C.P. No. 309
(18,412)
A.R.C. Technical Report

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Problems in Computation of Aerodynamic Loading on Oscillating Lifting Surfaces

By

H. C. Garner, B.A., and W. E. A. Acum, A.R.C.S., B.Sc.,

of the Aerodynamics Division, N.P.L.

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1957

Price 2s 6d net

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30th April, 1956

SUMMARY

The linearized theories of oscillating wings in subsonic and supersonic flow are formally discussed. The computability of the subsonic problem is considered when Mach number, frequency and aspect ratio are arbitrary and also when any one of these parameters takes extreme values. Methods of treating aerodynamic flutter problems are briefly outlined. The most general subsonic problem for a rectangular wing, already solved numerically on a desk machine, should now be examined with a view to mechanized computation. Special theories, particularly suited to high-speed computation, should, if possible, be extended to arbitrary frequency and Mach number.

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1./

1. Introduction

In considering the perturbation of a uniform stream of velocity U due to an oscillating wing, it is usual to neglect the squares of increments in velocity, of wing thickness and amplitude of oscillation. Viscous effects are also ignored, so that the problem is specified by the planform of the wing, the mode of oscillation, the non-dimensional frequency and M, the Mach number of the stream. It is required to evaluate the local phase and amplitude of the aerodynamic loading. According to the linearized theory the perturbation velocity potential, Φ, satisfies the equation (Ref. 1, p.328)

$$(1 - M^2)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{2M^2}{U}\frac{\partial^2\Phi}{\partial x\partial t} - \frac{M^2}{U^2}\frac{\partial^2\Phi}{\partial t^2} = 0, \qquad \ldots(1)$$

and the pressure at any point is given by

$$p - p_\infty = -\rho_\infty\left(\frac{\partial\Phi}{\partial t} + U\frac{\partial\Phi}{\partial x}\right). \qquad \ldots(2)$$

When the stream is sonic ($M = 1$) the differential equation (1) simplifies and certain analytical solutions exist[2] (Mangler, 1952). In supersonic flow ($M > 1$), when the differential equation becomes hyperbolic, there is a class of solutions in closed form (§.4). No general solution of the problem exists when the flow is subsonic: even when $M = 0$, (1) reduces to Laplace's equation, but remains intractable.

An important feature of current theoretical research on subsonic oscillatory flow is the diversity of methods applicable to a given problem. The choice of method will depend on individual experience, facilities for computation, the required accuracy and the intended application. The various methods all involve further assumptions or approximations, so that (1) is never solved exactly for $M < 1$. Several simplifying assumptions are considered in §.2; the various numerical processes and approximations are discussed in §.3. Most methods use collocation, conditions of tangential flow being satisfied at, say, n positions on the wing. This involves arbitrary choice of the n positions and of n complex functions to represent the phase and amplitude of the pressure. The main effort of computation lies in expressing these boundary conditions as a set of n complex linear simultaneous equations, which are then solved by routine methods for the unknown coefficients of the pressure distribution.

The precise nature of the oscillatory motion of the wing is not important. Whether the problem to be solved is that of a pitching wing or an oscillating partial-span control, the matrix of the simultaneous equations is usually the same. Moreover, it is not highly significant whether the problem is symmetrical (pitching) or antisymmetrical (rolling).

2. General Equations in Subsonic Flow

2.1 Differential Equation and Boundary Conditions

The differential equation for the perturbation velocity potential, Φ, is given in equation (1), and is subject to the following four boundary conditions:

(a) On the planform S in the plane $z = 0$

$$\frac{\partial\Phi}{\partial z} = \frac{\partial Z(x, y, t)}{\partial t} + U\frac{\partial Z(x, y, t)}{\partial x}, \qquad \ldots(3)$$

where $z = Z(x, y, t)$ represents the surface of the wing.

(b)/

(b) In the wake

$$\partial\Phi/\partial t + U\,\partial\Phi/\partial x = 0. \qquad \ldots(4)$$

This expresses the fact that the pressure is continuous across the wake, and follows from equation (2) coupled with the observation that $\Phi(x, y, z, t) = -\Phi(x, y, -z, t)$.

(c) At the trailing edge the velocity is finite.

(d) Infinitely far from the wing and wake the disturbance tends to zero; Φ is such that it represents an outgoing disturbance [see footnote to equation (7)].

Equation (1) may be simplified by the substitution

$$\Phi = \psi(x, y, z)\exp\left[\frac{i M^2\omega}{U(1 - M^2)}\right]\exp(i\omega t); \qquad \ldots(5)$$

ψ then satisfies

$$(1 - M^2)\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{M^2\omega^2}{U^2(1 - M^2)}\psi = 0. \qquad \ldots(6)$$

Even so, the differential equation does not provide a solution to the problem specified above, because of the nature of the boundary conditions, (b) in particular. Direct numerical methods, such as relaxation, do not appear to have been attempted++.

2.2 Integral Equations

As shown in Ref. 3, §.3, the upwash at a point on the wing may be expressed in terms of l, the distribution of lift over the wing, by the equation*

$$\frac{w(x, y, 0)}{U} = -\frac{1}{8\pi}\lim_{z\to 0}\iint_S l(x', y')\exp\left[\frac{-i\omega(x - x')}{U}\right]\times \int_{-\infty}^{x-x'}\exp\left[\frac{i\omega\xi}{U\beta^2}\right]\frac{\partial^2}{\partial z^2}\left\{\frac{1}{r}\exp\left[-\frac{i\omega Mr}{U\beta^2}\right]\right\}d\xi\,dx'\,dy', \qquad \ldots(7)$$

where

$$\left.\begin{aligned} l &= (p_b - p_a)/\tfrac{1}{2}\rho_\infty U^2 \\ r^2 &= \xi^2 + \beta^2(y - y')^2 + \beta^2 z^2 \end{aligned}\right\}.$$

Suffices a and b denote values on the upper and lower surfaces respectively. An alternative integral equation4 (W. P. Jones, 1951) is

$$w = \frac{1}{4\pi}\lim_{z\to 0}\iint_{S+W}(\Phi_a - \Phi_b)\exp\left[\frac{i\omega M^2\xi}{U\beta^2}\right]\frac{\partial}{\partial z^2}\left\{\frac{1}{r}\exp\left[-\frac{i\omega Mr}{U\beta^2}\right]\right\}dx'\,dy', \qquad \ldots(8)$$

where/

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*The minus sign in the term $\exp[-i\omega Mr/U\beta^2]$ is implied by the boundary condition (d) in §.2.1.

++This approach is being examined in the Mathematics Division, N.P.L.

where $\xi = (x - x')$ and the integral is now taken over the wake, W, as well as the wing, S.

Equation (7) may be written

$$\frac{w}{U} = -\frac{1}{8\pi}\iint_S l(x', y') \cdot K(x - x', y - y'; \nu, M)\, dx'\, dy', \quad \ldots(9)$$

where $\nu = \omega d/U$ and d is any convenient length. The form of the kernel K defined by equations (7) and (9) is awkward for calculation because of the infinite range of integration. This difficulty has been eliminated by Watkins, Runyan and Woolston[5] (1954), who have shown that

$$\begin{aligned} K(x_o, y_o) = \frac{1}{d^2}\exp(-i\nu x_o)\Bigg\{ & \frac{-\nu}{|y_o|}K_1(\nu|y_o|) - \frac{i\pi\nu}{2|y_o|}\left[I_1(\nu|y_o|) - L_1(\nu|y_o|)\right] \\ & + \frac{iM\nu|y_o| + \beta}{M\beta y_o^2}\exp\left(\frac{-iM\nu|y_o|}{\beta}\right) \\ & - \nu^2\int_0^{M/\beta}\sqrt{1 + \tau^2}\,\exp(-i\nu|y_o|\tau)\,d\tau \\ & - \frac{Mx_o + \sqrt{x_o^2 + \beta^2 y_o^2}}{My_o^2\sqrt{x_o^2 + \beta^2 y_o^2}}\exp\left[\frac{i\nu}{\beta^2}\left(x_o - M\sqrt{x_o^2 + \beta^2 y_o^2}\right)\right] \\ & + \frac{i\nu}{My_o^2}\int_0^{x_o}\exp\left[\frac{i\nu}{\beta^2}\left(\lambda - M\sqrt{\lambda^2 + \beta^2 y_o^2}\right)\right]d\lambda\Bigg\}, \end{aligned} \quad \ldots(10)$$

where $x_o = (x - x')/d$, $y_o = (y - y')/d$, K_1 and I_1 are Bessel functions, and L_1 is a modified Struve function (Ref. 6, p.329).

To solve the integral equation (9), it is necessary to use a collocation method. Then, given M and ν, K may be computed for the appropriate values of x_o and y_o on a desk machine. Pitching derivatives for a rectangular wing of aspect ratio A (wing span/mean chord) = 4 at M = 0.866 are determined in this way in Ref. 7. Eight loading functions and eight collocation points are taken on the half wing (of chord d); without further approximations the equations have been solved when ν = 0, 0.3 and 0.6. The results appear satisfactory, but the method of calculation is too lengthy for routine use if only desk machines are used.

2.3 Approximations to the Kernel

K, as given by (10), is still a rather cumbersome function of four variables, so approximate forms of the kernel are sought. These are obtained by considering extreme values of the parameters ν, M and A.

For ν small, Lance[8] (1954) has shown that (in the present notation)

$$\begin{aligned} K(x_o, y_o) = \frac{1}{d^2}\exp(-i\nu x_o)\Bigg\{ & -\frac{1}{y_o^2} - \frac{x_o}{y_o^2\sqrt{x_o^2 + \beta^2 y_o^2}} + \frac{i\nu}{\sqrt{x_o^2 + \beta^2 y_o^2}} \\ & - \frac{\nu^2}{2}\log\frac{\nu}{2(1 - M)}\left(\sqrt{x_o^2 + \beta^2 y_o^2} - x_o\right) \\ & - \frac{\nu^2}{2}\left[\gamma - \frac{1}{2} + \frac{i\pi}{2} - \frac{1}{\beta^2}\left(M - \frac{x_o}{\sqrt{x_o^2 + \beta^2 y_o^2}}\right)^2\right] + O(\nu^3/\beta^6)\Bigg\} \end{aligned} \quad \ldots(11)$$

The/

The assumption that $\nu^2 \log \nu, \nu^2, \ldots$ are negligible leads to the so-called "low-frequency" theory in Ref. 9. Lance[8] has also considered the case $\nu \to \infty$ and gives the equation

$$K(x_o, y_o) = - \frac{M\beta^2 \exp(i\nu[M^2 x_o - M\sqrt{x_o^2 + \beta^2 y_o^2}/\beta^2])}{\sqrt{x_o^2 + \beta^2 y_o^2}(\sqrt{x_o^2 + \beta^2 y_o^2} - Mx_o)} + O(\nu^{-1}) . \qquad \ldots(12)$$

M may similarly be varied. When $M \to 1$, K vanishes when $x_o \leqslant 0$. The general expression for K, when $x_o > 0$, is given in Ref. 5, equation (47a). The corresponding expansion in powers of ν is found in Ref. 8, equation (13). Neither of these approximations to the kernel is thought to be of practical use in computation. M = 0 gives the case of incompressible flow. The kernel then becomes[5]

$$K(x_o, y_o) = \frac{1}{d^2} \exp(-i\nu x_o) \left\{ - \frac{\nu}{|y_o|} K_1(\nu|y_o|) - \frac{i\pi\nu}{2|y_o|}[I_1(\nu|y_o|) - L_1(\nu|y_o|)] - \frac{x_o}{y_o^2\sqrt{x_o^2 + y_o^2}} \exp(i\nu x_o) + \frac{i\nu\sqrt{x_o^2 - y_o^2}}{y_o^2} \exp(i\nu x_o) + \frac{\nu^2}{y_o^2}\int_0^{x_o} \sqrt{\lambda^2 + y_o^2} \exp(i\nu\lambda) d\lambda \right\} . \qquad \ldots(13)$$

Methods, which have been proposed for dealing with wings for which the aspect ratio is very large or very small, have been reviewed by Eckhaus[10] (1954). Most of these theories on the oscillating finite wing are restricted to either lifting-line theory, incompressible flow or unswept wings. Apart from the very general treatment by Küssner[11] (1954), only the slender-wing theory of Merbt and Landahl[12,13] (1953) and the recently published strip theory by Eckhaus[14] (1955) need be mentioned here.[+] Both these methods apply to wings of arbitrary sweep and taper without restriction on ν or subsonic M. In Ref. 12, A^2 is neglected so that the first term of the differential equation (6) disappears; this approximation is not thought to be valid when $A > \frac{1}{2}$. Ref. 14, based on two-dimensional theory, would not be expected to apply when $A < 6$ or when ν is too low. Thus no reliable routine method of calculation exists for wings of present-day aircraft.

3. Computational Problems

3.1 Use of the Exact Kernel

Allen[15] (1953) has used the exact kernel in equation (7) to reduce the computation required to evaluate the downwash so that the major portion of the work lies in evaluating certain "influence functions", I, II, J and JJ.

$$I(X, Y, \lambda_1) = \frac{1}{\pi}\int_0^{\pi} \left(1 + \frac{2X - 1 + \cos\phi}{\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}}\right) \times \exp[-\tfrac{1}{2}i\lambda_1\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}](1 + \cos\phi)\, d\phi, \ldots(14)$$

where ϕ is the usual angular chordwise co-ordinate,

$$X = \{x - x_l(y')\}/c(y'), \quad Y = \beta|y - y'|/c(y')$$

and the frequency parameter $\lambda_1 = \frac{\nu M}{\beta^2} \frac{c(y')}{d}$.

II/

[+]See also the slender-wing theory of B. Mazelsby (J. Ae. Sci., July, 1956).

$$II(X, Y, \lambda_1, \lambda_1/M) = \int_{-\infty}^{X} I(X_0, Y, \lambda_1) \exp\left[\frac{i\lambda_1}{M}(X_0 - X)\right] dX_0 \ . \qquad ...(15)$$

J is obtained by replacing the factor $(1 + \cos\phi)$ in (14) by $(\cos\phi + \cos 2\phi)$, and JJ by replacing I by J in (15). Thus I and J are functions of three parameters, and II and JJ of four. The chief difficulty here lies in the infinite range of integration in equation (15).

Alternatively, if the form of K in equation (10) is used, only two functions I and J are required, given by the following equations (Ref. 3)

$$\left.\begin{aligned} I &= i_1 + i_2 + i_3 + i_4 \\ J &= j_1 + j_2 + j_3 + j_4 \end{aligned}\right\}, \qquad ...(16)$$

where

$$\left.\begin{aligned} i_p &= \frac{1}{\pi}\int_0^{\pi} G_p(1 + \cos\phi)\, d\phi \\ j_p &= \frac{4}{\pi}\int_0^{\pi} G_p(\cos\phi + \cos 2\phi)\, d\phi \end{aligned}\right\} (p = 1, 2, 3, 4), \qquad ...(17)$$

$$\left.\begin{aligned} G_1 &= \frac{\nu Y}{\beta}\frac{c(y')}{d} K_1\left(\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d}\right) + \frac{\pi i}{2}\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d}\left[I_1\left(\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d}\right) - L_1\left(\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d}\right)\right] \\ &\quad - \frac{\left(iM\cdot\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d} + \beta\right)}{M\beta}\exp\left(-i\frac{M}{\beta}\cdot\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d}\right) \\ G_2 &= \left(\frac{\nu Y}{\beta}\cdot\frac{c(y')}{d}\right)^2\int_0^{M/\beta}\sqrt{1+\tau^2}\exp\left\{-i\frac{\nu Y}{\beta}\frac{c(y')}{d}\tau\right\} d\tau \\ G_3 &= \frac{1}{M}\left[1 + \frac{M(2X - 1 + \cos\phi)}{\sqrt{(2X-1+\cos\phi)^2 + 4Y^2}}\right]\exp\left[\frac{i\nu}{2\beta^2}\cdot\frac{c(y')}{d}\left\{(2X - 1 + \cos\phi) - M\sqrt{(2X-1+\cos\phi)^2+4Y^2}\right\}\right] \\ G_4 &= -\frac{i\nu}{M}\int_0^{\frac{c(y')}{2d}(2x-1+\cos\phi)}\exp\left[\frac{i\nu}{\beta^2}\left(t - M\sqrt{t^2 + \left(\frac{c(y')}{d}\right)^2 Y^2}\right)\right] dt \end{aligned}\right\} ...(18)$$

Since G_1 and G_2 are independent of ϕ,

$$i_1 = G_1, \quad i_2 = G_2, \quad j_1 = j_2 = 0. \qquad ...(19)$$

i_4 and j_4 may be simplified by integration by parts to

i_4/

$$i_4 = \frac{i\nu}{2M\pi}\frac{c(y')}{d}\left\{\pi\int_{\cos^{-1}(1-2X)}^{\pi} \sin\phi \exp\left[\frac{i\nu}{2\beta^2}\frac{c(y')}{d}\left\{(2X - 1 + \cos\phi) - M\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}\right\}\right]d\phi - \int_0^{\pi}(\phi + \sin\phi)\sin\phi \times \exp\left[\frac{i\nu}{2\beta^2}\frac{c(y')}{d}\left\{(2X - 1 + \cos\phi) - M\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}\right\}\right]d\phi\right\}$$

$$j_4 = -\frac{2i\nu}{M\pi}\cdot\frac{c(y')}{d}\int_0^{\pi}(1 + \cos\phi)\sin^2\phi \exp\left[\frac{i\nu}{2\beta^2}\cdot\frac{c(y')}{d}\left\{(2X - 1 + \cos\phi) - M\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}\right\}\right]d\phi . \quad \ldots(20)$$

In these equations the infinite range of integration no longer appears and all the integrations are amenable to numerical treatment[7].

Both of the above methods use the exact kernel, so that, provided enough collocation points are used the integral equation may be solved as precisely as required. Finite integrals, similar to (20), would arise from equation (13) in the special case $M = 0$*.

3.2 Low-frequency Theories

With the approximation that $\nu^2 \log \nu, \nu^3, \ldots$ in (11) are negligible, it is possible to achieve routine numerical solutions for compressible subsonic flow. Two widely-used methods are Multhopp's[16] (1950) theory, developed in Ref. 9, and Falkner's[17] (1947) vortex-lattice theory, developed by Miss Lehrian[18] (1953).

Multhopp's theory involves the calculation of the influence functions

$$\left.\begin{aligned} i &= 1 + \frac{1}{\pi}\int_0^{\pi}\frac{(2X - 1 + \cos\phi)}{\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}}(1 + \cos\phi)d\phi \\ j &= \frac{4}{\pi}\int_0^{\pi}\frac{(2X - 1 + \cos\phi)}{\sqrt{(2X - 1 + \cos\phi)^2 + 4Y^2}}(\cos\phi + \cos 2\phi)d\phi \\ ii &= \int_{-\infty}^{X} i(X_0, Y)\,dX_0 \\ jj &= \int_{-\infty}^{X} j(X_0, Y)dX_0 . \end{aligned}\right\} \quad \ldots(21)$$

These functions, which depend on X and Y only, have been tabulated by Curtis[19] (1952). If more terms of (11) were included extra functions would have to be tabulated to account for them. The use of two collocation points on each chordwise section is envisaged, but, if more were required, further functions k, l, etc. would be introduced by replacing the factor $(1 + \cos\phi)$ or $4(\cos\phi + \cos 2\phi)$ in (21)

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*This line of approach has been developed in Structures Department, R.A.E, Farnborough.

by $(\cos 2\phi + \cos 3\phi)$, $(\cos 3\phi + \cos 4\phi)$, etc. The possibility of computing these in the low-frequency case has been considered by Alway[20] (1954), who has expressed these influence functions in terms of simple iterative operations suitable for an electronic digital computer, the computation time being 1-2 seconds.

The use of a vortex lattice to evaluate w does introduce some unknown irreducible error, but gives good comparisons with the results of Multhopp's theory, as shown in Ref. 9. In the special case of incompressible flow, Miss Lehrian[21] (1954) has extended the vortex-lattice theory to arbitrary frequency. The most general influence function is the downwash at a point (x_1, y_1) due to an oscillatory doublet-strip of width $2d$ with the mid-point of its leading edge as origin:

$$\frac{W(X_1, Y_1, \nu)}{U} = \frac{1}{4\pi}\Big[H(-X_1, Y_1 + 1) - H(-X_1, Y_1 - 1) + i\nu\exp(-i\nu X_1)\int_{-X}^{\infty}\exp(-i\nu X)\{H(X, Y_1 - 1) - H(X, Y_1 + 1)\}dX\Big] \quad \ldots(22)$$

with

$$H(X_1, Y_1) = \frac{1}{X_1} + \frac{1}{Y_1} - \frac{\sqrt{X_1^2 + Y_1^2}}{X_1 Y_1}, \quad X_1 = x_1/d, \; Y_1 = y_1/d.$$

This quantity arises from the integral over the wake in equation (8) and has been tabulated by the Mathematics Division, N.P.L.[22] (1952). A treatment of the problem for moderately small values of ν, suggested by W. P. Jones[4], is to approximate to equation (8) by replacing the factor

$$\frac{\partial^2}{\partial z^2}\left\{\frac{1}{r}\exp\left[-\frac{i\omega Mr}{U\beta^2}\right]\right\} \text{ by } \frac{\partial^2}{\partial z^2}\left\{\frac{1}{r}\right\}.$$

It can be shown that the error in w is then of order $\frac{M^2\nu^2}{\beta^4}\log\frac{M\nu}{\beta^2}$. Subject to this error, results applicable to compressible subsonic flow can be obtained from an equivalent problem in incompressible flow. Recent calculations on this basis have been carried out for rectangular wings[23]. Comparisons with the results in Ref. 7 suggest that at high M Jones' approximation is only valid for a very restricted range of ν, but this may perhaps be improved by an iterative procedure. The Structures Department of R.A.E., Farnborough have developed a method of handling the numerical work on a high-speed computing machine, the influence function W/U being evaluated from an expansion in ν. The possible usefulness of the vortex-lattice theory as a general routine method should not be overlooked.

4. General Equations in Supersonic Flow

The integral equation for linearized supersonic flow analogous to equation (7) is,

$\frac{w}{U}$/

$$\frac{w}{U}(x, y, 0) = -\frac{1}{8\pi}\iint_S l(x', y')K(x - x', y - y')dx'dy',$$

where

$$K(x - x', y - y') = 2 \lim_{z \to 0} \exp\left[\frac{-i\omega(x - x')}{U}\right]\int_{\beta R}^{x-x'} \exp\left[\frac{-i\omega\lambda}{U\beta^2}\right] \times \frac{\partial^2}{\partial z^2}\left[\frac{\cos\left(\frac{M\omega}{U\beta^2}\sqrt{\lambda^2 - \beta^2 R^2}\right)}{\sqrt{\lambda^2 - \beta^2 R^2}}\right] d\lambda, \quad \ldots(23)$$

$R = \sqrt{(y - y')^2 + z^2}$, $\beta = \sqrt{M^2 - 1}$ and S is that part of the wing inside the forward Mach cone of $(x, y, 0)$. Watkins and Berman[24] (1955) give a general treatment of equation (23).

Another integral relation which holds for $M > 1$ is[25] (Evvard, 1950)

$$\Phi(x, y, z, t) = -\frac{1}{2\pi}\iint_S \frac{w(x', y', t - \tau_a) + w(x', y', \tau - \tau_b)}{\sqrt{(x - x')^2 - \beta^2(y - y')^2 - \beta^2 z^2}} dx'dy',$$

where

$$\tau_a = \frac{(x - x')M}{\beta^2 a_\infty} + \frac{\sqrt{(x - x')^2 - \beta^2(y - y')^2 - \beta^2 z^2}}{\beta^2 a_\infty},$$

$$\tau_b = \frac{(x - x')M}{\beta^2 a_\infty} - \frac{\sqrt{(x - x')^2 - \beta^2(y - y')^2 - \beta^2 z^2}}{\beta^2 a_\infty}, \quad \ldots(24)$$

and S here represents that part of the plane $z = 0$ intercepted by the forward Mach cone. This will give the pressure distribution in closed form in the case of wings with entirely supersonic edges.

The differential equation (1) still holds, though since $M > 1$ it is now hyperbolic, and the boundary conditions are slightly changed. A treatment of some problems using the differential equations is given by Stewartson[26] (1950).

The methods adopted for solving such equations differ widely and bear little relation to those used in subsonic flow.[+] In many cases, such as the solutions obtained for rectangular, delta and arrowhead wings (Refs. 27 to 30), the computing consists merely in substituting in formulae for the required quantities. The derivatives found in these papers are given as power series in ν; terms containing $\log \nu$ no longer appear. Indeed analytical treatments of particular planforms are commoner than general numerical methods.

5. Aeroelastic Problems

5.1 Subsonic Flow

The aerodynamic loading of rigid lifting surfaces in subsonic flow is normally solved by a collocation method, which relates certain loading functions $\bar{l}$ to downwash functions $\bar{w}$ by a matrix equation with complex coefficients

$$\bar{w} = A\,\bar{l}\,. \quad \ldots(25)$$

The/

[+] See, however, the work of Richardson, discussed in the Addendum.

The downwash is no longer prescribed by equation (3), when the wing is supposed to deform under load; structural deformation then gives an additional term in $\partial \Phi/\partial z$, which is linearly dependent on the aerodynamic loading and expressible in matrix form as

$$\overline{w}' = E \overline{l} , \qquad \ldots(26)$$

so that instead of equation (25)

$$\overline{w} + \overline{w}' = A \overline{l} . \qquad \ldots(27)$$

Equations (26) and (27) lead to the formal solution

$$\overline{l} = (A - E)^{-1} \overline{w} . \qquad \ldots(28)$$

In general, the elements of the matrix A depend on the frequency parameter ν; in a flutter problem, separate calculations would be necessary for selected values of ν. The task of evaluating the separate matrices A remains the most formidable part of the computation.

In the special case of low frequency, the aerodynamic problem reduces to that of Multhopp's[9] theory. A significant advance is made by Richardson[31] (1954), who chooses loading functions $\overline{l}$ at, say, N specific chordwise locations instead of the basic two-dimensional load distributions that lead to the influence coefficients in equation (21). The real and imaginary parts of his influence coefficients are linear functions of i, j, etc. and ii, jj, etc., respectively. By careful choice of the chordwise locations, Richardson arrives at approximate influence coefficients

$$\left.\begin{aligned} (K_{\nu n})_{rq} &= 1 + \frac{X_r - X_q}{\{(X_r - X_q)^2 + (Y_\nu - Y_n)^2\}^{\frac{1}{2}}} \\ (KK_{\nu n})_{rq} &= (X_r - X_q) + \{(X_r - X_q)^2 + (Y_\nu - Y_n)^2\}^{\frac{1}{2}} \end{aligned}\right\} . \qquad \ldots(29)$$

The accuracy of these approximate values increases as N increases and is stated to appear reasonable for N = 4.

5.2 Supersonic Flow

The theoretical study of aeroelastic effects in supersonic flow has mainly led to analytical treatment of particular planforms. For example, solutions for rectangular[32] and triangular[33] wings are extensions of those for rigid wings in Refs. 27 and 29.

A general treatment of the flutter problem for wings with subsonic or supersonic leading edges is proposed by Pines, Dugundji and Neuringer[34] (1955). The wing is represented by a grid of square boxes and the influence of one box B on another is determined by the quantity

$$R + iI = -\frac{1}{\pi}\left[i + \frac{U}{\omega}\frac{\partial}{\partial x}\right]\iint_B \frac{1}{R'} \exp\left\{-\frac{i\omega M^2}{U(M^2 - 1)}(x - x')\right\} \times$$

$$\times \cos \frac{\omega M R'}{U(M^2 - 1)} \, dx' dy' , \qquad \ldots(30)$$

where

$$R' = \sqrt{(x - x')^2 - (M^2 - 1)(y - y')^2} .$$

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These pressure influence coefficients are evaluated by approximation to the exponential and cosine terms, but an exact expression involving infinite series of Bessel functions and arc sines, due to C. E. Watkins, is given in Ref. 35. Equation (30) only applies when the box lies completely within the forward Mach cone of the receiving point. When the leading edge is subsonic, Evvard's[25] method is used to give a correction term. The procedure of Ref. 34 is well suited to high-speed computing machinery, but is only valid for $M \geq \sqrt{2}$. Rectangular or diamond-shaped boxes are suggested means of extending the range of Mach number.

6. Concluding Remarks

Practical methods are required for estimating the forces on oscillating wings of the type used on modern high-speed aircraft, that is of medium aspect ratio and possibly high sweepback. In supersonic flow, analytical formulae exist for a wide range of particular planforms (§.4). In subsonic flow, practical methods exist only when the frequency parameter ν is small (§.3.2). Extension to higher frequencies is theoretically possible (§.3.1).

The method described in Ref. 3 appears to be the most promising; in fact, calculations using desk machines have been carried out for a rectangular wing of aspect ratio $A = 4$ at $M = 0.866$ with $\nu = 0.3$ and 0.6, the computation time being four months for each frequency. This theory will probably cover the practical range of ν unless M is very near unity, when the assumptions of linearized theory are invalid and accurate solutions are of doubtful merit. The method is uneconomic, unless the calculation can be mechanized, particularly the evaluation of the influence functions in equations (14) to (20).

Simplifications result from considering extreme values of the parameters ν, A and M (§.2.3). Equation (12) for $\nu \rightarrow \infty$ probably only applies to values of ν outside the practical range. High-aspect-ratio theory $(A > 6)$ and low-aspect-ratio theory $(A < \frac{1}{5})$ are inapplicable to the type of wings mentioned above. The approximation to the kernel when $M = 1$ is not thought to be of practical use. The low-speed case $(M = 0)$ appears to be satisfactorily covered by vortex-lattice theory though it could be treated by methods similar to those of Ref. 3. The possibility of extending vortex-lattice theory to cover the practical range of $\nu M/(1 - M^2)$ remains to be studied (§.3.2).

In many respects wing elasticity does not affect the basic aerodynamic computation in subsonic or supersonic flow. However, the low-frequency subsonic theory of Ref. 31 (§.5.1) and the box-grid method of Ref. 34 for $M \geq \sqrt{2}$ (§.5.2) both appear to lead to calculations well suited to programming on computing machinery. Though approximate as desk computations, these methods when mechanized should become accurate within the framework of linearized theory. Extensions of Ref. 31 to higher frequencies and of Ref. 34 to Mach numbers below $\sqrt{2}$ are among the most important problems in theoretical flutter research.

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ADDENDUM

Three reports (Refs. 36, 37 and 38) have come to the notice of the authors since the paper was written.

Runyan and Woolston[36] have applied the form of the kernel derived in Ref. 5 for general frequency parameter, ν, and subsonic Mach number, M, to wings of arbitrary plan form. This paper appears to offer a good chance of providing a practicable routine method of calculation by collocation. The assumed loading involves the replacement of a continuous chordwise loading by discrete loads determined similarly to those used in vortex lattice theory (Refs. 21 and 23). It is assumed that the upwash on the wing is represented accurately enough when the kernel is expanded and 6th and higher powers of the frequency parameter are neglected. A modification of the method to the case $M = 1$ is also given. The results of calculations on a rectangular wing of aspect ratio $A = 2$ are presented for the complete subsonic range and $\nu = 0.44$. Calculations by the method of Ref. 7 for the same wing are being made at the N.P.L. for $M = 0.866$ and $\nu = 0.3$ and $\nu = 0.6$. The mechanization of these calculations is being considered by the Mathematics Division of the N.P.L.

With regard to the general equations of supersonic flow, Richardson[37] has presented a unified approach for both subsonic and supersonic Mach numbers. In contrast with Refs. 27 to 30 he has suggested a collocation method for any plan form in supersonic flow. Ref. 37 is an important step towards the extension of Ref. 31 to higher frequencies. The evaluation of the kernel function in Appendix I of Ref. 37 is being programmed for high-speed computing machinery in the Structures and Mathematical Services Departments of the Royal Aircraft Establishment.

Another important problem in flutter research, the extension of Ref. 34 to Mach numbers below $\sqrt{2}$, has been carried out by Ta Li[38], who includes the case of a subsonic leading edge. He has obtained highly satisfactory results for two-dimensional flutter coefficients when $M \geqslant 1.1$.

C.P. No. 309
(18,412)
A.R.C. Technical Report

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