$$

where $\bar{E}_{22} = E_{22} + E_{23}\theta/\beta = K_\beta(1 - \phi\theta/\beta)$.

Equations (A.16), (A.17) thus become independent of θ, except in so far as $\bar{E}_{22}$ is a function of θ/β. At the same time equation (A.18), which is independent of q, provides a solution (after putting $\ddot{\theta} = -\omega^2\theta$) for θ/β and hence for $\bar{E}_{22}$ in terms of the frequency ω, *viz.*,

$$\bar{E}_{22} = K_\beta \frac{\omega^2}{\omega^2 - \omega_0^2} \qquad \left(\text{where } \omega_0^2 = \frac{K_\theta}{A_{33}}\right). \quad \text{.. (A.19)}$$

The complete ternary (q, β, θ) system thus reduces effectively to a binary (q, β) system in which the stiffness coefficient $\bar{E}_{22}$ is a function of frequency.

Equation (A.18) is in fact identical with equation (1), and $\bar{E}_{22}$ is of course the same as $\bar{K}_\beta$ (*see* equation (5)). The above analysis for the case $A_{13} = A_{31} = 0$ thus corresponds exactly with that followed in the main text.

For the case where the inertia coupling A_{13}, A_{31} is present, however, the system cannot be reduced so simply. Putting $\ddot{q} = -\omega^2 q$, etc., in equations (A.13), (A.14) and (A.15), equation (A.15) then gives

$$\theta = -\frac{E_{32}\beta - \omega^2 A_{31} q}{E_{33} - \omega^2 A_{33}} . \quad \text{.. (A.20)}$$

Substituting for θ from (A.20), equations (A.13) and (A.14) then become

$$\bar{A}_{11}\ddot{q} + \bar{A}_{12}\ddot{\beta} + E_{11}q = Q_q \quad \ldots \quad (A.21)$$

$$\bar{A}_{21}\ddot{q} + A_{22}\ddot{\beta} + \bar{E}_{22}\beta = Q_\beta \quad \ldots \quad (A.22)$$

where

$$\bar{A}_{11} = A_{11}\ddot{\beta} - \frac{\omega^2 A_{13}^2}{A_{33}(\omega^2 - \omega_\theta^2)}$$

$$\bar{A}_{12} = \bar{A}_{21} = A_{12} + \frac{E_{23}A_{13}}{A_{33}(\omega^2 - \omega_\theta^2)}$$

and $\bar{E}_{22}$ is again as given by (A.19).

The ternary (q, β, θ) system is thus reduced effectively to a binary (q, β) system in which the effect of the stick movement is not only to make the stiffness coefficient $\bar{E}_{22}$ variable with frequency, but also to provide contributions to the inertia coefficients $\bar{A}_{11}$, $\bar{A}_{12}$. $\bar{A}_{21}$ that are functions of both the frequency and the inertia coupling A_{13}, A_{31}.

APPENDIX III

Circuit Inertia Effects

The system considered in Appendix II—fuselage-bending–elevator-rotation–stick-movement—is again considered here. In this case, however, the bob-weight is deleted but circuit inertia effects included. It is assumed that there are no circuit masses large enough to introduce additional degrees of freedom of appreciable amplitude; or to introduce appreciable inertia coupling with the aircraft mode (q), as in the case of the bob-weight of Appendix II.

With these assumptions, the kinetic energy of the system can be written

$$T = \tfrac{1}{2}I_\theta\dot{\theta}^2 + \sum_c \tfrac{1}{2}m(g_1\dot{\beta} + g_2\dot{\theta})^2 + \sum_a \tfrac{1}{2}m(z\dot{q})^2 + \sum_e \tfrac{1}{2}m(z\dot{q} + x\dot{\beta})^2 \quad \ldots \quad (A.23)$$

where g_1, g_2 are gear ratios appropriate to a point in the circuit, defined as in the main text (section 2). Suffix c attached to the summation sign denotes summation over the circuit.

The potential energy, as before, is

$$V = \tfrac{1}{2}K_\theta(\theta - \beta/\phi)^2 + \tfrac{1}{2}E_{11}q^2 \quad \ldots \quad (A.24)$$

The flutter equations (A.13), (A.14), (A.15) then become (with $A_{13} = A_{31} = 0$)

$$A_{11}\ddot{q} + A_{12}\ddot{\beta} + E_{11}q = Q_q \quad \ldots \quad (A.25)$$

$$A_{11}\ddot{q} + \bar{A}_{22}\ddot{\beta} + A_{23}\ddot{\theta} + E_{22}\beta + E_{23}\theta = Q_\beta \quad \ldots \quad (A.26)$$

$$A_{32}\ddot{\beta} + \bar{A}_{33}\ddot{\theta} + E_{32}\beta + E_{33}\theta = 0 \quad \ldots \quad (A.27)$$

where

$$\bar{A}_{22} = A_{22} + \sum_c mg_1^2$$

$$\bar{A}_{33} = A_{33} + \sum_c mg_2^2 = I_\theta'$$

$$A_{23} = A_{32} = \sum_c mg_1g_2 = I_c/\phi$$

and the unbarred coefficients (excepting A_{23}, A_{32}) are as in Appendix II.

Circuit inertia thus adds known contributions to the direct inertia coefficients A_{22}, A_{33} and introduces an inertia coupling A_{23}, A_{32} between the degrees of freedom β and θ.

Putting $\ddot{\beta} = -\omega^2\beta$, $\ddot{\theta} = -\omega^2\theta$ in (A.27) gives

$$\theta = -\frac{E_{32} - \omega^2 A_{32}}{E_{33} - \omega^2 \bar{A}_{33}}\beta \qquad \text{(A.28)}$$

Substituting for θ from (*A*.28), equations (A.25) and (A.26) then become

$$A_{11}\ddot{q} + A_{12}\ddot{\beta} + E_{11}q = Q_q \qquad \text{(A.29)}$$

$$A_{21}\ddot{q} + \bar{A}_{22}\ddot{\beta} + \bar{E}_{22}\beta = Q_\beta \qquad \text{(A.30)}$$

where now

$$\bar{E}_{22} = \frac{\omega^2}{\omega^2 - \omega_\theta^2}\left[E_{22} - 2E_{23}\frac{A_{23}}{\bar{A}_{33}} + \frac{\omega^2 A_{23}{}^2}{\bar{A}_{33}}\right]$$

$$= K_\beta \frac{\omega^2}{\omega^2 - \omega_\theta^2}\left[1 + \frac{2I_c}{I_\theta'} + \left(\frac{\omega I_c}{\omega_0 I_\theta'}\right)^2\right] \qquad \text{(A.31)}$$

and $\omega_0^2 = K_\theta/\bar{A}_{33}$. $\qquad$ (A.32)

The ternary (q, β, θ) system is thus again reduced to a binary (q, β) system in which the effective circuit stiffness $\bar{K}_\beta$ $(= \bar{E}_{22})$ is a function of frequency. The effect of circuit inertia is to add a known contribution to the control-surface inertia $\bar{A}_{22}$, and to change the value of the effective stiffness $\bar{E}_{22}$ from that of (A.19) to that of (A.31), the difference being a function of the inertia coupling A_{23}, A_{32} between stick and control surface.

For simplicity, the above demonstration of circuit inertia effects has been made for a direct-control system. Similar principles, however, apply to a spring-tab system. In the latter case, circuit inertia adds known contributions to the direct and cross inertias of control surface and tab, and modifies the corresponding effective stiffnesses $\bar{E}_{rs}$ by amounts dependent upon the inertia couplings introduced by the circuit inertia between stick and control surface and between stick and tab.

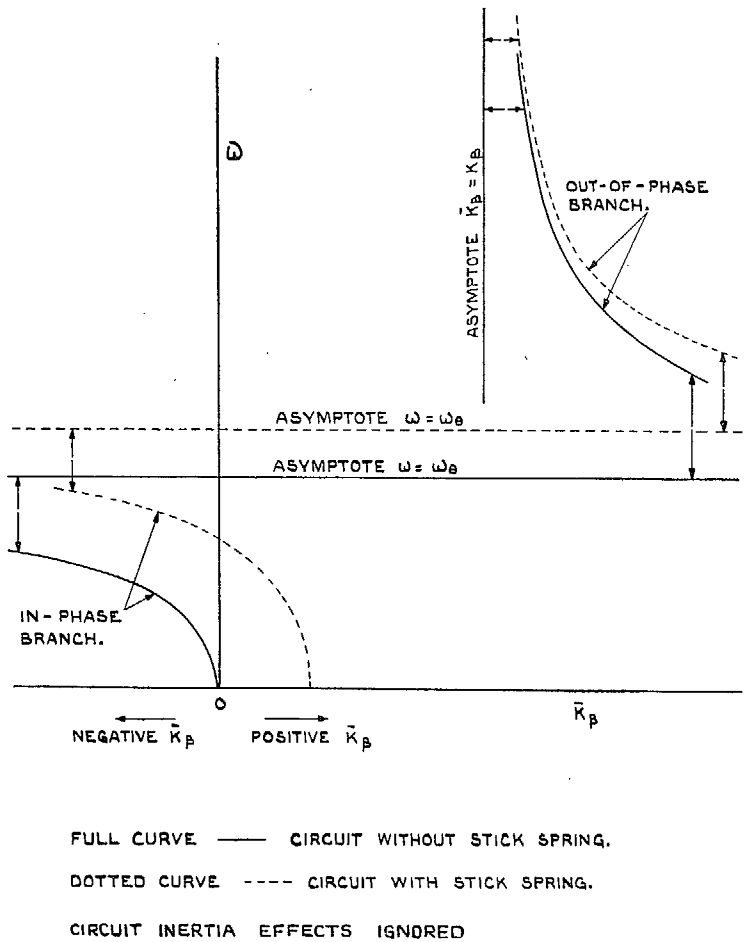


FIG. 1. Circuit curves for system with and without stick spring.

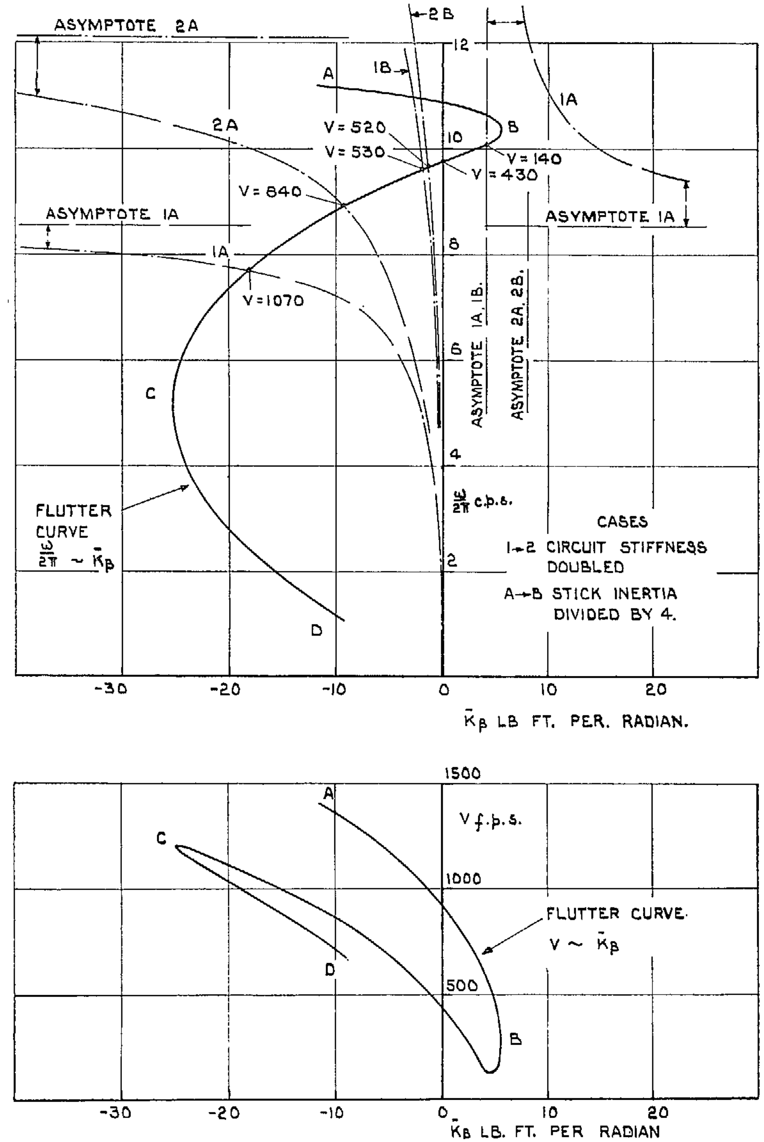


FIG. 2. Stick-free flutter of a hypothetical elevator system.

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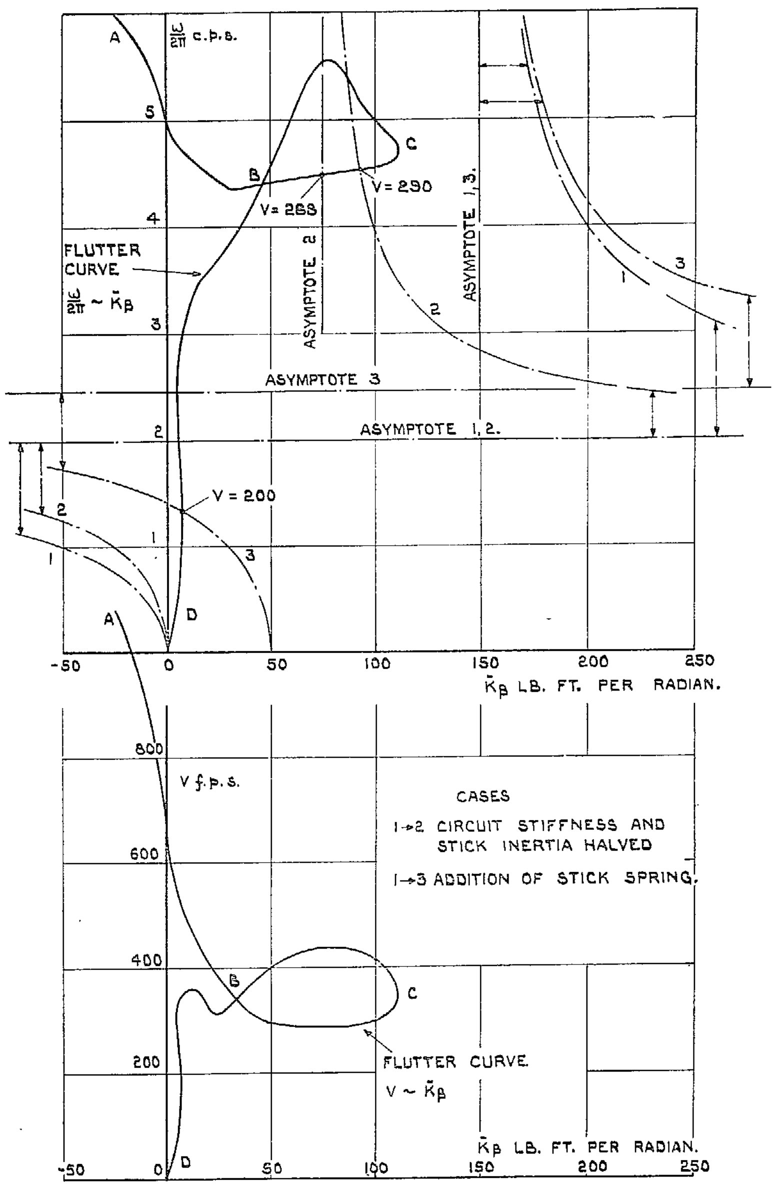

FIG. 3. Stick-free flutter of an aileron servo-tab system.

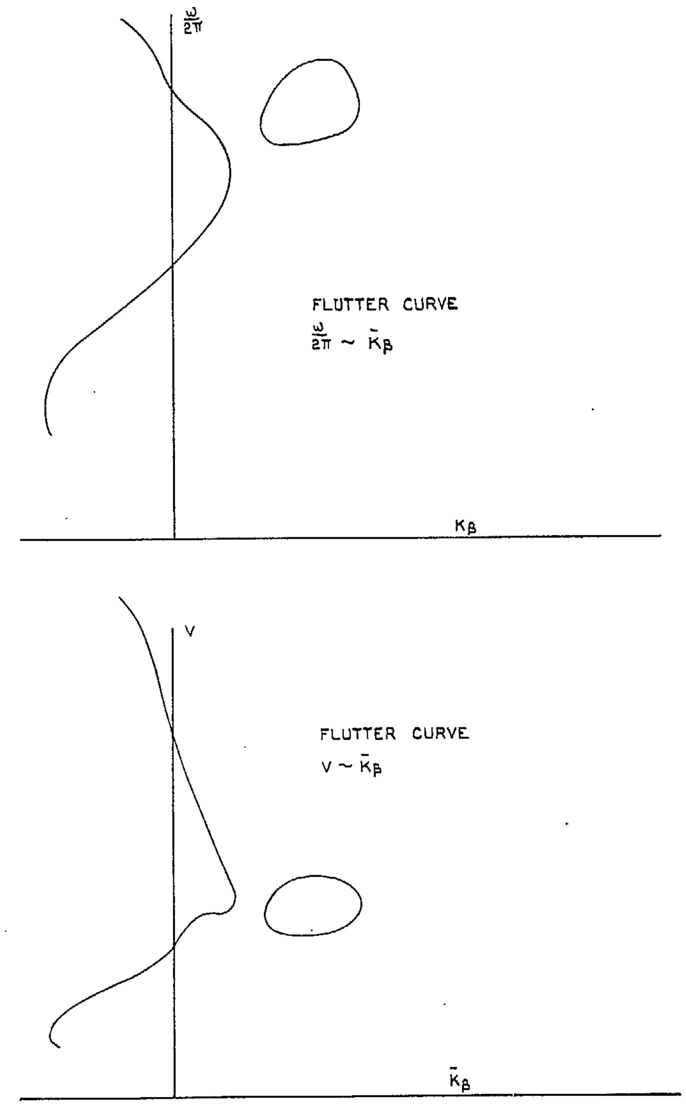

FIG. 4. Possible effect of wing structural damping on flutter curves of Fig. 3.

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