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Note on the Southwell Method for Estimating Critical Loads

By

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of the Engineering Division, N.P.L.

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Summary.—(*a*) *Purpose of Note.*—To draw attention to certain restrictions on the use of the ' Southwell plot ' to estimate critical loads in cases differing in conditions from those which the method was first proposed.

(*b*) *Range of Note.*—The effects on the ' Southwell plot ' of variation of stress distribution, of elastic failure of the material and of other variation of critical stress, however it may be occasioned, is examined.

(*c*) *Conclusions.*—The ' Southwell plot ' is strictly applicable only to deflections which go to infinity at a definite critical load. In other cases the plot usually over-estimates the buckling load ; but the error should seldom be important.

1. *Introduction.*—In tests on structures liable to fail by instability it is often convenient to record the gradual development of the lateral deflection δ in the buckling mode against values of the applied load P. If then values of δ are plotted against values of δ/P, the relation is often nearly linear and the slope of this line sometimes affords an accurate estimate of the critical load P_0.

The theory on which the plot is based was founded specifically on an assumed hyperbolic relation between δ and P (Southwell, 1932), namely $(\delta+\delta_0)(P_0-P)=P_0\delta_0$, or $P_0\delta-P\delta_0-P\delta=0$. Dividing by P we have $P_0(\delta/P) - \delta = \delta_0$. Thus the relation between δ and (δ/P) is linear, the slope of the line, when δ is plotted against (δ/P) as abscissa, is P_0 and the intercept on the axis of δ is $-\delta_0$. If the value of P is in fact less than the value indicated by the hyperbolic relation $P = P_0\delta/(\delta + \delta_0)$, the values of (δ/P) are increased and the slope of the $\delta - (\delta/P)$ line is correspondingly reduced, and *vice versa*. Accordingly the method should strictly be used only in cases to which the hyperbolic relation applies, but this restriction is frequently overlooked, and the plot is used in cases for which it was not intended. In such cases a moderately good linear relation between δ and δ/P may still be obtained but the slope of the line may not correspond closely to the critical value P_0 of the applied load.

The application of the Southwell method to the whole range of buckling problems has previously been considered by Donnell[4]. He showed that the method was strictly valid only if the differential equation governing the deformation were linear, so that the buckling of plates, in which ' part at least of the extensional strains are proportional to the square of the normal displacement ' should be properly excluded. Donnell showed, however, that the error introduced by non-linear behaviour as also by elastic failure of material could sometimes be negligible, and he concluded that application of the ' Southwell plot ' need not be too rigorously restricted.

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In comparison with Donnell's work, the present analysis represents an attempt to generalize his conclusions and to restate then in forms more readily applicable to practical cases; in respect of the buckling of plates, by reference to the 'slope after buckling,' η (*see* below), and by comparison with the load deformation curve in which buckling is evidenced by this change of slope; in respect of elastic failure, by reference to specific forms of stress-strain curves and by examination of the general effect of variation of the critical load, however that variation may be occasioned. Finally an attempt is made to postulate specific rules by which the validity of the 'Southwell plot' may be judged.

No reference is made in the present note to the effect of exaggeration of initial irregularities other than that corresponding to the final buckled form. This effect is treated by Donnell, and nothing of value could be added to his analysis except by reference to specific experiments. It may, however, be remarked that the results of a few tests made on pin ended struts having large initial deformations in three half-waves suggested that this effect of initial irregularities in distorting the Southwell plot should usually be unimportant.

2. *Buckling of Plates.*—In the buckling of plates of which the edges are supported (Cox, 1945), the strain energy function may be expressed in the form

$$f_e^2 + f_c w^2 - f_e w^2 + \{1/4(1-\eta)\} w^4,$$

where $(w/\sqrt{E})$ is the relative (non-dimensional) amplitude of the buckle in some definite form, so that w^2 has the dimensions of E, the Young's modulus, that is, the dimensions of a stress.

In this formula

- η is a constant less than unity,
- f_e is the applied edge stress (proportional to the average strain),
- f_c is the critical value of f_e at which buckling begins.

In order to cater for the Southwell plot, account must be taken of the initial amplitude w_0 in the buckled form[1] when the modified form of the strain energy function becomes

$$f_e^2 + f_c (w - w_0)^2 - f_e(w^2 - w_0^2) + \{1/4(1-\eta)\} (w^2 - w_0^2)^2. \quad \text{(2)}$$

Differentiation of the expression (2) with respect to f_e and w leads to the relations

$$f_a = f_e - \tfrac{1}{2}(w^2 - w_0^2) \quad \text{(3)}$$

and

$$f_c(w - w_0) - f_e w + \{1/2(1-\eta)\} w(w^2 - w_0^2) = 0 \quad \text{(4)}$$

where f_a is the average stress in the plate.

Dividing relation (3) by f_c and (4) by $f_c w_0$ and writing $(w - w_0)/w_0 = x$ and $w_0^2/f_c = A$ leads to the formulæ

$$\frac{f_a}{f_c} = \frac{x}{x+1} + \frac{A\eta x(x+2)}{2(1-\eta)} \quad \text{(5)}$$

and

$$\frac{f_e}{f_c} = \frac{x}{x+1} + \frac{Ax(x+2)}{2(1-\eta)} \quad \text{(6)}$$

so that

$$x/(f_a/f_c) = 1 \Big/ \left\{ \frac{1}{x+1} + \frac{A\eta(x+2)}{2(1-\eta)} \right\} \quad \text{(7)}$$

In Fig. 1a, values of f_a/f_c are plotted against values of f_e/f_c for the cases $\eta = \frac{1}{2}$ with $A = 0{\cdot}01$, $0{\cdot}001$ and $0{\cdot}0001$, and in Fig. 1b, values of $(w - w_0)/\sqrt{f_c}$ $(= x\sqrt{A})$, corresponding to δ in section 1, are plotted against values of $(w - w_0)\sqrt{f_c/f_a}$ $[= x\sqrt{A}/(f_a/f_c)]$, corresponding to δ/P of section 1.

In order to facilitate comparison of Figs. 1a and b lines representing $f_e/f_c = 1$ and $f_a/f_c = 1$ are shown dotted on Fig. 1b.

If curves similar to those shown in Figs. 1a and b have been derived from experimental results, the value of f_c might be estimated either by drawing a straight line through the upper part of the curve in Fig. 1a and recording the intersection of this line with the line $f_a = f_e$ (the ' change of slope ' method) or by measuring the slope of the line in Fig. 1b (the Southwell method). Both methods must be to some extent subjective, and in estimating the values of f_c thus recorded only probable ranges in which the estimates may be expected to lie may be stated. In dealing with Fig. 1a, allowances have to be made for uncertainty as to the initial slope of the line which should represent $f_a = f_e$ and for the effect of modification of the buckled wave form as f_e is increased above f_c; the latter effect causes the upper part of the $f_a - f_e$ curve to become curved downwards, so that the estimate of the slope after buckling has to be based on a fairly short length of the line. In dealing with Fig. 1b, allowance has to be made for the inevitable uncertainty of the lower part of the curve when $w - w_0$ is very small and care must be taken to base the estimate on the deflections before f_e exceeds f_c or at least before f_a exceeds f_c. The irregularity of the Southwell plot at low values of $w - w_0$ is due primarily (that is apart from actual observational errors) to the effect of initial irregularities other than those which represent the final buckled form. These irregularities also affect the $f_a - f_e$ diagram but to a less extent and only in respect of the initial slope which should represent $f_a = f_e$.

From the table of estimates given in Fig. 1 it will be seen that the Southwell plot tends to over-estimate the value of f_c whereas the change of slope method tends to under-estimate it. Actually the errors by the two methods are likely to be of the same order, so that the mean of the two estimates should seldom be more than 5 per cent in error. In fact if $A < 1/1000$ this mean value may usually be within 2 to 3 per cent of the true value.

It is not easy to describe the relation of the value of A to the actual amplitude of the initial irregularities, but if the maximum amplitude of the initial wave is W_0 and its wavelength is 2λ, $W_0{}^2 = E(\pi W_0/2\lambda)^2$. Then $A = K(W_0/t)^2$ where t is the thickness of the plate and K is numerical constant of order unity. Thus, even for $A = 1/100$, W_0 is of order $t/10$ and $A = 10^{-4}$ corresponding to W_0 of order $t/100$ represents a high standard of finish of the test panel. In practice values of $A < 1/1000$ may be expected to be realised by good workmanship.

If $\eta = 0$ the second term of equation (5) disappears and formula (7) reduces to $x/(f_a/f_c) = x + 1$, which represents a true linear plot whatever the value of A. In general the accuracy of the Southwell plot increases as η is reduced. In application to cases of flexure or to other cases for which η is zero or small, the outstanding merit of the Southwell method is that it thus provides a close estimate of the value of f_c particularly when the presence of large initial irregularities prevents a close approach to this stress in the actual test. This is the case for which the method was first proposed[1].

The most important conclusion to be drawn from the present examination of cases for which $\eta \neq 0$ is that the Southwell method fails to retain this merit. When the initial irregularities are small, the method is accurate; but then it is difficult to use and the change of slope method is relatively easier. When the initial irregularities are large the Southwell method may seriously over-estimate the value of the critical load. In such cases the mean of the two estimates by the Southwell method and the change of slope method may afford a reasonably accurate value of the critical load.

3. *Effect of Elastic Failure.*—The effect of elastic failure in a buckled plate (or other structure in which the stress is not uniform) is so complicated that to discuss this effect attention will be restricted to the case of a simple strut buckling under uniform compression. Moreover, it will be assumed that the initial deflection of the strut is small, so that the variation of stress across its section remains slight nearly up to the true critical stress. On this basis, if the initial deflection is y_0, the added (*i.e.*, the measured) deflection y under stress f is $fy_0/\{(\pi^2E'k^2/l^2) - f\}$, where E' is the tangent modulus of the material at the stress f and k is the radius of gyration of the strut section. This relation may be written in the form

$$(y/y_0) = (f/f_0)/\{(E'/E) - (f/f_0)\} \qquad (8)$$

where $f_0 = \pi^2Ek^2/l^2$ is the Euler stress.

From the form (8) it is clear that the initial portion of the Southwell plot, for values of f so low that $E' = E$, must be a straight line of slope f_0. On the other hand the true critical stress f_c is defined by $(E_c/E) - (f_c/f_0) = 0$, when E_c is the tangent modulus at the stress f_c and the final stage in the approach to this critical condition may be effected more indirectly by reduction of E_c/E than directly by the increase of f towards f_c. Thus the upper part of the Southwell plot will always tend to the slope f_c.

These considerations are illustrated in Figs. 2a and b; Fig. 2a describes the behaviour of a material having a slow yield and Fig. 2b that of a material having an abrupt yield. From these diagrams it will be seen that the reliability of the Southwell plot depends entirely on how far it is extended beyond the proportional limit of the material. When this limit is fairly low the plot up to 90 per cent of the true critical stress affords an estimate of that stress which is only 1 or 2 per cent above the true value; but when the proportional limit of the material is high a plot even up to 97 per cent of the true critical stress may still over-estimate its value by nearly 10 per cent. From Figs. 2a and b it may also be concluded that the Southwell method of determining the value of f_c is feasible for material having a slow yield, for which it may be expected to be moderately accurate; but that for a material having an abrupt yield the method is likely to fail completely.

Fortunately the whole question of material properties may be evaded by referring instead to the values of $(y/y_0)/(f/f_0)$. Taking the intersection of the Southwell line (in this case 'curve') with the axis of abscissæ as unity, it suffices to postulate that the plot shall extend up to about 5 on this axis. Provided that this condition be fulfilled the estimate of f_c afforded by the upper part of this curve (say from 3 to 5 on the axis of abscissæ) is unlikely to be more than 5 per cent in excess of the true value. When this condition is not fulfilled the actual maximum value of f reached at collapse is probably not more than 5 per cent below the true value of f_c, unless the initial deflection of the strut were so large that the stresses induced by the moment $f_c y_0$ were comparable with f_c, that is unless y_0 were comparable with k^2/h where h is the half-depth of the (symmetrical) section. Or, if f_m is the highest value of f at which a deflection reading is obtained and if f_s is the slope of the Southwell plot up to that point, the geometric mean $(f_m f_s)^{1/2}$ appears usually to exceed f_c by about 3 to 5 per cent; but a more detailed investigation would be necessary to establish the reliability of this method of estimation.

4. *General Effect of Variation of the Critical Load.*—The case of elastic failure represents a condition under which the critical load decreases as the stress is increased; but the critical load may vary with the stress for other reasons. For instances in the case of torsional instability of a stringer attached to thin sheet the critical load may increase with the stress because the support afforded by the sheet may increase as membrane stresses are developed in it. This effect has been examined by Farrar and others of the Bristol Aeroplane Co.[3] and this specific case will not be considered here. Instead the general effect of variation of the critical load will be examined.

If the critical stress varies linearly with the applied stress f, the formula (8) may be written in the form

$$(y/y_0) = f/(f_0 + af - f), \quad \ldots \quad (9)$$

so that the critical stress increases from f_0 to $f_0/(1 - a) = f_1$ as f increases from zero up to f_1. Then formula (9) may be written

$$(1 - a)(y/y_0) = f/(f_1 - f) \quad \ldots \quad (9a)$$

and

$$(1 - a)(y/y_0)(f_1/f) = f_1/(f_1 - f) = 1 + (1 - a)(y/y_0). \quad \ldots \quad (9b)$$

Thus the Southwell plot is a straight line of slope f_1 through the point $(y/y_0)(f_1/f) = 1/(1 - a)$, and this plot indicates the correct value f_1 of the critical stress.

This conclusion can be applied to any mode of continuous variation of the critical stress by drawing the variation of f_c with f (Fig. 3a) and noting that the slope of the Southwell plot (Fig. 3b) is equal to the stress at which the tangent to the $f_c - f$ curve meets the line $f_c = f$. Thus, if the $f_c - f$ curve is concave downwards (Fig. 3a), the Southwell plot tends to over-estimate the value of the critical stress; whereas if it is concave upward, the Southwell plot tends to under-estimate. It will be seen that Figs. 2a and b are special cases of Figs. 3a and b.

In general, unless the variation of f_c with f is very irregular, as f approaches f_c the Southwell plot should afford a moderately accurate estimate of the true value f_1 of f_c at failure.

5. *Conclusions.*—From this survey of the effects of three types of disturbing influences on the Southwell method of determining critical loads, it appears that the method should afford moderately close estimates of the true critical loads in most ordinary cases. Its error will usually be in excess of the true value and where this error is due to elastic failure of the material the geometric mean between the highest stress at which a deflection is recorded and the value indicated by the Southwell plot up to that final deflection may often be very close to the true value.

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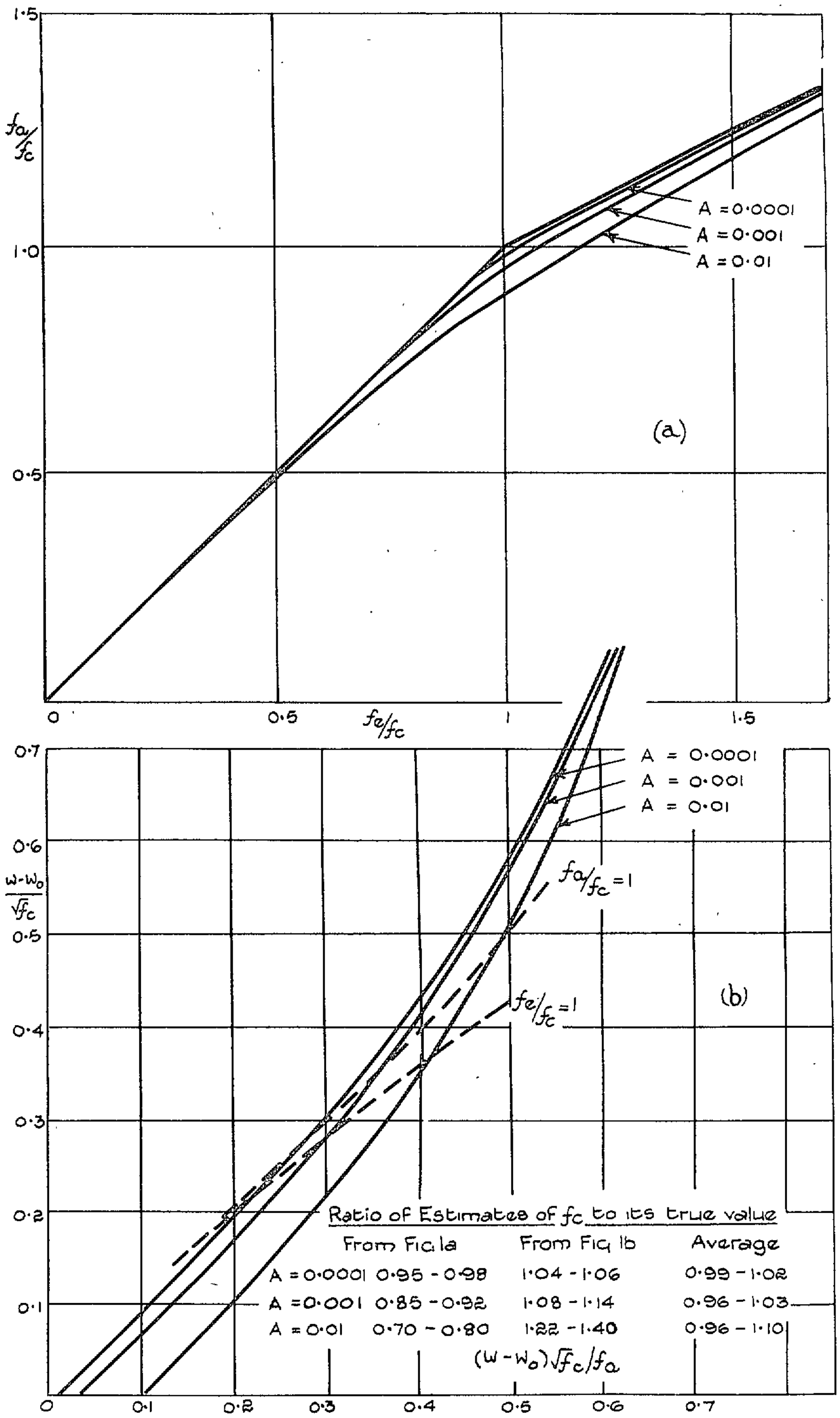

FIG. 1. Comparison of the Southwell method with the change-of-slope method.

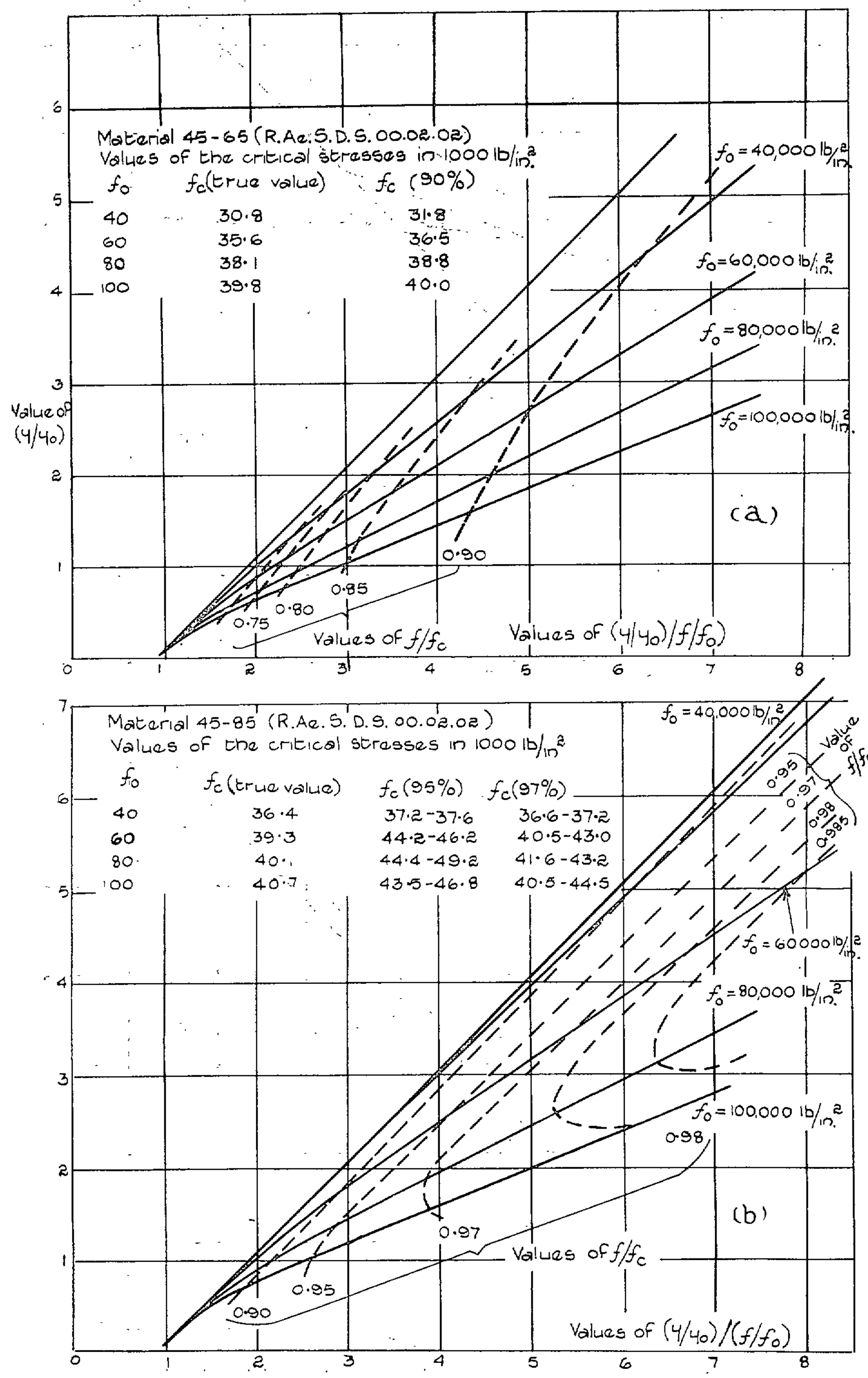

FIG. 2. Effect of elastic failure of a strut.

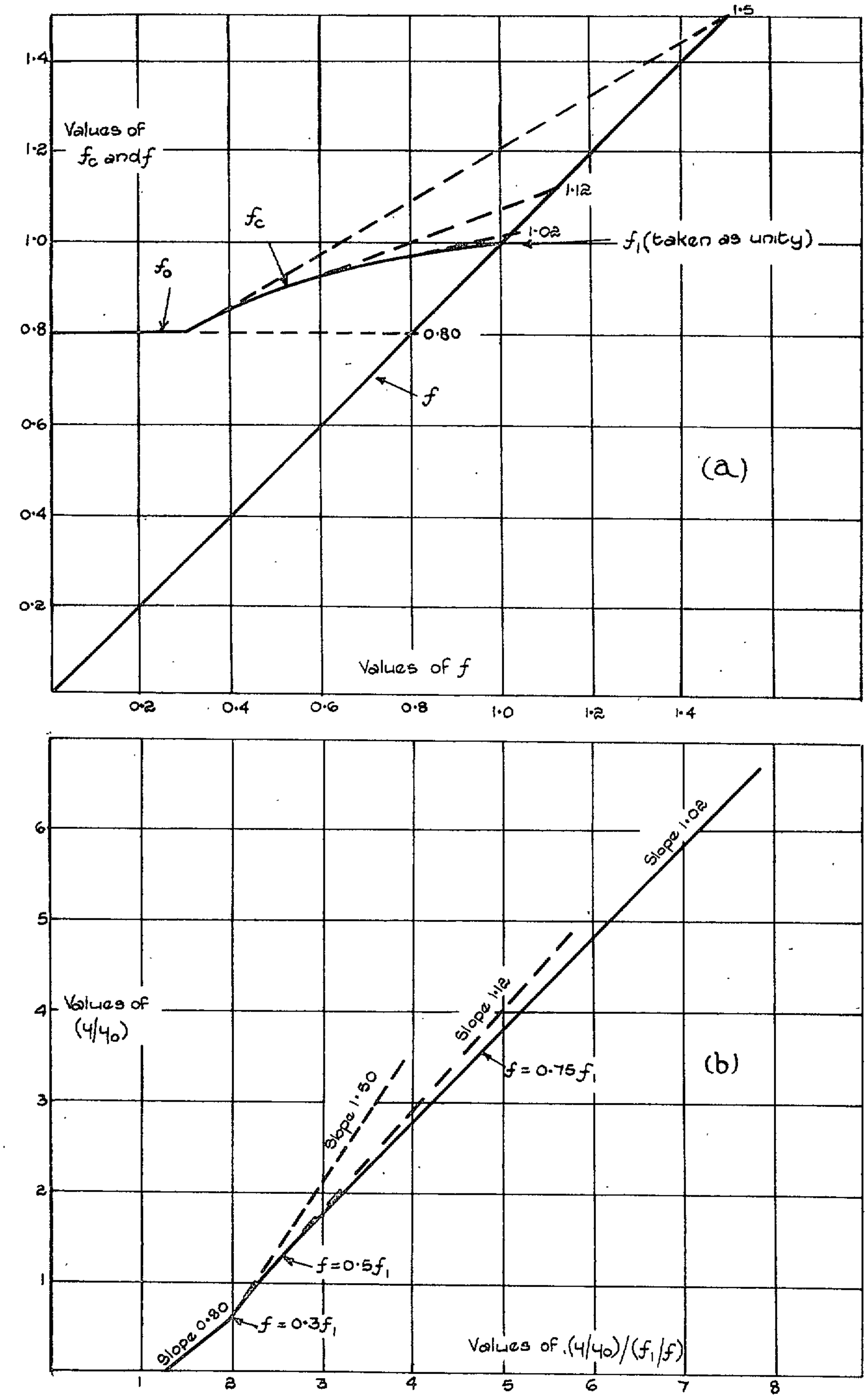

FIG. 3. General effect of variation of the critical load

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