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1. *Summary.*—The aerodynamic forces on rectangular wings of various aspect ratios describing simple harmonic oscillations of small amplitude in a supersonic air stream are determined. Linearized theory is used and numerical solutions are derived by the method of 'Relaxation'.

The problem is formulated in section 4 and in section 6 it is reduced to one of finding a series of conical flow solutions. Only a few terms of this series need be determined since the process converges quickly for the range of values of the frequency parameter considered. This range is believed to cover most of the practical supersonic flutter values.

Moment coefficients for a range of Mach numbers and various frequency parameter values were calculated and they are tabulated and plotted at the end of the report. The coefficients are referred to the leading-edge axis position but can be referred to any other axis by the usual formulae.

For a range of Mach numbers in two-dimensional flow, the aerodynamic damping for pitching oscillation can be negative for certain positions of the axis of pitch oscillation and this implies instability (R. & M. 2140[1] and 2194[14]). The results of this report show that aspect ratio has a stabilizing effect for axes less than about $0 \cdot 7$ of the chord downstream of the leading edge, but has the opposite effect for axes nearer than this to the trailing edge.

2. *Introduction.*—The linearized two-dimensional theory of a thin aerofoil oscillating in a supersonic stream predicts that for certain combinations of Mach number, frequency parameter, and positions of the axis of oscillation, the damping moment on a pitching aerofoil may be negative, that is it will give rise to instability (R. & M. 2140[1] and 2194[14]).

In this report the case of an oscillating rectangular wing is considered but the method of solution can be applied to more general plan forms.

The same problem has been considered in the U.S.A. by Garrick and Rubinow[2] but their treatment of the wing tips is unsatisfactory in that it ignores the interference of the upper and lower surfaces. Evvard[3] has given a method of solution for wings of a wide class of plan forms involving the evaluation of a number of integrals. Watkins[4] has obtained an approximate solution for the rectangular wing retaining only terms of the first degree in the frequency parameter, and gives some numerical results. Refs. 2, 3 and 4 use the method of determining a source distribution over the wing to satisfy the boundary conditions.

In this country Temple and Stewartson have used the Laplace and Fourier transform methods to derive formulae for the forces on oscillating rectangular and delta wings[5,6]. W. P. Jones

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(R. & M. 2655[7]) has suggested an iterative method of solution based on the use of known solutions for the steady case.

Of these Watkins (*loc. cit.*) is the only one to give numerical results and these are restricted to small values of the frequency parameter. The method of this report gives results which probably cover most of the practical flutter range.

Some allowance may be made for two-dimensional effects, which are important at the lower values of M, by replacing the two-dimensional thin-wing theory derivative coefficients by more accurate values if these are known (*e.g.*, by Jones' theory, R. & M. 2749[13]).

No experimental data are available for comparison but Bratt of the National Physical Laboratory is considering the possibility of measurements of pitching-moment damping on rectangular and delta wings.

It seems probable that the distorting wing may be treated by the present method and this possibility is being considered.

3. *List of Symbols Used.*

ϕ	Velocity potential of the disturbance due to the wing
x_1, y_1, z_1	Are space co-ordinates
t_1	Time measured from some instant
V	Velocity of stream
V_0	Speed of sound
c	Chord of wing
s	Semi-span of wing
A	Aspect-ratio $= 2s/c$
X, Y, Z	Non-dimensional co-ordinates
$T =$	V/ct_1 Non-dimensional time
M	Mach number $= V/V_0$
μ	Mach angle, $\sin \mu = 1/M$
$p =$	$2\pi \times$ frequency of oscillation
$\phi' =$	$\phi \exp(-ipt_1)$
$\lambda =$	pc/V Frequency parameter
$\Phi =$	$\phi' \exp(i\lambda \sec^2 \mu \cot \mu X)$
$k =$	$\lambda \sec \mu$
$L(\ )$	A differential operator defined in equation (5a)
$z_0 =$	$z_0' e^{ipt_1}$ Is such that the leading edge of the wing is depressed by cz_0
$\alpha =$	$\alpha' e^{ipt_1}$ Angle of incidence of wing
$\eta =$	Y/X
$\zeta =$	Z/X

R and θ	are defined by $\eta = R\cos\theta$, $\zeta = R\sin\theta$
ϱ	$= \log R$
$\Phi_n(n = 0, 1, 2, \ldots)$ $\Phi_{a,b}\,(a, b = 0, 1, 2, \ldots)$	potentials of various conical flows (*see* equations 13 and 14)
$f_{a,b}$	(*see* equations (13) and (14))
Δ_p	Pressure due to disturbance
L	Lift due to disturbance
$\mathscr{M}$	Pitching moment due to disturbance. ($\mathscr{M} > 0$ when it tends to raise the leading edge and depress the trailing edge)

l_z, $l_{\dot z}$, l_α, $l_{\dot\alpha}$, m_z, $m_{\dot z}$, m_α, $m_{\dot\alpha}$, δl_z, $\delta l_{\dot z}$, δl_α, $\delta l_{\dot\alpha}$, δm_z, $\delta m_{\dot z}$, δm_α, $\delta m_{\dot\alpha}$, $\tilde l_z$, $\tilde l_{\dot z}$, $\tilde l_\alpha$, $\tilde l_{\dot\alpha}$, $\tilde m_z$, $\tilde m_{\dot z}$, $\tilde m_\alpha$, $\tilde m_{\dot\alpha}$ are defined in equations (29) to (31).

4. *Statement of the Problem.*—The linearized theory of supersonic flow leads to the equation

$$\frac{\partial^2\phi}{\partial t_1^2} + 2V\frac{\partial^2\phi}{\partial x_1\,\partial t_1} + V^2\frac{\partial^2\phi}{\partial x_1^2} = V_0^2\left[\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial y_1^2} - \frac{\partial^2\phi}{\partial z_1^2}\right] \qquad \ldots \quad (1)$$

for the velocity potential, ϕ, of a small disturbance caused by a wing oscillating in an air stream of velocity V, where V_0 is the speed of sound, and x_1, y_1, z_1, are co-ordinates relative to fixed rectangular axes (x_1 increasing in the direction of flow), and t_1 represents time.

Let c be the chord of the rectangular wing and

$$x_1 = Xc\cot\mu, \quad y_1 = cY, \quad z = cZ, \quad t_1 = cT/V \qquad \ldots \quad (2)$$

where $\sin\mu = 1/M$, $M = V/V_0$. $(0 < \mu < \pi/2)$.

Then in these non-dimensional co-ordinates X, Y, Z, T, equation (1) becomes

$$M^2\frac{\partial^2\phi}{\partial T^2} + \frac{2M^2}{\cot\mu}\frac{\partial^2\phi}{\partial X\,\partial T} = \frac{\partial^2\phi}{\partial Y^2} + \frac{\partial^2\phi}{\partial Z^2} - \frac{\partial^2\phi}{\partial X^2}. \qquad \ldots \quad (3)$$

The motion is now restricted to be simple harmonic, so that

$$\phi = \phi'\,e^{ipt_1} = \phi'\,e^{i\lambda t}$$

where $\lambda = pc/V$, and ϕ' is a function of x_1, y_1, z_1 only and hence of X, Y, Z only.

Let $\phi' = \Phi(X, Y, Z)\exp(-i\lambda\sec^2\mu\cot\mu X)$. $\qquad \ldots \quad (4)$

Then

$$L(\Phi) = k^2\Phi \qquad \ldots \quad (5)$$

where $k = \lambda\sec\mu$, and

$$L(\Phi) = \frac{\partial^2\Phi}{\partial Y^2} + \frac{\partial^2\Phi}{\partial Z^2} - \frac{\partial^2\Phi}{\partial X^2}. \qquad \ldots \quad (5a)$$

This report deals with a thin rectangular wing situated symmetrically in the stream (Fig. 1). The origin of the co-ordinates was taken to be at the intersection of the leading edge and the wing tip and their directions as shown in Fig. 1.

The Mach cone of any point (X_0, Y_0, Z_0) is that part of the cone

$$(X - X_0)^2 = (Y - Y_0)^2 + (Z - Z_0)^2$$

which lies in the region $X > X_0$, *i.e.*, downstream of its vertex.

The wave front is the envelope of the Mach cones of points on the leading edge. Fig. 3 shows the shape of the wave front, the closed curve ABHFEGA being its section by a plane X = constant, downstream of the leading edge. The circles BHFK and AJEG are the section of the Mach cones of the intersections of the leading edge and wing tips. On the wave front $\Phi = 0$, and moreover it is known that if (l,m,n) are the direction cosines of the normal to the wave front at any point on it, then the derivative of Φ in the direction whose direction cosines are $(-l,m,n)$, (the co-normal) is also zero at that point.

The boundary conditions on the wing are known since $\partial\Phi/\partial Z$ is known on the wing by the motion which it is prescribed to have. It is assumed that these conditions may be applied not in the displaced position of the wing but on its projection on the plane $Z = 0$, since the displacement of the wing is assumed to be small.

The motion of the wing is defined by z_0, and α, where cz_0 is the depression of the leading edge and α, the angle of incidence. The motion of the wing is assumed to be simple harmonic so that

$$z_0 = z' \, e^{ipt_1} \quad \text{and} \quad \alpha = \alpha' \, e^{ipt_1} .$$

Then the boundary condition on the wing is given by

$$\frac{\partial \phi}{\partial z_1} = - V\alpha - c\dot{z}_0 - x_1\dot{\alpha} \text{ (compare R. \& M. 2140[1], equation A.18)}$$

$$= - \alpha' \, e^{ipt_1} \, V - z_0' \, e^{ipt_1} \, ip - x_1 ip\alpha' \, e^{ipt_1} .$$

In terms of the transformed variables this may be written

$$\frac{\partial \Phi}{\partial Z} = - Vc \exp (i\lambda \sec^2 \mu \cot \mu X)\Big[i\lambda z_0' + \alpha'(1 + i\lambda \cot \mu X)\Big] \qquad \ldots \quad (6)$$

and hence $\partial\Phi/\partial Z$ may be expanded as a power series in X. This fact is used later.

It follows from the symmetry of the boundary conditions that Φ is symmetrical about the plane of symmetry of the wing, and hence we need consider only half of the region inside the wave front.

In the region inside the wave front, but outside the Mach cones of the points of intersection of the leading edge and the wing tips, the solution for Φ is that associated with an infinite wing of the same section performing the same motion. It will be assumed that these two Mach cones do not intersect upstream of the trailing edge, and then in the regions whose sections are ABKJ and EFKJ (Fig. 3), Φ is given by the two-dimensional solution

$$\Phi(X,Z) = \pm \int_0^{X-Z} \left(\frac{\partial \Phi}{\partial Z}\right)_{X_0, 0} J_0[k\{(X - X_0)^2 - Z^2\}^{1/2}] \, dX_0 \qquad \ldots \quad (7)$$

where J_0 is the Bessel function of order zero, and $(\partial\Phi/\partial Z)_{X_0, 0}$ denotes the Z-derivative of Φ at the point $X = X_0$ on the wing. The negative sign refers to the region $Z > 0$ and the positive to $Z < 0$. (*See* R. & M. 2655[7], equation 38).

Since the boundary conditions are the same on the upper and lower surface of the wing it follows that $\Phi(Z) = -\Phi(-Z)$ and hence $\Phi = 0$ for $Z = 0$ in the region $Y > 0$, since the continuity of pressure across the surface, ($Z = 0$, $Y > 0$), implies the continuity of Φ. Thus it is necessary to consider the region $Z \geqslant 0$ only.

5. *The Equations of Conical Flow.*—The potentials considered in the report can be expressed in the form

$$\Phi = X^n f(Y/X, Z/X) \qquad \ldots \qquad (8)$$

$$= X^n f(\eta, \zeta) \qquad \text{where } \eta = Y/X, \ \zeta = Z/X,$$

and n is a positive integer.

Then, by forming the required derivatives, it may be shown that

$$L(\Phi) \equiv \frac{\partial^2\Phi}{\partial Y^2} + \frac{\partial^2\Phi}{\partial Z^2} - \frac{\partial^2\Phi}{\partial X^2} = X^{n-2}\left\{\frac{\partial^2 f}{\partial\eta^2} + \frac{\partial^2 f}{\partial\zeta^2} - \left[n(n-1)f - 2(n-1)\left(\eta\frac{\partial f}{\partial\eta} + \zeta\frac{\partial f}{\partial\zeta}\right)\right.\right.$$

$$\left.\left. + \eta^2\frac{\partial^2 f}{\partial\eta^2} + 2\eta\zeta\frac{\partial^2 f}{\partial\eta\,\partial\zeta} + \zeta^2\frac{\partial^2 f}{\partial\zeta^2}\right]\right\}. \qquad \ldots \qquad (9)$$

In particular if $n = 1$

$$L(\Phi) = X^{-1}\left\{\frac{\partial^2 f}{\partial\eta^2} + \frac{\partial^2 f}{\partial\zeta^2} - \left[\eta^2\frac{\partial^2 f}{\partial\eta^2} + 2\eta\zeta\frac{\partial^2 f}{\partial\eta\,\partial\zeta} + \zeta^2\frac{\partial^2 f}{\partial\zeta^2}\right]\right\}.$$

When the independent variables are transformed by putting $\eta = R\cos\theta$, $\zeta = R\sin\theta$, it follows that

$$L(\Phi) = X^{n-2}\left\{\frac{\partial^2 f}{\partial R^2} + \frac{1}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\frac{\partial^2 f}{\partial\theta^2} - n(n-1)f + 2(n-1)R\frac{\partial f}{\partial R} - R^2\frac{\partial^2 f}{\partial R^2}\right\}$$

$$= X^{n-2}\left\{\frac{\partial^2 f}{\partial R^2}(1 - R^2) + \left(\frac{1}{R} + 2(n-1)R\right)\frac{\partial f}{\partial R} + \frac{1}{R}\frac{\partial^2 f}{\partial\phi^2} - n(n-1)f\right\}. \qquad \ldots \qquad (10)$$

The unit circle B'H'F'K' in the (η, ζ)-plane corresponds to the circle BHFK in Fig. 3 and the interior of the first transforms into the interior of the second.

Let $\qquad \varrho = \log R$, then

$$L(\Phi) = X^{n-2}\,\mathrm{e}^{-2\varrho})\left\{(1 - \mathrm{e}^{2\varrho})\frac{\partial^2 f}{\partial\varrho^2} + (2n-1)\frac{\partial f}{\partial\varrho} + \frac{\partial^2 f}{\partial\theta^2} - n(n-1)\mathrm{e}^{2\varrho} f\right\}. \qquad \ldots \qquad (11)$$

The unit circle in the (η, ζ)-plane transforms into the semi-infinite strip, $\varrho < 0$, $-\pi \leqslant \theta \leqslant \pi$ in the (ϱ, θ)-plane.

Solutions of type (8) are such that along any straight line passing through the origin they vary as the nth power of the distance from the origin. In particular if $n = 1$, the derivatives of Φ are constant along such lines ; this is the form associated with steady flow.

6. *Reduction of Φ to a Sum of Solutions of Conical Type.*—Φ must satisfy $L(\Phi) = k^2\Phi$, and $\Phi = 0$ on the wave front. Also $\partial\Phi/\partial Z$ is prescribed as a function of X on the wing and this function can be expanded as a power series in X. Since all the conditions imposed on Φ are linear it is permissible to consider each power of X separately and then form a solution by adding suitable multiples of the functions thus obtained.

Thus let Φ_n be a solution of (5) satisfying all the boundary conditions for Φ except (6) which is replaced by $\partial\Phi_n/\partial Z = -X^n/n!$ on the wing. $(n = 0,1,2,\ldots)$.

The solution in the region where the flow is two dimensional is given by

$$\Phi_n(X, Z) = \int_0^{X-Z} \frac{X_0^{\ n}}{n\,!} J_0\left[k\left((X-X_0)^2 - Z^2\right)^{1/2}\right] dX_0 \quad \ldots \quad (12)$$

and hence Φ_n must assume this value on the arc BK (Fig. 3).

Let

$$\Phi_n = X^{n+1} f_{n,0}\left(\frac{Y}{X}, \frac{Z}{X}\right) + k^2 X^{n+3} f_{n,1}\left(\frac{Y}{X}, \frac{Z}{X}\right)$$

$$+ k^4 X^{n+5} f_{n,2}\left(\frac{Y}{X}, \frac{Z}{X}\right) + \ldots \quad \ldots \quad (13)$$

$$= \Phi_{n,0} + k^2\Phi_{n,1} + k^4\Phi_{n,2} + \ldots \text{ say} \quad \ldots \quad (14)$$

where the f's and Φ's are functions which will be determined later, and impose the conditions

$$\left.\begin{aligned} L(\Phi_{n,0}) &= 0 \\ L(\Phi_{n,1}) &= \Phi_{n,0} \\ L(\Phi_{n,2}) &= \Phi_{n,1} \end{aligned}\right\} . \quad \ldots \quad (15)$$

Then equations (15) ensure that $L(\Phi_n) = k^2\Phi_n$ i.e., Φ_n satisfies (5).

It is possible to satisfy conditions (15) by functions of the type given in (8) since both sides of any equation of (15) contain the same power of X, which may be removed leaving a differential equation with η, ζ (or ϱ and θ) as independent variables, connecting two successive f's.

$$\text{By (12), } \Phi_n(X,Z) = \int_0^{X-Z} \frac{X_0^{\ n}}{n\,!}\left\{1 - \frac{k^2}{2^2}\left[(X-X_0)^2 - Z^2\right] + \frac{k^4}{2^2 4^2}\left[(X-X_0)^2 - Z^2\right]^2 - \ldots\right\} dX_0$$

$$= \frac{(X-Z)^{n+1}}{(n+1)\,!} + k^2 P_{n+3} + k^4 P_{n+5} + \ldots$$

where P_{n+3}, P_{n+5}, $\ldots$ are homogeneous polynomials in X and Z of degrees $n+3$, $n+5$, $\ldots$.

$$\text{Hence } \Phi_n(X, Z) = X^{n+1}\frac{(1-Z/X)^{n+1}}{(n+1)\,!} + k^2 X^{n+3} Q_{n+3} + k^4 X^{n+5} Q_{n+5} + \ldots$$

where Q_{n+3}, Q_{n+5}, $\ldots$ are polynomials in $\zeta = Z/X$.

In order to ensure Φ_n assumes the value given by (12) on the arc BK it is necessary to impose the conditions

$$f_{n,0}(\eta,\zeta) = \frac{(1-\zeta)^{n+1}}{(n+1)\,!}, \qquad f_{n,1}(\eta,\zeta) = Q_{n+3}(\eta,\zeta),$$

$$f_{n,2}(\eta,\zeta) = Q_{n+5}(\eta,\zeta) \ldots$$

on the arc B′K′ in the (η,ζ)-plane.

On D′H′ and the arc B′H′, $f_{n,0}$, $f_{n,1}$, $f_{n,2}$, . . . are all prescribed to be zero, so that $\Phi_n = 0$ on DH and BH.

The boundary condition on the wing is satisfied by making $\dfrac{\partial \Phi_{n,0}}{\partial Z} = \dfrac{X_n}{n\,!}$, and $\dfrac{\partial \Phi_{n,1}}{\partial Z} = 0$, $\dfrac{\partial \Phi_{n,2}}{\partial Z} = 0, \ldots$ on $Z = 0,\ Y < 0$. In terms of f's and η and ζ this implies $\dfrac{\partial f_{n,0}}{\partial \zeta} = -\dfrac{1}{n\,!}$, $\dfrac{\partial f_{n,1}}{\partial \zeta} = \dfrac{\partial f_{n,2}}{\partial \zeta} = \ldots = 0$.

Thus for any n the determination of Φ_n is reduced to the solution of an infinite chain of differential equations in η and ζ.

However it is proved in Appendix I that in the case of the rectangular wing

$$\Phi_{m+1} = \int_{M(Y,Z)}^{X} \Phi_n \, dX \text{ where } X = M(Y,Z) \text{ is the equation of the wave front.} \quad \ldots (16)$$

Thus it was necessary only to calculate Φ_0 and then determine Φ_1, Φ_2, ... by repeated integration.

In the particular case $n = 0$, the previous results become:—

$$\begin{aligned}\Phi_0 &= \Phi_{0,0} + k^2\Phi_{0,1} + k^4\Phi_{0,2} + \ldots \\ &= X f_{0,0}(\eta,\zeta) + X^3k^2f_{0,1}(\eta,\zeta) + X^5k^4f_{0,2}(\eta,\zeta) + \ldots \quad \ldots (17)\end{aligned}$$

and $\dfrac{\partial \Phi}{\partial Z} = -1$ on the wing, i.e., $\dfrac{\partial f_{0,0}}{\partial \zeta} = -1$, $\dfrac{\partial f_{0,1}}{\partial \zeta} = 0$, $\dfrac{\partial f_{0,2}}{\partial \zeta} = 0$, $\quad \ldots (18)$

On the arc BK (and in the region where the flow is two-dimensional),

$$\begin{aligned}\Phi_0 &= \int_0^{X-Z} J_0\left[k\Big((X-X_0)^2 - Z^2\Big)^{1/2}\right] dX_0 \\ &= \int_0^{X-Z} \left\{1 - \frac{k^2}{2^2}\Big[(X-X_0)^2 - Z^2\Big] + \frac{k^4}{2^2 4^2}\Big[(X-X_0)^2 - Z^2\Big]^2 - \ldots\right\} dX_0 \\ &= (X - Z) - \frac{k^2}{12}X^3\left\{1 - 3\Big(\frac{Z}{X}\Big)^2 + 2\Big(\frac{Z}{X}\Big)^3\right\} \\ &\quad + k^4\frac{X^5}{960}\left\{3 - 10\Big(\frac{Z}{X}\Big)^2 + 15\Big(\frac{Z}{X}\Big)^4 - 8\Big(\frac{Z}{X}\Big)^5\right\} - \ldots .\end{aligned}$$

Hence on B′K′, $f_{0,0} = 1 - \zeta$, $\quad f_{0,1} = \dfrac{-1}{12}\left\{1 - 3\zeta^2 + 2\zeta^3\right\}$,

$$f_{0,2} = \frac{1}{960}\left\{3 - 10\zeta^2 + 15\zeta^4 - 8\zeta^5\right\}, \ldots . \quad \ldots (19)$$

The conditions (15) become (by equation 9)

$$
\left.
\begin{aligned}
&\frac{\partial^2 f_{0,0}}{\partial \eta^2} + \frac{\partial^2 f_{0,0}}{\partial \zeta^2} - \left[\eta^2 \frac{\partial^2 f_{0,0}}{\partial \eta^2} + 2\eta\zeta \frac{\partial^2 f_{0,0}}{\partial \eta\, \partial \zeta} + \zeta^2 \frac{\partial^2 f}{\partial \zeta^2}\right] = 0 \\
&\frac{\partial^2 f_{0,1}}{\partial \eta^2} + \frac{\partial^2 f_{0,1}}{\partial \zeta^2} - \left[\eta^2 \frac{\partial^2 f_{0,1}}{\partial \eta^2} + 2\eta\zeta \frac{\partial^2 f_{0,1}}{\partial \eta\, \partial \zeta} + \zeta^2 \frac{\partial^2 f}{\partial \zeta^2}\right] + 4\left(\eta \frac{\partial f_{0,1}}{\partial \eta} + \zeta \frac{\partial f_{0,1}}{\partial \zeta}\right) - 6 f_{0,1} = f_{0,0}
\end{aligned}
\right\} \quad \ldots (20)
$$

and so on.

There are, of course, sets of equations corresponding to (20) with (R,θ) or (ϱ,θ) as independent variables and the solution is required in the region $\eta^2 + \zeta^2 \leqslant 1$ (or the corresponding regions in the (R,θ) and (ϱ,θ)-planes).

7. *Determination of* $f_{0,0}, f_{0,1}, f_{0,2}, \ldots$.—The equations (20) are all elliptic partial differential equations the solutions of which are to be found in a semi-circular region B′H′D′K′ in the (η,ζ)-plane (or on the ϱ,θ-plane in a semi-infinite strip) on every point of whose boundary f or its gradient are known. This is the kind of problem which can be tackled by the method of Relaxation (*see* Appendix II for a brief description) and this method was used in the present case. (*See* Appendix II.)

The values of $f_{0,0}, f_{0,1}, f_{0,2}$ are shown in Figs. **5** to **7** as functions of ϱ and θ. From the rate at which the $f_{0,n}$'s decreased with n it was concluded that $f_{0,3}$ was negligible to three decimal places.

Since $\Phi_{0,0}$ (and hence $f_{0,0}$) corresponds to the case of a rectangular wing in steady supersonic flow it was possible to check the solution obtained with the analytical solution (*see* for example Ref. 8, appendix (B)). The difference was found to be not greater than 0·004 (*i.e.*, less than $\frac{1}{2}$ per cent) at the points for which the values were compared. All these values were on the line $\theta = \pi$, since the analytical solution applies only to the surface of the wing.

After $f_{0,0}$, $f_{0,1}$, and $f_{0,2}$ had been calculated, the values of $\Phi_{0,0}$, $\Phi_{0,1}$, $\Phi_{0,2}$ were computed in the plane $Z = 0$ by (17) and then integrated to obtain $\Phi_{1,0}$, $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,0}$, $\Phi_{2,1}$, $\Phi_{2,2}$, and so on, up to $\Phi_{7,0}$, $\Phi_{4,1}$, $\Phi_{2,2}$, all the others being negligible so far as results given in this report are concerned. Outside the Mach cones of the intersections of the leading edge and wing tips the Φ's were derived from (12).

It is necessary to know the values of Φ on the wing only since the pressure is determined by this alone.

In order to determine the forces acting on the wing as a whole it is necessary to integrate spanwise, so the integrals

$$
\begin{aligned}
&\int_{-X}^{0} \frac{\partial \Phi_{0,0}}{\partial X}\, dY, \qquad \int_{-X}^{0} \Phi_{0,0}\, dY, \ldots \qquad \int_{-X}^{0} \Phi_{7,0}\, dY, \\
&\int_{-X}^{0} \frac{\partial \Phi_{0,1}}{\partial X}\, dY, \qquad \int_{-X}^{0} \Phi_{0,1}\, dY, \ldots \qquad \int_{-X}^{0} \Phi_{4,1}\, dY \\
\text{and } &\int_{-X}^{0} \frac{\partial \Phi_{0,2}}{\partial X}\, dY, \qquad \int_{-X}^{0} \Phi_{0,2}\, dY, \ldots \qquad \int_{-X}^{0} \Phi_{2,2}\, dY \text{ were calculated for } Z = 0.
\end{aligned}
$$

Their values are given in Table 1. The range of integration is spanwise from the point where the flow ceases to be two-dimensional to the wing tip.

It is not necessary to differentiate numerically in order to obtain $\int_{-X}^{0} \frac{\partial \Phi_{0,0}}{\partial X} dY$ since we have

$$\Phi_{0,0}(X,Y,Z) = X f_{0,0}\left(\frac{Y}{X}, \frac{Z}{X}\right) \text{ so that } \frac{\partial \Phi_{0,0}}{\partial X} = f_{0,0}\left(\frac{Y}{X}, \frac{Z}{X}\right) - \frac{Y}{X}\frac{\partial f_{0,0}}{\partial \eta} - \frac{Z}{X}\frac{\partial f_{0,0}}{\partial \zeta},$$

$$\frac{\partial \Phi_{0,0}}{\partial Y} = \frac{\partial f_{0,0}}{\partial \eta}$$

and it follows that $\int_{-X}^{0} \frac{\partial \Phi_{0,0}}{\partial X} dY = X\left[2 \int_{-1}^{0} f_{0,0}(\eta,0)\, d\eta - 1\right]$

and similarly for $\frac{\partial \Phi_{0,1}}{\partial X}$, $\frac{\partial \Phi_{0,2}}{\partial X}$.

$$\int_{-X}^{0} \Phi_{0,0}(X,Y,0)\, dY = X^2 \int_{-1}^{0} f_{0,0}(\eta,0)\, d\eta \text{ and similarly for the others.}$$

8. *The Calculation of the Forces Acting on the Wing.*—The pressure due to the disturbance is given by

$$\Delta p = -\varrho\left(\frac{\partial \phi}{\partial t_1} + V\frac{\partial \phi}{\partial x_1}\right)$$

$$= \frac{-\varrho V}{c} \tan\mu \exp\left(-i\lambda \sec^2\mu \cot\mu X + i\lambda T\right)\left[\frac{\partial \Phi}{\partial X} - i\lambda \tan\mu \Phi\right] \qquad \text{.. (21)}$$

and since $\Phi(-Z) = -\Phi(Z)$ it follows that the lift on an element of area dA is $-2\Delta p\, dA$.

Then supposing for the moment that Δp is known all over the wing the lift, L, is given by

$$L = \int_0^c 2\left[\int_{-s}^{0} -2\Delta p\, dy_1\right] dx_1 = -4c^2 \cot\mu \int_0^{\tan\mu}\left[\int_{-s/c}^{0} \Delta p\, dY\right] dX.$$

Similarly $\mathscr{M}$ the moment about the leading edge, is given by

$$\mathscr{M} = 4c^3 \cot^2\mu \int_0^{\tan\mu} X\left[\int_{-s/c}^{0} \Delta p\, dY\right] dX .$$

$\mathscr{M}$ is positive if it tends to raise the leading edge and depress the trailing edge.

These are the forces acting on the complete wing of span $2s$ and chord c.

In order to determine Δp, we expand $\partial\Phi/\partial Z$ as given by equation (6) in a power series in X.

Suppose $\partial \Phi/\partial Z = -Vc \sum_0^{\infty} A_n X^n/n!$ .. .. (22)

where the A_n's are constant (in general complex). These A_n's will be linear in z_0' and α'.

Then since $\partial \Phi_n/\partial Z = -X^n/n!$ on the wing,

$$\Phi = Vc \sum_0^{\infty} A_n \Phi_n$$

and $$\partial\Phi/\partial X = Vc\left\{A_0\,\partial\Phi_0/\partial X + A_1\Phi_0 + A_2\Phi_1 + \ldots\right\} \quad \ldots \quad (23)$$

(since $\partial\Phi_{n+1}/\partial X = \Phi_n$).

Hence, by (21) $\Delta p = -\varrho V^2 \tan\mu \exp(-i\lambda\sec^2\mu\cot\mu X + i\lambda T)\left[A_0\dfrac{\partial\Phi_0}{\partial X} + \sum_0^\infty B_n\Phi_n\right]$ .. (24)
when the B_n's are new complex constants.

Hence $$\int_{-X}^0 \Delta p\,dY = -\varrho V^2\tan\mu\exp(-i\lambda\sec^2\mu\cot\mu X + i\lambda T)$$
$$\times\left[A_0\int_{-X}^0\frac{\partial\Phi_0}{\partial X}\,dY + \sum_0^\infty B_n\int_{-X}^0\Phi_n\,dY\right] \quad \ldots \quad (25)$$

and $\displaystyle\int_{-X}^0\frac{\partial\Phi_0}{\partial X}\,dY$, $\displaystyle\int_{-X}^0\Phi_0\,dY, \ldots$ can be expanded as power series in X by (14) and the values given in Table 1.

Hence, collecting up the terms in the square bracket,

$$\int_{-X}^0\Delta p\,dY = -\varrho V^2\tan\mu\exp(-i\lambda\sec^2\mu\cot\mu X + i\lambda T)\sum_0^\infty C_nX^n \quad \ldots \quad (26)$$

where the C_n's are constants.

In the region where the flow is two-dimensional it follows from (12) that

$$\Phi_n(X, 0) = \int_0^X\frac{X_0^{\,n}}{n!}J_0[k(X - X_0)]\,dX_0$$

so that $$\frac{\partial\Phi_0}{\partial X} = 1 - \frac{k^2X^2}{2^2} + \frac{k^4}{2^24^2}X^4 - \ldots \quad = J_0(kX)$$

$$\Phi_0 = X - \frac{k^2}{2^2}\frac{X^3}{3} + \frac{k^4}{2^24^2}\frac{X^5}{5} - \ldots$$

$$\Phi_1 = \int_0^X\Phi_0\,dX = \frac{X_2}{2} - \frac{k^3}{2^2}\frac{X^4}{3\cdot4} + \ldots$$

and so on for all the Φ_n's.

When these expressions are substituted in (24) it follows that

$$\Delta p = -\varrho V^2\exp(-i\lambda\sec^2\mu\cot\mu X + i\lambda T)\tan\mu\sum_0^\infty C_n{}'X^n \quad \ldots \quad (27)$$

where the $C_n{}'$'s are complex constants.

Over the part of the wing where the flow is two-dimensional, Δp is known, and since it is independent of y, it follows that

$$\int_{-s/c}^{-X}\Delta p\,dY = \left(\frac{s}{c} - X\right)\Delta p. \quad \ldots \quad (28)$$

Then the expressions for L and $\mathcal{M}$ reduce to sums of integrals of the form

$$\int_0^{\tan\mu} X^n \cos(\lambda \sec^2\mu \, \cot\mu X)\, dX \text{ and } \int_0^{\tan\mu} X^n \sin(\lambda \sec^2\mu \, \cot\mu X)\, dX.$$

These integrals were computed by expanding the integrand in a power series and integrating term by term.

The lift and moment acting on the wing are thus found in the forms

$$L = \varrho V^2 c^2 \, e^{i\lambda T}\Big[z_0'\Big\{A(l_z + i\lambda l_{\dot{z}}) + (\delta l_z + i\lambda\, \delta l_{\dot{z}})\Big\}$$

$$+ \alpha'\Big\{A(l_\alpha + i\lambda l_{\dot{\alpha}}) + (\delta l_\alpha + i\lambda\, \delta l_{\dot{\alpha}})\Big\} \quad \ldots\ldots\ldots\ldots (29)$$

and

$$\mathcal{M} = \varrho V^2 c^3 \, e^{i\lambda T}\Big[z_0'\Big\{A(m_z + i\lambda m_{\dot{z}}) + (\delta m_z + i\lambda\, \delta m_{\dot{z}})\Big\}$$

$$+ \alpha'\Big\{A(m_\alpha + i\lambda m_{\dot{\alpha}}) + (\delta m_\alpha + i\lambda\, \delta m_{\dot{\alpha}})\Big\}\Big]. \quad \ldots\ldots\ldots (30)$$

In these equations l_z, $l_{\dot{z}}$, l_α, $l_{\dot{\alpha}}$, m_z, $m_{\dot{z}}$, m_α and $m_{\dot{\alpha}}$ are the two-dimensional thin-wing derivative coefficients, and δl_z, $\delta l_{\dot{z}}$, δl_α, $\delta l_{\dot{\alpha}}$, δm_z, $\delta m_{\dot{z}}$, δm_α, $\delta m_{\dot{\alpha}}$ which are defined by these equations may be regarded as corrections for the effect of the tips.

Write $\bar{l}_z = l_z + \delta l_z / A$, and corresponding expressions for the other coefficients.

Then (29) and (30) may be written

$$L = \varrho V^2 c^2 \, e^{i\lambda T} A\Big[z_0'\Big\{\bar{l}_z + i\lambda \bar{l}_{\dot{z}}\Big\} + \alpha'\Big\{\bar{l}_\alpha + i\lambda \bar{l}_{\dot{\alpha}}\Big\}\Big] \quad \ldots\ldots\ldots\ldots (31)$$

and

$$\mathcal{M} = \varrho V^2 c^3 \, e^{i\lambda T} A\Big[z_0'\Big\{\tilde{m}_z + i\lambda \tilde{m}_{\dot{z}}\Big\} + \alpha'\Big\{\tilde{m}_\alpha + i\lambda \tilde{m}_{\dot{\alpha}}\Big\}\Big]. \quad \ldots\ldots\ldots\ldots (32)$$

Evidently $\bar{l}_z$, etc., are functions of A, the aspect ratio.

Since it has been assumed that the Mach cones of the tips do not intersect upstream of the trailing edge, A is restricted to be greater than $2 \tan\mu$.

All these coefficients refer to the leading edge but if it is desired to take an axis hc downstream of the trailing edge, the corresponding coefficients may be calculated by using the equations

$$z_h = z_0 + h\alpha, \qquad \mathcal{M}_h = \mathcal{M} + hcL, \qquad L_h = L, \quad \ldots\ldots\ldots\ldots (33)$$

where cz_h is the depression of the new axis and $\mathcal{M}_h$ is taken in the same sense as $\mathcal{M}$ about it.

In the actual computations only a finite number of terms of any of the infinite series were retained, the rest being negligible.

If more accurate values of the two-dimensional coefficients are known, from theory (or even from experiment) (*e.g.*, those taking thickness effects into account), more accurate estimates of

L and $\mathscr{M}$ might be obtained by using these in (29) and (30), or possibly by multiplying l_z, etc., by the ratio of the more accurate two-dimensional values to the thin-wing two-dimensional values.

9. *Numerical Results.*—The coefficients $\delta l_z, \ldots \delta m_{\dot{\alpha}}$ were computed for $\lambda = 0 \cdot 2$, $M = 1 \cdot 2 (0 \cdot 2) 1 \cdot 8$; $\lambda = 0 \cdot 4$, $M = 1 \cdot 2(0 \cdot 2)1 \cdot 8$; and $\lambda = 0 \cdot 6$, $M = 1 \cdot 4(0 \cdot 2)2 \cdot 0$.

The coefficients $l_z, \ldots m_{\dot{\alpha}}$, also appeared in the course of the calculations.

The results are given in Table 2, and are plotted on Figs. 8 to 15.

In the case of a wing oscillating about a fixed leading edge (i.e., $z_0 = 0$) the damping moment is given by

$$- \varrho V^2 c^3 \alpha' \lambda (m_{\dot{\alpha}} + \delta m_{\dot{\alpha}}) = - \varrho V^2 c^3 \alpha' \lambda A \bar{m}_{\dot{\alpha}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

so that positive values of $\bar{m}_{\dot{\alpha}}$ imply negative damping.

The values of $\bar{m}_{\dot{\alpha}}$ have been plotted in Figs. 16 to 18 and from these it will be seen that with the leading edge as axis the negative damping predicted by two-dimensional theory occurs in the finite case but is reduced by aspect ratio.

If the matter is referred to an axis distance hc downstream of the leading edge it follows that

$$\mathscr{M}_h = \varrho V^2 c^3 e^{i\lambda T} A \left[z_h' \left\{ (\bar{m}_z)_h + i\lambda (\bar{m}_{\dot{z}})_h \right\} + \alpha' \left\{ (\bar{m}_{\alpha})_h + i\lambda (\bar{m}_{\dot{\alpha}})_h \right\} \right].$$

This equation defines $(\bar{m}_z)_h$, etc., where in particular

$$(\bar{m}_{\dot{\alpha}})_h = \bar{m}_{\dot{\alpha}} + h(\bar{l}_{\dot{\alpha}} - \bar{m}_{\dot{z}}) - h^2 \bar{l}_{\dot{z}}$$

$$= m_{\dot{\alpha}} + h(l_{\dot{\alpha}} - \bar{m}_{\dot{z}}) - h^2 l_{\dot{z}} + [\delta m_{\dot{\alpha}} + h(\delta l_{\dot{\alpha}} - \delta m_{\dot{z}}) - h^2 \, \delta l_{\dot{z}}]/A \text{ by equations (33).}$$

In order to discover the effect of the position of the axis on the damping effect of the tips the quantity $-[\delta m_{\dot{\alpha}} + h(\delta l_{\dot{\alpha}} - \delta m_{\dot{z}}) - h^2 \, \delta l_{\dot{z}}]$ was plotted for various values of M and λ (Figs. 19 to 21).

The damping moment for a wing pitching about the new axis is

$$- \varrho V^2 c^3 \alpha \lambda A (\bar{m}_{\dot{\alpha}})_h$$

so that if $-[\delta m_{\dot{\alpha}} + h(\delta l_{\dot{\alpha}} - \delta m_{\dot{z}}) - h^2 \, \delta l_{\dot{z}}]$ is positive the tip effect increases damping and *vice versa*.

Figs. 19 to 21 show that the damping effect of the tips becomes negative for axes near the trailing edge, the point of change from positive to negative being almost independent of M and varying only slightly with λ.

Comparison with Other Results.—In order to obtain a check on the process the computations were also carried out for the case $M = \sqrt{2}$, $\lambda = \frac{1}{2}$, with the slight difference that the two-dimensional mid-chord derivatives were computed instead of the leading-edge derivatives. These were compared with the mid-chord derivatives given by Temple and Jahn (R. & M. 2140[1]), and found to agree to three decimal places.

A further comparison was made with the result given by Temple[5]. The case considered was again $M = \sqrt{2}$, $\lambda = \frac{1}{2}$, the wing having a simple vertical oscillation ($\alpha' = 0$), and aspect ratio 2. The results agreed numerically though the signs were different, but it is beleived that Temple's result as given has the wrong sign.

Comparison of a case with Watkin's result[4] tends to the conclusion that this is satisfactory for small values of λ.

APPENDIX I

Proof that $\Phi_{m+1} = \int_{M(Y,Z)}^{X} \Phi_m \, dX$

Let $\quad X = M(Y,Z)$ be the equation of the wave front.

If
$$\Phi_1 = \int_{M(Y,Z)}^{X} \Phi_0(\xi, Y, Z)\, d\xi\,,$$

then
$$\frac{\partial \Phi_1}{\partial Z} = \int_M^X \frac{\partial \Phi_0}{\partial Z}(\xi, Y, Z)\, d\xi - \frac{\partial M}{\partial Z}\Phi_0[M(Y,Z), Y, Z]$$

$$= \int_M^X \frac{\partial \Phi_0}{\partial Z}(\xi, Y, Z)\, d\xi \qquad \text{(A.1)}$$

since $\Phi_0 = 0$ on the wave front.

$$\frac{\partial^2 \Phi_1}{\partial Z^2} = \int_M^X \frac{\partial^2 \Phi_0}{\partial Z^2}(\xi, Y, Z)\, d\xi - \frac{\partial M}{\partial Z}\cdot\frac{\partial \Phi_0}{\partial Z}[M(Y,Z), Y, Z]$$

Similarly
$$\frac{\partial^2 \Phi_1}{\partial Y^2} = \int_M^X \frac{\partial^2 \Phi_1}{\partial Y^2}(\xi, Y, Z)\, d\xi - \frac{\partial M}{\partial Y}\cdot\frac{\partial \Phi_0}{\partial Y}[M(Y,Z), Y, Z]$$

also
$$\frac{\partial \Phi_1}{\partial X} = \Phi_0(X, Y, Z)\,, \quad \frac{\partial^2 \Phi_1}{\partial X^2} = \frac{\partial \Phi_0}{\partial X}(X, Y, Z)\,.$$

Hence
$$L(\Phi_1) - k^2\Phi_1 \equiv \frac{\partial^2 \Phi_1}{\partial Y^2} + \frac{\partial^2 \Phi_1}{\partial Z^2} - \frac{\partial^2 \Phi_1}{\partial X^2} - k^2\Phi_1$$

$$= \int_M^X \left(\frac{\partial^2 \Phi_0}{\partial Y^2} + \frac{\partial^2 \Phi_0}{\partial Z^2} - k^2\Phi_0\right)_{\xi, Y, Z} d\xi - \frac{\partial M}{\partial Y}\cdot\frac{\partial \Phi_0}{\partial Y}(M, Y, Z)$$

$$- \frac{\partial M}{\partial Z}\cdot\frac{\partial \Phi_0}{\partial Z}(M, Y, Z) - \frac{\partial \Phi_0}{\partial X}(X, Y, Z)\,.$$

But $\qquad L(\Phi_0) = k^2\Phi_0$, hence

$$L(\Phi_1) - k^2\Phi_1 = \int_{M(Y,Z)}^{X} \frac{\partial^2\Phi_0}{\partial X^2}(\xi,Y,Z)\,d\xi - \frac{\partial M}{\partial Y}\cdot\frac{\partial\Phi_0}{\partial Y}(M,Y,Z) - \frac{\partial M}{\partial Z}\cdot\frac{\partial\Phi_0}{\partial Z}(M,Y,Z)$$

$$- \frac{\partial\Phi_0}{\partial X}(X,Y,Z)$$

$$= -\frac{\partial\Phi_0}{\partial X}(M,Y,Z) - \frac{\partial M}{\partial Y}\cdot\frac{\partial\Phi_0}{\partial Y}(M,Y,Z) - \frac{\partial M}{\partial Z}\cdot\frac{\partial\Phi}{\partial Z}(M,Y,Z) \quad .$$

Hence $L(\Phi_1) - k^2\Phi_1$ is equal to the scalar product of the vectors $(-1, -\partial M/\partial Y, -\partial M/\partial Z)$ and $(\partial\Phi_0/\partial X, \partial\Phi_0/\partial Y, \partial\Phi_0/\partial Z)_{M(Y,Z),Y,Z}$, but the first of these is along the co-normal to the wave front, $X - M(Y,Z) = 0$, and the second is the gradient of Φ_0, i.e., $L(\Phi_1) - k^2\Phi_1$ is equal to the derivative of Φ_0 along the co-normal.

But since Φ_0 is a solution this gradient is zero and hence $L(\Phi_1) - k^2\Phi_1 = 0$, and similarly $L(\Phi_2) - k^2\Phi_2 = 0$ and so on.

Moreover $\quad \dfrac{\partial\Phi_1}{\partial Z} = \displaystyle\int_{M(Y,Z)}^{X} \frac{\partial\Phi_0}{\partial Z}(\xi,Y,Z)\,d\xi \qquad$ by (A.1) .

Now for $Y \leqslant 0$, $Z = 0$, (i.e., on the wing), $\partial\Phi_0/\partial Z = -1$, and therefore

$$\frac{\partial\Phi_1}{\partial Z}(X,Y,0) = \int_{M(Y,0)}^{X} (-1)\,d\xi = -X + M(Y,0) = -X(Y \leqslant 0)$$

so that $\qquad \dfrac{\partial\Phi_1}{\partial Z} = -X$ on the wing.

Repeating this process it follows that $\partial\Phi_n/\partial Z = -X^n/n!$ on the wing.

For $Z = 0$, $Y \geqslant 0$, $\Phi_0 \equiv 0$, and hence $\Phi \equiv 0$ in the same region, and so on for all the Φ_n's.

Obviously Φ_1, Φ_2, ... are all zero on the wave front.

Thus the Φ_n's obtained in the wing satisfy all the boundary conditions and $L(\Phi_n) = k^2\Phi_n$.

APPENDIX II

A brief description of the Relaxation Method

As applied in the present case this method uses two facts

(i) The derivatives of any function may be expressed to any required degree of accuracy by a linear combination of its values at a discrete set of points, provided that it satisfies certain not very restrictive conditions.

(ii) When the expressions thus obtained are substituted into the linear differential equation, and its boundary conditions, the resulting set of linear equations can be solved (again to any required degree of accuracy) by a method of successive approximation.

If $f(x)$ is any function of x, where values are known at the points $(x_0 \pm nh)$, $(n = 0, \pm 1, \pm 2, \ldots$ and a table of its values is formed, and hence a table of differences, as below

$$
\begin{array}{llllll}
f(x_0 - 2h) & & \delta^2_{-2} & & \delta^4_{-2} & \ldots \\
 & \delta_{-3/2} & & \delta^3_{-3/2} & & \\
f(x_0 - h) & & \delta^2_{-1} & & \delta^4_{-1} & \ldots \\
 & \delta_{-1/2} & & \delta^3_{-1/2} & & \\
f(x_0) & & \delta_0 & & \delta^4_0 & \ldots \\
 & \delta_{1/2} & & \delta^3_{1/2} & & \\
f(x_0 + h) & & \delta^2_1 & & \delta^4_1 & \ldots \\
 & \delta_{3/2} & & \delta^3_{3/2} & & \\
f(x_0 + 2h) & & \delta^2_2 & & \delta^4_2 & \ldots
\end{array}
$$

then
$$hf'(x_0) = \tfrac{1}{2}(\delta_{-1/2} + \delta_{1/2}) - 1/6 \times \tfrac{1}{2}(\delta^3_{-1/2} + \delta^3_{1/2}) + 1/30 \times \tfrac{1}{2}(\delta^5_{-1/2} + \delta^5_{1/2}) - \ldots \quad \text{(A.2)}$$

$$h^2 f''(x_0) = \delta^2_0 - 1/12\, \delta^4_0 + 1/90\, \delta^6_0 - \ldots \quad \text{(A.3)}$$

and so on.

These formulae may be derived by differentiating Stirling's formula (*see* Ref. 9, p. 64). For a rigorous discussion of the subject and in particular of the conditions under which these expressions converge see Steffensen (Ref. 10, pp. 60-71).

In these expressions the first terms may be regarded as being an approximate value of the derivative, and the remaining terms being the corresponding correction.

Consider the values of the dependent variable at a lattice of points covering the region in which the solution is required, as in Fig. 5. Substituting from (A.2) and (A.3) into the differential equation, and its boundary condition when this contains a derivative, a set of linear equations (one for each point) is obtained, connecting the values at points of the lattice. These are now solved by adjusting the values until the conditions are satisfied to the required degree of accuracy.

For a full account of this method see Southwell[11] and Fox[12].

As ϱ tends to $-\infty$ the differential equations tend to the Laplace equation $\partial^2 f/\partial \varrho^2 + \partial^2 f/\partial \theta^2 = 0$, it is possible to infer the manner in which f tends to 0 as ϱ tends to $-\infty$ by considering the solutions of this in an infinite strip. This avoids the necessity for considering very large values of $-\varrho$.

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TABLE 1

Values of Certain Integrals

$$\int_{-X}^{0} \frac{\partial \Phi_{0,0}}{\partial X}(X,Y,0)\,dY = +0\cdot 49994\,X\,, \quad \int_{-X}^{0} \frac{\partial \Phi_{0,1}}{\partial X}(X,Y,0)\,dY = -0\cdot 1657\,X^{3}\,,$$

$$\int_{-X}^{0} \frac{\partial \Phi_{0,2}}{\partial X}(X,Y,0)\,dY = +0\cdot 1122\,X^{5}$$

$$\int_{-X}^{0} \Phi_{0,0}(X,Y,0)\,dY = +0\cdot 74997\,X^{2}\,, \quad \int_{-X}^{0} \Phi_{0,1}(X,Y,0)\,dY = -0\cdot 06225\,X^{4}\,,$$

$$\int_{-X}^{0} \Phi_{0,2}(X,Y,0)\,dY = +0\cdot 002391\,X^{6}$$

$$\int_{-X}^{0} \Phi_{1,0}(X,Y,0)\,dY = +0\cdot 41667\,X^{3}\,, \quad \int_{-X}^{0} \Phi_{1,1}(X,Y,0)\,dY = -0\cdot 01661\,X^{5}\,,$$

$$\int_{-X}^{0} \Phi_{1,2}(X,Y,0)\,dY = +0\cdot 000417\,X^{7}$$

$$\int_{-X}^{0} \Phi_{2,0}(X,Y,0)\,dY = +0\cdot 14582\,X^{4}\,, \quad \int_{-X}^{0} \Phi_{2,1}(X,Y,0)\,dY = -0\cdot 00346\,X^{6}\,,$$

$$\int_{-X}^{0} \Phi_{2,2}(X,Y,0)\,dY = +0\cdot 0000625\,X^{8}$$

$$\int_{-X}^{0} \Phi_{3,0}(X,Y,0)\,dY = +0\cdot 03750\,X^{5}\,, \quad \int_{-X}^{0} \Phi_{3,1}(X,Y,0)\,dY = -0\cdot 000589\,X^{8}\,,$$

$$\int_{-X}^{0} \Phi_{4,0}(X,Y,0)\,dY = +0\cdot 007637\,X^{6}\,, \quad \int_{-X}^{0} \Phi_{4,1}(X,Y,0)\,dY = -0\cdot 000085\,X^{9}\,,$$

$$\int_{-X}^{0} \Phi_{5,0}(X,Y,0)\,dY = +0\cdot 001288\,X^{7}\,,$$

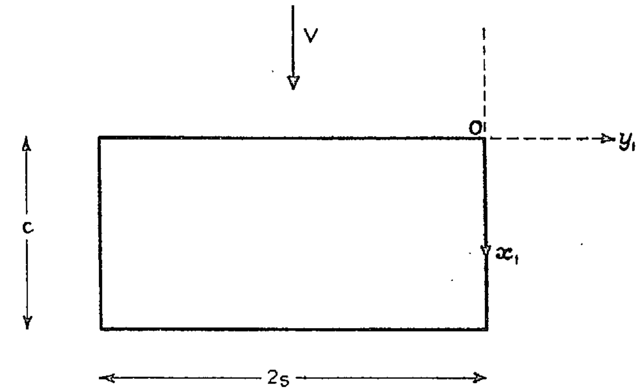
$$\int_{-X}^{0} \Phi_{6,0}(X,Y,0)\,dY = +0\cdot 0001855\,X^{8}\,,$$

$$\int_{-X}^{0} \Phi_{7,0}(X,Y,0)\,dY = +0\cdot 0000233\,X^{9}\,,$$

TABLE 2

Leading-Edge Derivative Coefficients

	M	δl_z	$\delta l_{\dot{z}}$	δl_α	$\delta l_{\dot{\alpha}}$	δm_z	$m_{\dot{z}}$	δm_α	$\delta m_{\dot{\alpha}}$
$\lambda = 0 \cdot 2$									
	1·2	−0·156	−2·034	−2·074	+3·201	+0·116	+1·325	+1·357	−2·381
	1·4	−0·042	−1·006	−1·016	+0·705	+0·031	+0·666	+0·674	−0·528
	1·6	−0·019	−0·629	−0·634	+0·270	+0·014	+0·417	+0·421	−0·202
	1·8	−0·011	−0·441	−0·443	+0·131	+0·008	+0·293	+0·295	−0·098
$\lambda = 0 \cdot 4$									
	1·2	−0·504	−1·445	−1·584	+2·568	+0·365	+0·862	+0·972	−1·858
	1·4	−0·156	−0·904	−0·944	+0·654	+0·115	+0·585	+0·617	−0·484
	1·6	−0·074	−0·593	−0·612	+0·258	+0·055	+0·389	+0·404	−0·192
	1·8	−0·043	−0·423	−0·434	+0·127	+0·032	+0·279	+0·288	−0·095
$\lambda = 0 \cdot 6$									
	1·4	−0·311	−0·756	−0·840	+0·575	+0·227	+0·469	+0·535	−0·420
	1·6	−0·155	−0·538	−0·578	+0·239	+0·115	+0·345	+0·377	−0·176
	1·8	−0·093	−0·396	−0·420	+0·120	+0·069	+0·257	+0·276	−0·089
	2·0	−0·062	−0·304	−0·320	+0·068	+0·046	+0·199	+0·211	−0·051
	M	l_z	$l_{\dot{z}}$	l_α	$l_{\dot{\alpha}}$	m_z	$m_{\dot{z}}$	m_α	$m_{\dot{\alpha}}$
$\lambda = 0 \cdot 2$									
	1·2	+0·126	+2·803	+2·846	−1·696	−0·083	−1·349	−1·382	+1·102
	1·4	+0·041	+1·999	+2·013	−0·020	−0·027	−0·989	−0·999	+0·011
	1·6	+0·020	+1·585	+1·591	+0·293	−0·013	−0·788	−0·793	−0·196
	1·8	+0·012	+1·328	+1·332	+0·372	−0·008	−0·662	−0·665	−0·248
$\lambda = 0 \cdot 4$									
	1·2	+0·390	+2·292	+2·442	−1·127	−0·241	−0·979	−1·089	+0·653
	1·4	+0·150	+1·880	+1·933	+0·043	−0·097	−0·901	−0·940	−0·039
	1·6	+0·076	+1·537	+1·563	+0·310	−0·050	−0·753	−0·772	−0·210
	1·8	+0·045	+1·303	+1·318	+0·379	−0·030	−0·643	−0·655	−0·254
$\lambda = 0 \cdot 6$									
	1·4	+0·287	+1·712	+1·820	+0·137	−0·179	−0·779	−0·858	−0·114
	1·6	+0·154	+1·465	+1·520	+0·337	−0·099	−0·699	−0·740	−0·231
	1·8	+0·094	+1·264	+1·297	+0·390	−0·061	−0·615	−0·639	−0·262
	2·0	+0·062	+1·111	+1·133	+0·395	−0·040	−0·545	−0·561	−0·264



The axis of z_1 is perpendicularly out of the paper through O

FIG. 1. Plan of wing.

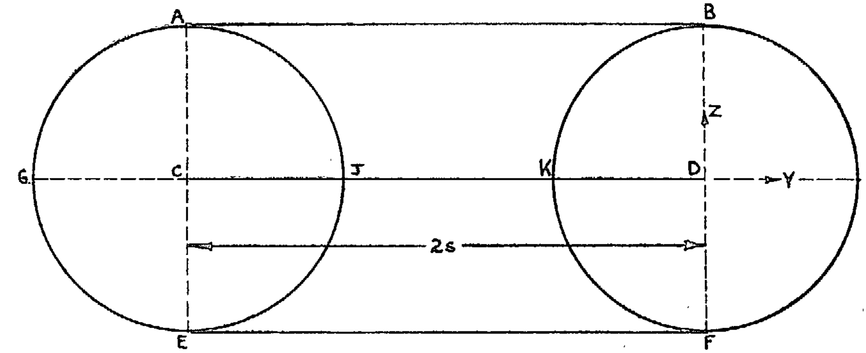


FIG. 3. Section by any plane. X = constant > 0.

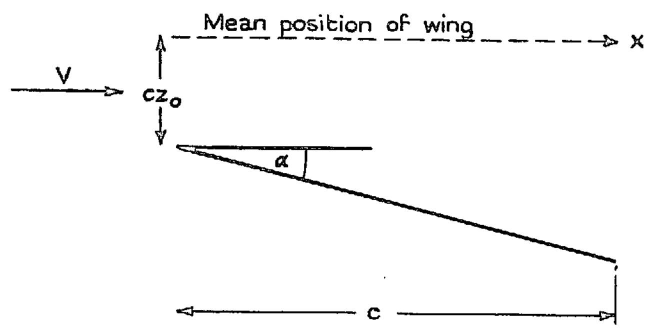


FIG. 2. Mode of vibration of wing.

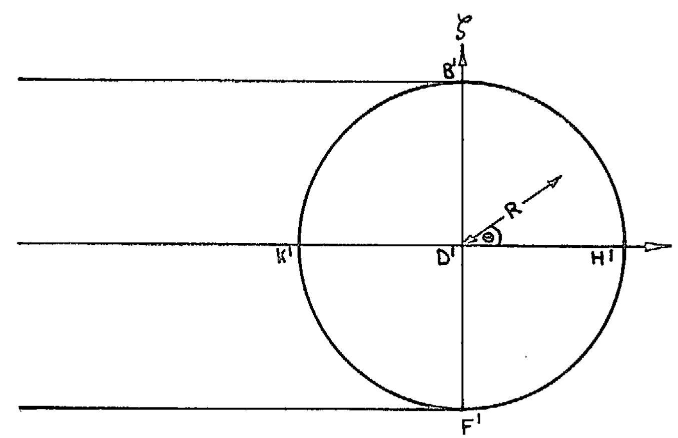


FIG. 4. The (η,ζ)-plane.

155	202	262	338	435	557	627	704	785	868	945	1000	$\theta=\pi$
					500	554	609	664	712	745	741	
143	183	232	292	361	439	478	515	545	563	555	500	
					378	404	424	434	427	389	293	
122	154	192	234	278	319	334	341	336	312	254	134	
					264	271	269	255	221	157	34	
97	119	145	172	198	214	215	208	189	155	95	0	
					169	167	157	138	107	60	0	
66	81	97	112	124	129	125	115	99	73	38	0	
					93	89	81	68	48	24	0	
34	41	48	56	60	60	57	51	42	30	14	0	
					29	28	24	20	14	7	0	
0	0	0	0	0	0	0	0	0	0	0	0	$\theta=0$

$\rho=-\pi$ $\rho=-\frac{\pi}{2}$ $\rho=0$

FIG. 5. Values of $f_{0,0} \times 10^3$.

	−14	−18	−24	−32	−43	−50	−58	−67	−75	−81	−83	$\theta=\pi$
					−42	−49	−56	−63	−69	−72	−69	
	−13	−17	−22	−30	−39	−45	−50	−54	−56	−53	−42	
					−35	−39	−42	−43	−41	−33	−17	
	−12	−15	−19	−24	−30	−33	−34	−33	−29	−19	−4	
					−25	−26	−26	−24	−19	−10	0	
	−9	−12	−15	−18	−20	−20	−20	−17	−13	−6	0	
					−16	−15	−15	−12	−8	−3	0	
	−6	−8	−10	−11	−12	−11	−10	−8	−5	−2	0	
					−8	−8	−7	−5	−3	−1	0	
	−3	−4	−5	−5	−5	−5	−4	−3	−2	−1	0	
					−2	−2	−2	−2	−1	−0	0	
0	0	0	0	0	0	0	0	0	0	0	0	$\theta=0$

$\rho=-\pi$ $\rho=-\frac{\pi}{2}$ $\rho=0$

FIG. 6. Values of $f_{0,1} \times 10^3$.

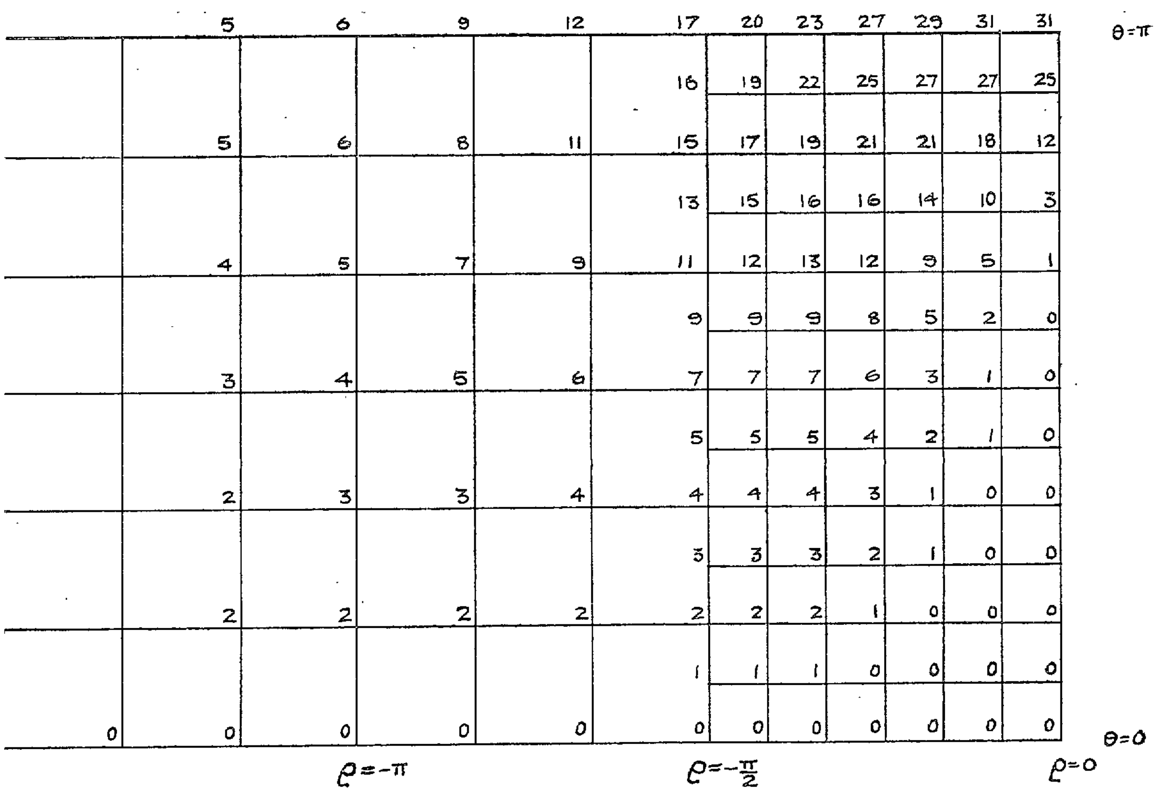


FIG. 7. Values of $f_{0,2} \times 10^4$.

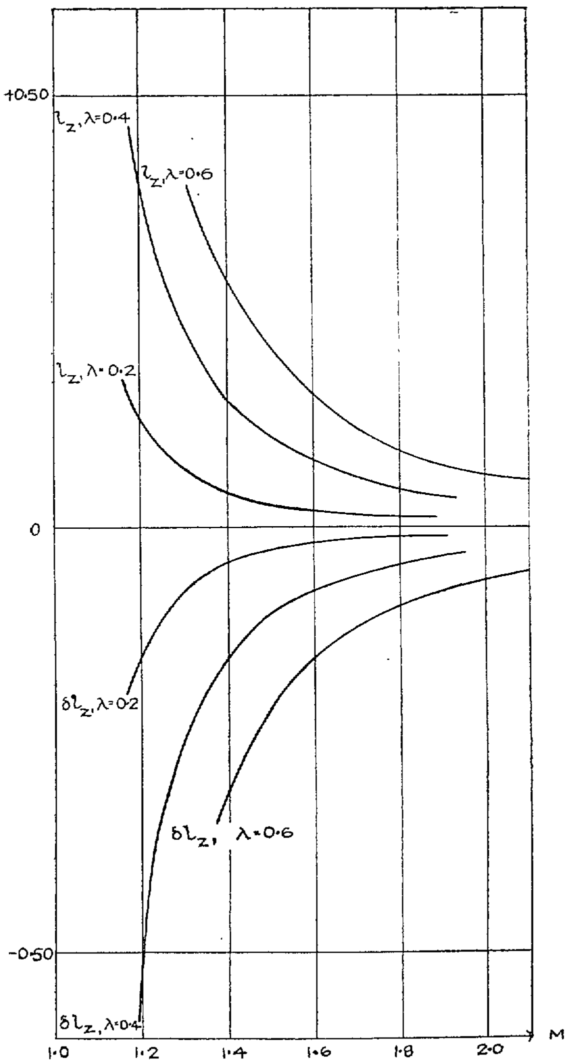


FIG. 8. Variation of l_z and δl_z with M and λ.

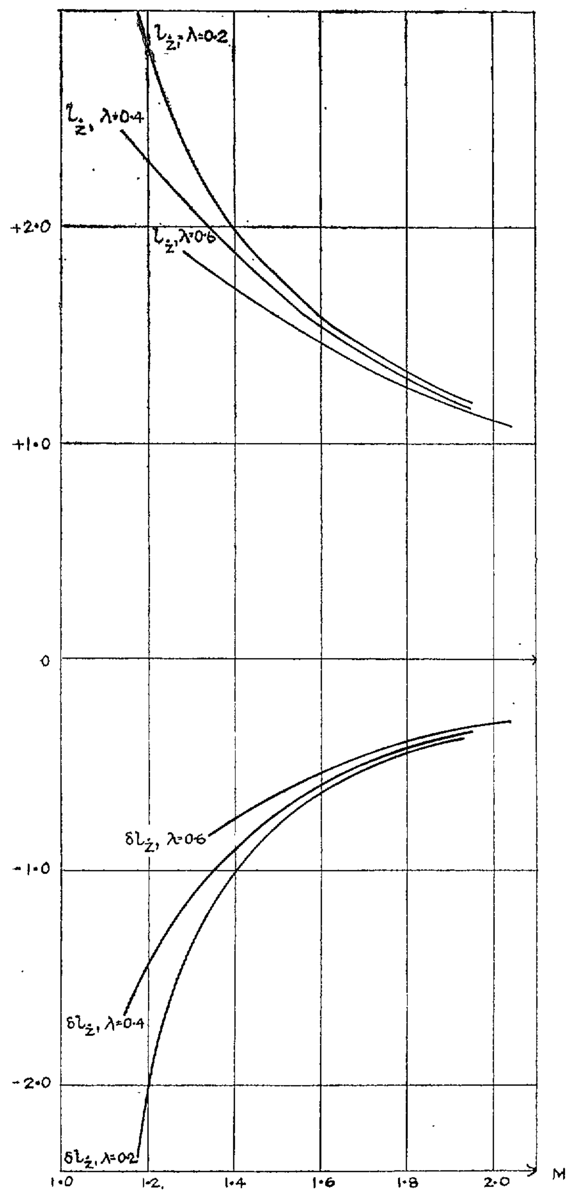


FIG. 9. Variation of $\delta l_{\dot{z}}$ and $l_{\dot{z}}$ with M and λ.

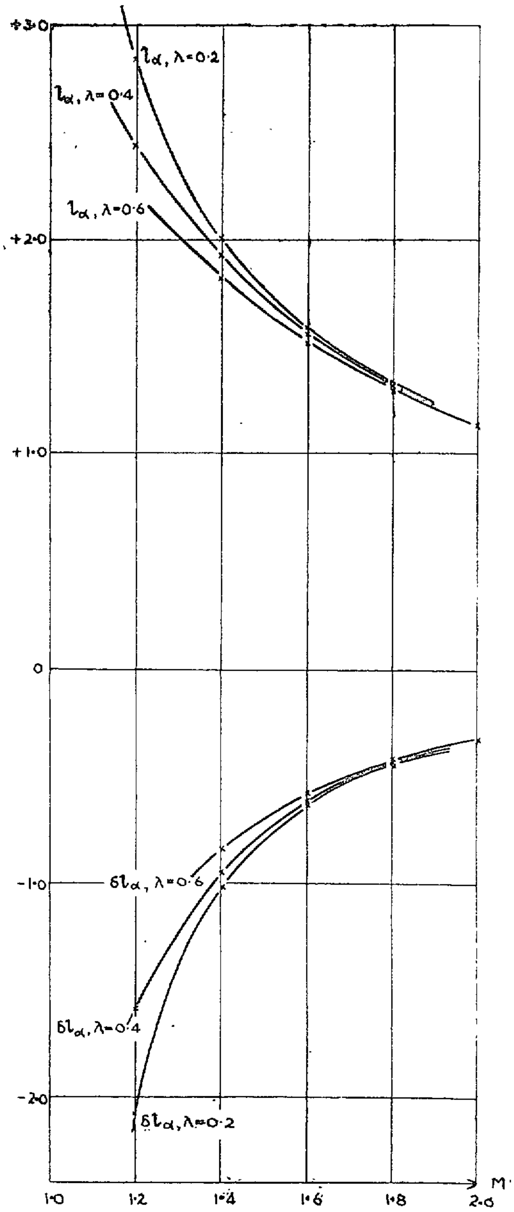


FIG. 10. Variation of δl_α and l_α with M and λ.

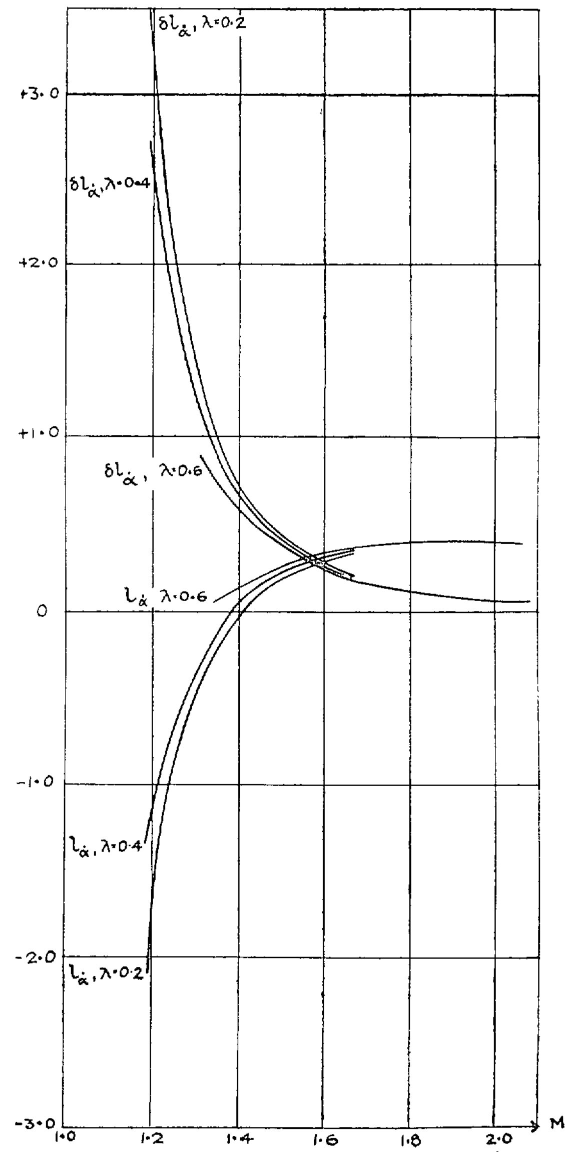


FIG. 11. Variation of $\delta l_{\dot\alpha}$ and $l_{\dot\alpha}$ with M and λ.

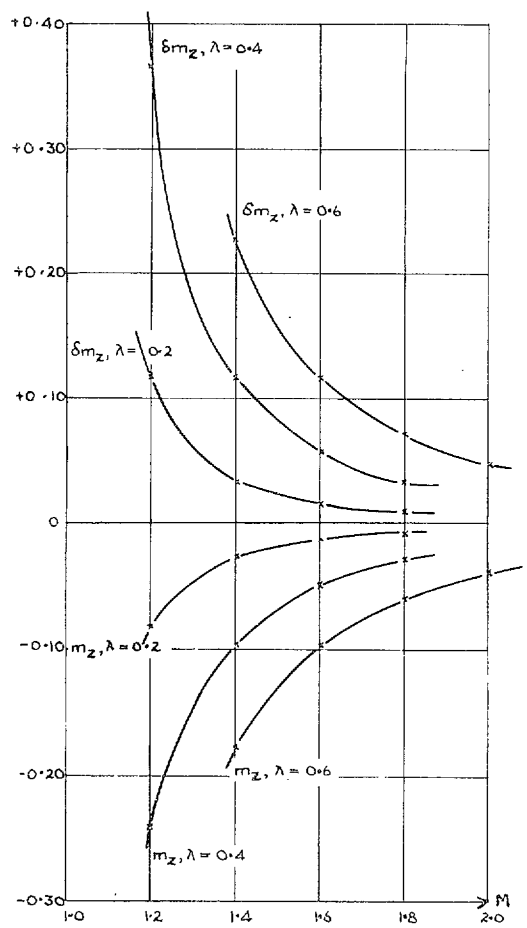


FIG. 12. Variation of m_z and δm_z with M and λ.

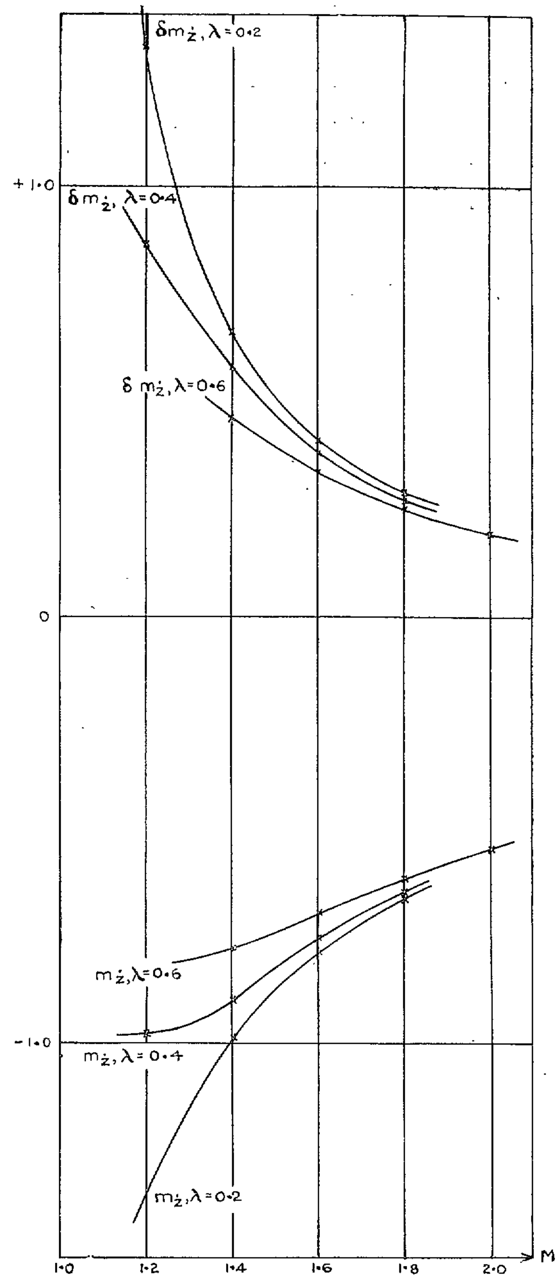


FIG. 13. Variation of $m_{\dot{z}}$ and $\delta m_{\dot{z}}$ with M and λ.

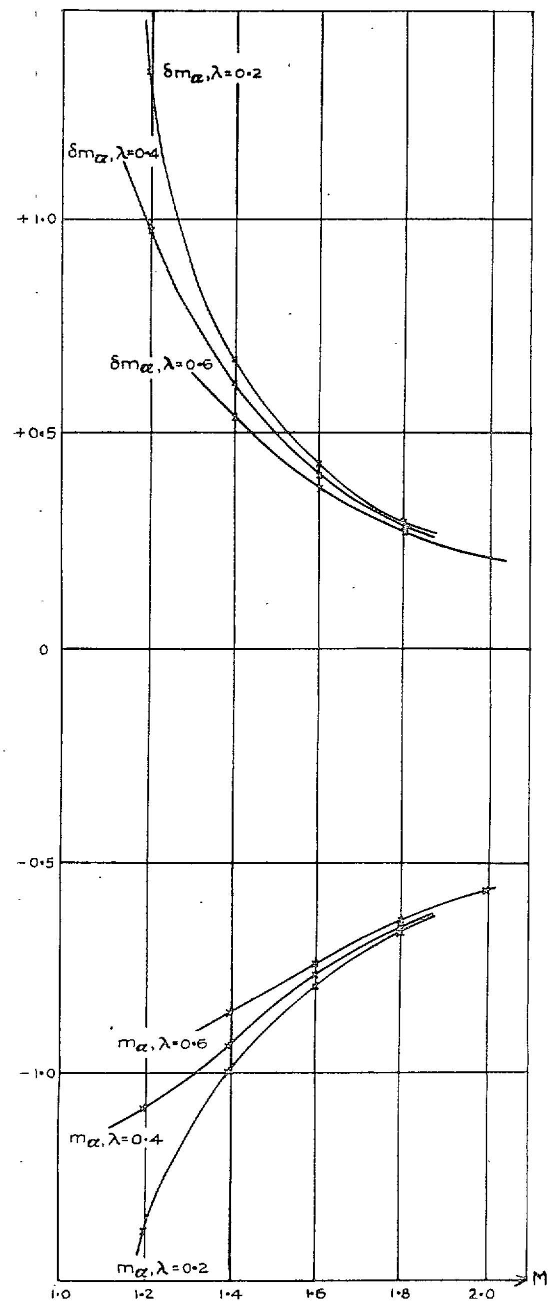


FIG. 14. Variation of δm_α and m_α with M and λ.

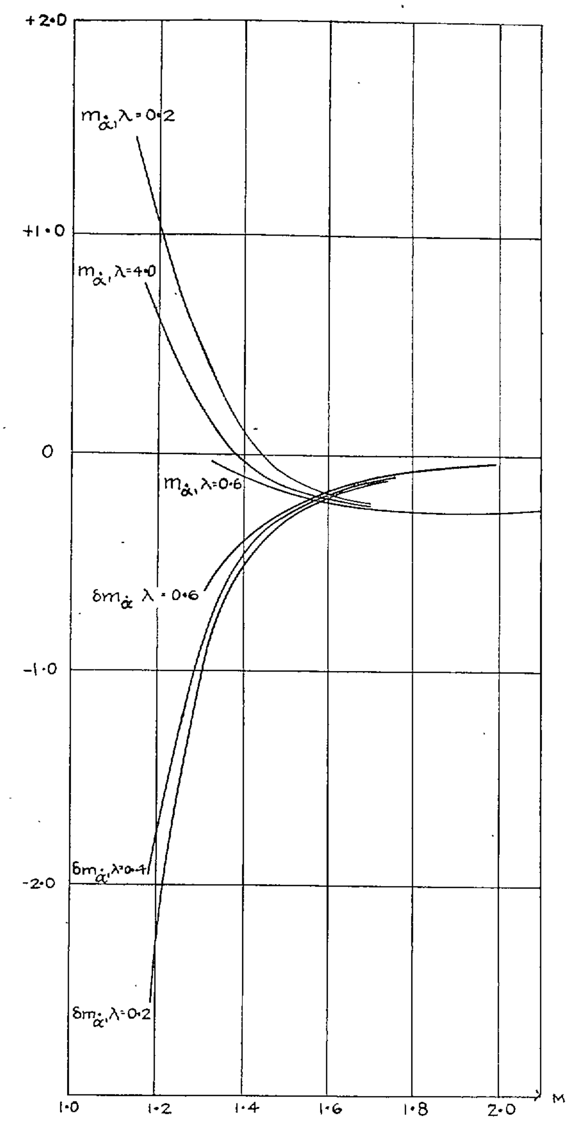


FIG. 15. Variation of $m_{\dot\alpha}$ and $\delta m_{\dot\alpha}$ with M and λ.
(*For* $\lambda = 4 \cdot 0$, *read* $\lambda = 0 \cdot 4$.)

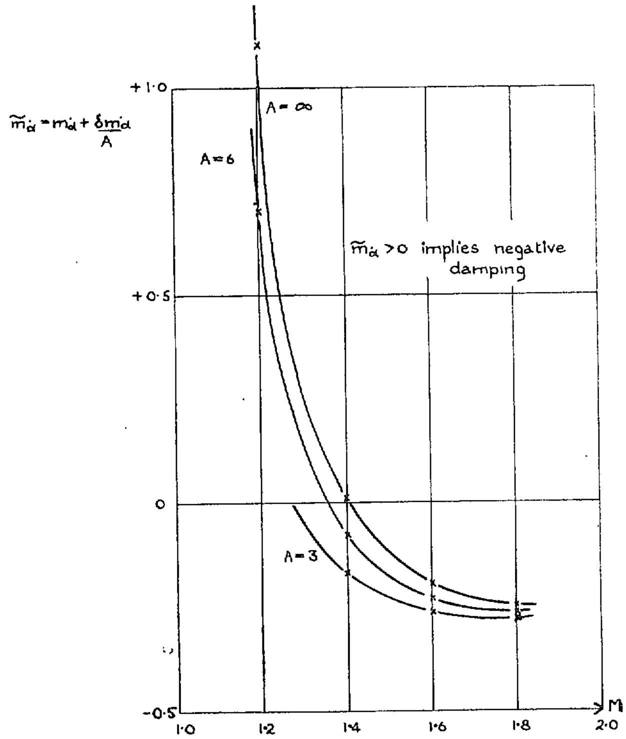


FIG. 16. Variation of $\widetilde{m}_{\dot\alpha} = m_{\dot\alpha} + \delta m_{\dot\alpha}/A$ with M and A ($\lambda = 0 \cdot 2$).

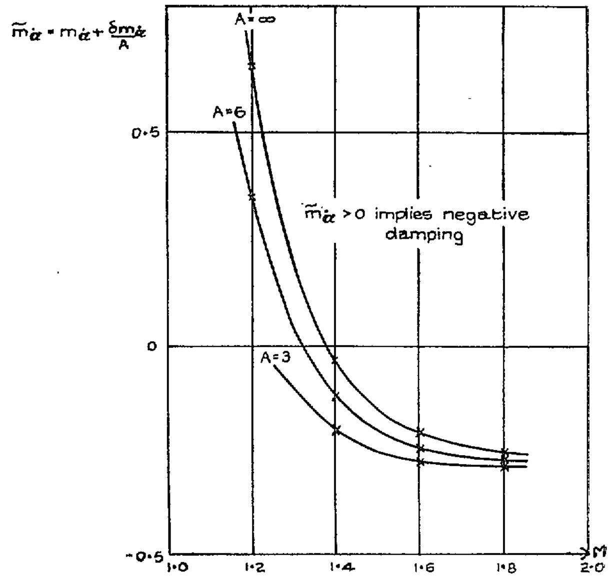


FIG. 17. Variation of $\widetilde{m}_{\dot\alpha}, = m_{\dot\alpha} + \delta m_{\dot\alpha}/A$ with M and $A(\lambda = 0 \cdot 4)$.

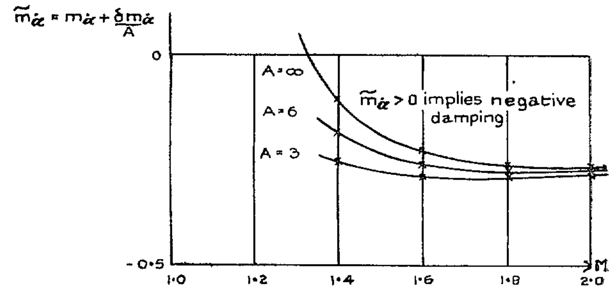


FIG. 18. Variation of $\widetilde{m}_{\dot\alpha}, = m_{\dot\alpha} + \delta m_{\dot\alpha}/A$ with M and $A(\lambda = 0 \cdot 6)$.

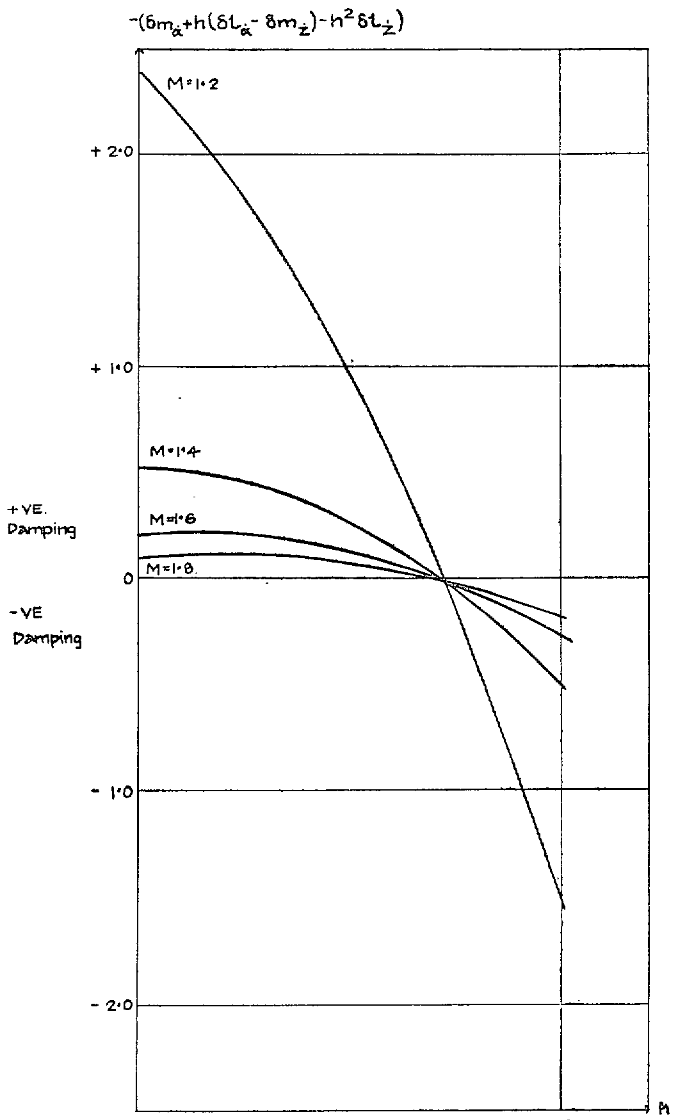


FIG. 19. Effect of axis position on damping effect of tip correction ($\lambda = 0 \cdot 2$).

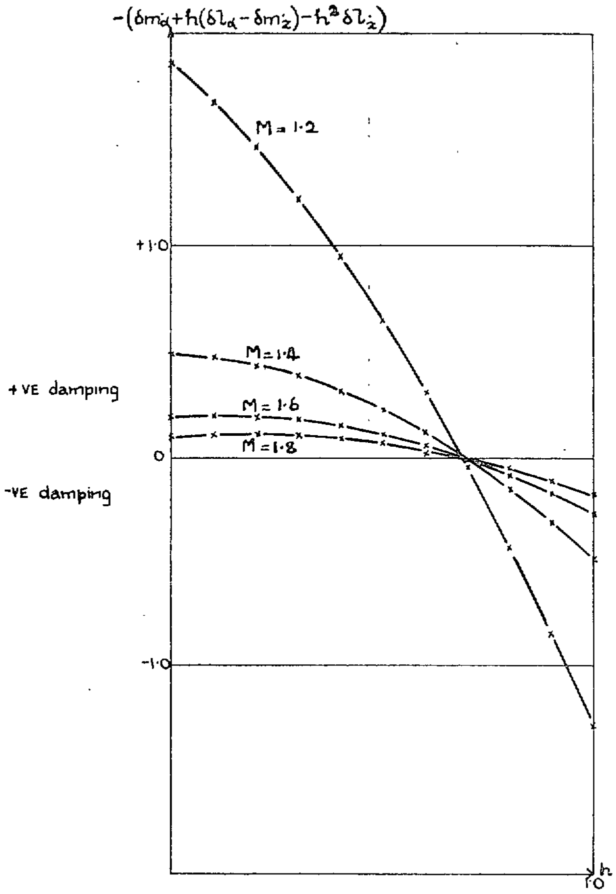


FIG. 20. Effect of axis position on damping effect of tip correction ($\lambda = 0 \cdot 4$).

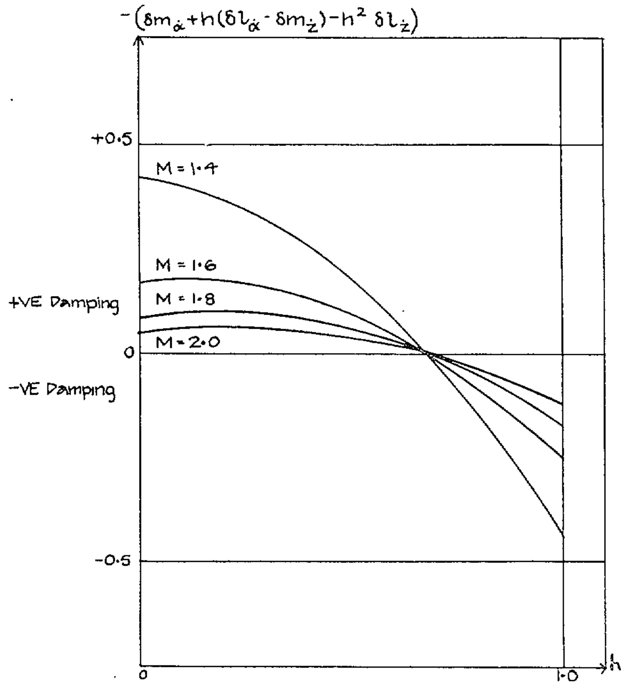


FIG. 21. Effect of axis position on damping effect of tip correction ($\lambda = 0 \cdot 6$).

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