**C.P. No. 303**
(17,952)
A.R.C. Technical Report

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# On the Stability of a Laminar Wake

*By*

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C.P. No. 303

# On the Stability of a Laminar Wake

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C. H. McKoen, B.Sc., Ph.D.*

Communicated by Dr. W. P. Jones

21st October, 1955

## SUMMARY

A perturbation method of solving the problem of stability in an unlimited field of flow is developed and used to investigate the stability of the laminar wake formed by a flat plate.

The inviscid problem of the wake formed by a flat plate is investigated, and the eigen-value of $\alpha$ for the neutral disturbance is found to be $\alpha_s = 4.0$.

A detailed account is given of the perturbation method which is developed. The necessary and sufficient condition that an integral of the small disturbance equation should satisfy the boundary conditions for the wake is established. This condition is found to lead to a simple determination of the $(\alpha,R)$ curve, and this curve is found for the neutral disturbances.

The method fails to predict a minimum critical Reynolds number, only because the approximations made in the above conditions are only valid for large Reynolds numbers.

## 1. Introduction

Laminar flow is regarded as stable if all velocity disturbances, caused accidentally in the fluid, tend ultimately to vanish and as unstable if any disturbance persists in time or tends to increase. The problem is to ascertain whether conditions exist under which any disturbance persists or tends to increase and, if so, to determine the characteristics of such disturbances for any given regime of flow.

Lord Rayleigh[1,2], first propounded the theory of stability based on infinitesimal disturbances, solving the inviscid problem for several types of velocity profile in a channel with parallel walls. The modern theory was instigated by Heisenberg[3], who showed that the fourth-order differential equation governing the disturbances had two slowly-varying integrals, sensible across the whole channel and unaffected by viscosity; and two rapidly varying integrals sensible only near the walls and very sensitive to the effects of viscosity. The first type are known as inviscid integrals, and are functions of the velocity profile. The second type are termed viscous integrals and do not vary appreciably with the velocity profile.

Mathematically/

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*This paper is an abridged version of the author's thesis at the University of London. It has been edited in the Aerodynamics Division of the National Physical Laboratory.

Mathematically a basic flow

$$u = U(y'), \quad v = 0 \qquad \ldots(1.1)$$

is considered and a small disturbance

$$u' = u'(x',y',t'), \quad v' = v'(x',y',t') \qquad \ldots(1.2)$$

is imposed on this.

The equation of continuity is

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \qquad \ldots(1.3)$$

and upon eliminating the pressure the Navier-Stokes equations lead to

$$\frac{\partial^2 u'}{\partial y'\partial t'} + U \frac{\partial^2 u'}{\partial x'\partial y'} + v' \frac{\partial^2 U}{\partial y'^2} - \frac{\partial^2 v'}{\partial x'\partial t'} - U \frac{\partial^2 v'}{\partial x'^2}$$

$$= \nu \left\{ \frac{\partial^3 u'}{\partial x'^2 \partial y'} + \frac{\partial^3 u'}{\partial y'^3} - \frac{\partial^3 v'}{\partial x'^3} - \frac{\partial^3 v'}{\partial x' \partial y'^2} \right\} . \qquad \ldots(1.4)$$

Equation (1.3) is formally satisfied by expressing $u'$ and $v'$ in terms of a stream function $\psi$. Hence

$$u' = \frac{\partial \psi}{\partial y'}, \quad v' = - \frac{\partial \psi}{\partial x'} . \qquad \ldots(1.5)$$

Assume $\psi$ to be of the form

$$\psi = \phi(y') \, e^{i\alpha'(x'-c't')} \qquad \ldots(1.6)$$

Substituting from (1.5) and (1.6) into (1.4)

$$(U - c')(\phi'' - \alpha'^2 \phi) - U'' \phi = \frac{\nu}{i\alpha'} (\phi'''' - 2\alpha'^2 \phi'' + \alpha'^4 \phi) . \qquad \ldots(1.7)$$

Converting to the dimensionless co-ordinates $y = y'/\delta$, the following is obtained:-

$$(w - c)(\phi'' - \alpha^2 \phi) - w''\phi = - \frac{i}{\alpha R} (\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi) , \qquad \ldots(1.8)$$

where/

where

$$\left.\begin{aligned} w &= U/U_0 , \\ c &= c'/U_0 , \\ \alpha &= \alpha'\delta , \\ R &= R_\delta = \frac{U_0\delta}{\nu} , \end{aligned}\right\} \qquad \ldots(1.9)$$

Equation (1.8) is the non-dimensional form of the small disturbance equation. The integrals of (1.8), in conjunction with the boundary conditions imposed by physical considerations determine the eigen-values of $\alpha$, $R$, $c$. The values of $\alpha$ and $R$ must be real, but $c$ may be complex. The disturbance is termed amplified, neutral or damped according as the imaginary part of $c$ is positive, zero or negative.

The inviscid integrals described above are solutions of the equation obtained by neglecting the viscous terms on the right-hand side of (1.8). Hence

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = 0 . \qquad \ldots(1.10)$$

Equation (1.10) has a singularity at $y = y_0$ where $w - c = 0$, and can be solved as a power series in $(y - y_0)$ by the method of Frobenius. The integrals obtained are

$$\left.\begin{aligned} \phi_1 &= z + a_2 z^2 + a_3 z^3 \ldots \\ \phi_2 &= 1 + b_1 z + b_2 z^2 \ldots + \frac{w_0''}{w_0'} \phi_1 \log(-z) \end{aligned}\right\} \qquad \ldots(1.11)$$

where

$$z = y - y_0 < 0 . \qquad \ldots(1.12)$$

Near to $y = y_0$ the term $(w - c)(\phi'' - \alpha^2\phi)$ is of the same order of magnitude as the terms neglected on the right-hand side of equation (1.8), and the above expansions are not valid in this region. The complete equation must therefore be considered in order to determine how the logarithmic term is transformed through the critical point. Tollmien[4] has shown by approximate methods that the term which equals

$$\frac{w_0''}{w_0'}$$

$$\frac{w_o''}{w_o'} \phi_1 \log(-z) \quad \text{for} \quad z < 0$$

transforms into

$$\frac{w_o''}{w_o'} \phi_1 \log z + \pi i \text{ for } \quad z > 0 \qquad \ldots(1.13)$$

The term $\pi i$ is essentially of positive sign because $w_o'$ is positive for wakes and boundary layers in $y > 0$. For a jet the term $\pi i$ would be of opposite sign because $w_o'$ is negative in $y > 0$. Tollmien's transformation has also been considered by alternative methods by Meksyn[5].

The methods developed by Tollmien[4] and Heisenberg[3] for boundary-layer problems cannot be directly applied to problems in an unlimited field of flow.

Firstly, the occurrence of the rapidly varying viscous integrals in the solution of the small disturbance equation is connected very closely with the presence of boundaries. Foote and Lin[6] have shown that they do not enter into the boundary condition equation, but that viscosity is only effective through the second-order approximations to the "inviscid" integrals.

Secondly, the velocity profiles all have at least one point of inflexion. Tollmien[7] has shown that, for the inviscid case, profiles with a point of inflexion are unstable. Thus there is a disturbance, with non-zero wave number $\alpha_s$, that is unstable for infinite Reynolds number. Tollmien has also shown that the neutral solution is that for which the wave-velocity $c_s$ is equal to the velocity at the point of inflexion of the profile.

The inviscid problem for the wake formed by a moving body was investigated by Hollingdale[8]. Using various approximations to the velocity profile, he obtained values of $\alpha$ in the region $\alpha = 4.0$ when referred to the effective half-width of the wake as unit. He also investigated experimentally the wakes formed by a flat plate and an aerofoil section. He observed a laminar and an oscillatory wake, and made an estimation of the critical Reynolds number below which the wakes were always laminar. This critical Reynolds number was 600 for the flat plate and 1000 for the aerofoil section.

Savic[9] solved the inviscid problem for the two-dimensional jet, determining the neutral wave-length and wave-velocity. He obtained good agreement between his results and experimental measurements on acoustically sensitive jets.

Two attempts have been made to solve the problem of stability in an unlimited field of flow with the effects of viscosity included, by Chiarulli[10] and Lessen[11]. In each case the inviscid integrals were expanded in powers of $(\alpha R)^{-1}$. Chiarulli put $\alpha = \alpha_s$ and then linearized in $(c - c_s)$ and $(\alpha R)^{-1}$. By this means the boundary condition equation was put in a form in which it could be solved for $(c - c_s)$ and $(\alpha R)^{-1}$. The tedious process of solving this equation was not attempted. Lessen evaluated numerically the first two terms in the series expansion, and then solved the boundary condition equation by trial and error. This process did not give a minimum critical Reynolds number, probably because more terms must be retained in the expansions of the "inviscid" integrals when $\alpha R$ is small.

In/

In the present work, starting from the known solution $\phi_s$ of the inviscid problem, having eigen-values $\alpha_s$ and $c_s$, the existence of a neighbouring viscous solution $\phi_1$ is assumed. The necessary and sufficient condition that $\phi_1$ should satisfy the boundary conditions imposed by physical considerations is obtained as an infinite integral to be zero. This condition determines the eigen-values $(\alpha, c, R)$. Although it is not found possible to do this in the general case, a solution valid for large $R$ is obtained by a linear perturbation about the known inviscid solution.

The method is applied to the problem of the wake formed by a flat plate. No minimum critical Reynolds number is predicted, because the approximations of linearizing are not valid at small enough Reynolds numbers. As the method requires the knowledge of the inviscid solution, the inviscid problem of the wake of a flat plate is first considered before the general theory is developed.

## 2. Inviscid Problem of the Wake Formed by a Flat Plate

Hollingdale[8] investigated this problem, but an error in his solution was found during the present investigation. As a knowledge of the inviscid solution is essential, the following alternative treatment is given.

Goldstein[12] gives as the first approximation to the velocity profile in the wake formed by a flat plate of length $l$,

$$w = 1 - a\, e^{-4y^2} \qquad \ldots(2.1)$$

$$\left.\begin{aligned} \text{where} \qquad y &= y'/\delta \\ \delta &= \{8\nu x'/U_0\}^{\frac{1}{2}} \\ \text{and} \qquad a &= \frac{2\beta}{\sqrt{\pi}}\left(\frac{x'}{l}\right)^{-\frac{1}{2}} \end{aligned}\right\} \qquad \ldots(2.2)$$

The inviscid equation (1.10) cannot conveniently be solved by using (2.1), therefore the velocity profile must be approximated. The most convenient approximation is

$$\left.\begin{aligned} e^{-4y^2} \sim f(y) &= A + B\cos ky & 0 < y < 0.5 \\ &= D(1-y)^2 & 0.5 < y < 1.0 \\ &= 0 & y > 1.0 \end{aligned}\right\} \qquad \ldots(2.3)$$

where $A$, $B$, $D$ and $k$ are such that

(i) $e^{-4y^2}$ and $f(y)$ have the same value at $y = 0$.

(ii) $e^{-4y^2}$ and $f(y)$ have the same point of inflexion.

(iii) $f(y)$ and $f'(y)$ are continuous at $y = 0.5$

$f(y)$ and $f'(y)$ are continuous at $y = 1$ because of the form of the approximation. With the above conditions

A/

$$\left.\begin{aligned} A &= 0.5950 \\ B &= 0.4050 \\ k &= 4.443 \\ D &= 1.334 \end{aligned}\right\} . \qquad \ldots(2.4)$$

The problem is now to solve the inviscid equation (1.10), where $w$ is given by (2.1) and (2.3), subject to the boundary conditions

$$\left.\begin{aligned} &\phi \text{ an even function of } y, \\ &\text{or } \phi'(0) = 0, \end{aligned}\right\} \qquad \ldots(2.5)$$

and

$$\phi \to 0 \text{ as } y \to \infty . \qquad \ldots(2.6)$$

<u>Region I</u> $0 < y < 0.5$

By using the approximate expression for $w$, (1.10) is reduced to

$$\phi'' + (k^2 - \alpha^2)\phi = 0 \qquad \ldots(2.7)$$

the integrals of which are

$$\left.\begin{aligned} \phi_I^{(1)} &= \cos \omega y \\ \phi_I^{(2)} &= \sin \omega y \end{aligned}\right\} \qquad \ldots(2.8)$$

where

$$\omega = \sqrt{k^2 - \alpha^2} . \qquad \ldots(2.9)$$

<u>Region II</u> $0.5 < y < 1.0$

By using the approximate expression for $w$, (1.10) is reduced to

$$(z^2 - K)(\phi'' - \alpha^2\phi) - 2\phi = 0, \qquad \ldots(2.10)$$

where

$$z = 1 - y \text{ and } K = c/D . \qquad \ldots(2.11)$$

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By solving (5.12) by Frobenius' Method the following integrals are obtained:-

$$\left.\begin{aligned} \phi_{II}^{(1)} &= \cosh \alpha z + a_2 z^2 + a_4 z^4 \ \ldots \\ \phi_{II}^{(2)} &= \sinh \alpha z + a_3 z^3 + a_5 z^5 \ \ldots \end{aligned}\right\} \qquad \ldots(2.12)$$

where

$$\left.\begin{aligned} a_2 &= -1/K \\ a_4 &= -\alpha^2/6K \\ a_3 &= -\alpha/3K \\ a_5 &= -\alpha/15K^2 - \alpha^3/30K \end{aligned}\right\} . \qquad \ldots(2.13)$$

Region III $y > 1.0$

By using the approximate expression for $w$, (1.10) is reduced to

$$\phi'' - \alpha^2\phi = 0 , \qquad \ldots(2.14)$$

which has integrals

$$\left.\begin{aligned} \phi_{III}^{(1)} &= e^{-\alpha y} \\ \phi_{III}^{(2)} &= e^{\alpha y} \end{aligned}\right\} . \qquad \ldots(2.15)$$

Thus $\phi$ is given by

$$\left.\begin{aligned} \phi_I &= A \cos \omega y + B \sin \omega y && \text{in } 0 < y < 0.5 \\ \phi_{II} &= C\, \phi_{II}^{(1)} + D\, \phi_{II}^{(2)} && \text{in } 0.5 < y < 1 \\ \phi_{III} &= E\, e^{-\alpha y} + F\, e^{\alpha y} && \text{in } y > 1 \end{aligned}\right\} . \qquad \ldots(2.16)$$

The arbitrary constants are to be chosen so that the boundary conditions (2.5) are satisfied, and $\phi$ and $\phi'$ are continuous at $y = 0.5$ and $y = 1.0$.

If (2.5) is to be satisfied then

$$B = F = 0 . \qquad \ldots(2.17)$$

By continuity at $y = 1.0$, i.e., $z = 0$

$$\left.\begin{aligned} E\, e^{-\alpha} &= C , \\ -\alpha E\, e^{-\alpha} &= -\alpha D , \end{aligned}\right\} \qquad \ldots(2.18)$$

and/

and this may be written

$$\left.\begin{aligned} C &= D \ , \\ E &= C\,e^{\alpha} \ . \end{aligned}\right\} \qquad \ldots(2.19)$$

Then by (2.16), (2.17) and (2.19) $\phi$ is given by

$$\left.\begin{aligned} \phi_{I} &= A \cos \omega y \ , \\ \phi_{II} &= C(e^{\alpha z} + a_2 z^2 + a_3 z^3 \ \ldots) \ , \\ \phi_{III} &= C\,e^{\alpha z} \ . \end{aligned}\right\} \qquad \ldots(2.20)$$

For continuity at $y = 0.5$

$$\left.\begin{aligned} A \cos \omega/2 &= (\phi_{II})_{0.5} \ , \\ -\,\omega A \sin \omega/2 &= (\phi'_{II})_{0.5} \ , \end{aligned}\right\} \qquad \ldots(2.21)$$

or

$$\omega \tan \omega/2 = -\,[\phi'_{II}/\phi_{II}]_{0.5} \ . \qquad \ldots(2.22)$$

Equation (2.22) defines $\alpha_s$, the eigen-value of $\alpha$ for the neutral disturbance in the inviscid case. Substituting from (2.20)

$$\omega \tan \omega/2 = \frac{\alpha\, e^{\alpha/2} + a_2 + \frac{3}{4} a_3 \ \ldots}{e^{\alpha/2} + \frac{1}{4} a_2 + \frac{1}{8} a_3 \ \ldots} \ , \qquad \ldots(2.23)$$

where $\omega$ is given by (2.9) and $a_2$, $a_3$, etc., by (2.13). This equation can be solved graphically to give $\alpha_s = 4.0$. Then $\phi_s$ is given by (2.20), and the result is shown in Table I.

Table I. Inviscid Solution

| $y$ | $\phi_s$ | $\phi_s'$ |
|---|---|---|
| 0.0 | 1.000 | 0.000 |
| 0.1 | 0.981 | -0.370 |
| 0.2 | 0.926 | -0.726 |
| 0.3 | 0.837 | -1.056 |
| 0.4 | 0.716 | -1.345 |
| 0.5 | 0.569 | -1.585 |
| 0.6 | 0.413 | -1.215 |
| 0.7 | 0.297 | -0.906 |
| 0.8 | 0.209 | -0.661 |
| 0.9 | 0.143 | -0.480 |
| 1.0 | 0.100 | -0.362 |

## 3. Stability in Unlimited Field of Flow at Finite Reynolds Number

The basis of the present method is that of a perturbation of the known inviscid solution. Starting from the known inviscid integral $\phi_s$ with known eigen-values $\alpha_s$ and $c_s$, the existence of a viscous integral $\phi_1$, in the neighbourhood of $\phi_s$, with eigen-values $\alpha_1$ and $c_1$, in the neighbourhood of $\alpha_s$ and $c_s$, is assumed. From the equations satisfied by $\phi_1$ and $\phi_s$ the necessary and sufficient condition that $\phi_1$ should satisfy the boundary conditions is found. This condition determines the eigen-values of $\alpha_1$, $c_1$ and R.

The complete small disturbance equation is

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = - \frac{i}{\alpha R}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \; . \qquad \ldots(3.1)$$

In the equation satisfied by $\phi_1$ all the terms on the right-hand side of (3.1) are retained as perturbation terms except the fourth-order term $\phi''''$. The influence of this term on the solution will be accounted for when the complete fourth-order equation is considered. As the rapidly varying viscous integrals do not enter into the problem, this is a reasonable approximation to make. The equation satisfied by $\phi_1$ is therefore

$$(w - c)(\phi_1'' - \alpha_1^2\phi_1) - w''\phi_1 = - \frac{i}{\alpha_1 R}(- 2\alpha_1^2\phi_1'' + \alpha_1^4\phi_1) \; ,$$

which/

which may be re-written as

$$(w - c_1)(\phi_1'' - \alpha_1^2\phi_1) - \left(w'' + \frac{i\alpha_1^3}{R}\right)\phi_1 = 0 , \qquad \ldots(3.2)$$

where

$$c_1 = c + 2i\alpha_1/R . \qquad \ldots(3.3)$$

If $w - c_1 = 0$ at $y = y_0$ then $y_0$ is a complex point in the neighbourhood of the point of inflexion but not coincident with it. Thus (3.2) has a singularity at the critical point $y = y_0$, and so the integrals of the equation will contain irregular terms which will be modified in the region of the critical point. This modification will now be discussed.

Equation (3.2) can be solved as a power series in $z = y - y_0$ by the Method of Frobenius. The following integrals are obtained

$$\left.\begin{aligned} \phi_1^{(1)} &= z + a_2 z^2 + a_3 z^3 \ldots \\ \phi_1^{(2)} &= 1 + b_1 z + \ldots + \left\{\frac{w_0''}{w_0'} + \frac{i\alpha_1^3}{Rw_0'}\right\} \phi_1^{(1)} \log(-z) \end{aligned}\right\} . \qquad \ldots(3.4)$$

The problem, as before, is to determine the modification of terms of the form $z^r \log(- z)$ near the critical point.

The complete small-disturbance equation (3.1) may be written

$$(w - c_1)(\phi_1'' - \alpha_1^2\phi_1) - \left(w'' + \frac{i\alpha_1^3}{R}\right)\phi_1 = - \frac{i}{\alpha_1 R}\phi_1'''' . \qquad \ldots(3.5)$$

Define

$$\left.\begin{aligned} y - y_0 &= \epsilon\,\eta , \\ \epsilon &= - (\alpha_1 R w_0')^{-\frac{1}{3}} , \end{aligned}\right\} \qquad \ldots(3.6)$$

where $y_0$ and therefore $\eta$ is complex. Transform (3.5) to the new variable $\eta$, and retain only terms in $\epsilon$. Thus

$$\phi_1'''' + i\,\eta\,\phi_1'' = i\epsilon\left(\frac{w_0''}{w_0'} + \frac{i\alpha_1^3}{Rw_0'}\right)\phi_1 . \qquad \ldots(3.7)$$

Now/

Now equations (3.4) and (3.7) are similar to those obtained in the work of Tollmien[4] and Meksyn[5], except that $\dfrac{w_0''}{w_0'}$ is replaced by $\left(\dfrac{w_0''}{w_0'} + \dfrac{i\alpha_1^3}{Rw_0'}\right)$. The determination of the transformation through the critical point is therefore exactly similar to the work of Tollmien and Meksyn. Hence

$$\left(\frac{w_0''}{w_0'} + \frac{i\alpha_1^3}{Rw_0'}\right) \phi_1 \log(-z) \quad \text{for} \quad y < R(y_0)$$

transforms into

$$\left(\frac{w_0''}{w_0'} + \frac{i\alpha_1^3}{Rw_0'}\right) \phi_1 \{\log z + \pi i\} \quad \text{for} \quad y > R(y_0) . \qquad \ldots(3.8)$$

## 4. Conditions Consequent upon Boundary Conditions

In this section the necessary and sufficient condition that $\phi_1$, an integral of (3.2), should satisfy the boundary conditions for a wake is established. These boundary conditions are

$$\left.\begin{aligned} \phi'(0) &= 0 \\ \phi \to 0 \text{ as } y &\to \infty \end{aligned}\right\} . \qquad \ldots(4.1)$$

It is more convenient, however, to have the condition at infinity in an equivalent form at a finite value of $y$ and then proceed to the limit. Take the width of the wake as $\sigma$, so that for $y > \sigma$ $w = 1$ and $\phi = A\,e^{-\alpha y}$. If $\phi$ and $\phi'$ are continuous at $y = \sigma$, then

$$\phi(\sigma) = A\,e^{-\alpha\sigma}$$

$$\phi'(\sigma) = -\alpha A\,e^{-\alpha\sigma}$$

and hence

$$\phi'(\sigma) + \alpha\,\phi(\sigma) = 0 . \qquad \ldots(4.2)$$

(4.2) will be taken as the boundary condition at the edge of the wake. Then there exists an inviscid integral $\phi_s$, with eigen-values $\alpha_s$, $c_s$, satisfying

$$(w - c_s)(\phi_s'' - \alpha_s^2\phi_s) - w''\phi_s = 0 , \qquad \ldots(4.3)$$

and/

and with boundary conditions

$$\left.\begin{aligned} \phi_S'(0) &= 0 \\ \phi_S'(\sigma) + \alpha_S\,\phi(\sigma) &= 0 \end{aligned}\right\} . \qquad \ldots(4.4)$$

Assume the existence of a neighbouring viscous integral $\phi_1$ satisfying

$$(w - c_1)(\phi_1'' - \alpha_1^2\phi_1) - \left(w'' + \frac{i\alpha_1^3}{R}\right)\phi_1 = 0 \qquad \ldots(4.5)$$

and with boundary conditions

$$\left.\begin{aligned} \phi_1'(0) &= 0 . \\ \phi_1'(\sigma) + \alpha_1\phi_1(\sigma) &= 0 . \end{aligned}\right\} \qquad \ldots(4.6)$$

Then $(\phi_1 - \phi_S)$ satisfies the equation

$$(\phi_1 - \phi_S)'' - \alpha_S^2(\phi_1 - \phi_S) - \frac{w''}{w - c_S}(\phi_1 - \phi_S) = g \qquad \ldots(4.7)$$

where

$$g = \phi_1'' - \alpha_S^2\,\phi_1 - \frac{w''}{w - c_S}\phi_1 .$$

Substitute for $\phi_1''$ from (4.5); then

$$g = \frac{\Delta c_1\ w''}{(w - c_1)(w - c_S)}\phi_1 + \Delta\alpha_1^2\ \phi_1 + \frac{i\alpha_1^3}{R}\ \frac{\phi_1}{w - c_1} . \qquad \ldots(4.8)$$

Here

$$\left.\begin{aligned} \Delta\alpha_1^2 &= \alpha_1^2 - \alpha_S^2 \\ \Delta c_1 &= c_1 - c_S \end{aligned}\right\} . \qquad \ldots(4.9)$$

Define the operator

$$L \equiv \frac{d^2}{dy^2} - \alpha_S^2 - \frac{w''}{w - c_S} . \qquad \ldots(4.10)$$

Then

$$\left.\begin{aligned} L(\phi_1 - \phi_s) &= g \\ L(\phi_s) &= 0 \end{aligned}\right\} \qquad \ldots(4.11)$$

$$\therefore \quad \phi_s\, L(\phi_1 - \phi_s) - (\phi_1 - \phi_s)\, L(\phi_s) = \phi_s(\phi_1 - \phi_s)'' - (\phi_1 - \phi_s)\phi_s''$$

$$= \frac{d}{dy}\{\phi_s(\phi_1 - \phi_s)' - (\phi_1 - \phi_s)\phi_s'\}$$

$$= \frac{d}{dy}\{\phi_s\, \phi_1' - \phi_1 \phi_s'\} \, . \qquad \ldots(4.12)$$

Integrating (4.12)

$$\int_0^\sigma \{\phi_s L(\phi_1 - \phi_s) - (\phi_1 - \phi_s)L(\phi_s)\}\, dy = [\phi_s\phi_1' - \phi_1\phi_s']_0^\sigma \, . \qquad \ldots(4.13)$$

Then, using (4.4), (4.6) and (4.11), (4.13) becomes

$$\int_0^\sigma g\, \phi_s\, dy = (\alpha_s - \alpha_1)\, \phi_1(\sigma)\, \phi_s(\sigma) \, , \qquad \ldots(4.14)$$

which is a necessary condition that $\phi_1$ should satisfy the boundary conditions.

It must now be demonstrated that (4.14) is also a sufficient condition that $\phi_1$ should satisfy the boundary conditions. Let $\phi_1 - \phi_s = f$, then (4.7) becomes

$$f'' - \left(\alpha_s^2 + \frac{w''}{w - c_s}\right) f = g \, . \qquad \ldots(4.15)$$

But $\phi_s$ and say $\bar{\phi}$ are two complementary functions of (4.15). Hence it follows that

$$f = \phi_s \int_0^y \frac{g\, \bar{\phi}}{\Delta}\, dy - \bar{\phi} \int_0^y \frac{g\, \phi_s}{\Delta}\, dy \, , \qquad \ldots(4.16)$$

where

$$\Delta = \phi_s'\, \bar{\phi} - \bar{\phi}'\, \phi_s \, . \qquad \ldots(4.17)$$

Now/

Now

$$\left.\begin{aligned} \phi_s'' - \left(\alpha_s^2 + \frac{w''}{w - c_s}\right)\phi_s &= 0 \\ \bar{\phi}'' - \left(\alpha_s^2 + \frac{w''}{w - c_s}\right)\bar{\phi} &= 0 \end{aligned}\right\} \qquad \ldots(4.18)$$

Hence $\phi_s''\bar{\phi} - \bar{\phi}''\phi_s = 0$, and integration gives

$$\phi_s'\bar{\phi} - \bar{\phi}'\phi_s = \text{constant}^* . \qquad \ldots(4.19)$$

So (4.16) may be written as

$$f = \phi_1 - \phi_s = \frac{1}{\Delta}\left\{\phi_s \int_0^y g\,\bar{\phi}\,dy - \bar{\phi}\int_0^y g\,\phi_s\,dy\right\} . \qquad \ldots(4.20)$$

From (4.20) it follows that

$$(\phi_1 - \phi_s)_\sigma' + \alpha_s(\phi_1 - \phi_s)_\sigma = f'(\sigma) + \alpha_s f(\sigma)$$

$$= \frac{1}{\Delta}\left\{(\phi_s' + \alpha_s\phi_s)_\sigma \int_0^\sigma g\,\bar{\phi}\,dy - (\bar{\varphi}' + \alpha_s\bar{\phi})_\sigma \int_0^\sigma g\,\phi_s\,dy\right\} . (4.21)$$

Hence from (4.14) and (4.21)

$$(\phi_1' + \alpha_s\phi_1)_\sigma - (\phi_s' + \alpha_s\phi_s)_\sigma = \frac{1}{\Delta}\left\{(\phi_s' + \alpha_s\phi_s)_\sigma\int_0^\sigma g\,\bar{\phi}dy\right.$$

$$\left. - (\bar{\varphi}' + \alpha_s\bar{\phi})_\sigma(\alpha_s - \alpha_1)(\phi_1\ \phi_s)_\sigma\right\} \ldots.(4.22)$$

But

$$(\bar{\phi}' + \alpha_s\phi)_\sigma(\alpha_s - \alpha_1)(\phi_1\phi_s)_\sigma = (\alpha_s - \alpha_1)\{(\bar{\phi}'\phi_s\phi_1)_\sigma + \alpha_s(\bar{\phi}\phi_1\phi_s)_\sigma\}$$

$$= (\alpha_s - \alpha_1)\{(-\Delta + \phi_{s\sigma}'\bar{\phi}_\sigma)\phi_{1\sigma} + \alpha_s\bar{\phi}_\sigma(\phi_1\phi_s)_\sigma\}$$

$$= (\alpha_s - \alpha_1)\{-\Delta\phi_{1\sigma} + (\phi_1\bar{\phi})_\sigma(\phi_s' + \alpha_s\phi_s)_\sigma\} . \qquad \ldots(4.23)$$

Then/

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[*] At first sight this constant might appear to be zero by the boundary condition at infinity. But this is not so, for $\bar{\phi}$ will not satisfy the same boundary conditions as $\phi_s$.

Then from (4.22) and (4.23)

$$(\phi_1' + \alpha_1\phi_1)_\sigma = (\phi_s' + \alpha_s\phi_s)_\sigma \left\{1 + \frac{1}{\Delta}\int_0^\sigma g\,\bar{\phi}\,dy + \frac{\alpha_1 - \alpha_s}{\Delta}(\phi_1\bar{\phi})_\sigma\right\}$$

$$= 0 \quad \text{by } (4.4). \qquad \ldots(4.24)$$

Hence it follows that (4.14) is also a sufficient condition that $\phi'(\sigma) + \alpha_1\phi_1(\sigma) = 0$. Then by letting $\sigma \to \infty$ in (4.14)

$$\int_0^\infty g\,\phi_s\,dy = 0 . \qquad \ldots(4.25)$$

## 5. Solution of Boundary Condition Equation

The determination of the eigen-values by solution of (4.25) is as follows. Substitute in (4.25) the expression (4.8) for $g$. Then

$$\Delta c_1 \int_0^\infty \frac{w''}{(w - c_1)(w - c_s)}\,\phi_1\,\phi_s\,dy + \Delta\alpha_1^2 \int_0^\infty \phi_1\,\phi_s\,dy + \frac{i\alpha_1^3}{R}\int_0^\infty \frac{\phi_1\,\phi_s}{w - c_1}\,dy = 0 . \qquad \ldots(5.1)$$

Now the integrands of the first and third integral are infinite at the critical point $y = y_0$, therefore the modifications near to this point must be considered carefully. Consider an integration along the positive real axis of $y$ and split up the range of integration into three parts, so that

$$\int_0^\infty \frac{w''}{(w - c_1)(w - c_s)}\,\phi_1\,\phi_s\,dy = \int_0^{R(y_0)-K} + \int_{R(y_0)-K}^{R(y_0)+K} + \int_{R(y_0)+K}^{\infty} \qquad \ldots(5.2)$$

The first and third of these integrals are regular, and so can be evaluated. In the second integral make the approximations

$$\left.\begin{aligned} w''(y) &= w''(y_0) = w_0'' \\ w - c_s &= c_1 - c_s = \Delta c_1 \\ w - c_1 &= w_0'(y - y_0) \\ \phi_1\phi_s &= (\phi_1\phi_s)_0 \end{aligned}\right\} , \qquad \ldots(5.3)$$

Then/

Then

$$\int_{R(y_o)-K}^{R(y_o)+K} \frac{w''}{(w - c_s)(w - c_1)} \phi_1 \phi_s \, dy = \frac{w''_o}{w'_o} \frac{(\phi_1 \phi_s)_o}{\Delta c_1} \int_{R(y_o)-K}^{R(y_o)+K} \frac{dy}{y - y_o}$$

$$= \frac{w''_o}{w'_o} \frac{(\phi_1 \phi_s)_o}{\Delta c_1} \left[\log(y - y_o)\right]_{R(y_o)-K}^{R(y_o)+K} . \qquad \ldots(5.4)$$

Then, using the transformation (3.8) of the logarithmic terms, (5.4) becomes approximately

$$\int_{R(y_o)-K}^{R(y_o+K)} \frac{w''}{(w - c_s)(w - c_1)} \phi_1 \phi_s \, dy = \frac{w''_o}{w'_o} \frac{(\phi_1 \phi_s)_o}{\Delta c_1} \pi i . \qquad \ldots(5.5)$$

Similarly the other 'singular' integral of (5.1) may be dealt with, and so (5.1) becomes

$$\frac{w''_o}{w'_o} \pi i (\phi_1 \phi_s)_o - \frac{\alpha_1^3 \pi}{R w'_o} (\phi_1 \phi_s) + \Delta \alpha_1^2 \int_0^\infty \phi_1 \phi_s \, dy$$

$$+ \Delta c_1 \left\{ \int_0^{R(y_o)-K} + \int_{R(y_o)+K}^\infty \frac{w''}{(w - c_1)(w - c_s)} \phi_1 \phi_s \, dy \right\}$$

$$+ \frac{i \alpha_1^3}{R} \left\{ \int_0^{R(y_o)-K} + \int_{R(y_o)+K}^\infty \frac{\phi_1 \phi_s}{w - c_1} \, dy \right\} = 0 . \qquad \ldots(5.6)$$

Equation (5.6) cannot be solved as it stands. An approximate solution is obtained by taking $\phi_1 = \phi_s$, and by linearising in the small quantities $\Delta c_1$ and $1/R$. Hence if $y_o = y_s + \delta y_s$, then

$$c_1 = w(y_o) = w(y_s + \delta y_s)$$

$$= w(y_s) + \delta y_s \, w'_s$$

$$= c_s + \delta y_s \, w'_s$$

$$\text{or} \quad \delta y_s = \frac{\Delta c_1}{w'_s} . \qquad \ldots(5.7)$$

Also/

$$\left.\begin{aligned}\text{Also} \qquad w''_o &= w''(y_s + \delta y_s) \simeq \frac{\Delta c_1}{w'_s} w'''_s \\ \text{and} \qquad w'_o &= w'(y_s + \delta y_s) \approx w'_s\end{aligned}\right\} \qquad \ldots(5.8)$$

Then (5.6) reduces to

$$\Delta c_1 \frac{w'''_s}{w'_s} \pi i - \frac{\alpha_1^3 \pi}{R w'_s} + \frac{\Delta \alpha_1^2}{(\phi_s^2)_s} \int_0^\infty \phi_s^2 \, dy$$

$$+ \Delta c_1 \, E_1 + \frac{i\alpha^3}{R} E_2 = 0 \qquad \ldots(5.9)$$

where

$$\left.\begin{aligned}E_1 &= \frac{1}{(\phi_s^2)_s} \left\{ \int_0^{y_s - K} \frac{w''}{(w - c_s)^2} \phi_s^2 \, dy + \int_{y_s + K}^\infty \frac{w''}{(w - c_s)^2} \phi_s^2 \, dy \right\} \\ E_2 &= \frac{1}{(\phi_s^2)_s} \left\{ \int_0^{y_s - K} \frac{\phi_s^2}{w - c_s} \, dy + \int_{y_s + K}^\infty \frac{\phi_s^2}{w - c_s} \, dy \right\} .\end{aligned}\right\} \quad (5.10)$$

But for neutral disturbances

$$c_1 = c + 2i\alpha_1/R$$

$$= c_r + 2i\alpha_1/R$$

$$\therefore \quad \Delta c_1 = \Delta c_r + 2i\alpha_1/R . \qquad \ldots(5.11)$$

Substitute from (5.11) into (5.9), and equate real and imaginary parts to zero.

$$\left.\begin{aligned}- \frac{w'''_s}{w'^2_s} \frac{2\pi\alpha_1}{R} - \frac{\alpha_1^3 \pi}{R w'_s} + \Delta c_r \, E_1 + \Delta \alpha_1^2 \frac{1}{(\phi_s^2)_s} \int_0^\infty \phi_s^2 \, dy &= 0 \\ \frac{w'''_s}{w'^2_s} \pi \Delta c_r + \frac{2\alpha_1 E_1}{R} + \frac{\alpha_1^3 E_2}{R} &= 0\end{aligned}\right\} . \qquad \ldots(5.12)$$

Using $w$ and $\phi_s$ as defined in Section 2 the various terms have been evaluated by numerical integration for the wake formed by a flat plate. Then solving (5.12) for $\Delta c_r$ and $\Delta \alpha_1^2$,

$$\left.\begin{aligned} \Delta c_r &= 1.901 \frac{\alpha_1}{R} - 0.04486 \frac{\alpha_1^3}{R} \\ a\,\Delta\alpha_1^2 &= -160.1 \frac{\alpha_1}{R} + 4.423 \frac{\alpha_1^3}{R} \end{aligned}\right\} . \qquad \ldots(5.13)$$

Here "a" is as defined in (2.2). If, instead of expressing R in terms of δ, $R_\delta = \frac{U_0\delta}{\nu}$, it is expressed in terms of the length $l$ of the plate, $R_e = \frac{U_0 l}{\nu}$, then the equations (5.13) can easily be written as

$$e^{-\frac{1}{2}} - e^{-4y_0^2} = 1.270 \frac{\alpha_1}{R_l^{\frac{1}{2}}} - 0.0299 \frac{\alpha_1^3}{R_l^{\frac{1}{2}}} \qquad \ldots(5.14)$$

$$\Delta\alpha_1^2 = -107.0 \frac{\alpha_1}{R_l^{\frac{1}{2}}} + 2.95 \frac{\alpha_1^3}{R_l^{\frac{1}{2}}} \qquad \ldots(5.15)$$

(5.15) is the equation of the (α, R) curve, and may be written as

$$R_l = \left\{ \frac{107.0\,\alpha_1 - 2.95\,\alpha_1^3}{\alpha_1^2 - 16} \right\}^2 . \qquad \ldots(5.16)$$

The $(\alpha_1, R_l)$ curve (Fig. 1) gives no indication of a minimum critical Reynolds number below which all disturbances are stable, probably because the method of solving the boundary condition equation is not valid for small values of $R_l$. The curve is dotted below $R_l \approx 600$, the critical Reynolds number obtained experimentally by Hollingdale.

Thus the perturbation method leads to a simple determination of the (α, R) curve, valid for large values of R.

## Acknowledgements

The author would like to thank Dr. D. Meksyn of the Mathematics Department, Imperial College of Science and Technology, for many helpful suggestions during the writing of the Thesis of which this paper is an abridged version. The author would also like to acknowledge the help of Dr. N. Curle of the Aerodynamics Division of the National Physical Laboratory, in preparing this paper from the author's original Thesis.

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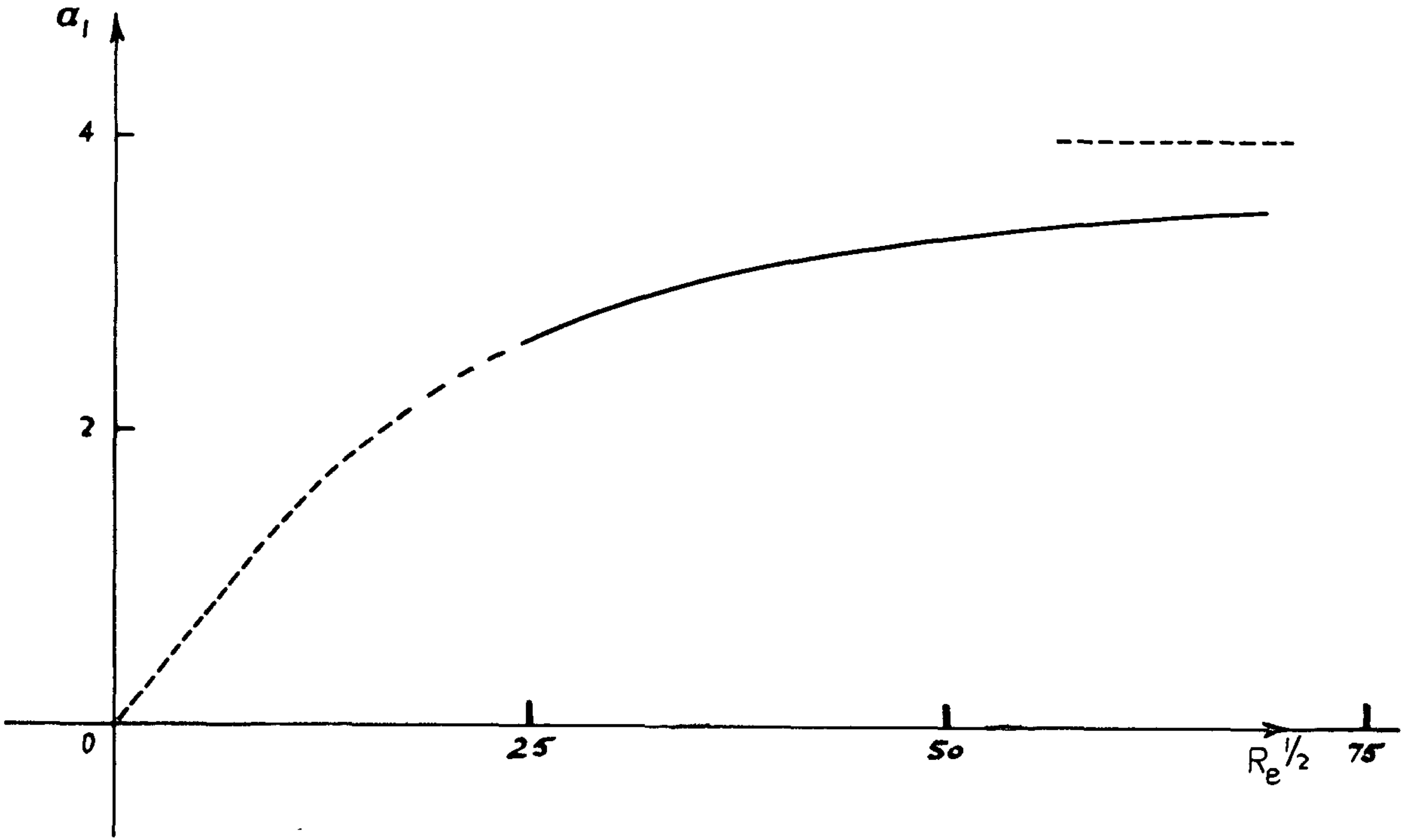
FIG. 1.



Curve of neutral stability.

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