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Some Preliminary Model Experiments on the Whirling of Shafts

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COMMUNICATED BY THE PRINCIPAL DIRECTOR OF SCIENTIFIC RESEARCH (AIR),
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Summary.—Experimental work is now being done to establish a basis for the solution of whirling problems on turbine and contra-rotating shaft systems in the design stage. This report is concerned primarily with the degree of accuracy to be expected from experiments on models.

In experiments here described results are obtained for a simple cantilever system which are in close agreement with theory. With more complicated systems the error is somewhat greater owing to practical effects not covered by theory, though still acceptable for most design purposes.

1. *Introduction.*—The need for an experimental approach to the solution of whirling problems has been discussed by the author (Current Paper 55, 1950[1]).

This report deals with some preliminary experiments designed to clarify the operating characteristics of a model test rig (Current Paper 55[1]) and to confirm the fundamental theory of shaft whirling. The Jeffcott theory (1919[2]) of shaft whirling has been chosen as a basis for these experiments. This theory, an abbreviated version of which is given in the report, deals with a simple shaft rotor system and forms a basis for all theoretical investigations of shaft whirling whether on simple or complex systems.

A cantilever shaft carrying a rotor of negligible inertia has been used for these experiments. Measured values of critical whirling speeds are compared with the calculated values, and also with the free transverse frequencies. The three frequencies are shown to be in very good agreement. The effect of increasing the clearance in the main bearing block on the whirling and natural frequencies has also been investigated and confirmation is obtained that all the frequencies are reduced by equal amounts, and hence that any reasonable clearance in these bearings will not affect the comparative results in future tests. Whilst these tests were being made a transient whirling phenomena was observed which was thought to be caused by friction between the shaft and the collet by which it is attached to the main driving shaft. Some records were taken to investigate the effect of this phenomena on the whirling of the shaft and are included in the report.

The experiments on the simple cantilever system were followed by a similar series of tests to investigate the effect of a flexible outrigger bearing on the whirling and free frequencies. In these experiments the additional bearing was supported by springs of symmetrical stiffness. The results of these tests show that the free transverse frequency of vibration is then higher than the critical whirling speed which indicates that the bearing imposes a greater constraint on the system in transverse vibration than in whirling. Similar discrepancies were found between the free vibrations and critical speeds when tests were made with the outrigger bearing supported by springs of unsymmetric stiffnesses.

* R.A.E. Report Structures 82, received January 25th, 1951.

The results show that for the simple cantilever system the critical whirling speeds can be calculated accurately, but the fitting of an outrigger bearing has the effect of decreasing the accuracy of the calculations. This is an example of the practical effects referred to in the previous report (Current Paper 55[1]). The outrigger bearing used in these experiments is self-aligning and theoretically should provide no constraint on the system either in the flexure or whirl whereas in practice the effect is to increase the free vibration frequency to about 5 per cent above the critical whirling speed and decrease the accuracy of calculations by about 2 per cent. These errors are small and have little practical significance in the case of the model but can become large in full-scale installations containing shafts supported by a number of bearings.

2. *The Fundamental Theory of Shaft Rotation.*—2.1. *The Critical Whirling Speed of a Simple Shaft Rotating in Rigid Bearings.*—For simplicity in illustrating the character of forced vibrations on a rotating shaft carrying an eccentric mass, consider the case of a light shaft supported freely in bearings and carrying a mass m whose c.g. is displaced by an amount h from the centre of the shaft. It is assumed that the moment of inertia of the mass is small. In accordance with the Jeffcott theory it is also assumed that if the shaft is statically deflected due to the mass of the rotor it will initially rotate about its deflected axis and not about the axis of the bearings. Let c be the load required at the rotor to produce unit deflection of the shaft at that point.

Let us now consider the motion of a cross-section at the point of attachment of the rotor.

At this point the shaft is subject to the following forces:—

(i) a restoring force equal to $c \times$ the distance of the elastic centre of the shaft from the axis of the bearings,

(ii) a damping force and

(iii) a disturbing effect produced by an impressed rotation combined with the fact that the centre of the mass is placed eccentrically with respect to the elastic centre.

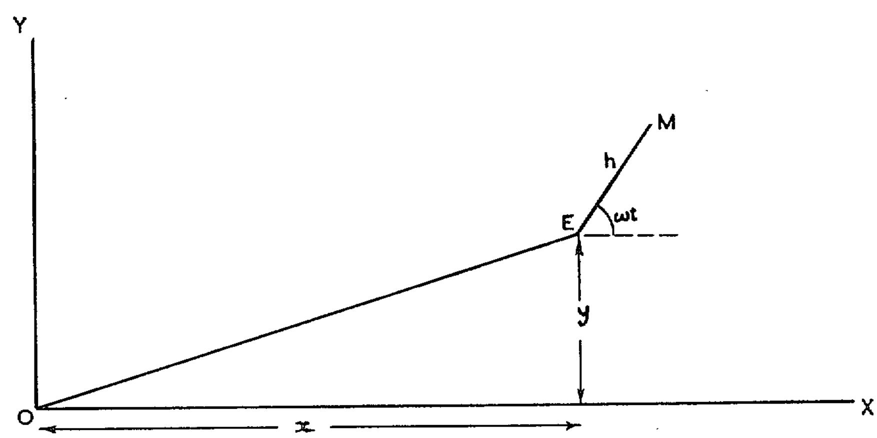


FIG. 1a.

In the figure let O be the intersection of the elastic centre of the shaft when at rest with the plane XY. Let x and y be the co-ordinates of E the elastic centre of the shaft at any instant and let M be the position of the centre-of-mass at the same instant.

It is given that the shaft rotates with angular velocity ω. This angular velocity determines the angle ωt which the line EM makes with the horizontal. Under these conditions the equation of motion of the rotor parallel to OX is:—

$$m \frac{d^2}{dt^2}(x + h \cos \omega t) + b \frac{dx}{dt} + cx = 0 . \qquad (1)$$

where b is the damping coefficient due to viscous resistance.

Therefore $m \dfrac{d^2x}{dt^2} + b \dfrac{dx}{dt} + cx = mh\omega^2 \cos \omega t$ . .. . (2)

The solution to this equation will be

$$x = A\mathrm{e}^{-bt/2m} \sin (qt + \alpha) + \frac{mh\omega^2}{\{(c - m\omega^2)^2 + b^2\omega^2\}^{1/2}} \cos (\omega t - \beta) \quad \text{.. ..} \quad (3)$$

where

$$\tan \beta = \frac{b\omega}{c - m\omega^2} \text{ and } q = \frac{(4mc - b^2)^{1/2}}{2m}$$

and corresponds to the free period of vibration of the shaft and A and α are arbitrary constants. Similarly the equation to motion parallel to OY is

$$\frac{md^2y}{dt^2} + b\frac{dy}{dt} + cy = mh\omega^2 \sin \omega t \,. \quad \text{.. .. ..} \quad (4)$$

and the solution is

$$y = A'\mathrm{e}^{-bt/2m} \sin (qt + \alpha') + \frac{mh\omega^2}{\{(c - m\omega^2)^2 + b^2\omega^2\}^{1/2}} \sin (\omega t - \beta) \,. \quad \text{.. ..} \quad (5)$$

The first term in the solution represents an oscillatory motion of amplitude

$$A\ \mathrm{e}^{-bt/2m}$$

or .. .. (6)

$$A'\ \mathrm{e}^{-bt/2m} \,.$$

This amplitude diminishes with increase in t so that the term becomes negligible. The second term persists and is a vibration of amplitude

$$\frac{mh\omega^2}{\{(c - m\omega^2)^2 + b^2\omega^2\}^{1/2}} \,.$$

From (3) and (5) neglecting the exponential term we have

$$x^2 + y^2 = \left[\frac{mh\omega^2}{\{(c - m\omega^2)^2 + b^2\omega^2\}^{1/2}}\right]^2 \left\{\cos^2 (\omega t - \beta) + \sin^2 (\omega t - \beta)\right\}$$

$$= \left[\frac{mh\omega^2}{\{(c - m\omega^2)^2 + b^2\omega^2\}^{1/2}}\right]^2 = \text{constant} \,. \quad \text{.. .. .. ..} \quad (7)$$

The motion is therefore circular round the axis through O of amplitude

$$\frac{mh\omega^2}{\{(c - m\omega^2)^2 + b^2\omega^2\}^{1/2}}$$

this amplitude is a maximum when

$$\omega = \frac{2c}{\{4mc - 2b^2\}^{1/2}}$$

or when damping is neglected

$$\omega = \sqrt{(c/m)}$$

which is the same frequency as the natural transverse frequency of the shaft. Therefore when the speed of rotation equals the natural frequency of the shaft in transverse vibration the shaft whirls at its maximum amplitude.

The phase of the displacement of the mass-centre relative to that of the elastic centre is given by β where

$$\tan \beta = \frac{b\omega}{c - m\omega^2}$$

when $\omega = 0$ $\quad\beta = 0$

when $\omega = \sqrt{(c/m)}$ $\quad\beta = \pi/2$

when $\omega \longrightarrow \infty$ $\quad\beta \longrightarrow \pi$

i.e., as shaft passes through its critical speed the centre of gravity of the rotor moves from a position remote from the centre-line of the bearings relative to the elastic centre, to a position between the elastic centre and bearing centre. As the damping coefficient b is usually very small it is obvious that this change of phase takes place almost entirely between values of ω slightly below and slightly above the critical speed. It is also seen that when ω becomes large relative to the critical speed the amplitude of whirl approaches a value h, *i.e.*, the c.g. of the rotor tends to rotate on the static elastic centre of the shaft. In most practical cases the damping factor is small and may be neglected.

2.2. *The Critical Speeds of a Shaft Rotating in Bearings having Unsymmetric Flexibilities.*—Consider a rotating shaft supported by rigid bearings at one end and at the other end by a bearing held by two pairs of springs of different stiffness in the horizontal and vertical directions. Let a rotor of mass m and eccentricity h be attached to the end of the shaft nearest the flexible support.

Let the horizontal stiffness of the shaft at the rotor be c_1 lb/in. and the vertical stiffness be c_2.

As in Fig. 1a (2.1) let O be the static elastic centre of the shaft, E be the displaced position of the elastic centre at time t and M the centre of gravity of the rotor. Then as in 2.1 the equations to the motion in the X and Y direction will be

$$m\ddot{x} + b_1\dot{x} + c_1x = mh\omega^2 \cos \omega t$$

$$\text{and } m\ddot{y} + b_1\dot{y} + c_2y = mh\omega^2 \sin \omega t$$

where b_1 and b_2 are the damping coefficients appropriate to the two axes.

The particular solutions to these equations are as before

$$x = \frac{mh\omega^2}{\{(c_1 - m\omega^2)^2 + b_1{}^2\omega^2\}^{1/2}} \cos (\omega t - \beta_1)\left(\tan \beta_1 = \frac{b_1\omega}{c_1 - m\omega^2}\right)$$

$$y = \frac{mh\omega^2}{\{(c_2 - m\omega^2)^2 + b_2{}^2\omega^2\}^{1/2}} \sin (\omega t - \beta_2)\left(\tan \beta_2 = \frac{b_2\omega}{c_2 - m\omega^2}\right)$$

or for any value of ω

$$x = A \cos (\omega t - \beta_1) = A \cos \beta_1, \cos \omega t + A \sin \beta_1 \sin \omega t$$

$$= A_1 \cos \omega t + A_2 \sin \omega t \qquad \ldots \quad (1)$$

similarly

$$y = B \sin (\omega t - \beta_2)$$

$$= B_1 \sin \omega t - B_2 \cos \omega t . \qquad \ldots \quad (2)$$

Eliminating ωt from (1) and (2) and substituting

$$A_1 = A \cos \beta_1,\ A_2 = A \sin \beta_1,\ B_1 = B \cos \beta_2,\ B_2 = B \sin \beta_2$$

we have

$$B^2x^2 + A^2y^2 - 2xyAB \sin (\beta_1 - \beta_2) = A^2B^2 \cos (\beta_1 - \beta_2)$$

which is an ellipse.

Similarly if damping is neglected we have

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 .$$

Therefore for the case of a shaft with unsymmetric bearing there will be two critical speeds given by $\omega = \sqrt{\frac{c_1}{m}}$ and $\omega = \sqrt{\frac{c_2}{m}}$ and the elastic centre of the shaft will describe an ellipse.

If $c_1 < c_2$ and $c_1 < m\omega^2$, *i.e.*, at speeds below the first critical both x and y are positive, *i.e.*, the shaft will whirl in the same direction as the rotation with the c.g. of the rotor on the outside (the position of the c.g. being proved in the same way as in 2.1). When $c_1 < m\omega^2 < c_2$, *i.e.*, between the two criticals, x will be negative and y positive, hence the shaft will whirl in the reverse direction to its rotation. When $m\omega^2 > c_2$, *i.e.*, above the second critical, both x and y are negative and the shaft will again whirl in the direction of rotation, but with the c.g. of the rotor between the elastic centre and the bearing centre.

Between the two criticals the motion is quite stable.

3.0. *The Model Shaft System.*—Photographs of the rig are shown in Fig. 13. The basic system consists of a cantilever shaft 12 in. long made from $\frac{1}{4}$ in. diameter silver steel rod. The shafts used were selected from standard rods, with particular care as to their straightness. The test piece is fitted by means of a split collet into the end of a 1 in. diameter driving shaft. This driving shaft is carried by two taper roller bearings set 6 in. apart in the main bearing block. By making the clearances in the taper roller bearings as small as practicable a condition was reached whereby it was possible to assume the end conditions for the cantilever to be perfectly encastre.

Provision is made to support the free end of the shaft by means of a flexible outrigger bearing 10 in. from the face of the collet. The outrigger bearing consists of a $\frac{1}{4}$ in. self-aligning bearing supported by four springs disposed at 90 deg about the axis of the shaft. The springs could be arranged to provide either equal or unequal stiffnesses in the two principal planes.

A Schrage motor is used to drive the test rig through a ten to one ratio step-up gear box giving a speed range of 0 to 20,000 r.p.m. if desired. A fine speed control is obtained by means of a friction brake fitted to the main bearing block between the two taper roller bearings. By setting the motor speed control to give a shaft speed slightly higher than the critical speed being investigated and then controlling the speed by means of the brake it is possible to explore whirl amplitudes and patterns at shaft speeds very close to the critical, where the whirl amplitude is very sensitive to shaft speed.

The accuracy of this method is evidenced by the small degree of scatter of the experimental points plotted near the critical speeds.

Preliminary investigations deal with the effects of unbalance of a rotor of small moments of inertia. The rotor used is made from aluminium, it is 2 in. diameter and composed of two eccentric rings, one inside the other so adjusted that by rotating the outer ring the c.g. of the complete rotor can be offset from the centre of the shaft, on which the rotor is fitted by as much as 0·2 in. The actual position of the c.g. can readily be fixed by means of a graduated scale engraved on the inner ring and corresponding to a datum line on the outer ring.

4.0. *Recording Equipment.*—The whirl paths at the free end of the shaft are recorded on film. This is done by means of a standard F.24 aerial camera carrying a K.24 lens on the end of an extension tube. This arrangement gives an optical enlargement of 5·6 times. The whirl amplitudes at the end of the shaft are kept to within approximately $\pm \frac{1}{4}$ in. by means of a guard ring behind the outrigger bearing, recorded amplitudes are therefore of the order of 1·25 in. maximum. For moderate amounts of rotor unbalance, at $\frac{1}{4}$ in. whirl amplitude, the slopes of the resonance curve above and below the critical speed are practically the same. It is therefore considered reasonable to assume that the critical speed lies midway between the values given by the two curves at this amplitude.

The recording technique found to be most suitable for the optical system described above was one whereby the rig was normally operated in subdued lighting and a small highly polished 'pip' on the centre of the shaft at the free end was intensely illuminated for a controlled period of time with an open shutter on the camera.

For this purpose the normal 'roller blind' shutter in the camera was removed and replaced by a solenoid operated shutter of the 'Venetian blind' type.

The light source is controlled by a relay system operated by sliding contacts on the motor shaft in such a way that, independent of shaft speed, the end of the shaft can be illuminated for any desired number of shaft revolutions up to five by operating the recording switch.

To facilitate the operation of the rig, the camera is also fitted with an automatic film winding mechanism operated from the control panel. To compare the shaft deflection forms for the static, transverse vibration and whirling cases a second light source has been designed to illuminate the whole length of the shaft along the top edge. Using a half plate camera a photograph is taken timed to include at least one vibration cycle. An enlargement four times shaft size is then obtained by projecting the plate negative on to a white screen, amplitudes at different points along the shaft being measured direct and plotted on an enlarged scale. The results thus obtained are found to be very satisfactory.

When plotting amplitude frequency curves it is essential to record the speed of the shaft simultaneously with the recording of the whirl amplitude. This is done by means of an inductance proximity pickup, placed behind the main bearing block near a clip which rotates with the shaft. The pickup is connected to a single channel cathode-ray tube recording trolley, speed records being taken at the same time as the whirl records. The shaft speed can then be measured against a fifty cycle per second timing mark.

5.0. *Experimental Results*.—Experiments have been carried out primarily to establish the operating characteristics of the test rig and by investigating the elementary whirling phenomena to provide a basis for the investigation of more complex whirling problems.

In the correlation between experiment and theory the effects of damping have been ignored as they have little effect on the critical whirling speeds. The main effect of damping is on the whirl amplitudes and phase relationships between the rotor c.g. and the shaft centre in the neighbourhood of the critical whirling speeds.

5.1. *The Simple Cantiliver Shaft* (*unsupported*).—Measurements were first taken to investigate the effect of rotor unbalance for the simple cantilever shaft. Amplitude-frequency curves have been plotted for various values of rotor unbalance (Fig. 1). It is seen that the unbalance of the rotor has no effect on the critical speed but increases the amplitude of the whirl at any particular shaft speed. The rotor eccentricities quoted in Fig. 1 must be modified to include the effective weight of the shaft. This may be shown to be equal to a mass of approximately $33/140 \times$ the weight of the shaft which may be concentrated at the rotor end of the shaft. For the system under consideration the weight of the rotor was $0 \cdot 095$ lb and the effective weight of the shaft $0 \cdot 040$ lb. The c.g. of the equivalent rotating mass is therefore obtained by reducing the rotor eccentricity by a factor $0 \cdot 095/0 \cdot 135 = 0 \cdot 7$ approx. It is seen from Fig. 1 that the whirl amplitude approaches the value of the rotor eccentricity as the speed becomes considerably greater than the critical speed.

The deflection forms for the static lateral vibration of the non-rotating shaft and for the whirling shaft were recorded and are shown compared with the corresponding calculated static deflection form in Fig. 2. An example of the recorded deflection form is shown in Fig. 9. The dynamic deflection forms are found to agree very well with the static form which confirms the theory and demonstrates the accuracy of the recording technique adopted.

Experiments were next carried out to compare the measured critical whirling speed with the measured natural frequency and the value of the frequency obtained by theoretical calculations based on the measured stiffness of the system.

Fig. 3 shows the amplitude-frequency curve for the cantilever shaft with minimum clearance in the main bearings for normal operating conditions and Fig. 4 is a similar curve with the axial clearance of the bearings increased by 0·010 in. In both cases the three frequencies are very nearly equal but increasing the bearing clearances is seen to reduce the frequencies by about 4 per cent. These experiments are not intended to be part of a general investigation on the effects of bearing clearances on the critical whirling speeds. They are part of the investigations designed to elucidate the operational characteristics of the test rig.

Whilst carrying out the above tests it was observed that when the shaft was running at speeds higher than the critical speed, any slight disturbance of the shaft gave rise to a transient vibration of increasing magnitude. The magnitude of vibration was seen to increase slowly at first until an amplitude of about 0·15 in. was reached after which the amplitude increased rapidly until the guard ring was struck. It was noticed that the initial instability was also a function of the initial amplitude caused by the disturbing force, except at a shaft speed of twice the critical speed, when the shaft was entirely unstable apparently under the action of internal disturbing forces arising from the driving mechanisms and bearings.

For shaft speeds below the critical speed the shaft was found to be quite stable although if subjected to a large disturbing force the subsequent transient whirl decays rapidly until an amplitude is reached slightly larger than the unbalance whirl amplitude after which the rate of decay is appreciably reduced resulting in a transient whirl of low decrement. Under these conditions the shaft was examined under stroboscopic light and it seemed that the whirl consisted of a circular whirl at shaft speed combining with a stationary elliptic whirl at the natural frequency of the shaft. As an illustration of this phenomena records of the whirl paths were taken at 5-second intervals from the time of application of a disturbing force at shaft speeds above and below the critical speed. The results are shown in Figs. 10 and 11. The transient whirls mentioned above are probably caused by forces arising from the internal hysteresis of the shaft and to a larger degree by forces arising from friction in the split collet in which the shaft is gripped.

These forces although small may be sufficient to force a whirl when the damping in the system is very small which was the case for the simple cantilever shaft system. It was noticeable that when the outrigger bearing was used, the damping in the bearing and spring support was sufficient to overcome the instability previously noted. Accounts of the effects of friction on the whirling of shafts have been given by Robertson[3] and Kimball[4]. It is not possible to probe this aspect further without devoting considerable time to carrying out extensive experiments which is not possible at present owing to the needs of the more primary objects of the investigation.

5.2. *The Cantilever Shaft Supported by a Flexible Outrigger Bearing of Symmetrical Stiffness.*— The outrigger bearing was situated at a distance of 10 in. from the fixed end of the shaft and served to alter appreciably the dynamic characteristics of the system. It was first necessary to recalculate the equivalent weight of the shaft using measured stiffnesses of shaft alone and shaft plus bearing. The method used is shown in Appendix I for particular values of these measured stiffnesses.

Tests were then carried out to compare measured critical whirling speeds and natural frequencies with calculated natural frequencies for the minimum practicable main bearing clearances and for increased clearances of 0·005 and 0·010 in. The amplitude frequency curves are shown in Figs. 5, 6 and 7.

It is seen from these results that the increased main bearing clearance has little effect on the critical speed and natural frequency, but with the outrigger bearing fitted there is a greater difference between the three frequencies under consideration, the greatest discrepancy being between the measured critical speed and the measured natural frequency, which was of the order of 4 per cent to 7 per cent. The calculated natural frequency was in all cases between the other two frequencies.

It is considered that this discrepancy is due to the functioning of the outrigger bearing, which may impose a greater constraint on the shaft when it is vibrating laterally than when it is whirling.

A series of model and full-scale experiments are to be carried out, which it is hoped will solve this problem and more practical problems related to the constraints in bearings.

5.3. *The Cantilever Shaft Supported by a Flexible Outrigger Bearing of Unsymmetric Stiffness.*—The flexible outrigger bearing was fitted with springs which were stiffer in the vertical plane than in the horizontal plane and records taken over a range of shaft speeds. The results are plotted in Fig. 8, typical records covering the speed range are shown in Fig. 12.

The resultant whirl is seen to be elliptical with two critical whirling speeds corresponding to the horizontal and vertical natural frequencies of the system. From a comparison of the measured critical speeds and natural frequencies and the calculated natural frequencies it is seen that as in the case of the symmetrically stiff outrigger bearing there is an appreciable difference (about 5 per cent) between the natural frequencies measured and calculated and the measured critical speeds which cannot be adequately explained.

With regard to the form of the whirl it is possible by using stroboscopic light to confirm that at speeds below the first critical and above the second critical the whirl is forwards at shaft speed, but between the two criticals, although the shaft is stable, the whirl is in the reverse direction, again at shaft frequency.

6. *Conclusions.*—The initial experiments using a small model rig for the investigation of shaft whirling problems have shown that the technique developed is very suitable for studying the fundamentals of the problem and obtaining accurate measurements of frequencies of vibration and critical whirling speeds, together with corresponding whirl forms and shaft deflection forms.

The results show that the Jeffcott theory for a simple cantilever system is basically correct and gives results in close agreement with experiment. Agreement between theory and experiment is only approximate, however, for the cantilever shaft with an outrigger flexible bearing.

An interesting transient whirl was encountered and is associated with friction effects.

APPENDIX I

The Calculation of the Equivalent Weight of a Cantilever Shaft fitted with a Flexible Outrigger Bearing

Available Data

Length of shaft e	$= 12$ in.
Weight of shaft	$= 0 \cdot 1671$ lb
Distance of outrigger bearing from fixed end a	$= 10$ in.
Measured stiffness at rotor	$= 31 \cdot 4$ lb/in.
Measured stiffness of shaft alone	$= 9 \cdot 5$ lb/in.
Elastic modulus for shaft material	$= 28 \cdot 4 \times 10^6$ lb/sq in.

1. *To find the Spring Stiffness of the Outrigger Bearing.*—Assuming a load of $31 \cdot 4$ lb to be placed at the rotor with the outrigger bearing removed.

Deflection under load,

$$y_c = \frac{31 \cdot 4 \times 1728}{3 \times 28 \cdot 4 \times \frac{\pi}{64} \times \frac{1}{256} \times 10^6}$$
$$= 3 \cdot 32 \text{ in.}$$

Therefore, the effect of the outrigger bearing is to reduce this deflection by 2·32 in. *i.e.*, if P is the force exerted by the outrigger bearing producing a deflection of 2·32 in. at the rotor, then

$$2{\cdot}32 = \frac{P \times 1728}{6 \times 28{\cdot}4 \times 10^6 \times \frac{\pi}{64} \times \frac{1}{256}} \frac{100}{144}(3 - 0{\cdot}8333)$$

or $P = 29{\cdot}2$ lb.

The deflection at the outrigger bearing for unit deflection at the rotor is given by

$$EI\frac{d^2y}{dx^2} = W(l - x) - P(a - x) \text{ from } x = 0 \text{ to } x = a\,.$$

Therefore

$$EIy = W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - P\left(\frac{ax^2}{2} - \frac{x^3}{6}\right).$$

For unit deflection at the rotor $W = 31{\cdot}4$ lb.

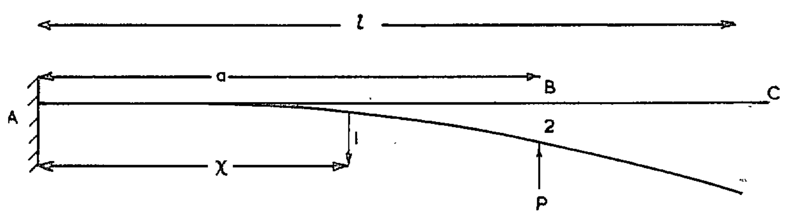
Therefore deflection at bearing ($x = a = 10$ in.)

$$= \frac{1}{EI}\, 31{\cdot}4(600 - 167) - 29{\cdot}2(500 - 167)$$

$$= 0{\cdot}710 \text{ in.}$$

Therefore spring stiffness $= \dfrac{29{\cdot}200}{0{\cdot}710} = 41{\cdot}1$ lb/in.

2. *To Find the Equivalent Weight of the Shaft.*—



Consider a uniformly loaded cantilever A, B, C with a flexible prop at B exerting an upward force P. Let ω = the load per unit run along the beam.

Let Y_{11} be deflection due to unit load at any point 1 on the beam
Y_{22} be deflection due to unit load at B
y_{11} be deflection due to unit load at any point 1 with springs removed
y_{22} be deflection due to unit load at B with springs removed
y_{12} be deflection at point 1 due to unit load at point 2

By the reciprocal theorem—

Let y_{21} be deflection at point 2 due to unit load at $1 = y_{12}$
S be stiffness of springs at B.

Then

$$Y_{11} = y_{11} - SY_{22}\,y_{12} \qquad (1)$$

and $$Y_{22} = y_{12} - SY_{22}\,y_{22}\,. \qquad (2)$$

From (2)

$$Y_{22} = \frac{y_{12}}{1 + Sy_{22}}$$

Therefore substituting in (1) for Y_{22}

$$Y_{11} = \frac{y_{11} + S(y_{22}y_{11} - y_{12}^2)}{1 + Sy_{22}}.$$

Using the Dunkerley theory for the natural frequency of a beam with a number of loads which in this case is applied to a continuously loaded beam we have

$$\frac{1}{\omega^2} = \int_0^l \frac{\omega}{g} Y_{11}\, dx.$$

Between A and B

$$y_{11} = \frac{x^3}{3EI}, \quad y_{22} = \frac{a^3}{3EI}, \quad y_{12} = \frac{x^2\,(3a - x)}{6EI},$$

these being standard deflection formulae for loaded beams.

$$\text{Therefore } Y_{11} = \frac{\dfrac{x^3}{3EI} + S\left(\dfrac{x^3a^3}{9(EI)^2} - \dfrac{x^4(3a - x)^2}{36(EI)^2}\right)}{1 + \dfrac{Sa^3}{3EI}},$$

from which we have

$$\int_0^l \frac{\omega}{g} Y_{11} = 0 \cdot 053w.$$

Between B and C

$$y_{11} = \frac{x^3}{3EI}, \quad y_{22} = \frac{a^3}{3EI}, \quad y_{12} = \frac{a^2(3x - a)}{6EI}$$

and

$$Y_{11} = \frac{\dfrac{x^3}{3EI} + S\left(\dfrac{a^3x^3}{9(EI)^2}\,\dfrac{a^4\,(3xa)^2}{36(EI)^2}\right)}{1 + \dfrac{Sa^3}{3EI}}$$

from which we have

$$\int_0^l \frac{w}{g} Y_{11} = 0 \cdot 054w.$$

$$\text{Therefore } \frac{1}{\omega^2} = \int_0^a \frac{w}{g} Y_{11}\, dx + \int_a^l \frac{w}{g} Y_{11}\, dx$$
$$= 0 \cdot 107w.$$

If W = equivalent weight of shaft

then

$$\frac{1}{\omega^2} = \frac{W}{S_c}$$

where S_c = stiffness of beam at c.

$$\text{Therefore} \quad 0 \cdot 107w = \frac{W}{S_c} = \frac{W}{31 \cdot 4}.$$

$$\text{Therefore} \quad W = 3 \cdot 36w$$
$$= 0 \cdot 038 \text{ lb},$$

and by adding $0 \cdot 038$ lb to the rotor allowance will be made for the weight of the shaft.

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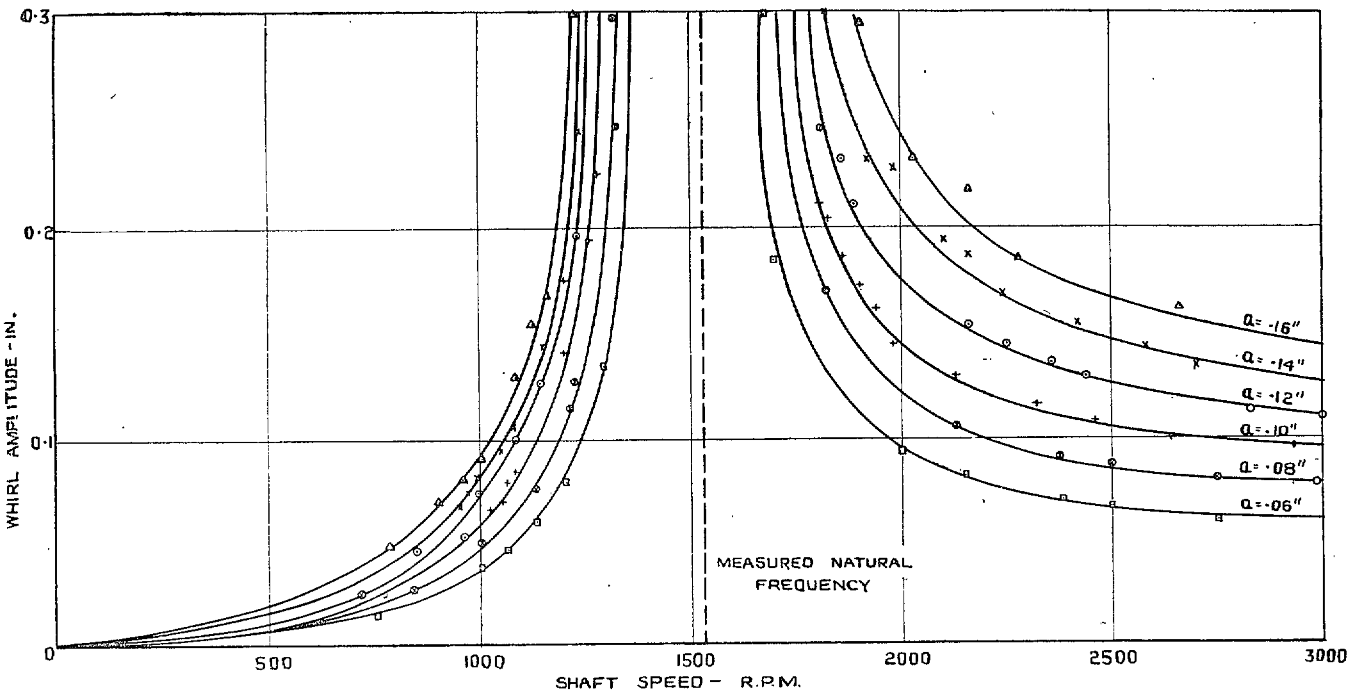


FIG. 1. Experimental whirl. Amplitude-frequency curves for various rotor eccentricities. Simple cantilever shaft.

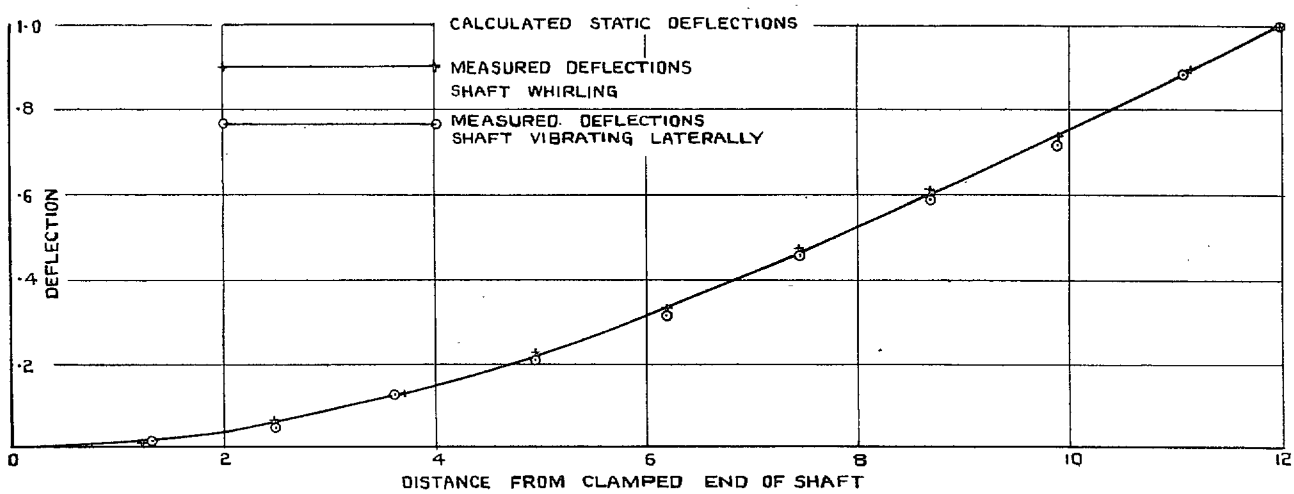


FIG. 2. Comparison between calculated static deflection form and measured whirling and vibrating deflection forms for simple cantilever shaft.

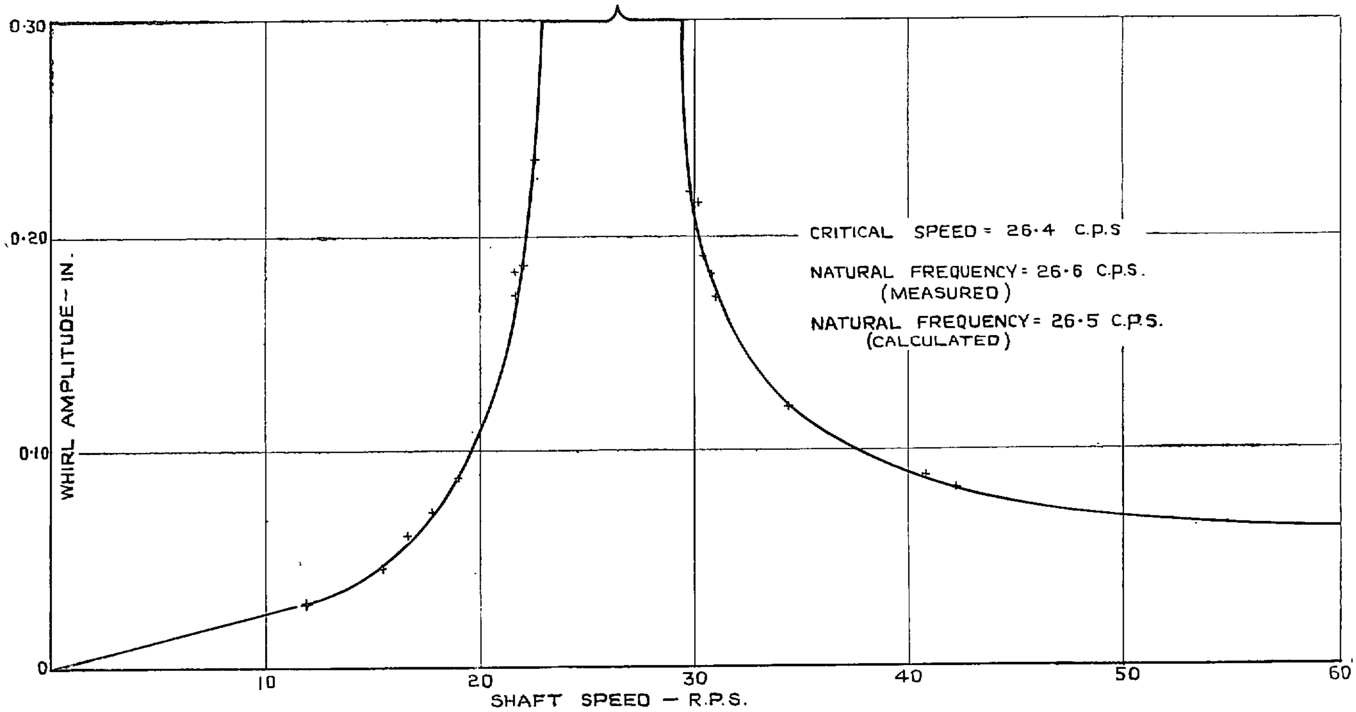


FIG. 3. Whirl amplitude-frequency curve for cantilever shaft with minimum practical bearing clearances.

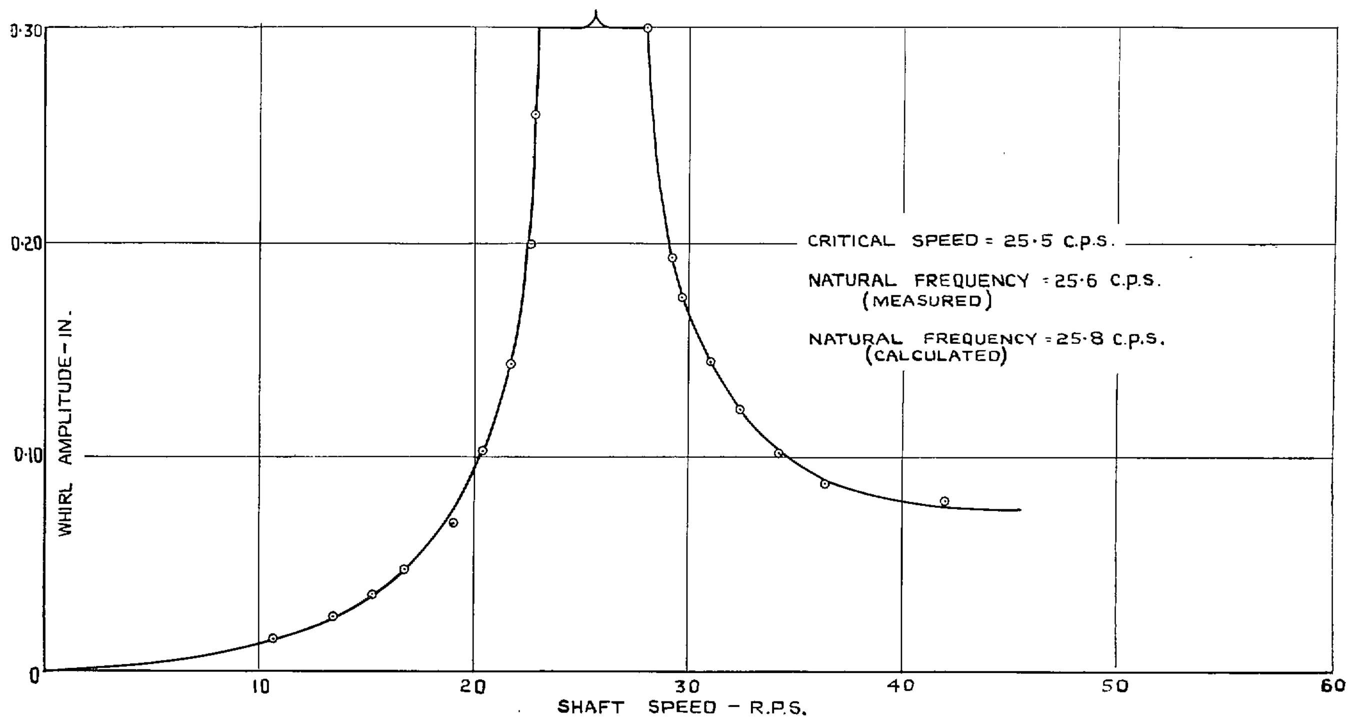


FIG. 4. Whirl amplitude-frequency curve for cantilever shaft with 0·010 in. bearing clearance.

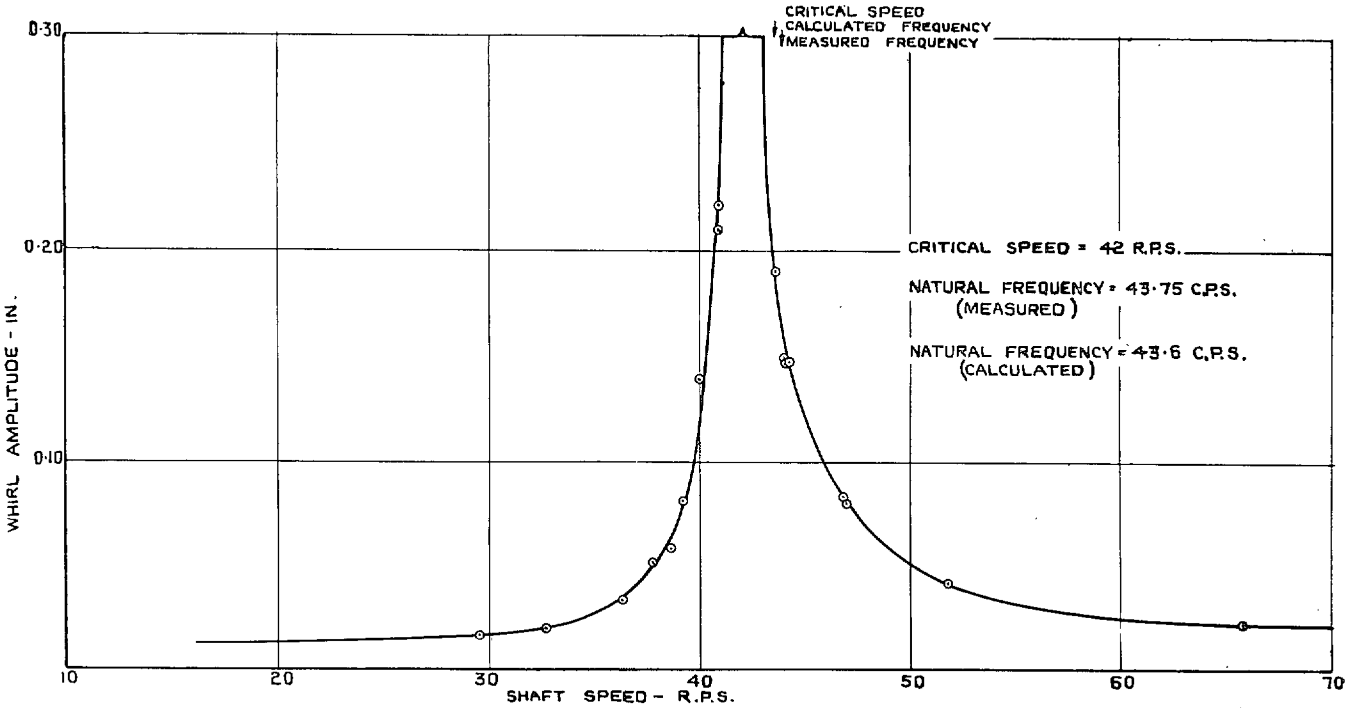


FIG. 5. Whirl amplitude-frequency curve for cantilever shaft supported by flexible self-aligning outrigger bearing minimum main bearing clearance.

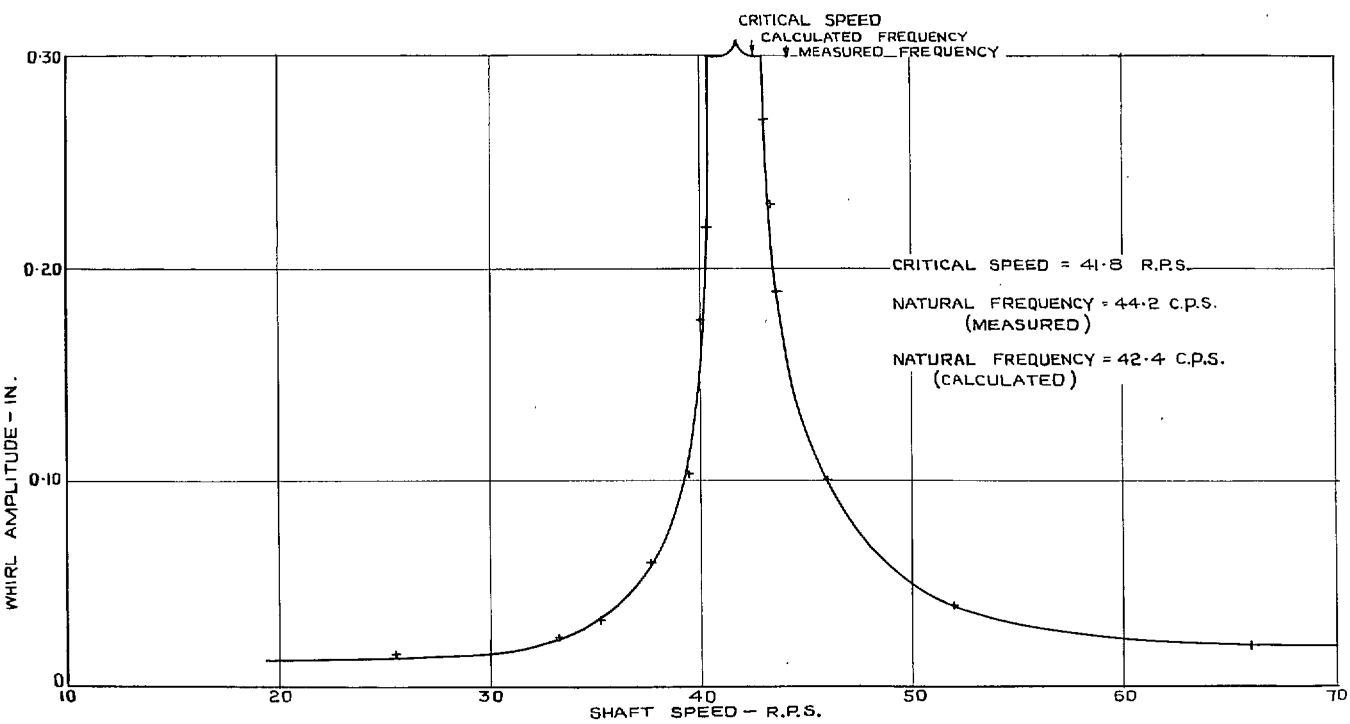


FIG. 6. Whirl amplitude-frequency curve for cantilever shaft supported by stiff flexible self-aligning outrigger bearing. Main bearing clearance 0·005 in.

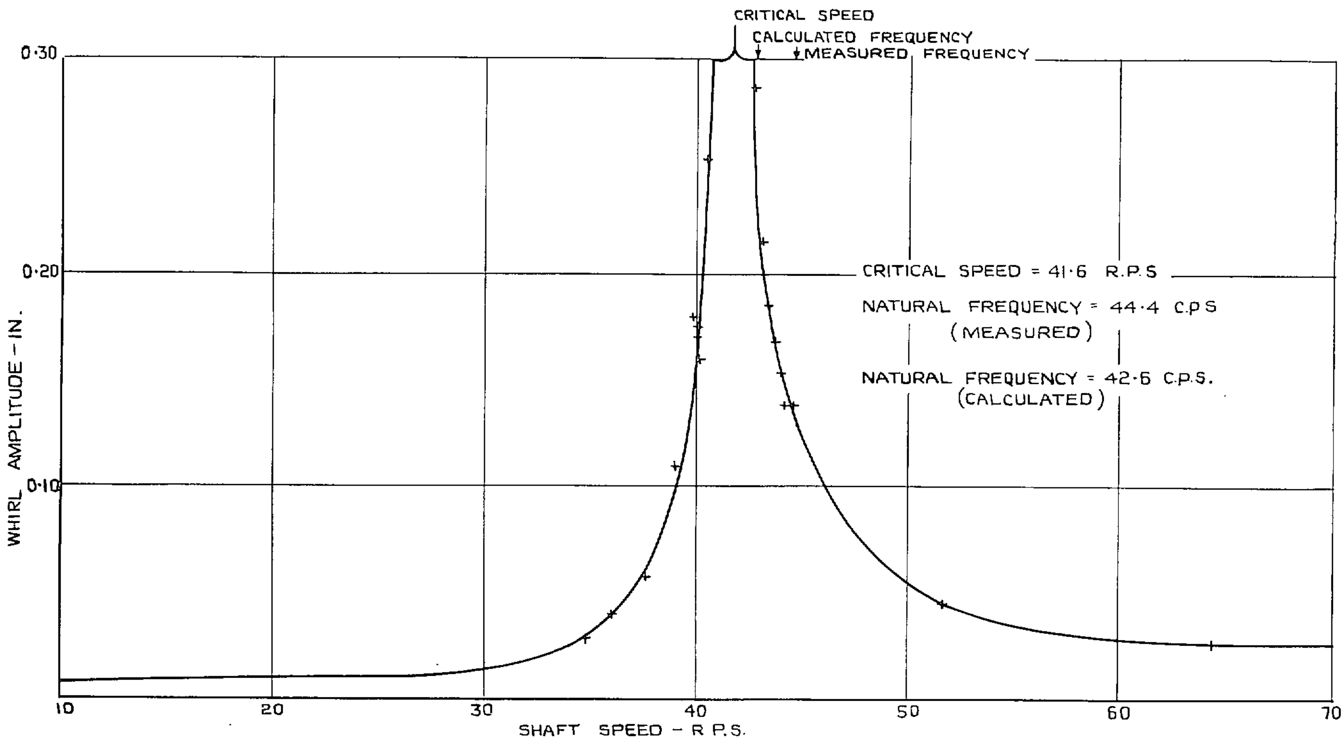


FIG. 7. Whirl amplitude-frequency curve for cantilever shaft supported by flexible self-aligning outrigger bearing. Main bearing clearance 0·010 in.

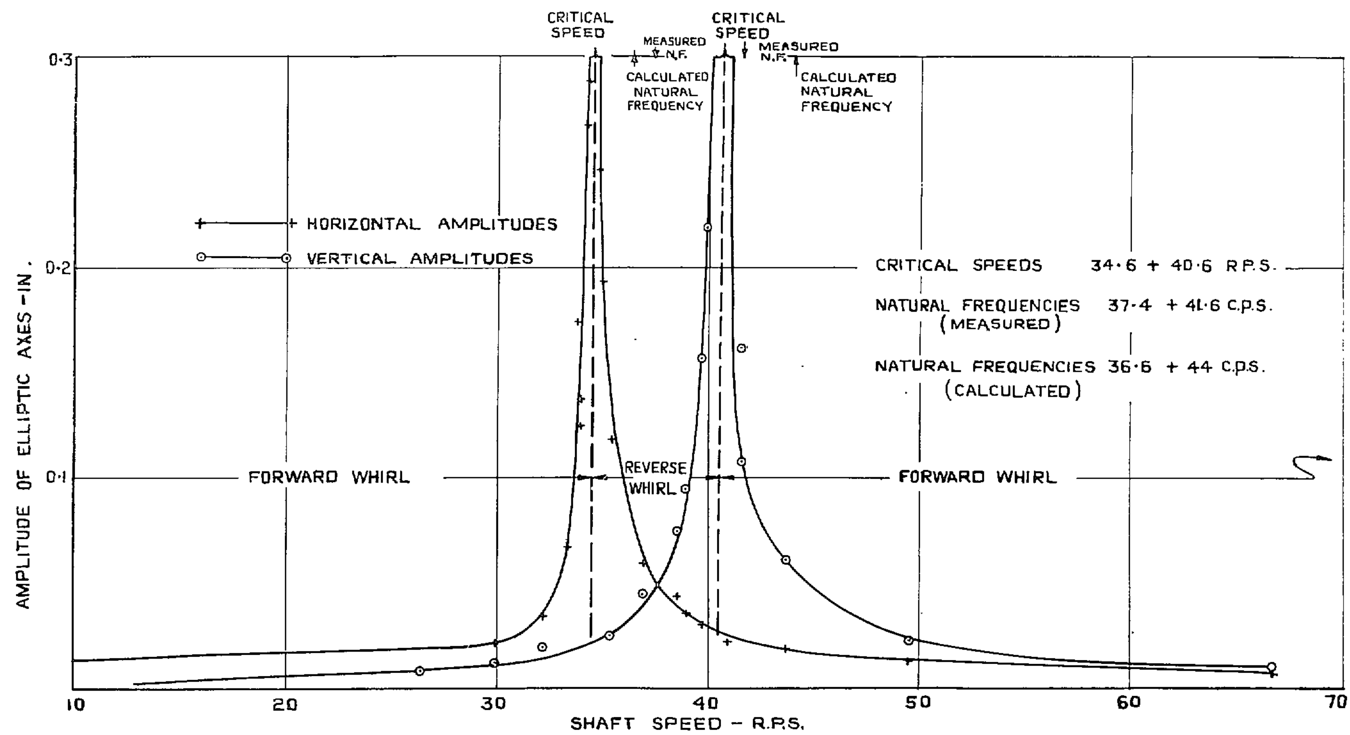


FIG. 8. The critical speeds of a cantilever shaft supported by a flexible outrigger bearing of unsymmetric stiffness.

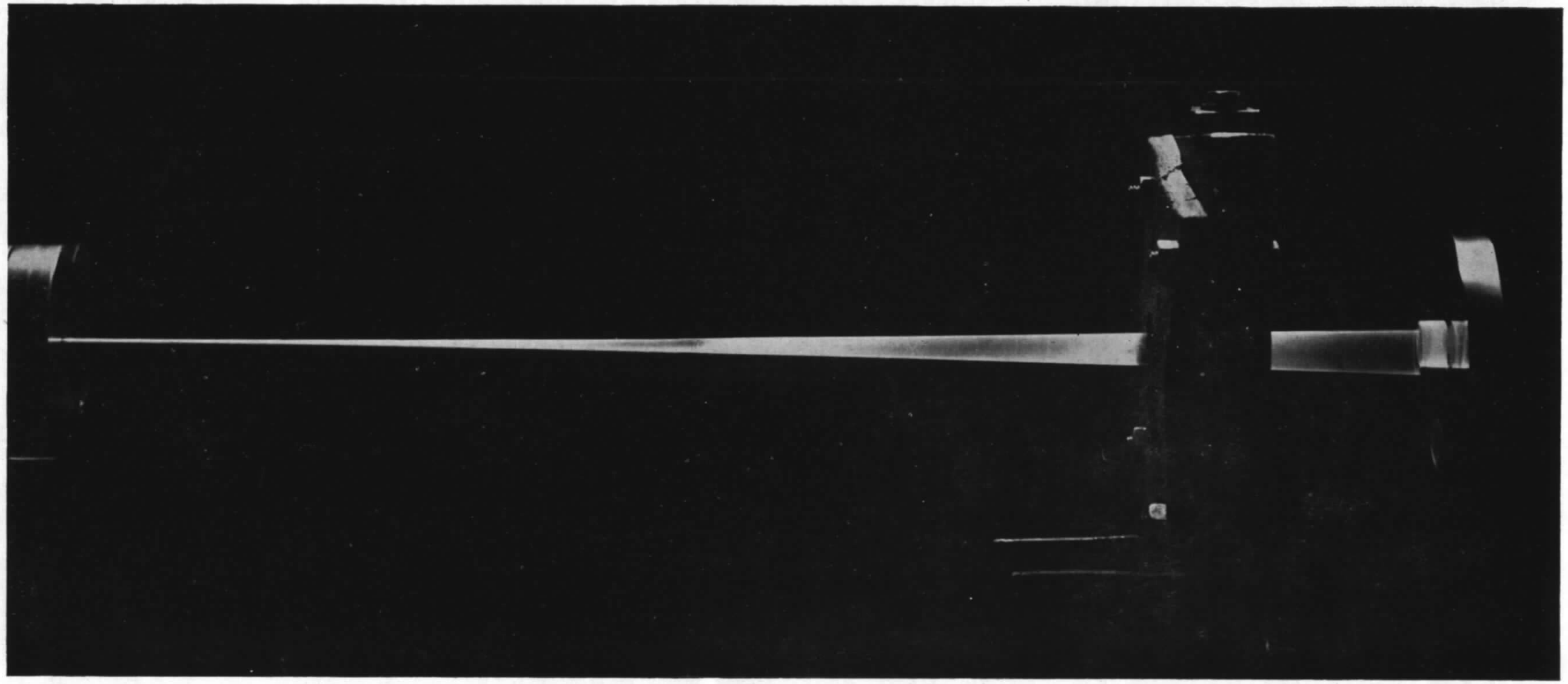
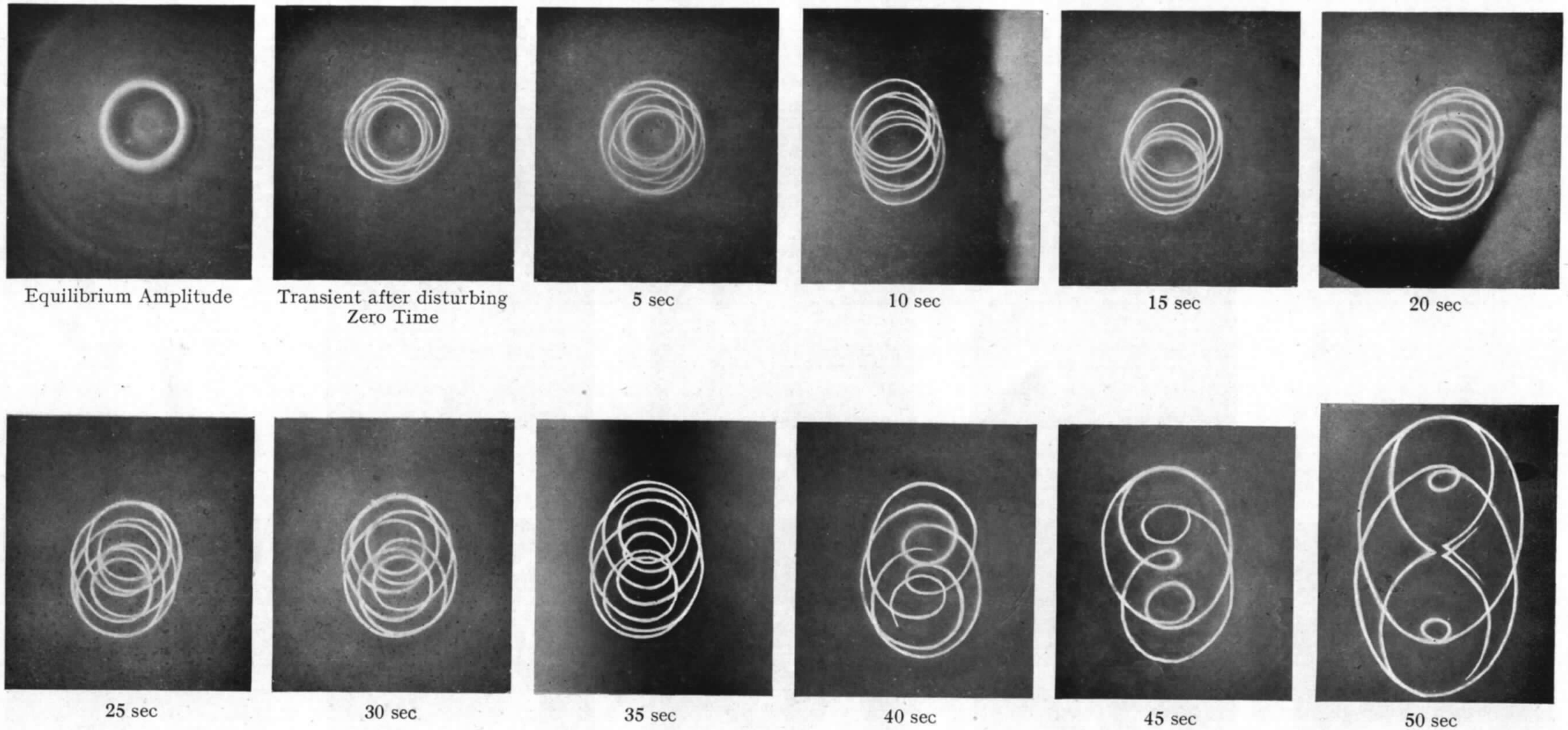
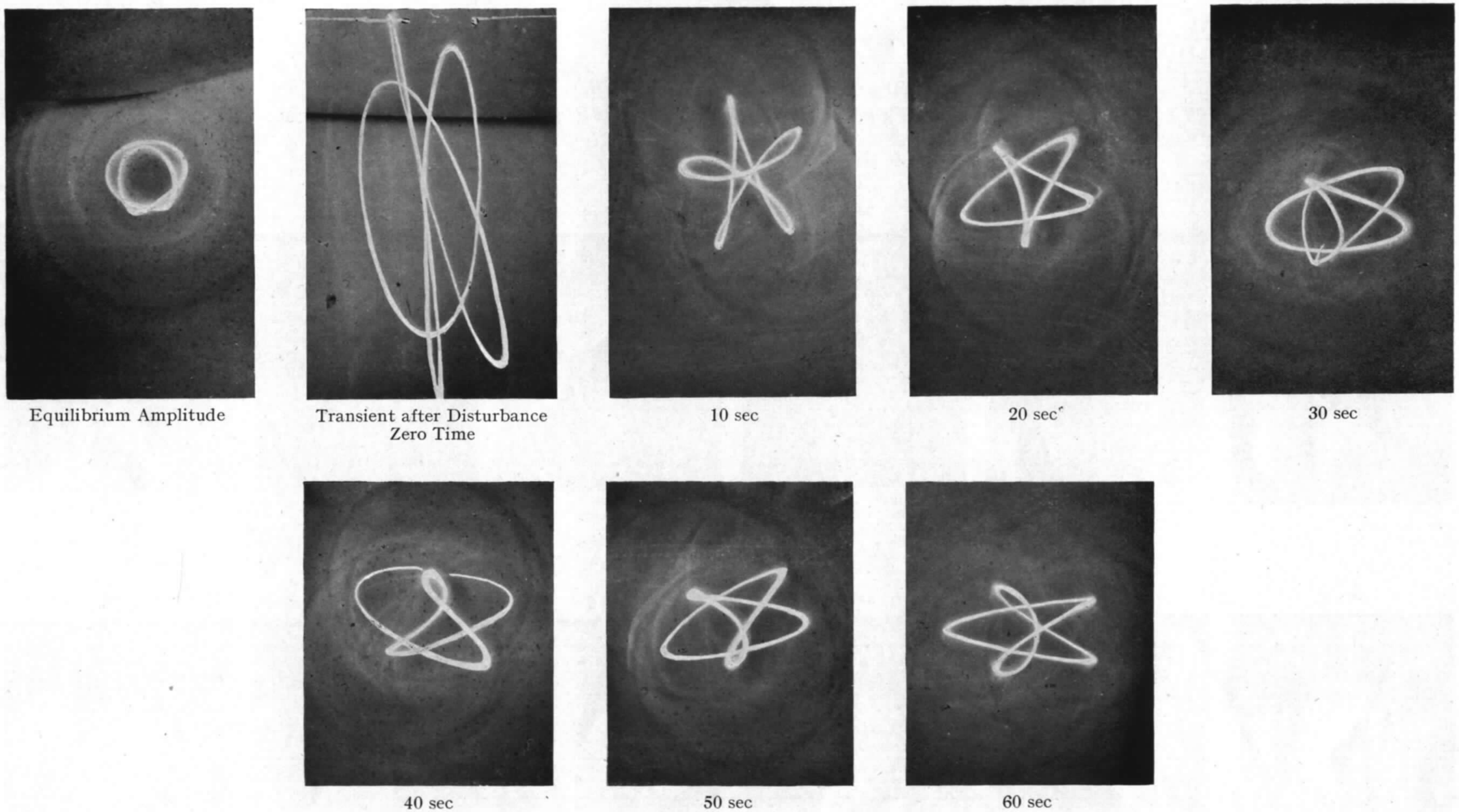


FIG. 9. Preliminary experiments on the whirling of shafts.



Critical speed 26·1 r.p.s. Shaft speed 58·6 r.p.s.

FIG. 10. Records at 5 sec intervals of transient whirl producing instability of a cantilever shaft carrying an unbalanced rotor, and rotating at a speed above its critical speed.



Critical speed 26·1 r.p.s. Shaft speed 14 r.p.s.

FIG. 11. Records at 10 sec intervals showing the decay of a transient whirl of a cantilever shaft carrying an unbalanced rotor, and rotating at a speed below its critical speed.

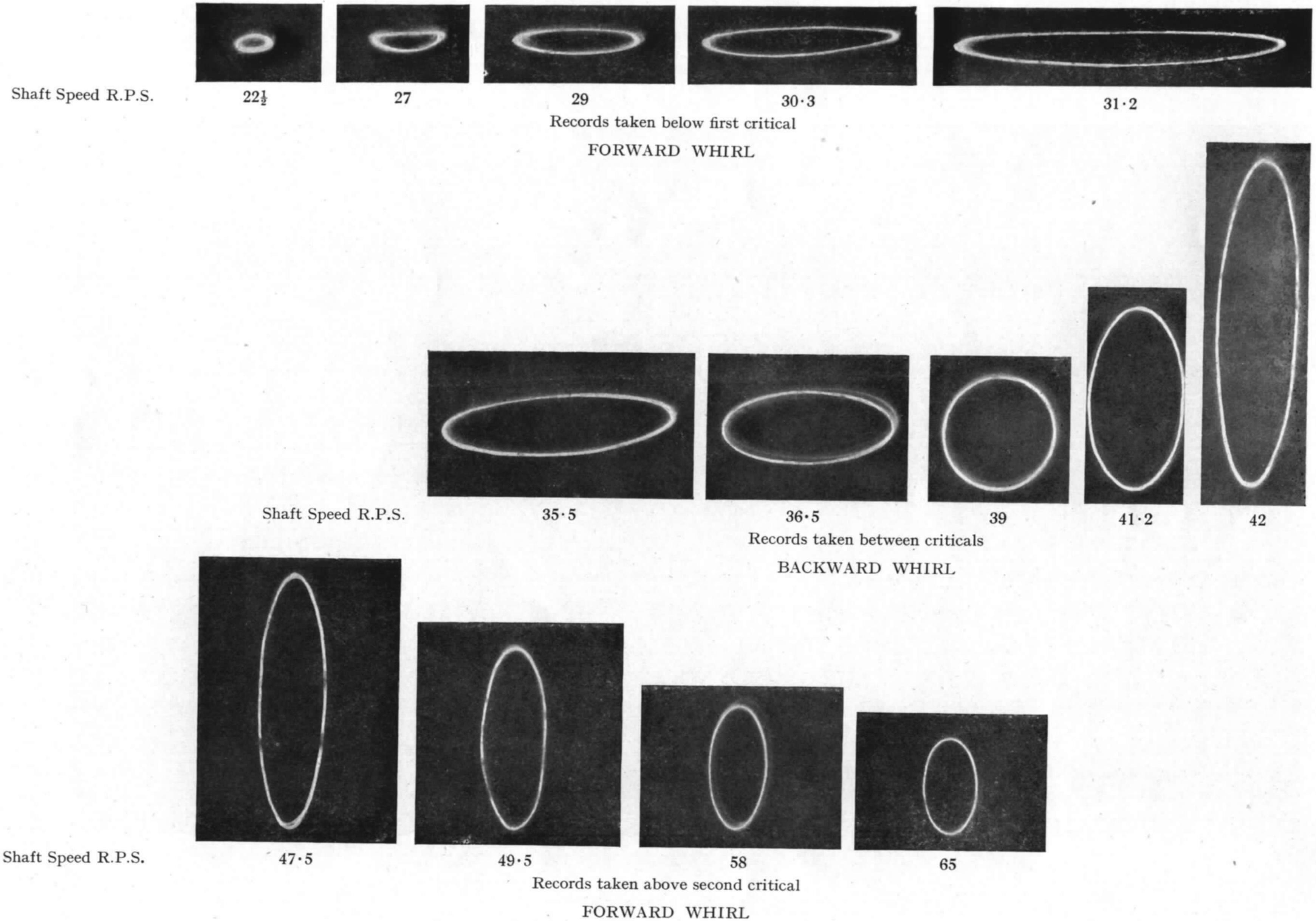


Fig. 12. Records of whirl amplitudes and paths of a rotating shaft supported by a flexible bearing of unsymmetric stiffness.

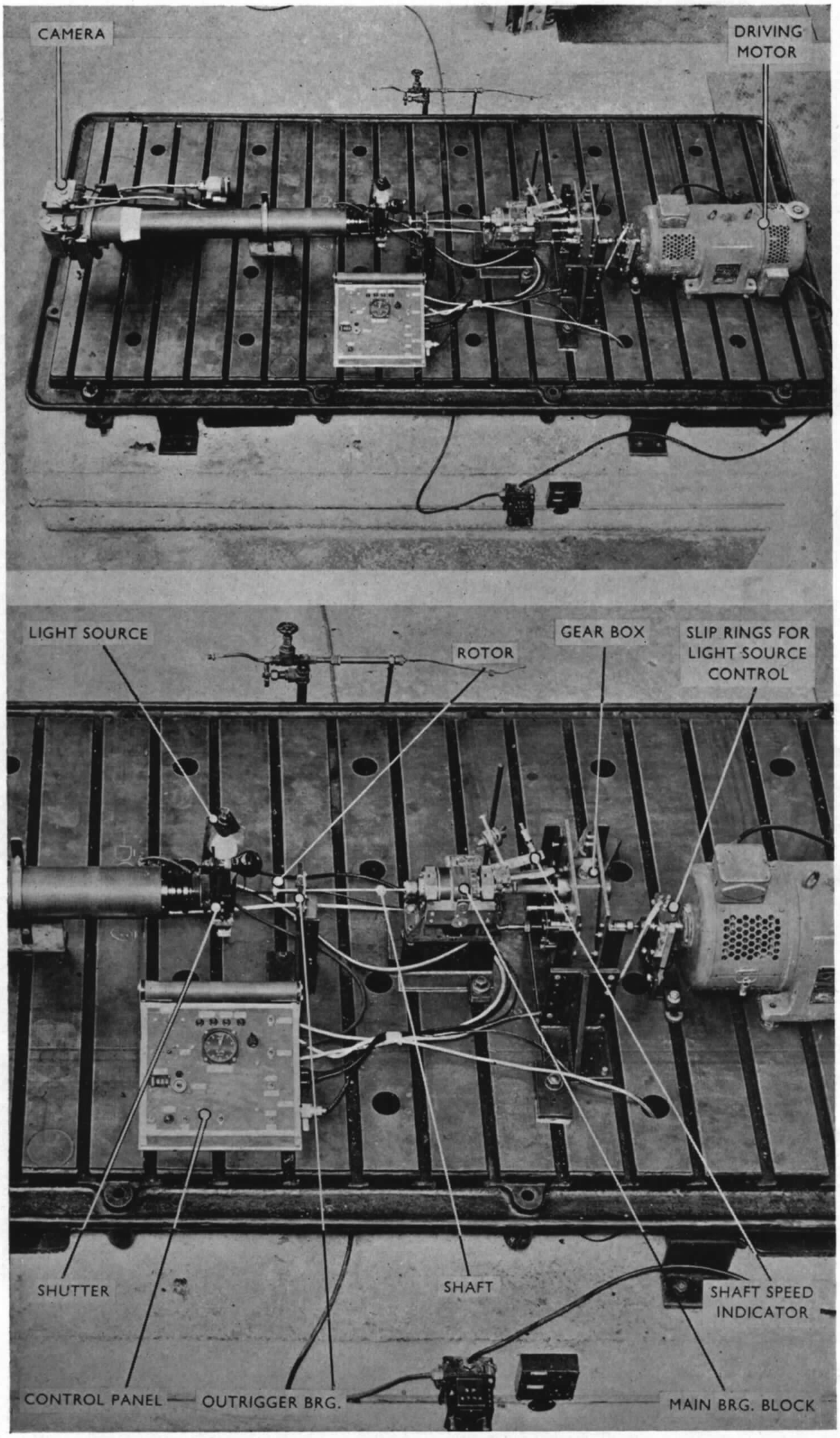


FIG. 13. Details of the experimental rig for the investigation of shaft whirling.

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