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An Experimental Investigation of Meksyn's Transonic Inviscid Flow Theory

By

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An Experimental Investigation of Meksyn's Transonic Inviscid Flow Theory

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19th November, 1955

SUMMARY

The solution of the two-dimensional transonic flow equations obtained by Meksyn (Proc.Roy.Soc. A, Vol. 220, pp.239-254, 1953) is critically examined for the flow past a bicusped aerofoil at zero lift. Comparisons with experimental results are made and it is clear from these that the method does not give useful results for mixed subsonic/supersonic flows.

CONTENTS

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1. Introduction

The problem of mixed subsonic/supersonic flows with shock waves is of great importance, but because of the non-linearity of the flow equations it has not been possible to obtain a satisfactory theoretical treatment of the problem. Therefore a paper by Meksyn[1] (1953) on the 'Integration of the equations of transonic flow in two-dimensions' has aroused considerable interest. In that paper a novel method is used to calculate transonic flows and the results are interpreted as indicating the first appearance of a shock wave on the surface of an aerofoil. A

knowledge/

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knowledge of the upper critical or 'characteristic' free-stream Mach number so obtained would be very useful. Therefore at the request of the Ministry of Supply, the Aerodynamics Division of the N.P.L. undertook an appraisal of the theory.

In its analytical form Meksyn's method can only be applied to aerofoils derived by exact transformations of a circle. As an example he calculated the solution for a 10 per cent thick bicusped aerofoil due to Kaplan[2] (1943). The bicusped aerofoil, and others derived from exact transformations, are of little practical importance, so the first consideration was whether Meksyn's theory could be extended to more conventional aerofoils. The author considered such an extension by relaxation methods and a short account of this treatment is given in the Appendix. It is shown there that such a solution is feasible but would be dependant on the fourth derivative of the low-speed velocity distribution for a given aerofoil; since this distribution could only be obtained numerically, the computations required to achieve useful results would be immense. So it was decided not to attempt a solution by relaxation methods until the usefulness of Meksyn's theory in relation to the bicusped aerofoil had been established.

To this end tests on a two-dimensional bump model have been carried out in the N.P.L. 9" x 3" high-speed wind tunnel. Also calculations additional to those given in Ref.1 were made so that the theoretical and experimental pressure distributions could be compared before the predicted first appearance of a shock wave. Results of the comparisons are discussed in Section 3, where it is shown that though the method is a satisfactory, if rather laborious, means of calculating subcritical pressure distributions, the results pertaining to the first appearance of shock waves do not accord with the experimental evidence. The possible causes of this failure of Meksyn's theory are discussed in Section 4.

2. Outline of Theory

The symbols used in this paper are defined in Section 7 and are the same as those of Ref.1.

The incompressible flow past an aerofoil is described by a complex potential $\gamma = \alpha + i\beta$ where α and β are functions of the space coordinates (x, y). $U_1 \phi(\alpha,\beta)$ is the velocity potential of the compressible flow past the aerofoil. It can been shown that ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} = \frac{U_1^2}{2C^2}\left\{\frac{\partial\phi}{\partial\alpha}\cdot\frac{\partial q^2}{\partial\alpha} + \frac{\partial\phi}{\partial\beta}\cdot\frac{\partial q^2}{\partial\beta}\right\}, \qquad \ldots\ldots(1)$$

where

$$q^2 = h^2\left\{\left(\frac{\partial\phi}{\partial\alpha}\right)^2 + \left(\frac{\partial\phi}{\partial\beta}\right)^2\right\}, \qquad \ldots\ldots(2)$$

and $h\ (\equiv|d\gamma/dz|)$ is the local speed in incompressible flow. On expansion, when terms of

$$O\left[(\partial\phi/\partial\beta)^2\right]$$

$\partial^2\phi/$

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} = \frac{M_1^2}{2}\left(\frac{\partial\phi}{\partial\alpha}\right)^3 \frac{\partial h^2}{\partial\alpha} + M_1^2 h^2 \left(\frac{\partial\phi}{\partial\alpha}\right)^2 \frac{\partial^2\phi}{\partial\alpha^2}$$

$$+ \frac{(k-1)\,M_1^2}{2}\,(q^2 - 1)\,\nabla^2\phi\ ; \qquad \ldots\ldots(3)$$

it is this equation that Meksyn solves.

With the assumption that compressibility produces only a perturbation effect on the incompressible flow, equation (3) becomes

$$\nabla^2\phi_o = \frac{M_1^2}{2}\cdot\left(\frac{\partial\phi}{\partial\alpha}\right)^3\cdot\frac{\partial h^2}{\partial\alpha} \qquad \ldots\ldots(4)$$

to a first approximation. It is now necessary to assume that $\partial\phi/\partial\alpha$ can be treated as a constant, then equation(4) can be integrated to give

$$\phi_o = \frac{M_1^2}{8}\left(\frac{\partial\phi}{\partial\alpha}\right)^3 \frac{\partial}{\partial\alpha}\left[\int\frac{dy}{dz}\cdot dy \times \int\frac{d\bar{y}}{d\bar{z}}\cdot d\bar{y}\right] .$$

After satisfying the condition $\partial\phi_o/\partial\beta = 0$ on the aerofoil and $\partial\phi_o/\partial\alpha = 1$ at infinity, the final result can be put in the form

$$\frac{\partial\phi_o}{\partial\alpha} = 1 + \frac{M_1^2}{8}\cdot\left(\frac{\partial\phi}{\partial\alpha}\right)^3 P(\alpha,\beta). \qquad \ldots\ldots(5)$$

Consider now the last term on the right hand side of equation(3). Substitution for $\nabla^2\phi$ from equation(4) with the aid of the relation $q^2 = h^2\,(\partial\phi/\partial\alpha)^2$ from equation(2), leads to the incremental result $\varphi = \phi_1 = \phi_1^{(1)} + \phi_1^{(2)}$, where

$$\nabla^2\phi_1^{(1)} = \frac{(k-1)\,M_1^4}{8}\left(\frac{\partial\phi}{\partial\alpha}\right)^5 \frac{\partial h^4}{\partial\alpha}, \qquad \ldots\ldots(6a)$$

and

$$\phi_1^{(2)} = -\frac{(k-1)\,M_1^2}{2}\cdot\phi_o\ . \qquad \ldots\ldots(6b)$$

The boundary conditions for ϕ_1 are

$$\frac{\partial\phi_1}{\partial\beta} = 0 \text{ on } \beta = 0\ ,$$

and

$$\frac{\partial\phi_1}{\partial\alpha} = 0 \text{ at infinity}\ .$$

Integration of equation(6a) similarly to equation(4) gives

$$\frac{\partial\phi_1^{(1)}}{\partial\alpha} = \frac{(k-1)\,M_1^4}{32}\left(\frac{\partial\phi}{\partial\alpha}\right)^5 Q(\alpha,\beta). \qquad \ldots\ldots(7)$$

There remains a further increment $\phi = \phi_2$ from the second term on the right hand side of equation(3). Substitution for $\partial^2\phi/\partial\alpha^2$ from the differential of equation(5) leads to the equation

$\nabla^2\phi_2$/

$$\nabla^2 \phi_2 = \frac{M_1^4}{8} \left(\frac{\partial\phi}{\partial\alpha}\right)^5 h^2 \frac{\partial P}{\partial\alpha}, \qquad \ldots\ldots(8)$$

which, when integrated similarly to equation(6a) under the same boundary conditions, gives

$$\frac{\partial\phi_2}{\partial\alpha} = \frac{M_1^4}{32} \left(\frac{\partial\phi}{\partial\alpha}\right)^5 R(\alpha,\beta). \qquad \ldots\ldots(9)$$

The complete solution is then given by

$$\phi = \phi_0 + \phi_1 + \phi_2 ,$$

so from equations(5), (6b), (7) and (9)

$$\frac{\partial\phi}{\partial x} = 1 + \frac{M_1^2}{8} \left(\frac{\partial\phi}{\partial\alpha}\right)^3 \left[1 - \frac{k-1}{2} M_1^2\right] P(\alpha,\beta) + \frac{M_1^4}{32} \left(\frac{\partial\phi}{\partial\alpha}\right)^5 S(\alpha,\beta), \quad \ldots(10)$$

where $S = (k - 1)Q + R.$

The functions $P(\alpha,\beta)$ and $S(\alpha,\beta)$ can be evaluated at any point in the field from the incompressible flow solution. Then for a given free-stream Mach number M_1, equation(10) can be solved for $\partial\phi/\partial\alpha$ to obtain the local velocity in a compressible flow.

It is found that for sufficiently low values of M_1, equation(10) always has two real positive roots, of which that closer to unity represents the solution of the flow equations compatible with the assumptions required for their integration. However, for any prescribed point in the flow field, the real positive roots approach each other as M_1 increases and coincide for some value of M_1, known as the characteristic Mach number. At higher values of M_1 there is no real positive solution to equation (10) at the prescribed point. Meksyn suggests that this breakdown of the method indicates the appearance of a shock wave in the flow.

3. Comparison with Experiment

It is mentioned in the Introduction that, because of the difficulty of applying Meksyn's method to a conventional aerofoil for which experimental results would be available, tests were made on the bicusped Kaplan aerofoil* treated algebraically by Meksyn. These tests were conducted in the N.P.L. 9" × 3" induced-flow high-speed tunnel on a half model of the aerofoil arranged as a bump of the bottom wall of the tunnel. Details of the wind-tunnel work will be given in a later paper; it is sufficient to note here that the tunnel-wall boundary layer upstream of the model was removed by suction so as to obtain flow conditions similar to those on the upper surface of a symmetrical aerofoil at zero incidence. It may also be mentioned that at the first appearance of a shock wave the solid blockage was estimated to give a correction of less than one per cent to the measured free-stream Mach number; this correction has not been applied, as it cannot be calculated accurately and would not materially affect the results.

The functions P, Q and R of Section 2 are calculated for positions on the aerofoil given by $x/c = 0.5$, 0.6 and 0.7; by symmetry their values at $x/c = 0.3$ and 0.4 are also known. The local Mach number M_l at these positions is calculated for a number of free-stream Mach numbers M_1 and the characteristic Mach number M_c

at/

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*The derivation and ordinates of this aerofoil are given in Ref.2.

at each point is evaluated. The complete results are shown in Table I and Fig.1.

The results for $M_1 = 0.70$ at the five positions on the aerofoil surface are shown plotted in terms of the pressure ratio p/p_0 in Fig.2, together with pressure distributions obtained by experiment and by the Polygon method of Woods[3] (1953). The latter distribution is shown because the Polygon method is known to give good agreement with aerofoil tests in the absence of circulation at subcritical Mach numbers*: it is thought that the comparison with the bump tests confirms that the removal of the upstream boundary layer is effective in producing the desired flow conditions. It is also clear from Fig.2 that Meksyn's method gives reasonable results for $M_1 = 0.70$. However, the important use of the method was to be the prediction of the first appearance of a shock wave in the flow. Clearly the position on the aerofoil surface giving the lowest characteristic Mach number is that at which the highest local velocity occurs in a subcritical flow. For the bicusped aerofoil this position is the mid-chord and calculations for that point give $M_c = 0.828$. In Fig.3 four measured pressure distributions are shown; the one obtained at $M_1 = 0.778$ does not include the pressure rise associated with a shock wave, whereas that obtained at $M_1 = 0.792$ clearly does. From these results it appears that a shock wave first appears in the flow at a free-stream Mach number of 0.78_5 at about 0.55c from the leading edge. Meksyn's predictions do not accord with these results; the apparently small excess of 0.04 in M_1 is important since the resultant increase in shock strength may induce boundary-layer separation with its attendant changes in the flow pattern. It should also be noted that in the real flow the shock first appears somewhat downstream of the mid-chord.

Calculated values of the characteristic Mach number at $x = 0.6c$ and $0.7c$ are 0.871 and 1.024 respectively. Because of the effects of shock-induced boundary-layer separation it was not possible to get reliable experimental results in this range, though the pressure distributions of Fig.3 afford some basis of comparison. Pearcey[4] (1955) has shown that one effect of the separation is to stabilise the shock position over a range of Mach number, so that the shock position 0.66c, shown by the pressure distribution at $M_1 = 0.855$, is upstream of that expected for the same M_1 in the absence of shock-induced separation. Even so, at this Mach number, lower than Meksyn's $M_c = 0.871$ for 0.6c, the shock is at 0.66c. The pressure distribution denoted 'Top' in Fig.3 was obtained with the tunnel running under choked conditions and corresponds approximately to $M_1 = 0.95$; the shock, as would be expected, has reached the trailing edge of the bump. Though this case is not strictly comparable with Meksyn's theory, it is clear that his value $M_c = 1.024$ for a 0.7c shock position is also unrealistic. However, as in these cases a solution does not exist over a part of the aerofoil, it is not unexpected that the characteristic Mach number has no significance.

4. Discussion

Meksyn compared the result which he obtained for the bicusped aerofoil with the work of Kaplan[2] on the same section. Kaplan suggests that the divergence of his series solution at the mid-chord for $M_1 > 0.83$ indicates the first appearance of a shock wave in the flow. This value is very close to Meksyn's lowest characteristic Mach number. However, as shown in Fig.4, the corresponding local Mach number distributions do not agree for $M_1 = 0.82$ˣ. From the results at five

chordwise/

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*A critical examination of this method will shortly be published by the present author.

ˣThe comparison is made for $M_1 = 0.82$ as Meksyn's solution is incomplete for $M_1 = 0.83$.

chordwise positions Meksyn's distribution appears to be peaked at the mid-chord; this is probably due to the rapid increase in local Mach number at a point as the characteristic Mach number of that point is approached. Fig.1 illustrates this feature of the solutions. That there will be a shock wave in the real flow at $M_1 = 0.82$ is obvious from the pressure distributions of Fig.3 and it is significant that the maximum local Mach numbers of about 1.25 predicted by both the Meksyn and Kaplan theories before the inclusion of a shock wave is considered necessary are both in excess of experimental values. It is usual for shock waves to be discernible in aerofoil tests when M_l exceeds about 1.1.

Meksyn's solutions for $M_1 = 0.70$, 0.75, 0.78 and 0.80 are shown in Fig.5 together with measured distributions for $M_1 = 0.709$ and 0.778. As mentioned earlier, the comparison at $M_1 = 0.70$ is satisfactory. At $M_1 = 0.78$, however, the theoretical results overestimate the measured values though there is no shock wave in this case. The results at $M_1 = 0.80$ show the tendency towards a peaked pressure curve at higher Mach numbers. It appears from these results that the approximations made in the theory are not valid when supersonic flow occurs.

Any solution of the transonic flow equations based on some form of linearization with respect to a subsonic flow ignores the change from an elliptic to hyperbolic type of equation in passing from subsonic to supersonic flow. Meksyn's assumption that terms of the order of $(\partial\phi/\partial\beta)^2$ can be neglected introduces an additional source of error when supersonic flow occurs, for stream tube behaviour in supersonic flow is radically different from that in incompressible flow. Linearized theories may give useful results with small supersonic regions provided that the assumption of small perturbations is plausible, but the breakdown of such methods cannot be expected to give any indication of shock wave formation. Refs. 5, 6 and 7 consider more exact forms of the transonic flow equations, and it is significant that discontinuites are introduced into the solutions when the local sonic velocity is exceeded in a small region of the flow.

5. Concluding Remarks

Meksyn's transonic flow theory cannot be accepted as a useful means of estimating the first appearance of shock waves in a mixed subsonic/supersonic flow. It is the author's opinion that any similar approach is unlikely to be successful. However, the assessment of Meksyn's work has led to a useful study of this important problem; clearly there is a need for further research.

A subsidiary result on the experimental side has been the investigation of boundary-layer influence on bump tests; this work will be published later. It is intended to construct a bump model of the bicusped aerofoil with facilities for distributed suction over the whole chord so that boundary-layer effects can be eliminated throughout the subsonic-speed range and shock-wave formation studied independently of boundary-layer interaction.

6. Acknowledgements

The calculations for this paper were made by Miss S. W. Skan and Mrs. J. Sindall of the Aerodynamics Division, N.P.L.

Mr. L. H. Tanner of the Aerodynamics Division gave much valuable advice and assistance in the experimental work.

7. Notation

C	local velocity of sound
c	aerofoil chord
h	$(\equiv \lvert dy/dz \rvert)$ local velocity in incompressible flow
k	ratio of specific heats (1.40 for air)
M_1, M_c, M_l	free-stream, characteristic and local Mach number respectively
P, Q, R, S	functions of (α, β) defined in equations (5), (7) (9) and (10)
p, p_0	local and stagnation static pressure
q	local velocity in compressible flow
U_1	velocity of free stream
x, y	rectangular coordinates in the physical plane
z	x + iy, complex coordinate
α, β	velocity potential, stream function for incompressible flow
γ	$\alpha + i\beta$
∇^2	$\partial^2/\partial\alpha^2 + \partial^2/\partial\beta^2$
ϕ	velocity potential for compressible flow
ϕ_0, ϕ_1, ϕ_2	contributions to ϕ in equations (4), (6) and (8)

References

No.	Author(s)	Title, etc.
1	D. Meksyn	Integration of the equations of transonic flow in two-dimensions. Proc.Roy.Soc. A, Vol.220, pp.239-254, 1953.
2	C. Kaplan	The flow of a compressible fluid past a curved surface. N.A.C.A. Rep.768, 1943.
3	L. C. Woods	The application of the Polygon method to the calculation of the compressible subsonic flow round two-dimensional profiles. A.R.C. C.P.115, 1952.
4	H. H. Pearcey	Some effects of shock-induced separation of turbulent boundary layers in transonic flow past aerofoils. Paper No.9 presented at the symposium on boundary-layer effects in aerodynamics, N.P.L., March-April, 1955. A.R.C.17,681. June, 1955.

No.	Author(s)	Title, etc.
5	H. W. Emmons	The theoretical flow of a frictionless, adiabatic, perfect gas inside of a two-dimensional hyperbolic nozzle. N.A.C.A. T.N.1003 (1946)
6	C. S. Sinnott	A method of computing subsonic and transonic plane flows. A.R.C. C.P.173 (1953)
7	J. R. Spreiter and A. Alksne	Theoretical prediction of pressure distributions on nonlifting airfoils at high subsonic speeds. N.A.C.A. T.N.3096 (1954)

APPENDIX

A Relaxation Treatment of Meksyn's Solution of the Transonic Flow Equations

The following treatment of the transonic flow equations is not suggested as the best means of obtaining a solution by relaxation methods. It was devised solely to indicate how Meksyn's theory could be applied to a geometrically defined aerofoil.

The derivation of the equations considered here is described in Section 2. It is convenient to take the (constant) coefficients of the functions of h on the right hand side of equations (4), (6a) and (8) as unity, with the boundary conditions for all three equations as

$$\frac{\partial\phi}{\partial\alpha} = 0 \quad \text{at infinity,}$$

$$\frac{\partial\phi}{\partial\beta} = 0 \quad \text{on } \beta = 0 \text{ (the aerofoil surface).}$$

The equations may then be written

$$\left.\begin{aligned} \nabla^2 \phi_0' &= \frac{\partial h^2}{\partial\alpha} \\ \nabla^2 \phi_1^{(1)'} &= \frac{\partial h^4}{\partial\alpha} \\ \nabla^2 \phi_2' &= h^2 \frac{\partial P}{\partial\alpha} \end{aligned}\right\} \qquad \text{.....(A.1)}$$

Their solutions corresponding respectively to equations (5), (7) and (9) are formally

$\partial\phi_0'/$

$$\left.\begin{aligned} \frac{\partial \phi_0'}{\partial \alpha} &= \frac{P}{4} \\ \frac{\partial \phi_1^{(1)'}}{\partial a} &= \frac{Q}{4} \\ \frac{\partial \phi_2'}{\partial \alpha} &= \frac{R}{4} \end{aligned}\right\} . \qquad \text{.....(A.2)}$$

It will facilitate solution by relaxation methods if equations (A.1) are differentiated with respect to α, giving

$$\left.\begin{aligned} \nabla^2 \psi_0 &= \frac{\partial^2 h^2}{\partial \alpha^2} \\ \nabla^2 \psi_1 &= \frac{\partial^2 h^4}{\partial \alpha^2} = 2\left\{h^2 \frac{\partial^2 h^2}{\partial \alpha^2} + \left(\frac{\partial h^2}{\partial \alpha}\right)^2\right\} \\ \nabla^2 \psi_2 &= \frac{\partial}{\partial \alpha}\left(h^2 \frac{\partial P}{\partial \alpha}\right) = \frac{\partial h^2}{\partial \alpha}\frac{\partial P}{\partial \alpha} + h^2 \frac{\partial^2 P}{\partial \alpha^2} \end{aligned}\right\} \qquad \text{.....(A.3)}$$

where $\frac{\partial \phi_0'}{\partial \alpha} = \psi_0$, $\frac{\partial \phi_1^{(1)'}}{\partial a} = \psi_1$ and $\frac{\partial \varphi_2'}{\partial \alpha} = \psi_2$.

The boundary conditions become

$$\left.\begin{aligned} \psi_i &= 0 \quad \text{at infinity} \\ \frac{\partial \psi_i}{\partial \beta} &= 0 \quad \text{on } \beta = 0 \end{aligned}\right\} \quad i = 0,\ 1,\ 2.$$

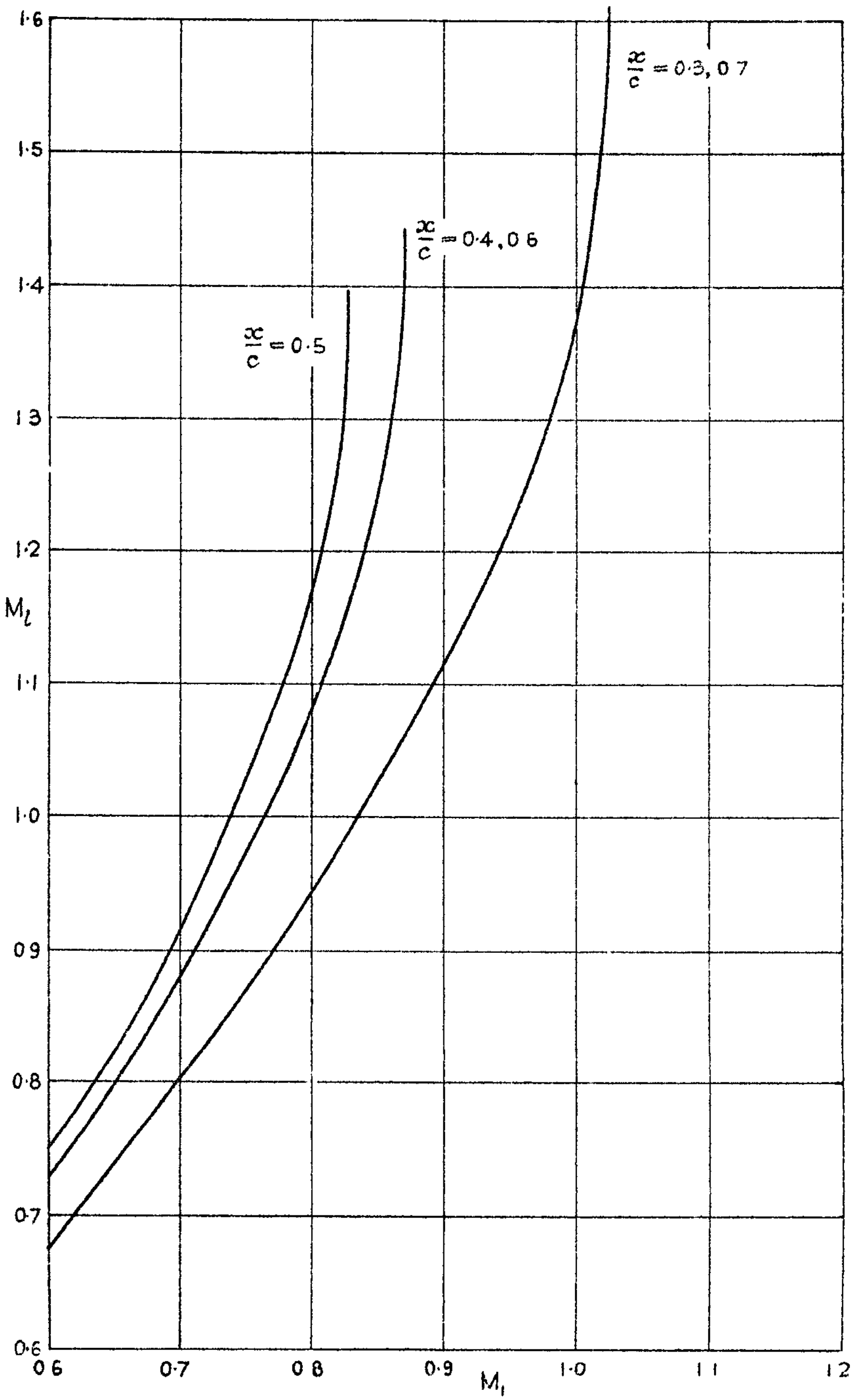
A relaxation solution of equations (A.3) would require the calculation, at each mesh point in the (α, β) plane, of the functions on the right hand sides of the equations. Hence the solutions for ψ_0 (= $P/4$) and ψ_1 are dependent on the second derivative of h, the velocity in incompressible flow. Furthermore the solution for P is required to such accuracy that its second derivative may be evaluated for use in the relaxation solution for ψ_2. To calculate usefully accurate values of these derivatives from a numerical incompressible-flow solution would require an immense amount of work.

TABLE I

Values of Meksyn's Functions P, Q, R and S, and M_c Calculated for the Bicusped Aerofoil

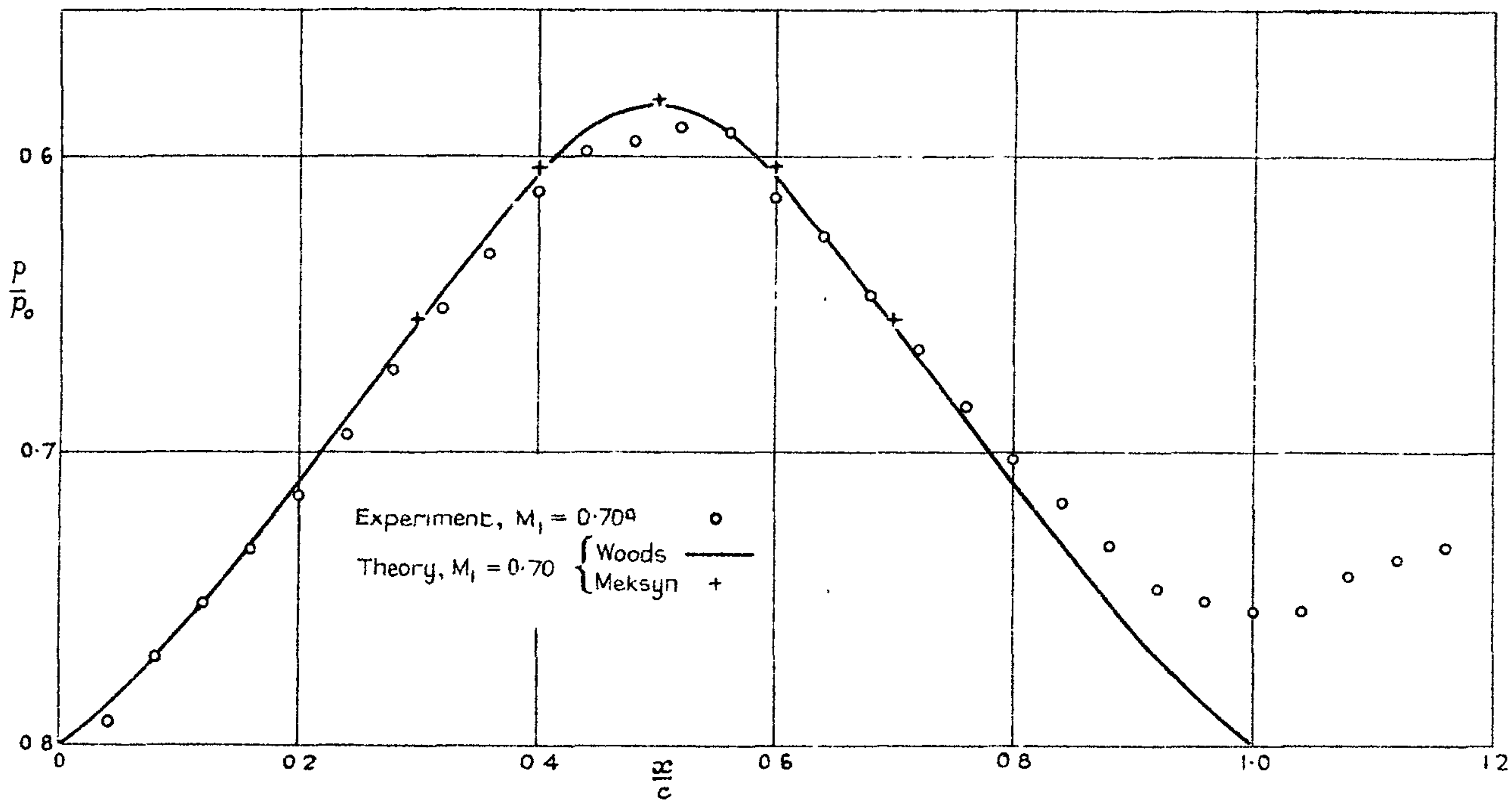
x/c	0.5	0.4, 0.6	0.3, 0.7
P	0.687287	0.588364	0.340280
Q	1.545228	1.284092	0.672774
R	2.827920	2.426042	1.474596
S	3.446011	2.939679	1.743706
M_c	0.8285	0.8707	1.0244

FIG 1



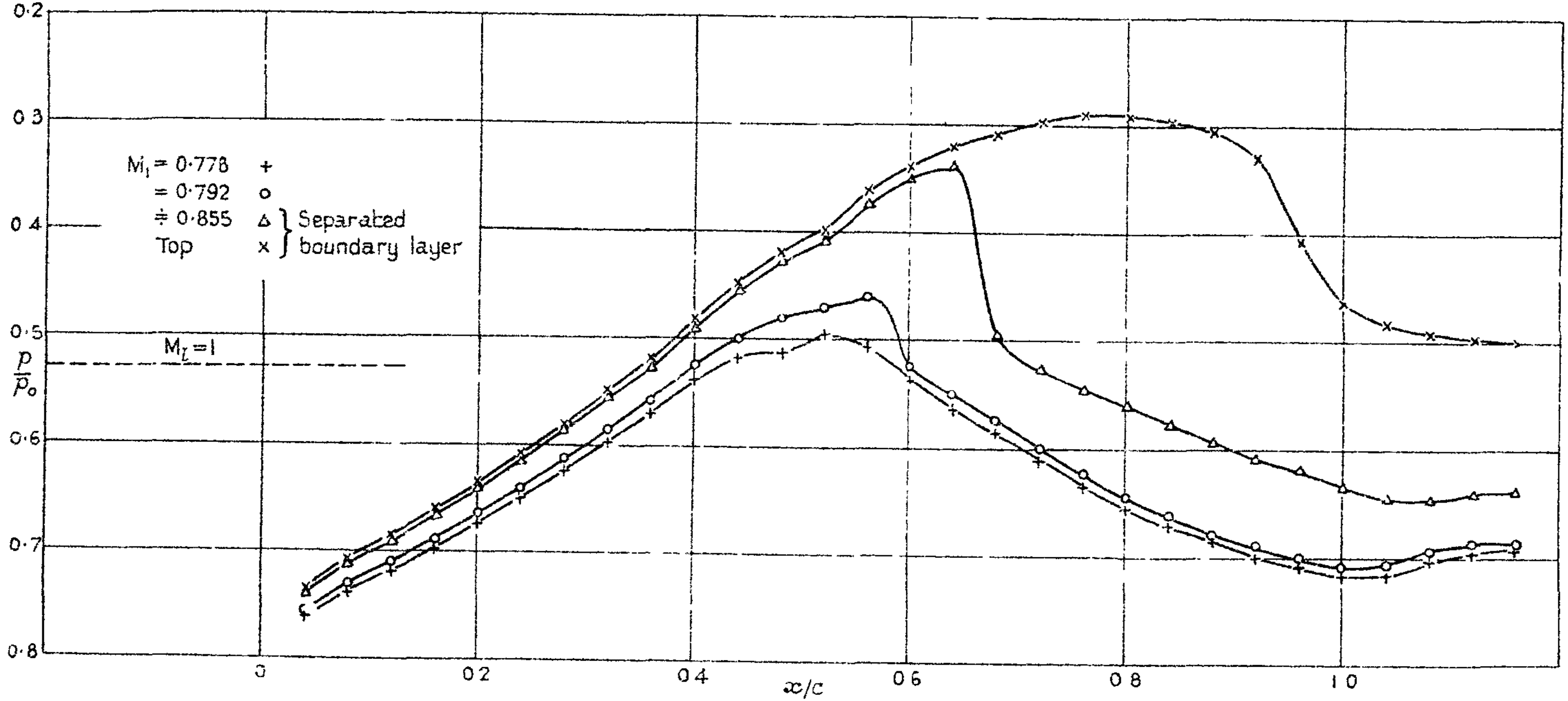
Variation of M_l with M_1 at three chordwise positions.

FIG. 2.



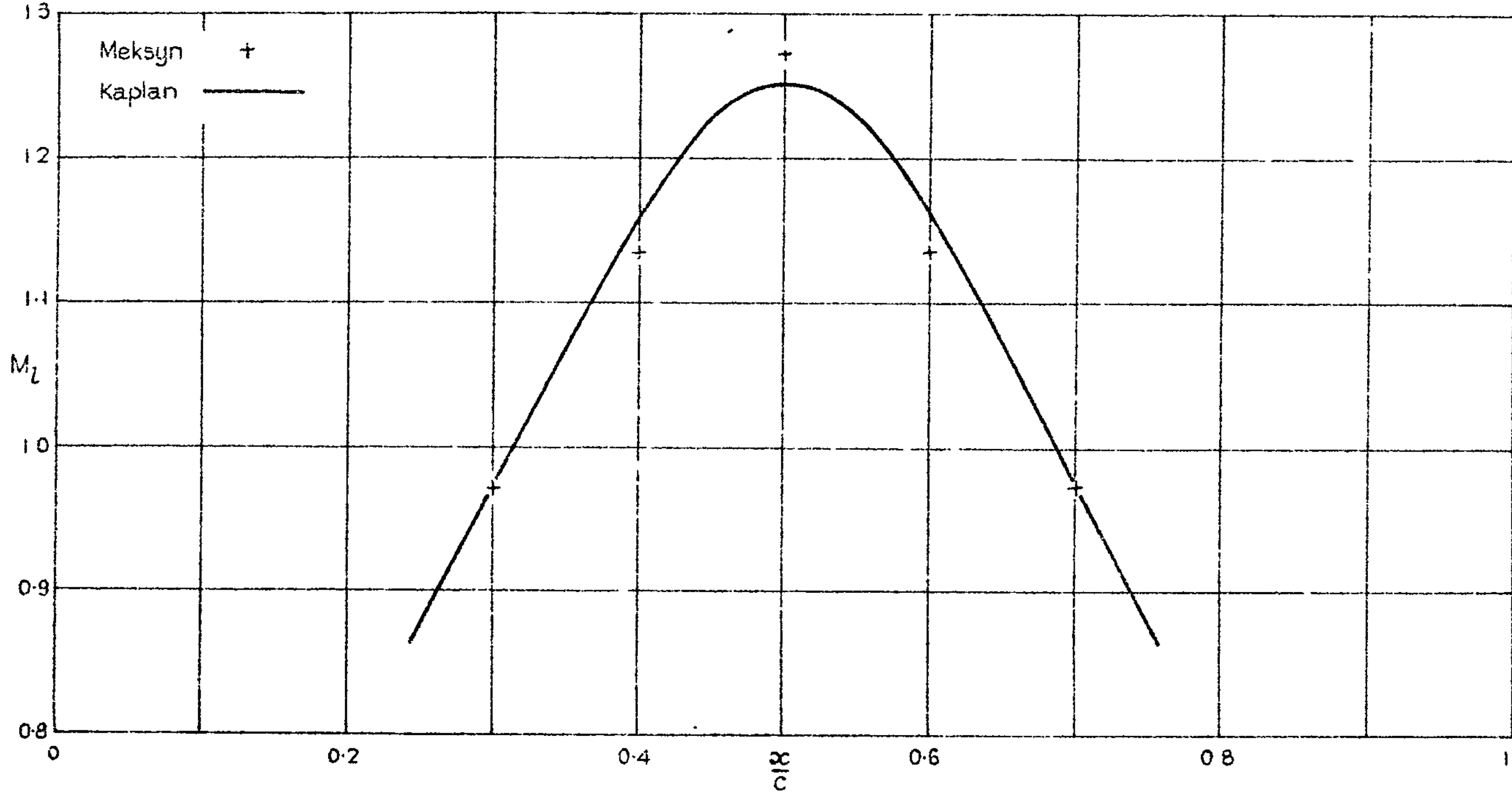
Comparison of pressure distributions at $M_1 = 0 \cdot 7$

FIG 3.

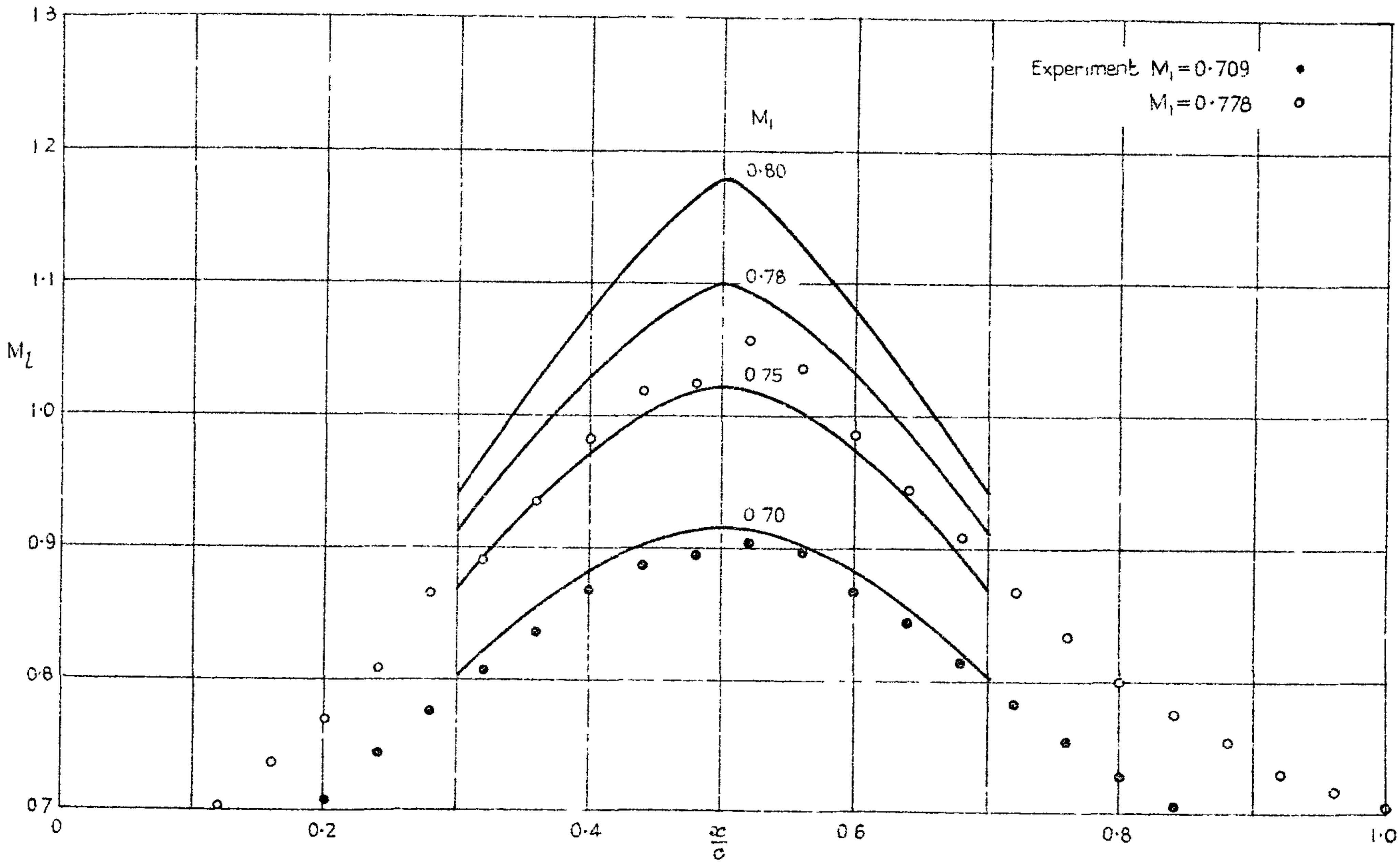


Pressure distributions for super-critical Mach numbers

FIG. 4.



Local Mach number distribution by Meksyn and Kaplan theories, $M_1 = 0.82$.



Comparison between theory and experiment at high Mach numbers

FIG 5

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