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EXPANSIONS IN SERIES OF CHEBYSHEV POLYNOMIALS OF A FUNCTION OCCURRING IN LINEAR UNSTEADY AERODYNAMIC THEORY

by

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SUMMARY

A procedure is described for obtaining expansions in series of Chebyshev polynomials of the function $\int_0^\infty \frac{e^{-i\alpha u} du}{(u^2 + 1)^{n+\frac{1}{2}}}$ for all real α and all integer $n \geqslant 0$. Numerical values are given of the coefficients of the series of Chebyshev polynomials obtained from a FORTRAN program. The leading coefficients are given to twelve significant decimal digits.

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LIST OF CONTENTS

1 INTRODUCTION

In linearised unsteady aerodynamics the function $S_n(\alpha)$, defined by formula (2-1) for $n \geqslant 1$, has to be evaluated for many values of α. Expansions of functions in terms of series of Chebyshev polynomials may be used for rapid evaluation of the functions. It is the purpose of this paper to use the procedure introduced by Clenshaw[1] to obtain such expansions of $S_n(\alpha)$. The form of these expansions are given in formulae (9-1), (9-2), (9-3) and (9-4). Numerical values of the coefficients of the series of Chebyshev polynomials were obtained from a FORTRAN program with double precision arithmetic.

2 DISCUSSION OF THE NATURE OF THE FUNCTION UNDER CONSIDERATION

In unsteady linearised aerodynamics we need to evaluate, for any real α, the complex function $S_n(\alpha)$, $n = 1,2$, which is defined for positive integral values of n by means of the integral

$$S_n(\alpha) = \int_0^\infty \frac{e^{-i\alpha u}}{(u^2+1)^{n+\frac{1}{2}}}\,du \quad . \tag{2-1}$$

For real $\alpha \geqslant 0$ we shall write

$$S_n(\alpha) = F_n(\alpha) + iG_n(\alpha) \tag{2-2}$$

where $F_n(\alpha)$ and $G_n(\alpha)$ are real functions of α.

For real $\alpha < 0$ we get, from (2-1) and (2-2),

$$S_n(\alpha) = F_n(-\alpha) - iG_n(-\alpha) \quad . \tag{2-3}$$

The function $S_0(\alpha)$ may also be defined, but since the integrand in (2-1) is not absolutely integrable over $(0, \infty)$ for $n = 0$ the formula

$$S_0(\alpha) = \lim_{\lambda\to\infty} \int_0^\lambda \frac{e^{-i\alpha u}}{(u^2+1)^{\frac{1}{2}}}\,du \tag{2-4}$$

can be used instead.

Even though $S_n(\alpha)$ is used in unsteady aerodynamics only for $n = 1$ and 2 we shall here describe a process for obtaining numerical values of $S_n(\alpha)$ at all the integer values $n = 0, 1, 2, 3, \ldots$

We could, of course, use the reduction formula

$$S_{n+1}(\alpha) = \frac{2n}{2n+1} S_n(\alpha) + \frac{\alpha^2}{4n^2 - 1} S_{n-1}(\alpha) + \frac{i\alpha}{4n^2 - 1} , \qquad n \geqslant 1 \qquad (2\text{-}5)$$

to obtain numerical values of $S_n(\alpha)$ at all the integer values $n = 2, 3, \ldots$ from the numerical values $S_0(\alpha)$ and $S_1(\alpha)$ but values so obtained would not be as accurate as those obtained directly when α is large.

The definition (2-1) of $S_n(\alpha)$ is, in fact, valid for complex values of α with $\mathrm{Im}(\alpha) \leqslant 0$, and even for $n = 0$ it is valid for complex values of α with $\mathrm{Im}(\alpha) < 0$. The functions $S_n(\alpha)$ so defined are regular functions of α in the complex half-plane $\mathrm{Im}(\alpha) < 0$. The domain of validity of the functions $S_n(\alpha)$ can be extended into the complex half-plane $\mathrm{Im}(\alpha) > 0$, but we must expect a singularity on the line $\mathrm{Im}(\alpha) = 0$ because the definition (2-1) of $S_n(\alpha)$ is not valid for $\mathrm{Im}(\alpha) > 0$.

We can show directly from the definition (2-1) of $S_n(\alpha)$ that $S_n(\alpha)$ satisfies the differential equation

$$\alpha \left\{ \frac{d^2 S_n(\alpha)}{d\alpha^2} - S_n(\alpha) \right\} - (2n - 1) \frac{dS_n(\alpha)}{d\alpha} = i \quad . \qquad (2\text{-}6)$$

This differential equation is shown, in the first place, to be satisfied only for $\mathrm{Im}(\alpha) < 0$, but by using analytic continuation we may show that it is satisfied for all $\alpha \neq 0$. At $\alpha = 0$ the differential equation has a regular singularity. If we confine ourselves to real α then we can show directly from the definition (2-1) of $S_n(\alpha)$ that for $n \neq 0$ the differential equation (2-6) is satisfied. For $n = 0$ we must first of allow α to have a small negative imaginary part and then, having shown that the differential equation (2-6) is satisfied we proceed to the limit of taking the negative imaginary part to be zero.

If we put

$$S_n(\alpha) = \alpha^n Q_n(\alpha) \quad , \tag{2-7}$$

substitute for $S_n(\alpha)$ from (2-7) into the differential equation (2-6) and divide the resulting equation by α^{n+1} we get the differential equation

$$Q_n''(\alpha) + \frac{1}{\alpha} Q_n'(\alpha) - \left(1 + \frac{n^2}{\alpha^2}\right) Q_n(\alpha) = \frac{i}{\alpha^{n+1}} \tag{2-8}$$

for $Q_n(\alpha)$. We recognise the differential equation (2-8) as a modified form of Bessel's differential equation with a non-zero right-hand side. This differential equation has the general solution (see *eg* Ref 2)

$$Q_n(\alpha) = \frac{2^{n-1} n!}{(2n)!} \pi \left\{ iL_n(\alpha) + C_n I_n(\alpha) + D_n K_n(\alpha) \right\} \tag{2-9}$$

where C_n and D_n are integration constants. The functions $I_n(\alpha)$ and $K_n(\alpha)$ are modified Bessel functions of order n and of the first and second kinds respectively and $L_n(\alpha)$ is a modified Struve function which is related to the Struve function $\mathbb{H}_{-n}(i\alpha)$ by means of the formula (see Ref 2)

$$L_n(\alpha) = (-1)^n i^{n-1} \mathbb{H}_{-n}(i\alpha) \tag{2-10}$$

For small values of $|\alpha|$ we may write (see Ref 2)

$$I_n(\alpha) = \frac{1}{n!}\left(\frac{\alpha}{2}\right)^n i_n(\alpha) \qquad n \geqslant 0 \quad , \tag{2-11}$$

$$\begin{aligned} K_n(\alpha) = {} & \frac{1}{2}\left(\frac{2}{\alpha}\right)^n \sum_{r=0}^{n-1} (-1)^r \frac{(n-r-1)!}{r!} \left(\frac{\alpha}{2}\right)^{2r} \\ & + \frac{(-1)^{n+1}}{n!}\left(\frac{\alpha}{2}\right)^n \left[\log\left(\frac{\alpha}{2}\right) + \gamma\right] i_n(\alpha) \\ & + \frac{(-1)^n}{n!}\left(\frac{\alpha}{2}\right)^n k_n(\alpha) \qquad n \geqslant 1 \quad , \end{aligned} \tag{2-12}$$

$$K_0(\alpha) = -\left[\log\left(\frac{\alpha}{2}\right) + \gamma\right] i_0(\alpha) + k_0(\alpha) \qquad (2\text{-}13)$$

and

$$L_n(\alpha) = (-1)^n \left(\frac{2}{\alpha}\right)^{n-1} \ell_n(\alpha) \qquad n \geqslant 1 \ , \qquad (2\text{-}14)$$

where $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ are even integral functions of α which have the power series expansions

$$i_n(\alpha) = \sum_{r=0}^{\infty} \frac{n!}{r!(n+r)!} \left(\frac{\alpha}{2}\right)^{2r} \qquad n \geqslant 0 \ , \qquad (2\text{-}15)$$

$$k_n(\alpha) = \frac{1}{2} \sum_{r=0}^{\infty} \frac{n!}{r!(n+r)!} (\psi_r + \psi_{n+r}) \left(\frac{\alpha}{2}\right)^{2r} \qquad n \geqslant 0 \ , \qquad (2\text{-}16)$$

and

$$\ell_n(\alpha) = \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^r}{\Gamma(r + 3/2)\Gamma(r + 3/2 - n)} \qquad n \geqslant 0 \ , \qquad (2\text{-}17)$$

where $$\psi_s = -\frac{1}{(s+1)} + \sum_{p=1}^{s+1} \frac{1}{p} \ , \qquad s = 0, 1, 2, \ldots, \qquad (2\text{-}18)$$

and

$$\gamma = 0.57721566490153286\ldots \qquad (2\text{-}19)$$

is Euler's constant.

If we put the expression (2-9) for $Q_n(\alpha)$ into (2-7) we get

$$S_n(\alpha) = \frac{2^{n-1}n!}{(2n)!}\pi\alpha^n\left\{iL_n(\alpha) + C_nI_n(\alpha) + D_nK_n(\alpha)\right\} . \tag{2-20}$$

The constants C_n and D_n still have to be determined and their values depend on the branch of the function $\log\left(\frac{\alpha}{2}\right)$ used in (2-12). We shall take the branch of $\log\left(\frac{\alpha}{2}\right)$ to be one which is real for real positive α. The branch line of $\log\left(\frac{\alpha}{2}\right)$ from $\alpha = 0$ must not enter the complex half plane $\mathrm{Im}(\alpha) < 0$ because $S_n(\alpha)$ is regular for $\mathrm{Im}(\alpha) < 0$.

If we use the expressions (2-11), (2-12), (2-13) and (2-14) for $I_n(\alpha)$, $K_n(\alpha)$, $K_0(\alpha)$ and $L_n(\alpha)$ in (2-20) we get

$$S_0(\alpha) = \tfrac{1}{2}\pi\left[\tfrac{1}{2}i\alpha\ell_0(\alpha) + C_0i_0(\alpha) + D_0k_0(\alpha) - D_0\left[\log\left(\frac{\alpha}{2}\right) + \gamma\right]i_0(\alpha)\right] \tag{2-21}$$

and

$$\begin{aligned} S_n(\alpha) = \tfrac{1}{2}\pi\Bigg[&(-1)^n\frac{2^{2n-1}n!}{(2n)!}(i\alpha)\ell_n(\alpha) + \frac{C_n}{(2n)!}\alpha^{2n}i_n(\alpha) \\ &+ (-1)^n\frac{D_n}{(2n)!}\alpha^{2n}k_n(\alpha) + (-1)^{n+1}\frac{D_n}{(2n)!}\alpha^{2n}\left[\log\left(\frac{\alpha}{2}\right) + \gamma\right]i_n(\alpha) \\ &+ \frac{2^{2n-1}n!}{(2n)!}D_n\sum_{r=0}^{n-1}(-1)^r\frac{(n-r-1)!}{r!}\left(\frac{\alpha}{2}\right)^{2r}\Bigg] . \end{aligned} \tag{2-22}$$

Directly from the integral representation (2-4) for $S_0(\alpha)$ and (2-1) for $S_n(\alpha)$, we can show that, for small real positive α,

$$S_0(\alpha) = -\left[\log\left(\frac{\alpha}{2}\right) + \gamma + \frac{i\pi}{2}\right] + O(\alpha) \tag{2-23}$$

and

$$S_1(\alpha) = 1 - i\alpha + \tfrac{1}{2}\alpha^2\left[\log\left(\frac{\alpha}{2}\right) + \gamma + \frac{i\pi}{2} - \tfrac{1}{2}\right] + O(\alpha^3) . \tag{2-24}$$

By comparing (2-21) and (2-23) we get immediately

$$C_0 = -i \tag{2-25}$$

and

$$D_0 = \frac{2}{\pi} \quad . \tag{2-26}$$

By comparing (2-22) for $n = 1$ and (2-24) we get immediately

$$C_1 = i \tag{2-27}$$

and

$$D_1 = \frac{2}{\pi} \quad . \tag{2-28}$$

If we use the expansions (2-21) and (2-22) for $S_n(\alpha)$, $n \geqslant 0$ in the reduction formula (2-5) and compare the coefficients of $\alpha^{2n+2} \log\left(\frac{\alpha}{2}\right)$ and α^{2n+2} respectively on the two sides of the formula, we get the equalities

$$D_{n+1} = -nD_n + (n+1)D_{n-1} \qquad n \geqslant 1 \tag{2-29}$$

and

$$\begin{aligned} C_{n+1} &+ \tfrac{1}{2}(-1)^{n+1}D_{n+1}\left\{\psi_0 + \psi_{n+1} - 2\gamma\right\} \\ &= nC_n + \tfrac{1}{2}(-1)^n nD_n\left\{\psi_1 + \psi_{n+1} - 2\gamma\right\} \\ &\quad + (n+1)C_{n-1} + \tfrac{1}{2}(-1)^{n-1}(n+1)D_{n-1}\left\{\psi_1 + \psi_n - 2\gamma\right\} \qquad n \geqslant 1 \quad . \end{aligned}$$

..... (2-30)

By using the starting values (2-25), (2-26), (2-27) and (2-28) for C_0, D_0, C_1, D_1 in (2-29) and (2-30) we get

$$C_n = (-1)^{n+1} i \tag{2-31}$$

and

$$D_n = \frac{2}{\pi} . \tag{2-32}$$

The formula (2-20) for $S_n(\alpha)$ therefore takes the final form

$$S_n(\alpha) = \frac{2^{n-1} n!}{(2n)!} \pi \alpha^n \left\{ \frac{2}{\pi} K_n(\alpha) + i(L_n(\alpha) + (-1)^{n+1} I_n(\alpha)) \right\} . \tag{2-33}$$

The functions $F_n(\alpha)$ and $G_n(\alpha)$ of formula (2-2), which are real for real positive α are obtained from (2-33), by using the formulae (2-11), (2-12), (2-13) and (2-14) for $I_n(\alpha)$, $K_n(\alpha)$, $K_0(\alpha)$ and $L_n(\alpha)$, in the forms

$$F_n(\alpha) = \frac{2^{2n-1} n!}{(2n)!} \sum_{r=0}^{n-1} (-1)^r \frac{(n-r-1)!}{r!} \left(\frac{\alpha}{2}\right)^{2r}$$

$$+ \frac{(-1)^{n+1}}{(2n)!} \alpha^{2n} \left[\log\left(\frac{\alpha}{2}\right) + \gamma\right] i_n(\alpha)$$

$$+ \frac{(-1)^n}{(2n)!} \alpha^{2n} k_n(\alpha) \qquad n \geq 1 , \tag{2-34}$$

$$F_0(\alpha) = -\left[\log \frac{\alpha}{2} + \gamma\right] i_0(\alpha) + k_0(\alpha) \tag{2-35}$$

and

$$G_n(\alpha) = (-1)^n \frac{2^{2n-2}}{(2n)!} \pi \alpha \ell_n(\alpha) + \frac{(-1)^{n+1}}{(2n)!} \frac{\pi}{2} \alpha^{2n} i_n(\alpha) \qquad n \geq 0 . \tag{2-36}$$

For later use we note that

$$i_n(0) = 1 \qquad n \geq 0 \tag{2-37}$$

and

$$k_n(0) = \tfrac{1}{2}\psi_n$$

$$= -\frac{1}{2(n+1)} + \frac{1}{2}\sum_{p=1}^{n+1}\frac{1}{p} \qquad n \geqslant 0 \quad . \tag{2-38}$$

We may deduce from the definitions (2-1) and (2-4) of $S_n(\alpha)$ for $n = 0, 1, 2, 3, \ldots$ the following asymptotic expansions for $F_n(\alpha)$ and $G_n(\alpha)$ of formula (2-2) for large real positive α . With the integer $p \geqslant 0$ arbitrary we get

$$F_n(\alpha) = \frac{2^n n!}{(2n)!}\alpha^n e^{-\alpha}\sqrt{\frac{\pi}{2\alpha}}\left\{F_{n,p}(\alpha) + \frac{1}{\alpha^p}\delta_{n,p}(\alpha)\right\} \qquad n \geqslant 0 \tag{2-39}$$

and

$$G_n(\alpha) = -\frac{1}{\alpha}\left\{G_{n,p}(\alpha) + \frac{1}{\alpha^{2p}}\varepsilon_{n,p}(\alpha)\right\} \qquad n \geqslant 0 \quad , \tag{2-40}$$

where

$$F_{n,0}(\alpha) = 1 \quad , \tag{2-41}$$

$$F_{n,p}(\alpha) = 1 + \sum_{r=1}^{p}\frac{1}{r!(8\alpha)^r}\prod_{s=1}^{r}\left(4n^2 - (2s-1)^2\right) \qquad p \geqslant 1 \tag{2-42}$$

and

$$G_{n,p}(\alpha) = \sum_{r=0}^{p}\frac{(2r)!}{2^{2r}r!}\frac{n!(2n+2r)!}{(2n)!(n+r)!}\frac{1}{\alpha^{2r}} \qquad p \geqslant 0 \quad . \tag{2-43}$$

The remainder functions $\delta_{n,p}(\alpha)$, $\varepsilon_{n,p}(\alpha)$, for any $p \geqslant 0$ have the behaviour

$$\delta_{n,p}(\alpha) = o(1) \tag{2-44}$$

and

$$\varepsilon_{n,p}(\alpha) \;=\; o(1) \qquad \text{for} \quad \alpha \to +\infty\ . \qquad (2\text{-}45)$$

The asymptotic expansions (2-39) and (2-40) for $F_n(\alpha)$ and $G_n(\alpha)$ respectively may be deduced also quite easily from the differential equation (2-6) provided that we use the knowledge that $F_n(\alpha) \to 0$ and $G_n(\alpha) \to 0$ as real $\alpha \to +\infty$. This latter knowledge is obtained from the integral representations (2-1) and (2-4).

The power series (2-15), (2-16) and (2-17) for $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ respectively have to be truncated at finite values of r in order to be able to evaluate from them numerical values of the functions $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$, and these finite values of r will depend upon the accuracy to which the numerical values of these functions are required and on the value of α under consideration. If we work numerically to a given number of significant figures, the accuracy with which we can evaluate the sums of the truncated series will decrease as real α increases. Thus, although these power series expansions are convergent for any finite value of $|\alpha|$, they cannot be used to give accurate numerical values of the functions $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ when real α becomes indefinitely large if we are limited in the number of significant figures used in the arithmetical operations. When working with a given number of significant figures there is a maximum value of real α , for each n , for which formulae (2-34) and (2-35), with $i_n(\alpha)$ and $k_n(\alpha)$ obtained from the power series expansions (2-15) and (2-16) respectively, can be used to obtain $F_n(\alpha)$ to within some prescribed $\varepsilon > 0$. Similarly there is a maximum value of real α , for each n , for which formula (2-36), with $i_n(\alpha)$ and $\ell_n(\alpha)$ obtained from the power series expansions (2-15) and (2-17) respectively, can be used to obtain $G_n(\alpha)$ to within the accuracy ε . The smaller real α is, the fewer terms, in general, will be needed in the truncations of the power series expansions (2-15), (2-16) and (2-17) for $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ to obtain $F_n(\alpha)$ and $G_n(\alpha)$ from formulae (2-34), (2-35) and (2-36) to within the accuracy ε .

For very large values of real α we can use the asymptotic formulae (2-39) and (2-40) to evaluate the numerical values of $F_n(\alpha)$ and $G_n(\alpha)$. Because of formula (2-39) we can, for given $\varepsilon > 0$, and given n and p , find $\alpha_1(n,p,\varepsilon) > 0$ such that

$$\left| F_n(\alpha) - \frac{2^n n!}{(2n)!} \alpha^n e^{-\alpha} \sqrt{\frac{\pi}{2\alpha}} F_{n,p}(\alpha) \right| < \varepsilon \tag{2-46}$$

whenever

$$\alpha > \alpha_1(n,p,\varepsilon) \quad . \tag{2-47}$$

If we take $\alpha_1(n,p,\varepsilon)$ for fixed n, p and ε to be the minimum quantity for which (2-46) is true under the condition (2-47), then $\alpha_1(n,p,\varepsilon)$ for fixed n and ε decreases, in general, as p is increased from zero up to a certain value of p and then increases as p is increased beyond this certain value. The minimum value of $\alpha_1(n,p,\varepsilon)$ for fixed n and ε, and all values of p, is then the minimum value of α for which $F_n(\alpha)$ may be obtained to within accuracy ε from formula (2-39). This accuracy may be somewhat reduced if we work numerically to a given number of significant figures. The higher α is, beyond the minimum value, the smaller will the value of p need to be, in general, for (2-46) to be true.

Likewise, because of formula (2-40) we can, for given $\varepsilon > 0$, and given n and p, find $\alpha_2(n,p,\varepsilon) > 0$ such that

$$\left| G_n(\alpha) + \frac{1}{\alpha} G_{n,p}(\alpha) \right| < \varepsilon \tag{2-48}$$

whenever

$$\alpha > \alpha_2(n,p,\varepsilon) \quad . \tag{2-49}$$

Again, there is a minimum value of α for which $G_n(\alpha)$, for each n, may be obtained to within accuracy ε from formula (2-40).

The maximum values of real α for which $F_n(\alpha)$ and $G_n(\alpha)$ can be evaluated to the given accuracy from formulae (2-34), (2-35) and (2-36) with $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ obtained from the power series expansions (2-15), (2-16) and (2-17) respectively, depend strongly on the number of significant figures used in the arithmetic, whereas the minimum values of α for which $F_n(\alpha)$ and $G_n(\alpha)$ can be evaluated to the given accuracy from formulae (2-39) and (2-40) are hardly dependent on the number of significant figures used in the arithmetic, provided that this number is greater than the number of significant figures required in the numerical values of the functions. The functions $F_n(\alpha)$ and $G_n(\alpha)$ cannot be evaluated to the given accuracy ε, for all α in $(0,\infty)$, using formulae (2-34), (2-35), (2-36), (2-39) and (2-40) as described above, if the number of significant figures used in the arithmetic is not sufficiently high. In other words, if we work to a given number of significant figures, then ε must be greater than a certain lower bound in order that $F_n(\alpha)$ and $G_n(\alpha)$ may be evaluated in the above manner to the given accuracy ε for all α in $(0,\infty)$. If ε is less than this lower bound then some other means of evaluating the functions $F_n(\alpha)$ and $G_n(\alpha)$ must be used, at least over the ranges of α for which the above method does not yield the required accuracy ε.

We shall expand the functions $F_n(\alpha)$ and $G_n(\alpha)$ for real $\alpha > 0$ in series of Chebyshev polynomials rather than in power series. Although ε will still have to be greater than a certain lower bound in order that $F_n(\alpha)$ and $G_n(\alpha)$ may be evaluated to the given accuracy ε, this lower bound should be less than the former lower bound, thus rendering the procedure involving expansion of functions in series of Chebyshev polynomials of wider application than that involving expansion of functions in power series. It may be true that with the number of significant figures available on a particular computing machine, functions may be evaluated to a sufficient accuracy for some applications from power series, but it would seem to be good practice to use another procedure which is capable of giving superior accuracy and is no more difficult to apply.

The power series (2-15), (2-16) and (2-17) for $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ are valid for all complex values of α and the asymptotic expansions (2-39) and (2-40) are also valid for complex values of α. On the other hand, the series of Chebyshev polynomials are valid only for real α. Accordingly we consider the differential equation (2-6) only for real α and split it up into its separate real and imaginary parts using (2-2) for real $\alpha \geq 0$ to get the two real differential equations

$$\alpha\left\{\frac{d^2F_n(\alpha)}{d\alpha^2} - F_n(\alpha)\right\} - (2n-1)\frac{dF_n(\alpha)}{d\alpha} = 0 \qquad (2\text{-}50)$$

and

$$\alpha\left\{\frac{d^2G_n(\alpha)}{d\alpha^2} - G_n(\alpha)\right\} - (2n - 1)\,\frac{dG_n(\alpha)}{d\alpha} = 1 \quad . \tag{2-51}$$

We wish to evaluate $F_n(\alpha)$ and $G_n(\alpha)$ for real α in the range $0 < \alpha < \infty$. We split the range α at $\alpha = A$ and use different methods of evaluations for $0 < \alpha \leqslant A$ and for $A \leqslant \alpha < \infty$.

3 THE CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials $T_p(z)$ are polynomials in z of degree p defined by the formula

$$T_p(z) = \cos\,(p\,\cos^{-1} z) \qquad p \geqslant 0 \tag{3-1}$$

The properties of $T_p(z)$ which we need for our development are easily deduced from the definition (3-1) and are as follows:

$$T_p(+\,1) = 1 \qquad p \geqslant 0 \tag{3-2}$$

$$T_p(-\,1) = (-\,1)^p \qquad p \geqslant 0 \tag{3-3}$$

$$T_{2p}(z) = T_p(2z^2 - 1) \qquad p \geqslant 0 \tag{3-4}$$

$$T_0(z) = 1 \tag{3-5}$$

$$zT_0(z) = T_1(z) \tag{3-6}$$

$$zT_p(z) = \tfrac{1}{2}T_{p-1}(z) + \tfrac{1}{2}T_{p+1}(z) \tag{3-7}$$

$$0 = T_0'(z) \tag{3-8}$$

$$T_0(z) = T_1'(z) \tag{3-9}$$

$$T_1(z) = \tfrac{1}{4}T_2'(z) \tag{3-10}$$

and

$$T_p(z) = \frac{T'_{p+1}(z)}{2(p+1)} - \frac{T'_{p-1}(z)}{2(p-1)} , \qquad p \neq 1 . \qquad (3\text{-}11)$$

The dash on the function $T'_p(z)$ indicates differentiation of $T_p(z)$ with respect to z .

4 EXPANSION OF A FUNCTION IN A SERIES OF CHEBYSHEV POLYNOMIALS

If $f(z)$ is an even function of z defined for z real in the range $-1 \leqslant z \leqslant 1$ then we can express $f(z)$ as a convergent series of Chebyshev polynomials

$$f(z) = {\sum_{r=0}^{\infty}}' f_r T_{2r}(z) \qquad (4\text{-}1)$$

provided $f(z)$ satisfies some simple conditions, such as that it is continuous and of bounded variation for z in $(-1,1)$. The dash on the summation sign in (4-1) indicates that the $r = 0$ term must be multiplied by $\frac{1}{2}$ before being inserted into the series.

By using the properties (3-6) and (3-7) of the Chebyshev polynomials $T_r(z)$ we can easily show that

$$z^2 f(z) = \frac{1}{4} {\sum_{r=0}^{\infty}}' (f_{(r-1)} + 2f_r + f_{r+1}) T_{2r}(z) . \qquad (4\text{-}2)$$

By using the properties (3-6), (3-7), (3-8), (3-9), (3-10) and (3-11) of the Chebyshev polynomials $T_r(z)$ we can easily show that

$$f'(z) = z {\sum_{r=0}^{\infty}}' f_r^{(1)} T_{2r}(z) \qquad (4\text{-}3)$$

where the coefficients $f_r^{(1)}$ are related to the coefficients f_r by means of the formulae

$$f_r = \frac{f_{r-1}^{(1)} - f_{r+1}^{(1)}}{8r} \qquad r \geqslant 1 \quad . \tag{4-4}$$

By differentiating formula (4-3) with respect to z we get

$$f''(z) = \sum_{r=0}^{\infty}{}' f_r^{(1)} T_{2r}(z) + z \frac{d}{dz} \sum_{r=0}^{\infty}{}' f_r^{(1)} T_{2r}(z)$$

$$= \sum_{r=0}^{\infty}{}' f_r^{(1)} T_{2r}(z) + z^2 \sum_{r=0}^{\infty}{}' f_r^{(2)} T_{2r}(z) \tag{4-5}$$

where the coefficients $f_r^{(2)}$ are related to the coefficients $f_r^{(1)}$ by means of the formulae

$$f_r^{(1)} = \frac{f_{r-1}^{(2)} - f_{r+1}^{(2)}}{8r} \qquad r \geqslant 1 \quad . \tag{4-6}$$

Then, on using formulation (4-2) for the second term on the right-hand side of (4-5) we get

$$f''(z) = \sum_{r=0}^{\infty}{}' \left\{ f_r^{(1)} + \tfrac{1}{4}\left(f_{|r-1|}^{(2)} + 2f_r^{(2)} + f_{r+1}^{(2)} \right) \right\} T_{2r}(z) \quad . \tag{4-7}$$

The infinite series on the right-hand side of formula (4-1) must be truncated to a finite series in order to carry out a numerical evaluation of the function $f(z)$. There will then be a small error in the evaluated numerical value, but by taking the number of terms retained in the truncated series to be sufficiently high this error can be made as small as we like. The numerical evaluation of the truncated series is easily and conveniently carried out by using the scheme described by Clenshaw[2]. Let

$$f(z,p) = \sum_{r=0}^{p}{}' f_r T_{2r}(z)$$

$$= \sum_{r=0}^{p}{}' f_r T_r(x) \tag{4-8}$$

where
$$x = 2z^2 - 1 \ . \tag{4-9}$$

Then Clenshaw's scheme is to put

$$\left. \begin{aligned} b_{p+2} &= 0 \\ b_{p+1} &= 0 \end{aligned} \right\} \tag{4-10}$$

$$b_{p-r} = 2xb_{p-r+1} - b_{p-r+2} + f_{p-r}, \qquad r = 0,1,2,\ldots,p \ . \tag{4-11}$$

Then

$$f(z,p) = \tfrac{1}{2}(b_0 - b_2) \tag{4-12}$$

as may be shown by application of properties (3-5), (3-6) and (3-7) of the Chebyshev polynomials $T_r(z)$.

If $g(z)$ is a function of z defined for z real in the range $-1 \leqslant z \leqslant 1$ then we can express $g(z)$ as a convergent series of Chebyshev polynomials

$$g(z) = \sum_{r=0}^{\infty}{}' g_r T_r(z) \tag{4-13}$$

provided $g(z)$ satisfies some simple conditions, such as, that it is continuous and of bounded variation for z in $(-1,1)$.

By using the properties (3-6) and (3-7) of the Chebyshev polynomials $T_r(z)$ we can easily show that

$$zg(z) = \frac{1}{2} \sum_{r=0}^{\infty}{}' (g_{|r-1|} + g_{r+1}) T_r(z) \tag{4-14}$$

and

$$z^2 g(z) = \frac{1}{4} \sum_{r=0}^{\infty}{}' (g_{|r-2|} + 2g_r + g_{r+2}) T_r(z) \quad . \tag{4-15}$$

By using the properties (3-6), (3-7), (3-8), (3-9), (3-10) and (3-11) of the Chebyshev polynomials $T_r(z)$ we can easily show that

$$g'(z) = \sum_{r=0}^{\infty}{}' g_r^{(1)} T_r(z) \tag{4-16}$$

and

$$g''(z) = \sum_{r=0}^{\infty}{}' g_r^{(2)} T_r(z) \tag{4-17}$$

where the coefficients $g_r^{(1)}$ are related to the coefficients g_r by means of the formulae

$$g_r = \frac{g_{r-1}^{(1)} - g_{r+1}^{(1)}}{2r} \qquad r \geqslant 1 \tag{4-18}$$

and the coefficients $g_r^{(2)}$ are related to the coefficients $g_r^{(1)}$ by means of the formulae

$$g_r^{(1)} = \frac{g_{r-1}^{(2)} - g_{r+1}^{(2)}}{2r} \qquad r \geqslant 1 \quad . \tag{4-19}$$

The numerical evaluation of a truncated series of Chebyshev polynomials for $g(z)$ is carried out by the method of Clenshaw in exactly the same way as described earlier for $f(z,p)$.

5 EXPANSION OF $F_n(\alpha)$ for $0 < \alpha \leqslant A$

We introduce the variable z by means of the formula

$$z = \frac{\alpha}{A} . \tag{5-1}$$

The range $0 < \alpha \leqslant A$ of α corresponds to the range $0 < z \leqslant 1$ of z.

In conformity with the expressions (2-34) for $F_n(\alpha)$, $n \geqslant 1$ and (2-35) for $F_0(\alpha)$ we put

$$F_n(\alpha) = \frac{2^{2n-1} n!}{(2n)!} \sum_{r=0}^{n-1} (-1)^r \frac{(n-r-1)!}{r!} \left(\frac{\alpha}{2}\right)^{2r}$$

$$+ \frac{(-1)^{n+1}}{(2n)!} \alpha^{2n} \left\{ j_n(z) \log (z) - p_n(z) \right\} \qquad n \geqslant 1 \tag{5-2}$$

and

$$F_0(\alpha) = - \left\{ j_0(z) \log (z) - p_0(z) \right\} \tag{5-3}$$

where the $j_n(z)$ and $p_n(z)$, $n \geqslant 0$ are even functions of z.

If we substitute for $F_n(\alpha)$ from (5-2) or (5-3) into the differential equation (2-50) we get, after simplification, the equation

$$z^2 \left\{ j_n''(z) + \frac{(2n+1)}{z} j_n'(z) - A^2 j_n(z) \right\} \log z$$

$$- z^2 \left\{ p_n''(z) + \frac{(2n+1)}{z} p_n'(z) - A^2 p_n(z) \right\}$$

$$+ 2z j_n'(z) + 2n j_n(z) - 2n = 0 . \tag{5-4}$$

Since equation (5-4) is valid for all z in $(0,1)$ we must have

$$j_n''(z) + \frac{(2n+1)}{z} j_n'(z) - A^2 j_n(z) = 0 \qquad (5\text{-}5)$$

and

$$z^2\left\{p_n''(z) + \frac{2n+1}{z} p_n'(z) - A^2 p_n(z)\right\} - 2zj_n'(z) - 2nj_n(z) + 2n = 0 \ . \qquad (5\text{-}6)$$

We know, from (2-34), (2-35), (5-2) and (5-3) that

$$j_n(0) = 1 \qquad (5\text{-}7)$$

and

$$p_n(0) = -\gamma - \log\left(\frac{A}{2}\right) - \frac{1}{2(n+1)} + \frac{1}{2}\sum_{p=1}^{n+1} \frac{1}{p} \ . \qquad (5\text{-}8)$$

The conditions (5-7) and (5-8) are sufficient to ensure that the even functions $j_n(z)$ and $p_n(z)$ that satisfy the pair of differential equations (5-5) and (5-6) are unique. We shall seek approximations to the functions $j_n(z)$ and $p_n(z)$ which are also even functions of z .

We denote the approximation to $j_n(z)$ by $j_n^{(a)}(z)$ and express it as the series of Chebyshev polynomials

$$j_n^{(a)}(z) = \sum_{r=0}^{\infty}{}' C_r T_{2r}(z) \qquad (5\text{-}9)$$

where $$C_r = 0 \qquad r \geq M_1 + 3 \qquad (5\text{-}10)$$

and M_1 is some positive integer. This approximation will be taken to satisfy a differential equation which is a slight modification of the differential equation (5-5). The precise form of this modified differential equation will appear later, equation (5-30).

We can write the first and second derivatives of $j_n^{(a)}(z)$ with respect to z in the forms

$$j_n^{(a)'}(z) = z\sum_{r=0}^{\infty}{}' C_r^{(1)} T_{2r}(z) \qquad (5\text{-}11)$$

and

$$j_n^{(a)''}(z) = \sum_{r=0}^{\infty}{}' \left\{ C_r^{(1)} + \tfrac{1}{4}\left(C_{(r-1)}^{(2)} + 2C_r^{(2)} + C_{r+1}^{(2)} \right) \right\} T_{2r}(z) \qquad (5\text{-}12)$$

where

$$C_r^{(1)} = 0 \qquad r \geqslant M_1 + 2 \qquad (5\text{-}13)$$

$$C_r^{(2)} = 0 \qquad r \geqslant M_1 + 1 \qquad (5\text{-}14)$$

$$C_r = \frac{C_{r-1}^{(1)} - C_{r+1}^{(1)}}{8r} \qquad r \geqslant 1 \qquad (5\text{-}15)$$

and

$$C_r^{(1)} = \frac{C_{r-1}^{(2)} - C_{r+1}^{(2)}}{8r} \qquad r \geqslant 1 \ . \qquad (5\text{-}16)$$

By using the expansions (5-9), (5-11) and (5-12) for $j_n^{(a)}(z)$, $j_n^{(a)'}(z)$ and $j_n^{(a)''}(z)$ respectively we get

$$j_n^{(a)''}(z) + \frac{(2n+1)}{z} j_n^{(a)'}(z) - A^2 j_n^{(a)}(z)$$

$$= \sum_{r=0}^{\infty}{}' \left\{ \tfrac{1}{4}\left(C_{|r-1|}^{(2)} + 2C_r^{(2)} + C_{r+1}^{(2)} \right) + 2(n+1)C_r^{(1)} - A^2 C_r \right\} T_{2r}(z) \ .$$

...... (5-17)

Let us put

$$\tfrac{1}{4}\left(C^{(2)}_{|r-1|} + 2C^{(2)}_{r} + C^{(2)}_{r+1}\right) + 2(n + 1)C^{(1)}_{r} - A^2 C_r = 0 \qquad 0 \leqslant r \leqslant M_1 - 1 \quad .$$

$$\ldots\ldots \text{(5-18)}$$

Then the differential equation (5-17) becomes

$$\begin{aligned} j_n^{(a)''}(z) &+ \frac{(2n + 1)}{z}\, j_n^{(a)'}(z) - A^2 j_n^{(a)}(z) \\ &= \left\{\tfrac{1}{4}\left(C^{(2)}_{|M_1-1|} + 2C^{(2)}_{M_1}\right) + 2(n + 1)C^{(1)}_{M_1} - A^2 C_{M_1}\right\} T_{2M_1}(z) \\ &\quad + \left\{\tfrac{1}{4}C^{(2)}_{M_1} + 2(n + 1)C^{(1)}_{M_1+1} - A^2 C_{M_1+1}\right\} T_{2M_1+2}(z) \\ &\quad - A^2 C_{M_1+2} T_{2M_1+4}(z) \quad . \end{aligned} \qquad \text{(5-19)}$$

If we use (5-16) to express $C^{(2)}_{r-1}$ in terms of $C^{(1)}_{r}$ and $C^{(2)}_{r+1}$ for $r \geqslant 1$ in relations (5-18) we get the equivalent relations

$$\tfrac{1}{2}\left(C^{(2)}_{r} + C^{(2)}_{r+1}\right) + 2(r + n + 1)C^{(1)}_{r} - A^2 C_r = 0 \qquad 0 \leqslant r \leqslant M_1 - 1 \quad \text{(5-20)}$$

which are a little more convenient to use than are the relations (5-18). The relation (5-20) for $r = 0$ is exactly the same as relation (5-18) for $r = 0$.

We now proceed as follows:

Put $\qquad C^{(2)}_{M_1+1} = 0$

from (5-15) determine $\qquad C^{(1)}_{M_1+1} = 8(M_1 + 2)C_{M_1+2}$

from (5-16) determine $\qquad C^{(2)}_{M_1} = 8(M_1 + 1)C^{(1)}_{M_1+1}$

from (5-15) determine $\qquad C^{(1)}_{M_1} = 8(M_1 + 1)C_{M_1+1}$

$p = 1(1)M_1$

from (5-16) determine $\qquad C^{(2)}_{M_1-p} = 8(M_1 - p + 1)C^{(1)}_{M_1-p+1} + C^{(2)}_{M_1-p+2}$

from (5-15) determine $\qquad C^{(1)}_{M_1-p} = 8(M_1 - p + 1)C_{M_1-p+1} + C^{(1)}_{M_1-p+2}$

from (5-20) determine

$$C_{M_1-p} = \frac{1}{2A^2}\left\{C^{(2)}_{M_1-p} + C^{(2)}_{M_1-p+1} + 4(M_1 + n - p + 1)C^{(1)}_{M_1-p}\right\} . \qquad \ldots\ldots \quad (5\text{-}21)$$

In this way we obtain in turn $C^{(1)}_{M_1+1}$, $C^{(2)}_{M_1}$, $C^{(1)}_{M_1}$, $C^{(2)}_{M_1-1}$, $C^{(1)}_{M_1-1}$, C_{M_1-1}, $C^{(2)}_{M_1-2}$, $C^{(1)}_{M_1-2}$, C_{M_1-2}, ..., $C^{(2)}_{1}$, $C^{(1)}_{1}$, C_1, $C^{(2)}_{0}$, $C^{(1)}_{0}$, C_0, as linear combinations of C_{M_1}, C_{M_1+1}, C_{M_1+2} .

A solution $j^{(a)}_n(z)$ of the differential equation (5-19) is therefore known in terms of C_{M_1}, C_{M_1+1}, C_{M_1+2} . The function $j^{(a)}_n(z)$ so obtained is arbitrary insofar as the coefficients C_{M_1}, C_{M_1+1}, C_{M_1+2} are arbitrary.

Because of the condition (5-7) we shall impose the condition

$$j_n^{(a)}(0) = 1 \quad . \tag{5-22}$$

By using the formula (5-9) for $j_n^{(a)}(z)$ we get

$$1 = \sum_{r=0}^{M_1+2} C_r T_{2r}(0)$$

$$= \sum_{r=0}^{M_1+2} C_r T_r(-1)$$

$$= \sum_{r=0}^{M_1+2} (-1)^r C_r \quad . \tag{5-23}$$

Formula (5-23) is a linear relation connecting C_{M_1}, C_{M_1+1}, C_{M_1+2} from which we can express C_{M_1} in terms of C_{M_1+1} and C_{M_1+2}.

The coefficients C_{M_1+1}, C_{M_1+2} are still arbitrary. We can choose them as we wish and we could choose them so that $j_n^{(a)''}(0)$ and $j_n^{(a)''''}(0)$ have imposed values. However, we prefer to choose these coefficients so that the right-hand side of the differential equation (5-19) reduces as much as possible. This is achieved by taking

$$C_{M_1+1} = 0 \tag{5-24}$$

and

$$C_{M_1+2} = 0 \quad . \tag{5-25}$$

It then follows (5-15) and (5-16) that

$$C_{M_1}^{(1)} = 0 \tag{5-26}$$

$$C_{M_1+1}^{(1)} = 0 \tag{5-27}$$

$$C^{(2)}_{M_1-1} = 0 \tag{5-28}$$

$$C^{(2)}_{M_1} = 0 \ . \tag{5-29}$$

The conditions (5-24), (5-25), (5-26), (5-27), (5-28) and (5-29) extend the conditions (5-10), (5-13) and (5-14).

We can now determine uniquely all the coefficients C_r in the definition (5-9) of the function $j_n^{(a)}(z)$. The differential equation (5-19) satisfied by $j_n^{(a)}(z)$ reduces to

$$j_n^{(a)''}(z) + \frac{(2n+1)}{z} j_n^{(a)'}(z) - A^2 j_n^{(a)}(z) = - A^2 C_{M_1} T_{2M_1}(z) \ . \tag{5-30}$$

The differential equation (5-30) is a slight modification of the differential equation (5-5) if $A^2 C_{M_1}$ is a very small number compared with unity and if this is so then the function $j_n^{(a)}(z)$ can be expected to be a good approximation to $j_n(z)$. The quality of the approximation depends on the smallness of the number $A^2 C_{M_1}$. It is found, in practice, that this number rapidly decreases as the positive integer M_1 is increased.

We shall take the approximation $p_n^{(a)}(z)$ to $p(z)$ to be given by the formula

$$p_n^{(a)}(z) = q_n^{(a)}(z) + \lambda j_n^{(a)}(z) \tag{5-31}$$

where $q_n^{(a)}(z)$ is any even function of z that satisfies the differential equation

$$z^2 \left\{ q_n^{(a)''}(z) + \frac{2n+1}{z} q_n^{(a)'}(z) - A^2 q_n^{(a)}(z) \right\}$$

$$- 2z j_n^{(a)'}(z) - 2n j_n^{(a)}(z) + 2n = 0 \tag{5-32}$$

and the number λ is chosen to satisfy

Then

$$z^2q_n^{(a)}(z) = \frac{1}{4}\sum_{r=0}^{\infty}{}' \left(q_{|r-1|} + 2q_r + q_{r+1}\right)T_{2r}(z) \tag{5-42}$$

$$zq_n^{(a)'}(z) = \frac{1}{4}\sum_{r=0}^{\infty}{}' \left(q^{(1)}_{|r-1|} + 2q^{(1)}_r + q^{(1)}_{r+1}\right)T_{2r}(z) \tag{5-43}$$

and

$$z^2q_n^{(a)''}(z) = \frac{1}{4}\sum_{r=0}^{\infty}{}' \left(q^{(1)}_{|r-1|} + 2q^{(1)}_r + q^{(1)}_{r+1}\right)T_{2r}(z)$$

$$+ \frac{1}{16}\sum_{r=0}^{\infty}{}' \left(q^{(2)}_{|r-2|} + 4q^{(2)}_{|r-1|} + 6q^{(2)}_r + 4q^{(2)}_{r+1} + q^{(2)}_{r+2}\right)T_{2r}|z|$$

...... (5-44)

From (5-11) we get

$$zj_n^{(a)'}(z) = \frac{1}{4}\sum_{r=0}^{\infty}{}' \left(C^{(1)}_{|r-1|} + 2C^{(1)}_r + C^{(1)}_{r+1}\right)T_{2r}(z) \quad . \tag{5-45}$$

By using the expansions (5-42), (5-43) and (5-44) for $z^2q_n^{(a)}(z)$, $zq_n^{(a)'}(z)$ and $z^2q_n^{(a)''}(z)$ respectively and the expansions (5-9) and (5-45) for $j_n^{(a)}(z)$ and $zj_n^{(a)'}(z)$ respectively we get

$$z^2\left\{q_n^{(a)''}(z) + \frac{(2n+1)}{z} q_n^{(a)'}(z) - A^2 q_n^{(a)}(z)\right\} - 2zj_n^{(a)'}(z) - 2nj_n^{(a)}(z) + 2n$$

$$= \sum_{r=0}^{\infty}{}' \left\{ \tfrac{1}{16}\left(q_{|r-2|}^{(2)} + 4q_{|r-1|}^{(2)} + 6q_r^{(2)} + 4q_{r+1}^{(2)} + q_{r+2}^{(2)}\right)\right.$$

$$+ \tfrac{1}{2}(n+1)\left(q_{|r-1|}^{(1)} + 2q_r^{(1)} + q_{r+1}^{(1)}\right) - \tfrac{1}{4}A^2\left(q_{|r-1|} + 2q_r + q_{r+1}\right)$$

$$\left. - \tfrac{1}{2}\left(C_{|r-1|}^{(1)} + 2C_r^{(1)} + C_{r+1}^{(1)}\right) - 2nC_r \right\} T_{2r}(z)$$

$$+ 2nT_0(z) \quad . \tag{5-46}$$

Let us put

$$\tfrac{1}{16}\left(3q_0^{(2)} + 4q_1^{(2)} + q_2^{(2)}\right) + \tfrac{1}{2}(n+1)\left(q_0^{(1)} + q_1^{(1)}\right) - \tfrac{1}{4}A^2(q_0 + q_1)$$

$$- \tfrac{1}{2}\left(C_0^{(1)} + C_1^{(1)}\right) - nC_0 + 2n = 0 \tag{5-47}$$

and

$$\tfrac{1}{16}\left(q_{|r-2|}^{(2)} + 4q_{r-1}^{(2)} + 6q_r^{(2)} + 4q_{r+1}^{(2)} + q_{r+2}^{(2)}\right)$$

$$+ \tfrac{1}{2}(n+1)\left(q_{r-1}^{(1)} + 2q_r^{(1)} + q_{r+1}^{(1)}\right) - \tfrac{1}{4}A^2\left(q_{r-1} + 2q_r + q_{r+1}\right)$$

$$- \tfrac{1}{2}\left(C_{r-1}^{(1)} + 2C_r^{(1)} + C_{r+1}^{(1)}\right) - 2nC_r = 0 \qquad 1 \leqslant r \leqslant M_1 \quad . \tag{5-48}$$

Then the differential equation (5-46) reduces to the differential equation (5-32).

If we multiply the rth equation in (5-48) by $(-1)^{r+1}$ and sum the resulting M_1 equations we get

$$\tfrac{1}{16}\left(3q_0^{(2)} + 4q_1^{(2)} + q_2^{(2)}\right) + \tfrac{1}{2}(n+1)\left(q_0^{(1)} + q_1^{(1)}\right) - \tfrac{1}{4}A^2(q_0 + q_1)$$

$$- \tfrac{1}{2}\left(C_0^{(1)} + C_1^{(1)}\right) + 2n\sum_{r=1}^{M_1}(-1)^r C_r = 0 \quad . \tag{5-49}$$

Now, according to (5-23), together with the equalities (5-24) and (5-25), we have

$$\sum_{r=1}^{M_1} (-1)^r C_r = 1 - \tfrac{1}{2}C_0 \quad . \tag{5-50}$$

If we substitute for $\sum_{r=1}^{M_1} (-1)^r C_r$ from (5-50) into (5-49) we recover the relation (5-47). The relation (5-47) is thus shown to be linearly dependent on the relations (5-48) and consequently is redundant. The reason for this linear dependence is connected with the fact that the differential equation (5-32) may be divided through by z^2 and the limit $z \to 0$ taken without any infinities occurring.

If we use (5-41) to express $q_{r-2}^{(2)}$ in terms of $q_{r-1}^{(1)}$ and $q_r^{(2)}$ for $r \geq 2$ and to express $q_{r+2}^{(2)}$ in terms of $q_{r+1}^{(1)}$ and $q_r^{(2)}$ for $r \geq 1$ in relations (5-48) we get the equivalent relations

$$\begin{aligned}&\tfrac{1}{4}\left(q_{r-1}^{(2)} + 2q_r^{(2)} + q_{r+1}^{(2)}\right) + \tfrac{1}{2}(n + r)q_{r-1}^{(1)} + (n + 1)q_r^{(1)} + \tfrac{1}{2}(n - r)q_{r+1}^{(1)} \\ &- \tfrac{1}{4}A^2\left(q_{r-1} + 2q_r + q_{r+1}\right) - \tfrac{1}{2}\left(C_{r-1}^{(1)} + 2C_r^{(1)} + C_{r+1}^{(1)}\right) - 2nC_r = 0 \\ &\qquad 1 \leqslant r \leqslant M_1\end{aligned} \tag{5-51}$$

which are a little more convenient to use than are the relations (5-48).

We now proceed as follows:

Put $\quad q^{(2)}_{M_1+1} = 0$

put $\quad q^{(1)}_{M_1+1} = 0$

put $\quad q_{M_1+1} = 0$

put $\quad q^{(2)}_{M_1} = 0$

put $\quad q^{(1)}_{M_1} = 0$

put $\quad q_{M_1} = 0$

$$p = 1(1)M_1$$

from (5-41) determine $\quad q^{(2)}_{M_1-p} = 8(M_1 - p + 1)q^{(1)}_{M_1-p+1} + q^{(2)}_{M_1-p+2}$

from (5-40) determine $\quad q^{(1)}_{M_1-p} = 8(M_1 - p + 1)q_{M_1-p+1} + q^{(1)}_{M_1-p+2}$

from (5-51) determine

$$\begin{aligned} q_{M_1-p} = & - 2q_{M_1-p+1} - q_{M_1-p+2} \\ & + \frac{1}{A^2}\Big\{ q^{(2)}_{M_1-p} + 2q^{(2)}_{M_1-p+1} + q^{(2)}_{M_1-p+2} \\ & \quad + 2(M_1 + n - p + 1)q^{(1)}_{M_1-p} + 4(n+1)q^{(1)}_{M_1-p+1} \\ & \quad - 2(M_1 - n - p + 1)q^{(1)}_{M_1-p+2} - 2C^{(1)}_{M_1-p} \\ & \quad - 4C^{(1)}_{M_1-p+1} - 2C^{(1)}_{M_1-p+2} - 8nC_{M-p+1}\Big\} \end{aligned}$$

...... (5-52)

In this way we obtain in turn $q^{(2)}_{M_1-1}$, $q^{(1)}_{M_1-1}$, q_{M_1-1}, $q^{(2)}_{M_1-2}$, $q^{(2)}_{M_1-2}$, $q^{(2)}_{M_1-2}$, ..., $q^{(2)}_1$, $q^{(1)}_1$, q_1, $q^{(2)}_0$, $q^{(1)}_1$, q_1 . Thus all the coefficients q_r in the definition (5-34) of the function $q^{(a)}_n(z)$ have been obtained and the resulting $q^{(a)}_n(z)$ satisfies the differential equation (5-32).

The function $p^{(a)}_n(z)$ defined by formula (5-31) then satisfies the differential equation

$$z^2\left\{p^{(a)''}_n(z) + \frac{2n+1}{z}\,p^{(a)'}_n(z) - A^2 p^{(a)}_n(z)\right\}$$
$$- 2zj^{(a)'}_n(z) - 2nj^{(a)}_n(z) + 2n = -\lambda A^2 C_{M_1} z^2 T_{2M_1}(z) \qquad (5\text{-}53)$$

which is obtained by combining the differential equations (5-32) and (5-30).

The differential equation (5-53) is a slight modification of the differential equation (5-6) if $A^2C_{M_1}$ and $\lambda A^2 C_{M_1}$ are very small numbers compared with unity and if this is so then the function $p^{(a)}_n(z)$ can be expected to be a good approximation to $p_n(z)$. It is found, in practice, that these numbers rapidly decrease as the positive integer M_1 is increased.

We may write

$$p^{(a)}_n(z) = \sum_{r=0}^{\infty}{}' p_r T_{2r}(z) \qquad (5\text{-}54)$$

where, according to (5-31), (5-9) and (5-34)

$$p_r = q_r + \lambda C_r \qquad r = 0,\ 1,\ 2,\ \ldots\ M_1$$
$$p_r = 0 \qquad r \geqslant M_1 + 1 \ . \qquad (5\text{-}55)$$

Let us now put

$$p^{(0)}_n(z) = p_n(z) \qquad (5\text{-}56)$$

and for $n \geqslant 1$, put

$$p_n^{(k)}(z) = Z_{k-1} - \frac{A^2 z^2}{(2n-2k+2)(2n-2k+1)} p_n^{(k-1)}(z) \qquad k = 1, 2, \ldots, n$$

$$\ldots\ldots \quad (5\text{-}57)$$

where
$$Z_0 = \frac{1}{(2n-1)} \qquad (5\text{-}58)$$

and

$$Z_k = \frac{2k}{(2n-2k-1)} Z_{k-1} \qquad k = 1, 2, \ldots \quad . \quad (5\text{-}59)$$

Then we may replace the expressions (5-2) and (5-3) for $F_n(\alpha)$ by the expression

$$F_n(\alpha) = p_n^{(n)}(z) + \frac{(-1)^{n+1}}{(2n)!} \alpha^{2n} j_n(z) \log(z) \qquad n \geqslant 0 \quad . \quad (5\text{-}60)$$

We can use the recurrence relations (5-57) and (5-59) to obtain an approximation $p_n^{(n,a)}(z)$ to $p_n^{(n)}(z)$ from the approximation $p_n^{(a)}(z)$ to $p_n(z)$. We use the formula (4-2) successively to get finally the expression $p_n^{(n,a)}(z)$ in the form

$$p_n^{(n,a)}(z) = \sum_{r=0}^{\infty}{}' d_r T_{2r}(z) \qquad (5\text{-}61)$$

where
$$d_r = 0 \qquad r \geqslant M_1 + n \quad . \quad (5\text{-}62)$$

We may therefore finally write an approximation $F_n^{(a)}(\alpha)$ to $F_n(\alpha)$ in the form

$$F_n^{(a)}(\alpha) = \sum_{r=0}^{M_1+n}{}' d_r T_{2r}\left(\frac{\alpha}{A}\right) + (-1)^{n+1} \frac{\alpha^{2n}}{(2n)!} \sum_{r=0}^{M_1}{}' c_r T_{2r}\left(\frac{\alpha}{A}\right) \log\left(\frac{\alpha}{A}\right)$$

$$\text{for} \quad 0 < \alpha \leqslant A \quad . \quad (5\text{-}63)$$

The function $F_n(\alpha)$ may be written in the form

$$F_n(\alpha) = \sum_{r=0}^{\infty}{}' D_r^{(n)}(A)T_{2r}\left(\frac{\alpha}{A}\right) + (-1)^{n+1}\frac{\alpha^{2n}}{(2n)!}\sum_{r=0}^{\infty}{}' C_r^{(n)}(A)T_{2r}\left(\frac{\alpha}{A}\right)\log\left(\frac{\alpha}{A}\right)$$

$$\text{for } 0 < \alpha \leqslant A \quad . \qquad (5\text{-}64)$$

The coefficients d_r for low values of r are approximations to the coefficients $D_r^{(n)}(A)$ and the coefficients C_r for low values of r are approximations to the coefficients $C_r^{(n)}(A)$. The value of the integer M_1 must be taken large enough for these approximations to be so good that $F_n(\alpha)$ can be evaluated to the desired accuracy from formula (5-63). Values of $D_r^{(n)}(A)$ and $C_r^{(n)}A$ obtained by this means are given for $A = 2$, 4 and 8, and $n = 0, 1, 2$ in the results section 9.

6 EXPANSION OF $G_n(\alpha)$ FOR $0 \leqslant \alpha \leqslant A$

Again we introduce the variable z by means of the formula

$$z = \frac{\alpha}{A} \quad . \qquad (6\text{-}1)$$

Then the range $0 \leqslant \alpha \leqslant A$ of α corresponds to the range $0 \leqslant z \leqslant 1$ of z.

In conformity with the expression (2-36) for $G_n(\alpha)$ we put

$$G_n(\alpha) = \alpha h_n(z) + \frac{(-1)^{n+1}}{(2n)!}\frac{\pi}{2}\alpha^{2n}j_n(z) \qquad (6\text{-}2)$$

where the $h_n(z)$ and $j_n(z)$, $n \geqslant 0$, are even functions of z. The functions $j_n(z)$, $n \geqslant 0$, are the same functions that occur in formulae (5-2) and (5-3).

If we substitute for $G_n(\alpha)$ from (6-2) into the differential equation (2-51) we get, after simplification, the equation

$$z^2h_n''(z) - (2n-3)zh_n'(z) - \left\{(2n-1) + A^2z^2\right\}h_n(z)$$
$$+ \frac{(-1)^{n+1}}{(2n)!}\frac{\pi}{2}A^{2n-1}z^{2n+1}\left\{j_n''(z) + \frac{(2n+1)}{z}j_n(z) - A^2j_n(z)\right\} = 1 \quad . \qquad (6\text{-}3)$$

On separating even and odd functions of z in the differential equation (6-3) we find that $h_n(z)$ and $j_n(z)$ satisfy the separate differential equations

$$z^2h_n''(z) - (2n - 3)zh_n'(z) - \left\{(2n - 1) + A^2z^2\right\}h_n(z) = 1 \qquad (6\text{-}4)$$

and

$$j_n''(z) + \frac{(2n + 1)}{z} j_n'(z) - A^2 j_n(z) = 0 \quad . \qquad (6\text{-}5)$$

Equation (6-5) is exactly the same as equation (5-5) and has the same solution.

We shall seek an approximation $h_n^{(a)}(z)$ to $h_n(z)$ in the form of a series of Chebyshev polynomials

$$h_n^{(a)}(z) = \sum_{r=0}^{\infty}{}' e_r T_{2r}(z) \qquad (6\text{-}6)$$

where $\qquad e_r = 0 \qquad r \geqslant M_2 + 6 \qquad (6\text{-}7)$

and M_2 is some positive integer. This approximation will be taken to satisfy a differential equation which is a slight modification of the differential equation (6-4). The precise form of this modified differential equation will appear later, equation (6-36).

We can write the first and second derivatives of $h_n^{(a)}(z)$ with respect to z in the forms

$$h_n^{(a)\prime}(z) = z \sum_{r=0}^{\infty}{}' e_r^{(1)} T_{2r}(z) \qquad (6\text{-}8)$$

and

$$h_n^{(a)\prime\prime}(z) = \sum_{r=0}^{\infty}{}' \left\{ e_r^{(1)} + \tfrac{1}{4}\left(e_{|r-1|}^{(2)} + 2e_r^{(2)} + e_{r+1}^{(2)} \right) \right\} T_{2r}(z) \qquad (6\text{-}9)$$

where $\qquad e_r^{(1)} = 0 \qquad r \geqslant M_2 + 5 \qquad (6\text{-}10)$

$\qquad e_r^{(2)} = 0 \qquad r \geqslant M_2 + 4 \qquad (6\text{-}11)$

$$e_r = \frac{e^{(1)}_{r-1} - e^{(1)}_{r+1}}{8r} \qquad r \geqslant 1 \tag{6-12}$$

and

$$e^{(1)}_r = \frac{e^{(2)}_{r-1} - e^{(2)}_{r+1}}{8r} \qquad r \geqslant 1 \quad . \tag{6-13}$$

By using the expansions (6-6), (6-8) and (6-9) for $h^{(a)}_n(z)$, $h^{(a)'}_n(z)$ and $h^{(a)''}_n(z)$ respectively we get

$$z^2 h^{(a)''}_n(z) - (2n - 3) z h^{(a)'}_n(z) - \left\{(2n - 1) + A^2 z^2\right\} h^{(a)}_n(z) - 1$$

$$= \sum_{r=0}^{\infty}{}' \left\{ \tfrac{1}{16}\left(e^{(2)}_{|r-2|} + 4e^{(2)}_{|r-1|} + 6e^{(2)}_r + 4e^{(2)}_{r+1} + e^{(2)}_{r+2}\right) \right.$$

$$- \tfrac{1}{2}(n - 2)\left(e^{(1)}_{|r-1|} + 2e^{(1)}_r + e^{(1)}_{r+1}\right) - (2n - 1)e_r$$

$$\left. - \tfrac{1}{4}A^2\left(e_{|r-1|} + 2e_r + e_{r+1}\right)\right\} T_{2r}(z) - T_0(z) \quad . \tag{6-14}$$

Let us put

$$\tfrac{1}{16}\left(3e^{(2)}_0 + 4e^{(2)}_1 + e^{(2)}_2\right) - \tfrac{1}{2}(n - 2)\left(e^{(1)}_0 + e^{(1)}_1\right) - \tfrac{1}{2}(2n - 1)e_0$$

$$- \tfrac{1}{4}A^2(e_0 + e_1) - 1 = 0 \tag{6-15}$$

and

$$\tfrac{1}{16}\left(e^{(2)}_{|r-2|} + 4e^{(2)}_{r-1} + 6e^{(2)}_r + 4e^{(2)}_{r+1} + e^{(2)}_{r+2}\right) - \tfrac{1}{2}(n - 2)\left(e^{(1)}_{r-1} + 2e^{(1)}_r + e^{(1)}_{r+1}\right)$$

$$- (2n - 1)e_r - \tfrac{1}{4}A^2(e_{r-1} + 2e_r + e_{r+1}) = 0 \qquad 1 \leqslant r \leqslant M_2 \quad . \tag{6-16}$$

Then the differential equation (6-14) becomes

$$
\begin{aligned}
&z^2 h_n^{(a)''}(z) - (2n-3)zh_n^{(a)'}(z) - \Big\{(2n-1) + A^2z^2\Big\}h_n^{(a)}(z) - 1 \\
&= \Bigg\{\tfrac{1}{16}\Big(e^{(2)}_{M_2-1} + 4e^{(2)}_{M_2} + 6e^{(2)}_{M_2+1} + 4e^{(2)}_{M_2+2} + e^{(2)}_{M_2+3}\Big) - \tfrac{1}{2}(n-2)\Big(e^{(1)}_{M_2} + 2e^{(1)}_{M_2+1} + e^{(1)}_{M_2+2}\Big) \\
&\qquad - (2n-1)e_{M_2+1} - \tfrac{1}{4}A^2\Big(e_{M_2} + 2e_{M_2+1} + e_{M_2+2}\Big)\Bigg\}T_{2M_2+2}(z) \\
&\quad + \Bigg\{\tfrac{1}{16}\Big(e^{(2)}_{M_2} + 4e^{(2)}_{M_2+1} + 6e^{(2)}_{M_2+2} + 4e^{(2)}_{M_2+3}\Big) - \tfrac{1}{2}(n-2)\Big(e^{(1)}_{M_2+1} + 2e^{(1)}_{M_2+2} + e^{(1)}_{M_2+3}\Big) \\
&\qquad - (2n-1)e_{M_2+2} - \tfrac{1}{4}A^2\Big(e_{M_2+1} + 2e_{M_2+2} + e_{M_2+3}\Big)\Bigg\}T_{2M_2+4}(z) \\
&\quad + \Bigg\{\tfrac{1}{16}\Big(e^{(2)}_{M_2+1} + 4e^{(2)}_{M_2+2} + 6e^{(2)}_{M_2+3}\Big) - \tfrac{1}{2}(n-2)\Big(e^{(1)}_{M_2+2} + 2e^{(1)}_{M_2+3} + e^{(1)}_{M_2+4}\Big) \\
&\qquad - (2n-1)e_{M_2+3} - \tfrac{1}{4}A^2\Big(e_{M_2+2} + 2e_{M_2+3} + e_{M_2+4}\Big)\Bigg\}T_{2M_2+6}(z) \\
&\quad + \Bigg\{\tfrac{1}{16}\Big(e^{(2)}_{M_2+2} + 4e^{(2)}_{M_2+3}\Big) - \tfrac{1}{2}(n-2)\Big(e^{(1)}_{M_2+3} + 2e^{(1)}_{M_2+4}\Big) - (2n-1)e_{M_2+4} \\
&\qquad - \tfrac{1}{4}A^2\Big(e_{M_2+3} + 2e_{M_2+4} + e_{M_2+5}\Big)\Bigg\}T_{2M_2+8}(z) \\
&\quad + \Bigg\{\tfrac{1}{16}\,e^{(2)}_{M_2+3} - \tfrac{1}{2}(n-2)e^{(1)}_{M_2+4} - (2n-1)e_{M_2+5} - \tfrac{1}{4}A^2\Big(e_{M_2+4} + 2e_{M_2+5}\Big)\Bigg\}T_{2M_2+10}(z) \\
&\qquad - \tfrac{1}{4}A^2 e_{M_2+5}T_{2M_2+12}(z) \quad . \qquad (6\text{-}17)
\end{aligned}
$$

If we use (6-13) to express $e^{(2)}_{r-2}$ in terms of $e^{(1)}_{r-1}$ and $e^{(2)}_{r}$ for $r \geqslant 2$ and to express $e^{(2)}_{r+2}$ in terms of $e^{(1)}_{r+1}$ and $e^{(2)}_{r}$ for $r \geqslant 1$ in relations (6-15) and (6-16) we get the equivalent relations

$$
\tfrac{1}{4}\Big(e^{(2)}_0 + e^{(2)}_1\Big) - \tfrac{1}{2}(n-2)e^{(1)}_0 - \tfrac{1}{2}(n-1)e^{(1)}_1 - \tfrac{1}{2}(2n-1)e_0 - \tfrac{1}{4}A^2(e_0 - e_1) - 1 = 0 \qquad (6\text{-}18)
$$

and

$$
\begin{aligned}
&\tfrac{1}{4}\Big(e^{(2)}_{r-1} + 2e^{(2)}_{r} + e^{(2)}_{r+1}\Big) + \tfrac{1}{2}(r-n+1)e^{(1)}_{r-1} - (n-2)e^{(1)}_{r} - \tfrac{1}{2}(r+n-1)e^{(1)}_{r+1} \\
&\quad - (2n-1)e_r - \tfrac{1}{4}A^2(e_{r-1} + 2e_r + e_{r+1}) = 0 \qquad 1 \leqslant r \leqslant M_2 \quad . \qquad (6\text{-}19)
\end{aligned}
$$

We now proceed as follows

from (6-12) determine $e^{(1)}_{M_2+4} = 8(M_2 + 5)e_{M_2+5}$

from (6-13) determine $e^{(2)}_{M_2+3} = 8(M_2 + 4)e^{(1)}_{M_2+4}$

from (6-12) determine $e^{(1)}_{M_2+3} = 8(M_2 + 4)e_{M_2+4}$

from (6-13) determine $e^{(2)}_{M_2+2} = 8(M_2 + 3)e^{(1)}_{M_2+3}$

from (6-12) determine $e^{(1)}_{M_2+2} = 8(M_2 + 3)e_{M_2+3}$

from (6-13) determine $e^{(2)}_{M_2+1} = 8(M_2 + 2)e^{(1)}_{M_2+2}$

from (6-12) determine $e^{(1)}_{M_2+1} = 8(M_2 + 2)e_{M_2+2}$

from (6-13) determine $e^{(2)}_{M_2} = 8(M_2 + 1)e^{(1)}_{M_2+1}$

from (6-12) determine $e^{(1)}_{M_2} = 8(M_2 + 1)e_{M_2+1}$

$p = 1(1)M_2$

from (6-13) determine $e^{(2)}_{M_2-p} = 8(M_2 - p + 1)e^{(1)}_{M_2-p+1} + e^{(2)}_{M_2-p+2}$

from (6-12) determine $e^{(1)}_{M_{2-p}} = 8(M_2 - p + 1)e_{M_2-p+1} + e^{(1)}_{M_2-p+2}$

from (6-19) determine

$$e_{M_2-p} = -2e_{M_2-p+1} - e_{M_2-p+2} + \frac{1}{A^2}\Big\{e^{(2)}_{M_2-p} + 2e^{(2)}_{M_2-p+1} + e^{(2)}_{M_2-p+2} + 2(M_2 - p - n + 2)e^{(1)}_{M_2-p} - 4(n - 2)e^{(1)}_{M_2-p+1} - 2(M_2 - p + n)e^{(1)}_{M_2-p+2} - 4(2n - 1)e_{M_2-p+1}\Big\} .$$

.... (6-20)

In this way we obtain in turn $e^{(1)}_{M_2+4}$, $e^{(2)}_{M_2+3}$, $e^{(1)}_{M_2+3}$, $e^{(2)}_{M_2+2}$, $e^{(1)}_{M_2+2}$, $e^{(2)}_{M_2+1}$, $e^{(1)}_{M_2+1}$, $e^{(2)}_{M_2}$, $e^{(1)}_{M_2}$, $e^{(2)}_{M_2-1}$, $e^{(1)}_{M_2-1}$, e_{M_2-1}, $e^{(2)}_{M_2-2}$, $e^{(1)}_{M_2-2}$, e_{M_2-2}, ..., $e^{(2)}_{1}$, $e^{(1)}_{1}$, $e^{(0)}_{1}$, $e^{(2)}_{0}$, $e^{(1)}_{0}$, $e^{(0)}_{0}$, as linear combinations of e_{M_2}, e_{M_2+1}, e_{M_2+2}, e_{M_2+3}, e_{M_2+4}, e_{M_2+5}.

When we substitute for $e^{(2)}_{0}$, $e^{(2)}_{1}$, $e^{(1)}_{0}$, $e^{(1)}_{1}$, e_0 and e_1 so obtained from the above procedure (6-20) into the relation (6-18) we get a linear relation connecting e_{M_2}, e_{M_2+1}, e_{M_2+2}, e_{M_2+3}, e_{M_2+4}, e_{M_2+5} from which we can express e_{M_2} in terms of e_{M_2+1}, e_{M_2+2}, e_{M_2+3}, e_{M_2+4}, e_{M_2+5}. A solution $h^{(a)}_n(z)$ of the differential equation (6-17) is therefore known in terms of e_{M_2+1}, e_{M_2+2}, e_{M_2+3}, e_{M_2+4}, e_{M_2+5}. The function $h^{(a)}_n(z)$ so obtained is arbitrary insofar as the coefficients e_{M_2+1}, e_{M_2+2}, e_{M_2+3}, e_{M_2+4}, e_{M_2+5} are arbitrary. We can choose these coefficients as we wish and we shall choose them so that the right-hand side of the differential equation (6-17) reduces as much as possible. This is achieved by taking

$$e_{M_2+1} = 0 \tag{6-21}$$

$$e_{M_2+2} = 0 \tag{6-22}$$

$$e_{M_2+3} = 0 \tag{6-23}$$

$$e_{M_2+4} = 0 \tag{6-24}$$

$$e_{M_2+5} = 0 \quad . \tag{6-25}$$

It then follows from (6-12) and (6-13) that

$$e^{(1)}_{M_2} = 0 \tag{6-26}$$

$$e^{(1)}_{M_2+1} = 0 \tag{6-27}$$

$$e^{(1)}_{M_2+2} = 0 \tag{6-28}$$

$$e^{(1)}_{M_2+3} = 0 \tag{6-29}$$

$$e^{(1)}_{M_2+4} = 0 \tag{6-30}$$

$$e^{(2)}_{M_2-1} = 0 \tag{6-31}$$

$$e^{(2)}_{M_2} = 0 \tag{6-32}$$

$$e^{(2)}_{M_2+1} = 0 \tag{6-33}$$

$$e^{(2)}_{M_2+2} = 0 \tag{6-34}$$

$$e^{(2)}_{M_2+3} = 0 \quad . \tag{6-35}$$

The conditions (6-21) to (6-35) extend the conditions (6-7), (6-10) and (6-11).

We can now determine uniquely all the coefficients e_r in the definition (6-6) of the function $h_n^{(a)}(z)$. The differential equation (6-17) satisfied by $h_n^{(a)}(z)$ reduces to

$$\begin{aligned} z^2 h_n^{(a)''}(z) - (2n-3) z h_n^{(a)'}(z) - \left\{(2n-1) + A^2 z^2\right\} h_n^{(a)}(z) - 1 \\ = -\tfrac{1}{4} A^2 e_{M_2} T_{2M_2+2}(z) \quad . \end{aligned} \tag{6-36}$$

The differential equation (6-36) is a slight modification of the differential equation (6-4) if $A^2 e_{M_2}$ is a small number compared with unity and if this is so then the function $h_n^{(a)}(z)$ can be expected to be a good approximation to $h_n(z)$. It is found, in practice, that this number rapidly decreases as the positive integer M_2 is increased.

We may therefore finally write an approximation $G_n^{(a)}(\alpha)$ to $G_n(\alpha)$ in the form

$$G_n^{(a)}(\alpha) \;=\; \alpha \sum_{r=0}^{M_2}{}' e_r T_{2r}\left(\frac{\alpha}{A}\right) + \frac{(-1)^{n+1}\pi}{2(2n)!}\,\alpha^{2n} \sum_{r=0}^{M_1}{}' C_r T_{2r}\left(\frac{\alpha}{A}\right)$$

$$\text{for} \quad 0 \leqslant \alpha \leqslant A \ . \qquad (6\text{-}37)$$

The function $G_n(\alpha)$ may be written in the form

$$G_n(\alpha) \;=\; \alpha \sum_{r=0}^{\infty}{}' E_r^{(n)}(A) T_{2r}\left(\frac{\alpha}{A}\right) + \frac{(-1)^{n+1}\pi}{2(2n)!}\,\alpha^{2n} \sum_{r=0}^{\infty}{}' C_r^{(n)}(A) T_{2r}\left(\frac{\alpha}{A}\right)$$

$$\text{for} \quad 0 \leqslant \alpha \leqslant A \ . \qquad (6\text{-}38)$$

The coefficients e_r for low values of r are approximations to the coefficients $E_r^{(n)}(A)$. The value of the integer M_2 must be taken large enough for these approximations to be so good and the value of the integer M_1, discussed in section 5, must be taken large enough for the approximation C_r to $C_r^{(n)}(A)$ to be so good that $G_n(\alpha)$ can be evaluated to the desired accuracy from formula (6-37).

Values of $E_r^{(n)}(A)$ and $C_r^{(n)}(A)$ obtained by this means are given for $A = 2, 4$ and 8, and $n = 0, 1, 2$ in the results section 9.

7 EXPANSION OF $F_n(\alpha)$ for $A \leqslant \alpha < \infty$

We introduce the variable z by means of the formula

$$z \;=\; \frac{2A}{\alpha} - 1 \ . \qquad (7\text{-}1)$$

The range $A \leqslant \alpha < \infty$ of α corresponds to the range $-1 < z \leqslant 1$ of z.

Since we know the form of $F_n(\alpha)$ for large real positive α from the asymptotic expansion (2-39) we can put

$$F_n(\alpha) \;=\; e^{-\alpha}\alpha^{n-\frac{1}{2}} f_n(z) \qquad (7\text{-}2)$$

where $f_n(z)$ is a function of bounded variation.

If we substitute for $F_n(\alpha)$ from (7-2) into the differential equation (2-50), we get, after simplification, the equation

$$4(z+1)^2 f_n''(z) + 8(z+1+2A)f_n'(z) - (4n^2-1)f_n(z) = 0 \ . \tag{7-3}$$

From the asymptotic expansion (2-39) and the form (7-2) of the function $F_n(\alpha)$ we get immediately

$$f_n(-1) = \frac{2^n n!}{(2n)!}\sqrt{\frac{\pi}{2}} \ . \tag{7-4}$$

We shall seek an approximation $f_n^{(a)}(z)$ to $f_n(z)$ in the form of a series of Chebyshev polynomials

$$f_n^{(a)}(z) = \sum_{r=0}^{\infty}{}' f_r T_r(z) \tag{7-5}$$

where $\qquad f_r = 0 \qquad r \geqslant M_3 + 4 \qquad$ (7-6)

and M_3 is some positive integer. This approximation will be taken to satisfy a differential equation which is a slight modification of the differential equation (7-3). The precise form of this modified differential equation will appear later, equation (7-29).

We can write the first and second derivatives of $f_n^{(a)}(z)$ with respect to z in the forms

$$f_n^{(a)\prime}(z) = \sum_{r=0}^{\infty}{}' f_r^{(1)} T_r(z) \tag{7-7}$$

and

$$f_n^{(a)\prime\prime}(z) = \sum_{r=0}^{\infty}{}' f_r^{(2)} T_r(z) \tag{7-8}$$

where $\qquad f_r^{(1)} = 0 \qquad r \geqslant M_3 + 3 \qquad$ (7-9)

$$f_r^{(2)} = 0 \qquad r \geqslant M_3 + 2 \tag{7-10}$$

$$f_r = \frac{f_{r-1}^{(1)} - f_{r+1}^{(1)}}{2r} \qquad r \geqslant 1 \tag{7-11}$$

and

$$f_r^{(1)} = \frac{f_{r-1}^{(2)} - f_{r+1}^{(2)}}{2r} \qquad r \geqslant 1 \quad . \tag{7-12}$$

By using the expansions (7-5), (7-7) and (7-8) for $f_n^{(a)}(z)$, $f_n^{(a)'}(z)$ and $f_n^{(a)''}(z)$ respectively we get

$$4(z+1)^2 f_n^{(a)''}(z) + 8(z+1+2A)f_n^{(a)'}(z) - (4n^2-1)f_n^{(a)}(z)$$

$$= \sum_{r=0}^{\infty}{}' \left\{\left(f_{|r-2|}^{(2)} + 4f_{|r-1|}^{(2)} + 6f_r^{(2)} + 4f_{r+1}^{(2)} + f_{r+2}^{(2)}\right) + 4\left(f_{|r-1|}^{(1)} + (2+4A)f_r^{(1)} + f_{r+1}^{(1)}\right) - (4n^2-1)f_r\right\} T_r(z) \quad . \tag{7-13}$$

Let us put

$$\left(f_{|r-2|}^{(2)} + 4f_{|r-1|}^{(2)} + 6f_r^{(2)} + 4f_{r+1}^{(2)} + f_{r+2}^{(2)}\right) + 4\left(f_{|r-1|}^{(1)} + (2+4A)f_r^{(1)} + f_{r+1}^{(1)}\right) - (4n^2-1)f_r = 0 \qquad 0 \leqslant r \leqslant M_3 - 1 \quad . \tag{7-14}$$

Then the differential equation (7-15) becomes

$$4(z+1)^2 f_n^{(a)''}(z) + 8(z+1+2A)f_n^{(a)'}(z) - (4n^2-1)f_n^{(a)}(z)$$

$$= \left\{\left(f_{M_3-2}^{(2)} + 4f_{M_3-1}^{(2)} + 6f_{M_3}^{(2)} + 4f_{M_3+1}^{(2)}\right) + 4\left(f_{M_3-1}^{(1)} + 2(1+2A)f_{M_3}^{(1)} + f_{M_3+1}^{(1)}\right) - (4n^2-1)f_{M_3}\right\} T_{M_3}(z)$$

$$+ \left\{\left(f_{M_3-1}^{(2)} + 4f_{M_3}^{(2)} + 6f_{M_3+1}^{(2)}\right) + 4\left(f_{M_3}^{(1)} + 2(1+2A)f_{M_3+1}^{(1)} + f_{M_3+2}^{(1)}\right) - (4n^2-1)f_{M_3+1}\right\} T_{M_3+1}(z)$$

$$+ \left\{\left(f_{M_3}^{(2)} + 4f_{M_3+1}^{(2)}\right) + 4\left(f_{M_3+1}^{(1)} + 2(1+2A)f_{M_3+2}^{(1)}\right) - (4n^2-1)f_{M_3+2}\right\} T_{M_3+2}(z)$$

$$+ \left\{f_{M_3+1}^{(2)} + 4f_{M_3+2}^{(1)} - (4n^2-1)f_{M_3+3}\right\} T_{M_3+3}(z) \quad . \tag{7-15}$$

If we use (7-12) to express $f_{r-2}^{(2)}$ in terms of $f_{r-1}^{(1)}$ and $f_r^{(2)}$ for $r \geq 2$, $f_{r-1}^{(2)}$ in terms of $f_r^{(1)}$ and $f_{r+1}^{(2)}$ for $r \geq 1$ and $f_{r+2}^{(2)}$ in terms of $f_{r+1}^{(1)}$ and $f_r^{(2)}$ for $r \geq 0$ and use (7-11) to express $f_{r-1}^{(1)}$ in terms of f_r and $f_{r+1}^{(1)}$ for $r \geq 1$ in the relations (7-14) we get the equivalent relations

$$8f_r^{(2)} + 8f_{r+1}^{(2)} + 8(r + 1 + 2A)f_r^{(1)} + 4f_{r+1}^{(1)} + \left\{4r(r + 1) - (4n^2 - 1)\right\}f_r = 0$$

$$0 \leq r \leq M_3 - 1 \quad . \qquad (7\text{-}16)$$

We now proceed as follows:

From (7-11) determine $f_{M_3+2}^{(1)} = 2(M_3 + 3)f_{M_3+3}$

from (7-12) determine $f_{M_3+1}^{(2)} = 2(M_3 + 2)f_{M_3+2}^{(1)}$

from (7-11) determine $f_{M_3+1}^{(1)} = 2(M_3 + 2)f_{M_3+2}^{(1)}$

from (7-12) determine $f_{M_3}^{(2)} = 2(M_3 + 1)f_{M_3+1}^{(1)}$

from (7-11) determine $f_{M_3}^{(1)} = 2(M_3 + 1)f_{M_3+1} + f_{M_3+2}^{(1)}$

$p = 1(1)M_3$

from (7-12) determine $f_{M_3-p}^{(2)} = 2(M_3 - p + 1)f_{M_3-p+1}^{(1)} + f_{M_3-p+2}^{(2)}$

from (7-11) determine $f_{M_3-p}^{(1)} = 2(M_3 - p + 1)f_{M_3-p+1} + f_{M_3-p+2}^{(1)}$

from (7-16) determine $f_{M_3-p} = \left\{8f_{M_3-p}^{(2)} + 8f_{M_3-p+1}^{(2)} + 8(M_3 - p + 1 + 2A)f_{M_3-p}^{(1)} + 4f_{M_3-p+1}^{(1)}\right\} \Big/ \left\{(4n^2 - 1) - 4(M_3 - p)(M_3 - p + 1)\right\}.$

...... (7-17)

In this way we obtain in turn $f^{(1)}_{M_3+2}$, $f^{(2)}_{M_3+1}$, $f^{(1)}_{M_3+1}$, $f^{(2)}_{M_3}$, $f^{(1)}_{M_3}$, $f^{(2)}_{M_3-1}$, $f^{(1)}_{M_3-1}$, f_{M_3-1}, $f^{(2)}_{M_3-2}$, $f^{(1)}_{M_3-2}$, f_{M_3-2}, ..., $f^{(2)}_1$, $f^{(1)}_1$, f_1, $f^{(2)}_0$, $f^{(1)}_0$, f_0 as linear combinations of f_{M_3}, f_{M_3+1}, f_{M_3+2}, f_{M_3+3} .

A solution $f^{(a)}_n(z)$ of the differential equation (7-15) is therefore known in terms of f_{M_3}, f_{M_3+1}, f_{M_3+2}, f_{M_3+3} . The function $f^{(a)}_n(z)$ so obtained is arbitrary insofar as the coefficients f_{M_3}, f_{M_3+1}, f_{M_3+2}, f_{M_3+3} are arbitrary.

Because of the condition (7-4) we shall impose the condition

$$f^{(a)}_n(-1) = \frac{2^n n!}{(2n)!}\sqrt{\frac{\pi}{2}} \quad . \tag{7-18}$$

By using the formula (7-5) for $f^{(a)}_n(z)$ we get

$$\begin{aligned} \frac{2^n n!}{(2n)!}\sqrt{\frac{\pi}{2}} &= \sum_{r=0}^{M_3+3} f_r T_r(-1) \\ &= \sum_{r=0}^{M_3+3} (-1)^r f_r \quad . \end{aligned} \tag{7-19}$$

Formula (7-19) is a linear relation connecting f_{M_3}, f_{M_3+1}, f_{M_3+2}, f_{M_3+3} from which we can express f_{M_3} in terms of f_{M_3+1}, f_{M_3+2} and f_{M_3+3} .

The coefficients f_{M_3+1}, f_{M_3+2}, f_{M_3+3} are still arbitrary. We can choose these coefficients as we wish and we shall choose them so that the right-hand side of the differential equation (7-15) reduces as much as possible. This is achieved by taking

$$f_{M_3+1} = 0 \tag{7-20}$$

$$f_{M_3+2} = 0 \tag{7-21}$$

$$f_{M_3+3} = 0 \quad . \tag{7-22}$$

It then follows from (7-11) and (7-13) that

$$f_{M_3}^{(1)} = 0 \tag{7-23}$$

$$f_{M_3+1}^{(1)} = 0 \tag{7-24}$$

$$f_{M_3+2}^{(1)} = 0 \tag{7-25}$$

$$f_{M_3-1}^{(2)} = 0 \tag{7-26}$$

$$f_{M_3}^{(2)} = 0 \tag{7-27}$$

$$f_{M_3+1}^{(2)} = 0 \ . \tag{7-28}$$

The conditions (7-20) to (7-28) extend the conditions (7-6), (7-9) and (7-10).

We can now determine uniquely all the coefficients f_r in the definition (7-5) of the function $f_n^{(a)}(z)$. The differential equation (7-15) satisfied by $f_n^{(a)}(z)$ reduces to

$$\begin{aligned} &4(z+1)^2 f_n^{(a)''}(z) + 8(z+1+2A) f_n^{(a)'}(z) - (4n^2-1) f_n^{(a)}(z) \\ &= \left\{ f_{M_3-2}^{(2)} + 4f_{M_3-1}^{(1)} - (4n^2-1) f_{M_3} \right\} T_{M_3}(z) \\ &= \left\{ 4M_3(M_3+2) - (4n^2-1) \right\} f_{M_3} T_{M_3}(z) \ . \end{aligned} \tag{7-29}$$

The differential equation (7-29) is a slight modification of the differential equation (7-3) if $\left\{ 4M_3(M_3+2) - (4n^2-1) \right\} f_{M_3}$ is a small number compared with unity and if this is so then the function $f_n^{(a)}(z)$ can be expected to be a good approximation to $f_n(z)$. It is found, in practice, that this number rapidly decreases as the positive integer M_3 is increased.

We may therefore finally write an approximation $F_n^{(a)}(\alpha)$ to $F_n(\alpha)$ in the form

$$F_n^{(a)}(\alpha) = e^{-\alpha}\alpha^{n-\frac{1}{2}} \sum_{r=0}^{M_3} f_r T_r\left(\frac{2A}{\alpha} - 1\right) \qquad \text{for} \quad A \leqslant \alpha < \infty \quad . \tag{7-30}$$

The function $F_n(\alpha)$ may be written in the form

$$F_n(\alpha) = e^{-\alpha}\alpha^{n-\frac{1}{2}} \sum_{r=0}^{\infty}{}' F_r^{(n)}(A) T_r\left(\frac{2A}{\alpha} - 1\right) \quad \text{for} \quad A \leqslant \alpha < \infty \quad . \tag{7-31}$$

The coefficients f_r for low values of r are approximations to the coefficients $F_r^{(n)}(A)$. The value of the integer M_3 must be taken large enough for these approximations to be so good that $F_n(\alpha)$ can be evaluated to the desired accuracy from formula (7-30). Values of $F_r^{(n)}(A)$ obtained by this means are given for $A = 2$, 4 and 8 and $n = 0, 1, 2$ in the results section 9.

8 EXPANSION OF $G_n(\alpha)$ FOR $A \leqslant \alpha < \infty$

We introduce the variable z by means of the formula

$$z = \frac{A}{\alpha} \quad . \tag{8-1}$$

The range $A \leqslant \alpha < \infty$ of α corresponds to the range $0 < z \leqslant 1$ of z .

Since we know the form of $G_n(\alpha)$ for large real positive α from the asymptotic expansion (2-40) we can put

$$G_n(\alpha) = \frac{1}{\alpha} g_n(z) \tag{8-2}$$

where $g_n(z)$ is an even function of z which is of bounded variation.

If we substitute for $G_n(\alpha)$ from (8-2) into the differential equation (2-51), we get, after simplification, the equation

$$z^4 g_n''(z) + (2n + 3)z^3 g_n'(z) + \left\{(2n + 1)z^2 - A^2\right\} g_n(z) = A^2 \quad . \tag{8-3}$$

We shall seek an approximation $g_n^{(a)}(z)$ to $g_n(z)$ in the form of a series of Chebyshev polynomials

$$g_n^{(a)}(z) = \sum_{r=0}^{\infty}{}' g_r T_{2r}(z) \tag{8-4}$$

where $$g_r = 0 \qquad r \geqslant M_4 + 6 \tag{8-5}$$

and M_4 is some positive integer. This approximation will be taken to satisfy a differential equation which is a slight modification of the differential equation (8-3). The precise form of this modified differential equation will appear later, equation (8-33).

We can write the first and second derivatives of $g_n^{(a)}(z)$ with respect to z in the forms

$$g_n^{(a)'}(z) = z \sum_{r=0}^{\infty}{}' g_r^{(1)} T_{2r}(z) \tag{8-6}$$

and

$$g_n^{(a)''}(z) = \sum_{r=0}^{\infty}{}' \left\{ g_r^{(1)} + \tfrac{1}{4}\left(g_{|r-1|}^{(2)} + 2g_r^{(2)} + g_{r+1}^{(2)} \right) \right\} T_{2r}(z) \tag{8-7}$$

where $$g_r^{(1)} = 0 \qquad r \geqslant M_4 + 5 \tag{8-8}$$

$$g_r^{(2)} = 0 \qquad r \geqslant M_4 + 4 \tag{8-9}$$

$$g_r = \frac{g_{r-1}^{(1)} - g_{r+1}^{(1)}}{8r} \qquad r \geqslant 1 \tag{8-10}$$

and

$$g_r^{(1)} = \frac{g_{r-1}^{(2)} - g_{r+1}^{(2)}}{8r} \qquad r \geqslant 1 \quad . \tag{8-11}$$

By using the expansions (8-4), (8-6) and (8-7) for $g_n^{(a)}(z)$, $g_n^{(a)\prime}(z)$ and $g_n^{(a)\prime\prime}(z)$ respectively we get

$$\begin{aligned}
& z^4 g_n^{(a)\prime\prime}(z) + (2n+3)z^3 g_n^{(a)\prime}(z) + \left\{(2n+1)z^2 - A^2\right\} g_n^{(a)}(z) - A^2 \\
&= \sum_{r=0}^{\infty}{}' \left\{ \tfrac{1}{64}\left(g_{|r-3|}^{(2)} + 6g_{|r-2|}^{(2)} + 15g_{|r-1|}^{(2)} + 20g_r^{(2)} + 15g_{r+1}^{(2)} + 6g_{r+2}^{(2)} + g_{r+3}^{(2)}\right)\right. \\
&\qquad + \tfrac{1}{8}(n+2)\left(g_{|r-2|}^{(1)} + 4g_{|r-1|}^{(1)} + 6g_r^{(1)} + 4g_{r+1}^{(1)} + g_{r+2}^{(1)}\right) \\
&\qquad \left. + \tfrac{1}{4}(2n+1)(g_{|r-1|} + 2g_r + g_{r+1}) - A^2 g_r \right\} T_{2r}(z) \\
&\quad - A^2 T_0(z) \quad .
\end{aligned} \tag{8-12}$$

Let us put

$$\begin{aligned}
& \tfrac{1}{64}\left(10g_0^{(2)} + 15g_1^{(2)} + 6g_2^{(2)} + g_3^{(2)}\right) \\
& + \tfrac{1}{8}(n+2)\left(3g_0^{(1)} + 4g_1^{(1)} + g_2^{(1)}\right) \\
& + \tfrac{1}{4}(2n+1)(g_0 + g_1) - \tfrac{1}{2}A^2 g_0 \;=\; A^2
\end{aligned} \tag{8-13}$$

and

$$\begin{aligned}
& \tfrac{1}{64}\left(g_{|r-3|}^{(2)} + 6g_{|r-2|}^{(2)} + 15g_{r-1}^{(2)} + 20g_r^{(2)} + 15g_{r+1}^{(2)} + 6g_{r+2}^{(2)} + g_{r+3}^{(2)}\right) \\
& + \tfrac{1}{8}(n+2)\left(g_{|r-2|}^{(1)} + 4g_{r-1}^{(1)} + 6g_r^{(1)} + 4g_{r+1}^{(1)} + g_{r+2}^{(1)}\right) \\
& + \tfrac{1}{4}(2n+1)(g_{r-1} + 2g_r + g_{r+1}) - A^2 g_r \;=\; 0 \qquad 1 \leqslant r \leqslant M_4 \quad .
\end{aligned} \tag{8-14}$$

Then the differential equation (8-12) becomes

$$z^4 g_n^{(a)''}(z) + (2n+3)z^3 g_n^{(a)'}(z) + \left\{(2n+1)z^2 - A^2\right\} g_n^{(a)}(z) - A^2$$

$$= \left\{\tfrac{1}{64}\left(g^{(2)}_{M_4-2} + 6g^{(2)}_{M_4-1} + 15g^{(2)}_{M_4} + 20g^{(2)}_{M_4+1} + 15g^{(2)}_{M_4+2} + 6g^{(2)}_{M_4+3}\right)\right.$$

$$+ \tfrac{1}{8}(n+2)\left(g^{(1)}_{M_4-1} + 4g^{(1)}_{M_4} + 6g^{(1)}_{M_4+1} + 4g^{(1)}_{M_4+2} + g^{(1)}_{M_4+3}\right)$$

$$\left. + \tfrac{1}{4}(2n+1)\left(g_{M_4} + 2g_{M_4+1} + g_{M_4+2}\right) - A^2 g_{M_4+1}\right\} T_{2M_4+2}(z)$$

$$+ \left\{\tfrac{1}{64}\left(g^{(2)}_{M_4-1} + 6g^{(2)}_{M_4} + 15g^{(2)}_{M_4+1} + 20g^{(2)}_{M_4+2} + 15g^{(2)}_{M_4+3}\right)\right.$$

$$+ \tfrac{1}{8}(n+2)\left(g^{(1)}_{M_4} + 4g^{(1)}_{M_4+1} + 6g^{(1)}_{M_4+2} + 4g^{(1)}_{M_4+3} + g^{(1)}_{M_4+4}\right)$$

$$\left. + \tfrac{1}{4}(2n+1)\left(g_{M_4+1} + 2g_{M_4+2} + g_{M_4+3}\right) - A^2 g_{M_4+2}\right\} T_{2M_4+4}(z)$$

$$+ \left\{\tfrac{1}{64}\left(g^{(2)}_{M_4} + 6g^{(2)}_{M_4+1} + 15g^{(2)}_{M_4+2} + 20g^{(2)}_{M_4+3}\right)\right.$$

$$+ \tfrac{1}{8}(n+2)\left(g^{(1)}_{M_4+1} + 4g^{(1)}_{M_4+2} + 6g^{(1)}_{M_4+3} + 4g^{(1)}_{M_4+4}\right)$$

$$\left. + \tfrac{1}{4}(2n+1)\left(g_{M_4+2} + 2g_{M_4+3} + g_{M_4+4}\right) - A^2 g_{M_4+3}\right\} T_{2M_4+6}(z)$$

$$+ \left\{\tfrac{1}{64}\left(g^{(2)}_{M_4+1} + 6g^{(2)}_{M_4+2} + 15g^{(2)}_{M_4+3}\right)\right.$$

$$+ \tfrac{1}{8}(n+2)\left(g^{(1)}_{M_4+2} + 4g^{(1)}_{M_4+3} + 6g^{(1)}_{M_4+4}\right)$$

$$\left. \tfrac{1}{4}(2n+1)\left(g_{M_4+3} + 2g_{M_4+4} + g_{M_4+5}\right) - A^2 g_{M_4+4}\right\} T_{2M_4+8}(z)$$

$$+ \left\{\tfrac{1}{64}\left(g^{(2)}_{M_4+2} + 6g^{(2)}_{M_4+3}\right) + \tfrac{1}{8}(n+2)\left(g^{(1)}_{M_4+3} + 4g^{(1)}_{M_4+4}\right)\right.$$

$$\left. + \tfrac{1}{4}(2n+1)\left(g_{M_4+4} + 2g_{M_4+5}\right) - A^2 g_{M_4+5}\right\} T_{2M_4+10}(z)$$

$$+ \left\{\tfrac{1}{64}\, g^{(2)}_{M_4+3} + \tfrac{1}{8}(n+2)g^{(1)}_{M_4+4} + \tfrac{1}{4}(2n+1)g_{M_4+5}\right\} T_{2M_4+12}(z) \quad . \tag{8-15}$$

If we use (8-11) to express $g_{r-3}^{(2)}$ in terms of $g_{r-2}^{(1)}$ and $g_{r-1}^{(2)}$ for $r \geqslant 3$, $g_{r-2}^{(2)}$ in terms of $g_{r-1}^{(1)}$ and $g_r^{(2)}$ for $r \geqslant 2$, $g_{r+2}^{(2)}$ in terms of $g_{r+1}^{(1)}$ and $g_r^{(2)}$ for $r \geqslant 0$ and $g_{r+3}^{(2)}$ in terms of $g_{r+2}^{(1)}$ and $g_{r+1}^{(2)}$ for $r \geqslant 0$ and use (8-10) to express $g_{r-2}^{(1)}$ in terms of g_{r-1} and $g_r^{(1)}$ for $r \geqslant 0$ and $g_{r+2}^{(1)}$ in terms of g_{r+1} and $g_r^{(1)}$ for $r \geqslant 0$ in the relations (8-13) and (8-14) we get the equivalent relations

$$\tfrac{1}{4}\left(g_0^{(2)} + g_1^{(2)}\right) + \tfrac{1}{4}(2n+3)g_0^{(1)} + \tfrac{1}{4}(2n+1)g_1^{(1)}$$
$$+ \tfrac{1}{4}(2n+1-2A^2)g_0 - \tfrac{1}{4}(2n-1)g_1 = A^2 \qquad (8\text{-}16)$$

and

$$\tfrac{1}{4}\left(g_{r-1}^{(2)} + 2g_r^{(2)} + g_{r+1}^{(2)}\right) + \tfrac{1}{4}(2n+3r+1)g_{r-1}^{(1)} + \tfrac{1}{2}(2n+3)g_r^{(1)}$$
$$+ \tfrac{1}{4}(2n-3r+1)g_{r+1}^{(1)} + \left\{(r-1)(n+r) + \tfrac{1}{4}(2n+1)\right\}g_{r-1} + \tfrac{1}{2}(2n+1-2A^2)g_r$$
$$+ \left\{(r+1)(r-n) + \tfrac{1}{4}(2n+1)\right\}g_{r+1} = 0 \qquad 1 \leqslant r \leqslant M_4 \quad . \qquad (8\text{-}17)$$

We now proceed as follows:

From (8-10) determine $g_{M_4+4}^{(1)} = 8(M_4+5)g_{M_4+5}$

from (8-11) determine $g_{M_4+3}^{(2)} = 8(M_4+4)g_{M_4+4}^{(1)}$

from (8-10) determine $g_{M_4+3}^{(1)} = 8(M_4+4)g_{M_4+4}$

from (8-11) determine $g_{M_4+2}^{(2)} = 8(M_4+3)g_{M_4+3}^{(1)}$

from (8-10) determine $g_{M_4+2}^{(1)} = 8(M_4+3)g_{M_4+3} + g_{M_4+4}^{(1)}$

from (8-11) determine $g_{M_4+1}^{(2)} = 8(M_4+2)g_{M_4+2}^{(1)} + g_{M_4+3}^{(2)}$

from (8-10) determine $g^{(1)}_{M_4+1} = 8(M_4 + 2)g^{(2)}_{M_4+2} + g^{(1)}_{M_4+3}$

from (8-11) determine $g^{(2)}_{M_4} = 8(M_4 + 1)g^{(1)}_{M_4+1} + g^{(2)}_{M_4+2}$

from (8-10) determine $g^{(1)}_{M_4} = 8(M_4 + 1)g_{M_4+1} + g^{(1)}_{M_4+2}$

$p = 1(1)M_3$

from (8-11) determine $g^{(2)}_{M_4-p} = 8(M_4 - p + 1)g^{(1)}_{M_4-p+1} + g^{(2)}_{M_4-p+2}$

from (8-10) determine $g^{(1)}_{M_4-p} = 8(M_4 - p + 1)g_{M_4-p+1} + g^{(1)}_{M_4-p+2}$

from (8-17) determine

$$g_{M_3-p} = \frac{-1}{\{4(M_3 - p)(M_3 + n - p + 1) + (2n + 1)\}}$$
$$\times \Big\{ g^{(2)}_{M_3-p} + 2g^{(2)}_{M_3-p+1} + g^{(2)}_{M_3-p+2} + (3M_3 + 2n - 3p + 4)g^{(1)}_{M_3-p}$$
$$+ 2(2n + 3)g^{(1)}_{M_3-p+1} - (3M_3 - 2n - 3_{p+2})g^{(1)}_{M_3-p+2}$$
$$+ 2(2n + 1 - 2A^2)g_{M_3-p+1}$$
$$+ [4(M_3 - p + 2)(M_3 - n - p + 1) + (2n + 1)]g_{M_3-p+2} \Big\} .$$

.... (8-18)

In this way we obtain, in turn, $g^{(1)}_{M_4+4}$, $g^{(2)}_{M_4+3}$, $g^{(1)}_{M_4+3}$, $g^{(2)}_{M_4+2}$, $g^{(1)}_{M_4+2}$, $g^{(2)}_{M_4+1}$, $g^{(1)}_{M_4+1}$, $g^{(2)}_{M_4}$, $g^{(1)}_{M_4}$, $g^{(2)}_{M_4-1}$, $g^{(1)}_{M_4-1}$, g_{M_4-1}, $g^{(2)}_{M_4-2}$, $g^{(1)}_{M_4-2}$, g_{M_4-2}, ..., $g^{(2)}_1$, $g^{(1)}_1$, g_1, $g^{(2)}_0$, $g^{(1)}_0$, g_1 as linear combinations of g_{M_4}, g_{M_4+1}, g_{M_4+2}, g_{M_4+3}, g_{M_4+4}, g_{M_4+5} .

Equation (8-16) has not been used in the procedure (8-18). When we substitute for $g^{(2)}_0$, $g^{(2)}_1$, $g^{(1)}_0$, $g^{(1)}_1$, g_0 and g_1 , as obtained from the procedure (8-18), into the equation (8-16) we get a linear equation connecting g_{M_4}, g_{M_4+1}, g_{M_4+2}, g_{M_4+3}, g_{M_4+4}, g_{M_4+5} .

By putting $z = 0$ in equation (8-3) we get

$$g_n(0) = -1 \quad . \tag{8-19}$$

However, a solution of (8-3), satisfying the condition (8-19), and which is finite for $0 \leqslant z \leqslant 1$, is not unique. We can get a unique solution by prescribing the value $g_n(1)$. Correspondingly we prescribe the value $g_n^{(a)}(1)$ to be

$$g_n^{(a)}(1) = AG_n^{(a)}(A - 0) \tag{8-20}$$

where $G_n^{(a)}(A - 0)$ is obtained from the approximation $G_n^{(a)}(\alpha)$ to $G_n(\alpha)$ in $0 \leqslant \alpha \leqslant A$ which was derived in section 6. This prescription leads to a second linear equation connecting g_{M_4}, g_{M_4+1}, g_{M_4+2}, g_{M_4+3}, g_{M_4+4}, g_{M_4+5} . We can use these two linear equations to express g_{M_4} and g_{M_4+1} in terms of g_{M_4+2}, g_{M_4+3}, g_{M_4+4}, g_{M_4+5} . A solution $g_n^{(a)}(z)$ of the differential equation (8-15) satisfying the prescribed value (8-20) is therefore known in terms of g_{M_4+2}, g_{M_4+3}, g_{M_4+4}, g_{M_4+5} . The function $g_n^{(a)}(z)$ so obtained is arbitrary insofar as the coefficients g_{M_4+2}, g_{M_4+3}, g_{M_4+4}, g_{M_4+5} are arbitrary. We can choose these coefficients as we wish and we shall choose them so that the right-hand side of the differential equation (8-15) reduces as much as possible. This is achieved by taking

$$g_{M_4+2} = 0 \tag{8-21}$$

$$g_{M_4+3} = 0 \tag{8-22}$$

$$g_{M_4+4} = 0 \tag{8-23}$$

$$g_{M_4+5} = 0 \quad . \tag{8-24}$$

It then follows from (8-10) and (8-11) that

$$g^{(1)}_{M_4+1} = 0 \tag{8-25}$$

$$g^{(1)}_{M_4+2} = 0 \tag{8-26}$$

$$g^{(1)}_{M_4+3} = 0 \tag{8-27}$$

$$g^{(1)}_{M_4+4} = 0 \tag{8-28}$$

$$g^{(2)}_{M_4} = 0 \tag{8-29}$$

$$g^{(2)}_{M_4+1} = 0 \tag{8-30}$$

$$g^{(2)}_{M_4+2} = 0 \tag{8-31}$$

$$g^{(2)}_{M_4+3} = 0 \quad . \tag{8-32}$$

The conditions (8-21) to (8-32) extend the conditions (8-5), (8-8) and (8-9).

We can now determine uniquely all the coefficients g_r in the definition (8-4) of the function $g_n^{(a)}(z)$. The differential equation (8-15) satisfied by $g_n^{(a)}(z)$ reduces to

$$\begin{aligned}
&z^4 g_n^{(a)''}(z) + (2n+3)z^3 g_n^{(a)'}(z) + \left\{(2n+1)z^2 - A^2\right\} g_n^{(a)}(z) - A^2 \\
&= \left\{\tfrac{1}{64}\left(g^{(2)}_{M_4-2} + 6g^{(2)}_{M_4-1}\right) + \tfrac{1}{8}(n+2)\left(g^{(1)}_{M_4-1} + 4g^{(1)}_{M_4}\right)\right. \\
&\qquad \left. + \tfrac{1}{4}(2n+1)\left(g_{M_4} + 2g_{M_4+1}\right) - A^2 g_{M_4+1}\right\} T_{2M_4+2}(z) \\
&\quad + \left\{\tfrac{1}{64}\, g^{(2)}_{M_4-1} + \tfrac{1}{8}(n+2)g^{(1)}_{M_4} + \tfrac{1}{4}(2n+1)g_{M_4+1}\right\} T_{2M_4+4}(z) \\
&= \left\{\left[M_4(M_4+n+1) + \tfrac{1}{4}(2n+1)\right] g_{M_4} + \left[(M_4+1)(3M_4+2n+4) + \tfrac{1}{4}(2n+1)\right.\right. \\
&\qquad \left.\left. - \tfrac{1}{2}A^2\right] g_{M_4+1}\right\} T_{2M_4+2}(z) \\
&\quad + \left\{(M_4+1)(M_4+n+2) + \tfrac{1}{4}(2n+1)\right\} g_{M_4} T_{2M_4+4}(z) \quad .
\end{aligned} \tag{8-33}$$

The differential equation (8-33) is a slight modification of the differential equation (8-3) if the numbers

$$\left[M_4(M_4+n+1) + \tfrac{1}{4}(2n+1)\right]g_{M_4} + \left[(M_4+1)(3M_4+2n+4) + \tfrac{1}{4}(2n+1) - \tfrac{1}{2}A^2\right]g_{M_4+1}$$

and

$$\left[(M_4 + 1)(M_4 + n + 2) + \tfrac{1}{4}(2n + 1)\right]g_{M_4}$$

are small compared with unity and if this is so then the function $g_n^{(a)}(z)$ can be expected to be a good approximation to $g_n(z)$. It is found, in practice, that these numbers rapidly decrease as the positive integer M_4 is increased.

We may therefore finally write an approximation $G_n^{(a)}(\alpha)$ to $G_n(\alpha)$ in the form

$$G_n^{(a)}(\alpha) = \frac{1}{\alpha}\sum_{r=0}^{M_4+1} g_r T_{2r}\left(\frac{A}{\alpha}\right) \qquad \text{for} \quad A \leqslant \alpha < \infty \quad . \tag{8-34}$$

The function $G_n(\alpha)$ may be written in the form

$$G_n(\alpha) = \frac{1}{\alpha}{\sum_{r=0}^{\infty}}' G_r^{(n)}(A) T_{2r}\left(\frac{A}{\alpha}\right) \qquad \text{for} \quad A \leqslant \alpha < \infty \quad . \tag{8-35}$$

The coefficients g_r for low values of r are approximations to the coefficients $G_r^{(n)}(A)$. The value of the integer M_4 must be taken large enough for these approximations to be so good that $G_n(\alpha)$ can be evaluated to the desired accuracy from formula (8-34). Values of $G_r^{(n)}(A)$ obtained by this means are given for $A = 2, 4$ and 8, and $n = 0, 1, 2$ in the results section 9.

9 RESULTS

For real $\alpha \geqslant 0$ the real functions $F_n(\alpha)$ and $G_n(\alpha)$ of α of formula (2-2) are obtained from formula (2-33) in the forms

$$F_n(\alpha) = \frac{2^n n!}{(2n)!}\alpha^n K_n(\alpha) \tag{9-1}$$

and

$$G_n(\alpha) = \frac{2^n n!}{(2n)!}\frac{\pi}{2}\alpha^n\left(L_n(\alpha) + (-1)^{n+1} I_n(\alpha)\right) \tag{9-2}$$

where $I_n(\alpha)$ and $K_n(\alpha)$ are modified Bessel functions of order n and of the first and second kinds respectively and $L_n(\alpha)$ is a modified Struve function[2].

The functions $F_n(\alpha)$ and $G_n(\alpha)$ have the following expansions in terms of Chebyshev polynomials.

$$F_n(\alpha) = \sum_{r=0}^{\infty}{}' D_r^{(n)}(A)T_{2r}\left(\frac{\alpha}{A}\right) + (-1)^{n+1}\frac{\alpha^{2n}}{(2n)!}\sum_{r=0}^{\infty}{}' C_r^{(n)}(A)T_{2r}\left(\frac{\alpha}{A}\right)\log\left(\frac{\alpha}{A}\right)$$

$$\text{for} \quad 0 \leqslant \alpha \leqslant A \; , \qquad (9\text{-}3)$$

$$G_n(\alpha) = \alpha\sum_{r=0}^{\infty}{}' E_r^{(n)}(A)T_{2r}\left(\frac{\alpha}{A}\right) + \frac{(-1)^{n+1}}{(2n)!}\frac{\pi}{2}\alpha^{2n}\sum_{r=0}^{\infty}{}' C_r^{(n)}(A)T_{2r}\left(\frac{\alpha}{A}\right)$$

$$\text{for} \quad 0 \leqslant \alpha \leqslant A \; , \qquad (9\text{-}4)$$

$$F_n(\alpha) = e^{-\alpha}\alpha^{n-\frac{1}{2}}\sum_{r=0}^{\infty}{}' F_r^{(n)}(A)T_r\left(\frac{2A}{\alpha} - 1\right) \qquad \text{for} \quad A \leqslant \alpha \leqslant \infty \qquad (9\text{-}5)$$

and

$$G_n(\alpha) = \frac{1}{\alpha}\sum_{r=0}^{\infty}{}' G_r^{(n)}(A)T_{2r}\left(\frac{A}{\alpha}\right) \qquad \text{for} \quad A \leqslant \alpha \leqslant \infty \; , \qquad (9\text{-}6)$$

where A is any positive quantity which we call the demarcation value of α, and the dash $'$ on the summation sign $\sum$ indicates that the quantity under the summation sign for $r=0$ is to be multiplied by $\frac{1}{2}$. The coefficients $D_r^{(n)}(A)$, $C_r^{(n)}(A)$, $E_r^{(n)}(A)$, $F_r^{(n)}(A)$ and $G_r^{(n)}(A)$ for $r = 0, 1, 2, \ldots,$ for a given value of A may be obtained by means of the procedures describes in sections 5, 6, 7 and 8, for any integer value n. Values of these coefficients when $A = 2, 4$ and 8, and $n = 0, 1$ and 2 have been obtained by these procedures and are given in the following tables. The FORTRAN program with double precision arithmetic gave values of the coefficients accurate to at least four more decimal places than are recorded in these tables.

r	$C_r^{(0)}(2)$	$C_r^{(1)}(2)$	$C_r^{(2)}(2)$
0	3.20584561362	2.56703598796	2.36638439931
1	0.63880962565	0.29507864030	0.18885410329
2	0.03685485969	0.01179748536	0.00575527693
3	0.00098287813	0.00023976274	0.00009433753
4	0.00001498365	0.00000294783	0.00000097108
5	0.00000014738	0.00000002428	0.00000000687
6	0.00000000101	0.00000000014	0.00000000004
7	0.00000000001	0.00000000000	0.00000000000
8	0.00000000000		

r	$D_r^{(0)}(2)$	$D_r^{(1)}(2)$	$D_r^{(2)}(2)$
0	-0.535327393234	1.52530022734	0.889508489353
1	0.344289899925	-0.35315596078	-0.172359859474
2	0.035979936515	-0.12261118082	0.057430675574
3	0.001264615411	-0.00697572386	0.008193527247
4	0.000022862121	-0.00017302890	0.000321511780
5	0.000000253479	-0.00000243341	0.000006168057
6	0.000000001905	-0.00000002213	0.000000071010
7	0.000000000010	-0.00000000014	0.000000000548
8	0.000000000000	-0.00000000000	0.000000000003
9			0.000000000000

r	$E_r^{(0)}(2)$	$E_r^{(1)}(2)$	$E_r^{(2)}(2)$
0	2.50156743335	-3.62686040785	0.090223522158
1	0.26062825773	-0.86466471676	0.434512274677
2	0.01004230874	-0.05265301734	0.058535673516
3	0.00020018002	-0.00144057353	0.002521317931
4	0.00000243208	-0.00002228887	0.000053495433
5	0.00000001987	-0.00000022136	0.000000676401
6	0.00000000012	-0.00000000153	0.000000005681
7	0.00000000000	-0.00000000001	0.000000000034
8		-0.00000000000	0.000000000000

r	$F_r^{(0)}(2)$	$F_r^{(1)}(2)$	$F_r^{(2)}(2)$
0	2.44030308207	2.72062619048	1.28419268187
1	-0.03144810131	0.10392373658	0.23331862013
2	0.00156988389	-0.00285781686	0.00872357116
3	-0.00012849550	0.00019521552	-0.00025006082
4	0.00001394981	-0.00001936198	0.00001789144
5	-0.00000183176	0.00000240648	-0.00000185219
6	0.00000027668	-0.00000035020	0.00000023919
7	-0.00000004660	0.00000005741	-0.00000003601
8	0.00000000857	-0.00000001035	0.00000000609
9	-0.00000000170	0.00000000202	-0.00000000113
10	0.00000000036	-0.00000000042	0.00000000023
11	-0.00000000008	0.00000000009	-0.00000000005
12	0.00000000002	-0.00000000002	0.00000000001
13	-0.00000000000	0.00000000001	-0.00000000000
14		-0.00000000000	

r	$G_r^{(0)}(2)$	$G_r^{(1)}(2)$	$G_r^{(2)}(2)$
0	-2.13217869983	-2.09643364580	-1.91194596134
1	-0.03217700555	0.06792205958	0.20712837099
2	0.02984899218	0.07490609971	0.06895623846
3	-0.00705161897	-0.03908076633	-0.06674389453
4	-0.00010761971	0.01103390443	0.03210856375
5	0.00129508722	0.00034669686	-0.00915804519
6	-0.00097613324	-0.00316179696	-0.00117500739
7	0.00051428094	0.00284228294	0.00421227330
8	-0.00020387361	-0.00181382702	-0.00403407960
9	0.00004311030	0.00091914922	0.00285774540
10	0.00002305925	-0.00033807218	-0.00166055679
11	-0.00004040266	0.00002613941	0.00076578034
12	0.00003686071	0.00010760420	-0.00020527751
13	-0.00002691184	-0.00014111748	-0.00009181465
14	0.00001698007	0.00012704050	0.00021365013
15	-0.00000928372	-0.00009615367	-0.00023371869
16	0.00000411161	0.00006418149	0.00020355054
17	-0.00000102277	-0.00003775024	-0.00015514772
18	-0.00000058649	0.00001857381	0.00010612667
19	0.00000125217	-0.00000605656	-0.00006467299
20	-0.00000137570	-0.00000123882	0.00003337879
21	0.00000122726	0.00000484729	-0.00001187962
22	-0.00000097313	-0.00000608037	-0.00000149056
23	0.00000070620	0.00000593157	0.00000874946
24	-0.00000047169	-0.00000509017	-0.00001178025
25	0.00000028623	0.00000399734	0.00001212772
26	-0.00000015078	-0.00000291057	-0.00001095266
27	0.00000005886	0.00000196133	0.00000906510
28	-0.00000000137	-0.00000120062	-0.00000698931

r	$G_r^{(0)}(2)$	$G_r^{(1)}(2)$	$G_r^{(2)}(2)$
29	-0.00000003078	0.00000063210	0.00000503433
30	0.00000004547	-0.00000023477	-0.00000335766
31	-0.00000004895	-0.00000002259	0.00000201682
32	0.00000004585	0.00000017285	-0.00000100847
33	-0.00000003941	-0.00000024595	0.00000029610
34	0.00000003175	0.00000026685	0.00000017129
35	-0.00000002419	-0.00000025507	-0.00000044754
36	0.00000001743	0.00000022507	0.00000058262
37	-0.00000001178	-0.00000018696	-0.00000061936
38	0.00000000733	0.00000014741	0.00000059228
39	-0.00000000398	-0.00000011046	-0.00000052787
40	0.00000000159	0.00000007828	0.00000044543
41	0.00000000002	-0.00000005172	-0.00000035827
42	-0.00000000101	0.00000003083	0.00000027498
43	0.00000000155	-0.00000001512	-0.00000020055
44	-0.00000000177	0.00000000391	0.00000013739
45	0.00000000177	0.00000000362	-0.00000008614
46	-0.00000000164	-0.00000000825	0.00000004629
47	0.00000000144	0.00000001069	-0.00000001667
48	-0.00000000121	-0.00000001155	-0.00000000422
49	0.00000000097	0.00000001133	0.00000001795
50	-0.00000000075	-0.00000001044	-0.00000002606
51	0.00000000056	0.00000000917	0.00000002990
52	-0.00000000039	-0.00000000773	-0.00000003067
53	0.00000000026	0.00000000628	0.00000002934
54	-0.00000000015	-0.00000000492	-0.00000002669
55	0.00000000008	0.00000000370	0.00000002331
56	-0.00000000002	-0.00000000266	-0.00000001964
57	-0.00000000002	0.00000000179	0.00000001599
58	0.00000000005	-0.00000000110	-0.00000001257
59	-0.00000000006	0.00000000056	0.00000000951
60	0.00000000007	-0.00000000016	-0.00000000685
61	-0.00000000007	-0.00000000013	0.00000000464
62	0.00000000006	0.00000000032	-0.00000000284
63	-0.00000000006	-0.00000000043	0.00000000143
64	0.00000000005	0.00000000049	-0.00000000036
65	-0.00000000004	-0.00000000050	-0.00000000042
66	0.00000000003	0.00000000049	0.00000000094
67	-0.00000000003	-0.00000000045	-0.00000000127
68	0.00000000002	0.00000000040	0.00000000144
69	-0.00000000002	-0.00000000035	-0.00000000150
70	0.00000000001	0.00000000029	0.00000000147
71	-0.00000000001	-0.00000000024	-0.00000000138
72	0.00000000000	0.00000000019	0.00000000125
73		-0.00000000015	-0.00000000110
74		0.00000000011	0.00000000094
75		-0.00000000008	-0.00000000079
76		0.00000000005	0.00000000064
77		-0.00000000003	-0.00000000051
78		0.00000000001	0.00000000039
79		-0.00000000000	-0.00000000028
80		-0.00000000001	0.00000000020

r	$G_r^{(0)}(2)$	$G_r^{(1)}(2)$	$G_r^{(2)}(2)$
81		0.00000000002	-0.00000000012
82		-0.00000000002	0.00000000007
83		0.00000000002	-0.00000000002
84		-0.00000000002	-0.00000000001
85		0.00000000002	0.00000000004
86		-0.00000000002	-0.00000000006
87		0.00000000002	0.00000000007
88		-0.00000000002	-0.00000000007
89		0.00000000001	0.00000000007
90		-0.00000000001	-0.00000000007
91		0.00000000001	0.00000000007
92		-0.00000000001	-0.00000000006
93		0.00000000001	0.00000000005
94		-0.00000000001	-0.00000000005
95		0.00000000000	0.00000000004
96			-0.00000000003
97			0.00000000003
98			-0.00000000002
99			0.00000000002
100			-0.00000000001
101			0.00000000001
102			-0.00000000001
103			0.00000000000

r	$C_r^{(0)}(4)$	$C_r^{(1)}(4)$	$C_r^{(2)}(4)$
0	10.3930183013	5.33276709459	3.96245000143
1	5.0602512067	1.91921769578	1.09780120523
2	0.9492999272	0.27251588793	0.12401460987
3	0.0905165805	0.02061784143	0.00773765351
4	0.0051467756	0.00096614639	0.00030756128
5	0.0001930879	0.00003073894	0.00000848241
6	0.0000051211	0.00000070665	0.00000017192
7	0.0000001009	0.00000001228	0.00000000267
8	0.0000000015	0.00000000017	0.00000000003
9	0.0000000000	0.00000000000	0.00000000000

r	$D_r^{(0)}(4)$	$D_r^{(1)}(4)$	$D_r^{(2)}(4)$
0	-2.07010521536	4.05596928486	-1.27162945068
1	0.57925185014	-0.11829410946	-0.75983364876
2	0.40087333324	-1.46724835571	0.94490368050
3	0.06130342160	-0.35471783977	0.46759026789
4	0.00457144436	-0.03569556020	0.07040584357
5	0.00020585246	-0.00201916844	0.00530641092
6	0.00000624313	-0.00007369308	0.00024217164
7	0.00000013654	-0.00000188403	0.00000743529
8	0.00000000226	-0.00000003566	0.00000016435
9	0.00000000003	-0.00000000052	0.00000000274
10	0.00000000000	-0.00000000001	0.00000000004
11		-0.00000000000	0.00000000000

r	$E_r^{(0)}(4)$	$E_r^{(1)}(4)$	$E_r^{(2)}(4)$
0	4.90866345562	-13.6449585986	9.10884949164
1	1.66839475339	-7.0679003896	6.58926863493
2	0.23044870859	-1.3713700036	1.94768516568
3	0.01715670552	-0.1333429331	0.26518165143
4	0.00079558551	-0.0076768959	0.01991274171
5	0.00002512773	-0.0002904768	0.00093490279
6	0.00000057454	-0.0000077518	0.00002988114
7	0.00000000994	-0.0000001535	0.00000069043
8	0.00000000013	-0.0000000023	0.00000001205
9	0.00000000000	-0.0000000000	0.00000000016
			0.00000000000

r	$F_r^{(0)}(4)$	$F_r^{(1)}(4)$	$F_r^{(2)}(4)$
0	2.47090781345	2.61822093104	1.04639456552
1	-0.01733468653	0.05488260137	0.10785139320
2	0.00049916079	-0.00087520483	0.00238173226
3	-0.00002457275	0.00003609633	-0.00004174267
4	0.00000165659	-0.00000222984	0.00000187777
5	-0.00000013857	0.00000017697	-0.00000012505
6	0.00000001362	-0.00000001679	0.00000001059
7	-0.00000000152	0.00000000183	-0.00000000106
8	0.00000000019	-0.00000000022	0.00000000012
9	-0.00000000003	0.00000000003	-0.00000000002
10	0.00000000000	-0.00000000000	0.00000000000

r	$G_r^{(0)}(4)$	$G_r^{(1)}(4)$	$G_r^{(2)}(4)$
0	-2.07704638677	-2.20275867159	-2.26114518420
1	-0.03811891413	-0.08759337069	-0.08758430284
2	0.00205735660	0.02059130191	0.05154088100
3	0.00133583904	0.00362532371	-0.00013081384
4	-0.00040242153	-0.00290859890	-0.00615912013
5	0.00000135756	0.00070097680	0.00287012995
6	0.00005271870	0.00010613794	-0.00048585002
7	-0.00002943689	-0.00019652550	-0.00029594622
8	0.00000887895	0.00011184366	0.00033304245
9	-0.00000009208	-0.00003753309	-0.00018763525
10	-0.00000196906	0.00000162860	0.00006586766
11	0.00000162254	0.00000890557	-0.00000302022
12	-0.00000086683	-0.00000847565	-0.00001790372
13	0.00000031946	0.00000520869	0.00001844388
14	-0.00000003999	-0.00000233692	-0.00001243241
15	-0.00000006125	0.00000059911	0.00000632377
16	0.00000007429	0.00000019495	-0.00000215099
17	-0.00000005558	-0.00000042253	-0.00000005929
18	0.00000003248	0.00000038583	0.00000090772
19	-0.00000001487	-0.00000026575	-0.00000100762
20	0.00000000425	0.00000014856	0.00000079217
21	0.00000000093	-0.00000006380	-0.00000050773
22	-0.00000000273	0.00000001341	0.00000026606
23	0.00000000278	0.00000001088	-0.00000009996
24	-0.00000000215	-0.00000001879	0.00000000415
25	0.00000000140	0.00000001810	0.00000004003
26	-0.00000000077	-0.00000001394	-0.00000005206
27	0.00000000033	0.00000000919	0.00000004726
28	-0.00000000007	-0.00000000518	-0.00000003583
29	-0.00000000006	0.00000000231	0.00000002361
30	0.00000000011	-0.00000000053	-0.00000001336
31	-0.00000000011	-0.00000000041	0.00000000595
32	0.00000000009	0.00000000080	-0.00000000125
33	-0.00000000006	-0.00000000084	-0.00000000132
34	0.00000000004	0.00000000072	0.00000000241
35	-0.00000000002	-0.00000000054	-0.00000000257
36	0.00000000001	0.00000000036	0.00000000225
37	-0.00000000000	-0.00000000022	-0.00000000174
38		0.00000000011	0.00000000121
39		-0.00000000003	-0.00000000075
40		-0.00000000001	0.00000000040
41		0.00000000003	-0.00000000015
42		-0.00000000004	-0.00000000000
43		0.00000000004	0.00000000009
44		-0.00000000003	-0.00000000012
45		0.00000000002	0.00000000013
46		-0.00000000002	-0.00000000011
47		0.00000000001	0.00000000009
48		-0.00000000001	-0.00000000007
49		0.00000000000	0.00000000004
50			-0.00000000003
51			0.00000000001

r	$G_r^{(0)}(4)$	$G_r^{(1)}(4)$	$G_r^{(2)}(4)$
52			-0.00000000000
53			-0.00000000000
54			0.00000000000
55			-0.00000000001
56			0.00000000001
57			-0.00000000001
58			0.00000000000

r	$C_r^{(0)}(8)$	$C_r^{(1)}(8)$	$C_r^{(2)}(8)$
0	255.466879624	64.9725594516	28.9707969754
1	190.494320173	45.3281540101	18.6527830678
2	82.489032744	17.3489794084	6.3067199703
3	22.274819242	4.0836376381	1.3038036594
4	4.011673760	0.6428649766	0.1812635131
5	0.509493365	0.0719638780	0.0180737062
6	0.047718749	0.0059982698	0.0013538182
7	0.003416332	0.0003857548	0.0000788968
8	0.000192469	0.0000196892	0.0000036766
9	0.000008738	0.0000008160	0.0000001401
10	0.000000326	0.0000000280	0.0000000044
11	0.000000010	0.0000000008	0.0000000001
12	0.000000000	0.0000000000	0.0000000000

r	$D_r^{(0)} 8$	$D_r^{(1)}(8)$	$D_r^{(2)}(8)$
0	-21.0576601774	116.109338821	-195.885440934
1	-4.5634335864	42.009912196	-90.282237104
2	8.0053688687	-43.100766931	60.497819474
3	5.2836328669	-39.591707756	80.721102981
4	1.5115356760	-14.060770629	36.310359055
5	0.2590844324	-2.879821291	9.065640544
6	0.0300807224	-0.389782914	1.456316380
7	0.0025363082	-0.037608348	0.163280020
8	0.0001627084	-0.002720562	0.013492644
9	0.0000082160	-0.000153070	0.000855157
10	0.0000003352	-0.000006890	0.000042861
11	0.0000000113	-0.000000254	0.000001741
12	0.0000000003	-0.000000008	0.000000058
13	0.0000000000	-0.000000000	0.000000002
			0.000000000

r	$E_r^{(0)}(8)$	$E_r^{(1)}(8)$	$E_r^{(2)}(8)$
0	55.0474921501	-372.619706447	745.239636537
1	38.0449554282	-279.527258869	589.918946483
2	14.4133068178	-122.152472131	293.155739975
3	3.3641911979	-33.264982053	93.384976387
4	0.5261166156	-6.032827358	19.844857559
5	0.0585897829	-0.770469771	2.943204561
6	0.0048633953	-0.072484811	0.317694398
7	0.0003117276	-0.005208214	0.025875685
8	0.0000158673	-0.000294292	0.001639103
9	0.0000006561	-0.000013394	0.000082815
10	0.0000000225	-0.000000501	0.000003408
11	0.0000000006	-0.000000016	0.000000116
12	0.0000000000	-0.000000000	0.000000003
			0.000000000

r	$F_r^{(0)}(8)$	$F_r^{(1)}(8)$	$F_r^{(2)}(8)$
0	2.48798130174	2.56379308344	0.937332182311
1	-0.00917485269	0.02832887813	0.051529284574
2	0.00014445509	-0.00024753707	0.000628142864
3	-0.00000401361	0.00000577197	-0.000006258705
4	0.00000015678	-0.00000020689	0.000000164093
5	-0.00000000777	0.00000000974	-0.000000006506
6	0.00000000046	-0.00000000056	0.000000000334
7	-0.00000000003	0.00000000004	-0.000000000021
8	0.00000000000	0.00000000000	0.000000000001
			-0.000000000000

r	$G_r^{(0)}(8)$	$G_r^{(1)}(8)$	$G_r^{(2)}(8)$
0	-2.01801261013	-2.05920385453	-2.10556259049
1	-0.00944907144	-0.03174217296	-0.05693946028
2	-0.00045461284	-0.00203112013	-0.00335879864
3	0.00000198940	0.00023199704	0.00114848463
4	0.00001488859	0.00011385109	0.00026274169
5	-0.00000010130	-0.00001923454	-0.00010983172
6	-0.00000102711	-0.00000682205	-0.00000601590
7	0.00000022192	0.00000364533	0.00001472816
8	0.00000004427	-0.00000034764	-0.00000442224
9	-0.00000004312	-0.00000036733	-0.00000038387
10	0.00000001216	0.00000022076	0.00000093643
11	0.00000000076	-0.00000005064	-0.00000043536

r	$G_r^{(0)}(8)$	$G_r^{(1)}(8)$	$G_r^{(2)}(8)$
12	-0.00000000244	-0.00000001203	0.00000007086
13	0.00000000127	0.00000001722	0.00000004713
14	-0.00000000031	-0.00000000861	-0.00000004756
15	-0.00000000006	0.00000000208	0.00000002266
16	0.00000000011	0.00000000050	-0.00000000512
17	-0.00000000006	-0.00000000088	-0.00000000186
18	0.00000000002	0.00000000056	0.00000000286
19	0.00000000000	-0.00000000022	-0.00000000186
20		0.00000000003	0.00000000077
21		0.00000000004	-0.00000000013
22		-0.00000000004	-0.00000000012
23		0.00000000003	0.00000000015
24		-0.00000000001	-0.00000000010
25		0.00000000000	0.00000000005
26			-0.00000000001
27			-0.00000000000
28			0.00000000001
29			-0.00000000001
30			0.00000000000

Values of $F_n(\alpha)$ and $G_n(\alpha)$ for $n = 0, 1, 2$, for a selection of values of α are as follows:

α	$F_0(\alpha)$	$G_0(\alpha)$
0	∞	-1.57079632679
1	0.42102443824	-0.87308424265
2	0.11389387275	-0.53745038906
3	0.03473950439	-0.36459259386
4	0.01115967609	-0.26840471551
5	0.00369109833	-0.21041554608
6	0.00124399433	-0.17271246880
7	0.00042479574	-0.14653670418
8	0.00014647071	-0.12736175065
9	0.00005088131	-0.11270609629
10	0.00001778006	-0.10112644070
11	0.00000624302	-0.09173512472
12	0.00000220083	-0.08395815487
13	0.00000077845	-0.07740790423
14	0.00000027614	-0.07181280030
15	0.00000009820	-0.06697661204
16	0.00000003499	-0.06275381750
17	0.00000001249	-0.05903409849
18	0.00000000447	-0.05573223363
19	0.00000000160	-0.05278130611
20	0.00000000057	-0.05012801666

α	$F_1(\alpha)$	$G_1(\alpha)$
0	1.00000000000	0.00000000000
1	0.60190723020	-0.46845081220
2	0.27973176363	-0.46728898961
3	0.12046929338	-0.37634317644
4	0.04993399555	-0.29174356448
5	0.02022306723	-0.22928450853
6	0.00806351831	-0.18560906849
7	0.00317927741	-0.15496532001
8	0.00124295369	-0.13288154121
9	0.00048273315	-0.11640802117
10	0.00018648773	-0.10369265741
11	0.00007172947	-0.09357737742
12	0.00002748909	-0.08532476963
13	0.00001050216	-0.07845132045
14	0.00000400168	-0.07262921268
15	0.00000152126	-0.06762868732
16	0.00000057715	-0.06328372054
17	0.00000021857	-0.05947107581
18	0.00000008264	-0.05609713464
19	0.00000003120	-0.05308935616
20	0.00000001177	-0.05039056921

α	$F_2(\alpha)$	$G_2(\alpha)$
0	0.66666666667	0.00000000000
1	0.54161296621	-0.26999528902
2	0.33834633942	-0.36145984516
3	0.18453137542	-0.34467323255
4	0.09280760282	-0.29265419235
5	0.04424119760	-0.23965255633
6	0.02030361081	-0.19628900465
7	0.00905784872	-0.16340971491
8	0.00395334417	-0.13897170808
9	0.00169561755	-0.12066994738
10	0.00071699390	-0.10667646165
11	0.00029962147	-0.09570161546
12	0.00012396568	-0.08687461347
13	0.00005085437	-0.07961281856
14	0.00002070874	-0.07352242786
15	0.00000837883	-0.06833169487
16	0.00000337093	-0.06384823994
17	0.00000134937	-0.05993220510
18	0.00000053772	-0.05647932206
19	0.00000021342	-0.05341007312
20	0.00000008439	-0.05066260063

10 CLOSING REMARKS

Expansions in terms of Chebyshev polynomials have been obtained in the forms (9-3), (9-4), (9-5) and (9-6) for the real functions $F_n(\alpha)$ and $G_n(\alpha)$ of the real variable $\alpha \geqslant 0$. Explicit expressions for $F_n(\alpha)$ and $G_n(\alpha)$ in terms of known functions are given in formulae (9-1) and (9-2).

Three values of the demarcation parameter A have been considered, namely $A = 2$, 4 and 8. The coefficients $C_r^{(n)}(8)$, $D_r^{(n)}(8)$ and $E_r^{(n)}(8)$ are quite large for low values of r compared with the values of the functions $F_n(\alpha)$ and $G_n(\alpha)$ except for those of $F_0(\alpha)$ near $\alpha = 0$. There is consequently a loss of accuracy in the values obtained for $F_n(\alpha)$ and $G_n(\alpha)$ for $0 \leqslant \alpha \leqslant 8$ when calculations are carried out from formulae (9-3) and (9-4) using only a small number of significant figures. The values obtained, however, are more accurate than the corresponding values obtained by using formulae (2-34), (2-35) and (2-36) and the power series expansions (2-15), (2-16) and (2-17) unless α is much smaller than 8. The coefficients $F_r^{(n)}(8)$ and $G_r^{(n)}(8)$ are not nearly as large for low values of r as are $C_r^{(n)}(8)$, $D_r^{(n)}(8)$ and $E_r^{(n)}(8)$ and there is therefore only a slight loss in accuracy in the values obtained for $F_n(\alpha)$ and $G_n(\alpha)$ for $8 \leqslant \alpha \leqslant \infty$ when calculations are carried out from formulae (9-5) and (9-6).

The coefficients $C_r^{(n)}(A)$, $D_r^{(n)}(A)$ and $E_r^{(n)}(A)$ are much smaller for $A = 2$ and 4 than they are for $A = 8$ for corresponding values of r and fewer of them need to be retained than for $A = 8$ to get a given accuracy in $F_n(\alpha)$ and $G_n(\alpha)$ for $0 \leqslant \alpha \leqslant A$. On the other hand the coefficients $F_r^{(n)}(A)$ and $G_r^{(n)}(A)$ are comparable for $A = 2$, 4 and 8 when $r = 0$ but their values decrease much more slowly as r increases when $A = 2$ and 4 than they do when $A = 8$ thus necessitating the retention of more of them than for $A = 8$ to get a given accuracy in $F_n(\alpha)$ and $G_n(\alpha)$ for $A \leqslant \alpha < \infty$. The choice of demarcation parameter A is a matter of compromise. Of course, it is possible to introduce series for an intermediate range of α not extending either to $\alpha = 0$ or $\alpha = \infty$, so that expansions in Chebyshev polynomials may be obtained which give $F_n(\alpha)$ and $G_n(\alpha)$ to a given accuracy without the need for retaining an excessive number of terms in any expansion.

From formula (9-1) for $n = 0$ and 1 we get

$$F_0(\alpha) = K_0(\alpha) \tag{10-1}$$

and

$$F_1(\alpha) = \alpha K_1(\alpha) \quad . \tag{10-2}$$

Clenshaw[3] has obtained expansions of $K_0(\alpha)$ and $K_1(\alpha)$ in terms of Chebyshev polynomials with demarcation parameter $A = 8$. His expansion for $K_0(\alpha)$ for $0 \leqslant \alpha \leqslant A$ is the same as our expansion (9-3) for $F_0(\alpha)$ and his coefficients agree with our coefficients $C_r^{(0)}(8)$ and $D_r^{(0)}(8)$ but he gives more significant decimal figures. His expansion for $K_0(\alpha)$ for $A \leqslant \alpha < \infty$ is different from our expansion for $F_0(\alpha)$ and his coefficients do not reduce to zero as rapidly as our coefficients $F_r^{(0)}(8)$ do as r increases. His expansion for $K_1(\alpha)$ for $0 \leqslant \alpha \leqslant A$ can be used in conjunction with the properties (3-6) and (3-7) of Chebyshev polynomials to give our expansion (9-3) for $F_1(\alpha)$. His expansion for $K_1(\alpha)$ for $A \leqslant \alpha < \infty$ when multiplied by α is different from our expansion for $F_1(\alpha)$ and his coefficients do not reduce to zero as rapidly as our coefficients $F_r^{(1)}(8)$ do as r increases.

Since $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ given by the power series expansions (2-15), (2-16) and (2-17) are integral functions of α it is possible to express them for $-A \leqslant \alpha \leqslant A$ as series of Chebyshev polynomials $T_{2r}\left(\frac{\alpha}{A}\right)$ by expressing α^{2r} in the above mentioned power series expansions as a series of Chebyshev polynomials. For practical numerical evaluation of the coefficients in these series of Chebyshev polynomials the power series need to be truncated to a finite number of terms, so again, only approximations to the coefficients in the series of Chebyshev polynomials are obtained, although these approximations become very good as the number of terms retained in the power series becomes very large. The resulting series of Chebyshev polynomials for the functions $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ may then be inserted into formulae (2-34), (2-35) and (2-36). The series (9-3) and (9-4) for $F_n(\alpha)$ and $G_n(\alpha)$ are then obtained. The process is, however, no easier to apply than the process described in this paper.

For the range $A \leqslant \alpha < \infty$ it is not possible to obtain the series (9-5) and (9-6) for $F_n(\alpha)$ and $G_n(\alpha)$ by applying a similar procedure to the asymptotic expansion (2-39) and (2-40) because these asymptotic expansions are divergent. However, the series (9-5) and (9-6) are obtained quite easily by applying the procedure described in this paper.

The series (9-3), (9-4), (9-5) and (9-6) are effectively finite series because the coefficients are known to only a finite number of decimal places. The evaluations of $F_n(\alpha)$ and $G_n(\alpha)$ from these series are very rapid. Unless

$\alpha \ll A$ the evaluations of $F_n(\alpha)$ and $G_n(\alpha)$ to a given accuracy are more rapid from the series (9-3) and (9-4) than from (2-34), (2-35) and (2-36) where the power series (2-15), (2-16) and (2-17) are used to evaluate $i_n(\alpha)$, $k_n(\alpha)$ and $\ell_n(\alpha)$ respectively.

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