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PROCUREMENT EXECUTIVE MINISTRY OF DEFENCE

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REPORTS AND MEMORANDA

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1977

PRICE £4·00 NET

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Summary

The inviscid flow field past a slender delta wing at incidence with leading-edge separation can be considered as conical. The shape of the resulting vortex sheet has been calculated by J. H. B. Smith and others. Here the behaviour of the sheet near the leading edge of the wing is investigated and an expansion of the solution in this neighbourhood is found by the application of certain theorems of the theory of complex functions. It is shown that in a cross-flow plane (normal to the undisturbed flow) the slope of the sheet can be expressed in powers of the square root of the arc length measured along the sheet. A related series expansion is found for the strength of the vortex sheet. The sheet is always tangential to the pressure side of the wing. On the suction side of a wing with thickness the flow is parallel to the leading edge, so that the strength of the vortex sheet at the leading edge is directly related to the overall circulation around the sheet.

* This work was done under the link between the University of Southampton and the R.A.E.
† Replaces R.A.E. Technical Report 74150—A.R.C. 35 901

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1. Introduction

In recent years nonlinear effects have played an important part in the calculation of the pressure distribution over wings. Vortex sheets which may extend from the trailing or leading edges of a lifting wing, have important effects on the aerodynamic properties. Although viscosity plays a large part in these flows, inviscid mathematical models have been shown to give valuable information on their behaviour.

Because of their relative simplicity, leading-edge vortex sheets extending from a slender delta wing have been studied in some detail[1,2,3,6]. These flows are conical so that the three-dimensional problem reduces to a two-dimensional one in the cross-flow plane. The theory of analytic functions and conformal mapping can be applied. Some information is available on the behaviour of this solution near the centre of the rolled-up vortex sheet[6].

In the present paper the behaviour of the sheet near the leading edge is studied in some detail. It is known already[5], that the sheet leaves the edge tangentially to the pressure side of the wing. By the application of certain theorems from the theory of analytic functions an expansion of the solution of this problem in the cross-flow plane, valid near the leading edge, is obtained. The slope of the sheet is expressed as a series ascending in half-powers of the arc length along the sheet. The coefficients of this series determine some of the coefficients in similar algebraic expansions (in terms of the arc length) of the distribution of the normal velocities along the sheet, the potential function and the tangential velocities along the sheet. These last determine the strength of the vortex sheet. The remaining coefficients in these expansions cannot be determined locally, but follow from a 'global' solution, usually only obtainable by numerical methods. On the suction side of a wing with thickness the sheet forms a finite angle with the surface. Here the relative cross-velocity is shown to be zero, so that (according to the pressure condition) the local strength of the vortex sheet at the leading edge determines the overall circulation round the sheet. In the limiting case of a wing of zero thickness this result cannot be deduced directly.

Since the shape of the vortex sheet is one of the unknown functions in this problem, it was decided to formulate the problem in another plane, which is obtained from the original cross-flow plane by conformal mapping. Here the two surfaces of the sheet and the surfaces of the wing are mapped on parts of the real axis. Both the potential function for the flow and the mapping function are determined and so the solution is found in parametric form.

The present report investigates only the case of an algebraic singularity at the leading edge and no other type of singularity is discussed. Within this class the possibility of a solution involving the square root of the arc length is shown.

The governing equations for the conical flow past a slender delta wing at incidence are summarised in Section 2. In Section 3 the mathematical background (e.g. the conformal mapping mentioned above) is explained and the leading terms of the various expansions are obtained. Some higher-order terms for functions on the sheet surface are enumerated in Sections 4 and 5, and corresponding expressions for the wing surface are given in Section 6. The results are summarised in Section 7.

2. Governing Equations

The equations we shall use are well known in slender body theory and have been derived in several sources. We quote the work of Smith[1,2].

The undisturbed stream flows with velocity U at small angle of incidence α' to the wing, and $Oxyz$ is a system of rectangular cartesian coordinates, with O at the wing apex, Ox directed along the wing centre line, Oy to starboard and Oz upward. Let k be $\tan\gamma$, where 2γ is the apex angle of the wing so that the local semi-span $s = kx$ (*see* Fig. 1).

For small angles of incidence we may use Ward's[3] development of slender body theory which uses axes related to the undisturbed stream. We consider only the subsonic case and write the potential as

$$U(x+b_0(x))+\Phi \tag{1}$$

where

$$2\pi b_0(x) = A'(x)\ln\left(\tfrac{1}{2}\beta\right)+\tfrac{1}{2}\int_x^1 A''(t)\ln(t-x)\,dt-\tfrac{1}{2}\int_0^x A''(t)\ln(x-t)\,dt$$

for a body of unit length. Here $\beta^2 = 1-M^2$ and $A(x)$ is the cross-sectional area of the body.

Then Φ is a solution of Laplace's equation

$$\Phi_{yy}+\Phi_{zz}=0 \tag{2}$$

and at large distances from the wing

$$\frac{\Phi}{U}\sim\alpha' z+\frac{A'(x)}{2\pi}\ln r \tag{3}$$

where $r^2=y^2+z^2$.

Also $Ux+\Phi$ satisfies the boundary conditions on the wing surface and the vortex sheet so we have no need to consider $b_0(x)$ further.

The boundary conditions are that the wing and vortex sheet should be stream surfaces of the three-dimensional flow, and that the pressure should be continuous across the vortex sheet. We introduce cylindrical polar coordinates (x, r, θ), where r is measured from the wing centre line and θ from the starboard tip. We consider the trace of the starboard vortex sheet in the cross-flow plane, $x=$ constant, and let ϕ be the angle between the radius and the tangent at any point, σ the arc length along the trace and n the inward-drawn normal, so that (x, σ, n) is a right-handed system (*see* Fig. 2).

If the equation of the sheet is $S(x, r, \theta)=0$, the condition for it to be a stream surface is $\mathbf{V}\cdot\nabla S=0$, where $\mathbf{V}=\nabla(Ux+\Phi)$. In accordance with the linearisation assumptions $\Phi_x \ll U\cos\alpha$ and the condition becomes

$$US_x+\Phi_r S_r+\frac{\Phi_\theta S_\theta}{r^2}=0. \tag{4}$$

In the cross-flow plane

$$\frac{S_\theta}{S_r}=-\frac{dr}{d\theta} \tag{5}$$

and for any plane curve

$$\frac{dr}{d\theta}=r\cot\phi \tag{6}$$

so that[1,2]

$$\Phi_n=U\sin\phi\frac{S_x}{S_r}. \tag{7}$$

For conical flow S is a function of r/x and θ only, so that

$$\Phi_n=-kU\left(\frac{r}{s}\right)\sin\phi \tag{8}$$

on the vortex sheet. There is a similar condition on the lower and upper surfaces of the wing

$$\Phi_n=-kU\sin\Omega_2, \qquad \Phi_n=kU\sin\Omega_1. \tag{9}$$

By slender-body theory we have

$$C_p=-\frac{2\Phi_x}{U}-\frac{1}{U^2}\left(\Phi_y^2+\Phi_z^2\right)+\alpha^2,$$

or for conical flow,

$$C_p=\frac{2}{Ux}\left(y\Phi_y+z\Phi_z-\Phi\right)-\left(\Phi_y^2+\Phi_z^2\right)\Big/U^2+\alpha^2. \tag{10}$$

We denote the difference across the sheet by D, *e.g.*

$$D\Phi = (\Phi)_{\text{upper}} - (\Phi)_{\text{lower}}.$$

Since $D\Phi_n = 0$ the condition $DC_p = 0$ leads to

$$D\Phi = D(\Phi_\sigma)\left\{r \cos \phi - \frac{s}{kU}(\Phi_\sigma)_m\right\}, \tag{11}$$

where $(\Phi_\sigma)_m$ is the mean of the tangential velocities on either side of the sheets. Also we require smooth outflow at the leading edges, i.e.

$$\Phi_y \quad \text{and} \quad \Phi_z \quad \text{are finite for} \quad z = 0, \qquad y = \pm s. \tag{12}$$

We define

$$r^* = \frac{r}{s}, \qquad n^* = \frac{n}{s}, \qquad \sigma^* = \frac{\sigma}{s}, \qquad \Phi^* = \frac{\Phi}{kUs} \tag{13}$$

and obtain our two conditions (8) and (11) in non-dimensional form

$$\frac{\partial \Phi}{\partial n} = -r \sin \phi \tag{14}$$

$$D\Phi = D\left(\frac{\partial \Phi}{\partial \sigma}\right)\left\{r \cos \phi - \left(\frac{\partial \Phi}{\partial \sigma}\right)_m\right\}. \tag{15}$$

Here the stars have been omitted.

3. First-Order Solution

Since the boundary conditions along the sheet have to be applied along a curve, the slope of which is as yet undefined, we may use the method proposed by Mangler and Sells[4] to map the whole region onto a strip (*see* Fig. 3b) bounded by straight lines. We note that points on the sheet surface are mapped onto two points, corresponding to the two faces of the sheet. It is also possible to map into the upper half-plane as shown in Fig. 3c with the points on the sheet mapped onto points on the real axis. For ease of reference we shall use the latter case in our analysis and as suggested in the diagrams we shall deal with the starboard leading edge. Our argument is very similar to that used by Craggs, Mangler and Zamir[7].

We use $Z^* = X^* + iY^*$ as a complex variable in the new half-plane ($Y^* \geq 0$) and denote the coordinates in the physical plane (Fig. 3a) by $\zeta = y + iz$. We denote the conformal mapping function by $\zeta = \zeta(Z^*)$ and define

$$\frac{d\zeta}{dZ^*} = e^{h^* + i\psi^*} \tag{16}$$

where h^*, ψ^* are real.

On the sheet surface ψ^* is the slope of the sheet in the cross-sectional plane, x = constant (angle ψ in Fig. 2). We consider the function

$$\ln \frac{d\zeta}{dZ^*} = h^* + i\psi^*, \tag{17}$$

which is analytic in the upper half-plane. If ψ^* is known on the boundary, in this case the real axis, we may apply Poisson's integral (*vide* Moretti[8]) to calculate h^* on the boundary.

Therefore, for small values of X^*, the function $h^*(X^*)$ consists of a regular function

$$h^* - h_1^* = H_0^* + H_1^* X^* + \dots\ , \tag{18}$$

which arises from the distant part of the integral, and a local contribution h_1^* which may be singular if ψ^* is singular at $X^* = 0$.

We are interested in the behaviour close to the leading edge and shall therefore work in variables local to it. We use $Z = X + iY$ as a complex variable, origin at C, for the 'upper leading edge' and $\bar{Z} = \bar{X} + i\bar{Y}$ as a complex variable, origin at C', for the 'lower leading edge'. We shall apply this notation to all variables, for example, $h^*(Z^*) = h(Z)$ near the upper leading edge and $h^*(Z^*) = \bar{h}(\bar{Z})$ near the lower leading edge.

For small X, $\bar{X}$ we assume expansions of the form

$$\psi_u = \Omega_2 + A_1 X^{\lambda_1}, \qquad \psi_l = \Omega_2 + B_1 \bar{X}^{\mu_1} \tag{19}$$

on the sheet ($X, \bar{X} > 0$) where $0 < \lambda_1 < 1$, $0 < \mu_1 < 1$ and $\psi = \Omega_2$ at the leading edge.

On the wing ($X, \bar{X} < 0$) we have (*see* Fig. 2)

$$\psi_u = -\Omega_1; \qquad \psi_l = \Omega_2. \tag{20}$$

Knowing the local behaviour of ψ enables us to find the behaviour of h_1 near the leading edge. This is dealt with in Appendix A, giving, from equations (A-2), (A-5)

$$\left.\begin{array}{l} h_1 = \begin{cases} -K \log X + P_1 X^{\lambda_1} + \ldots & (X > 0) \\ -K \log |X| - A_1 \operatorname{cosec} \lambda_1 \pi |X|^{\lambda_1} + \ldots & (X < 0) \end{cases} ; \\ \bar{h}_1 = \begin{cases} Q_1 \bar{X}^{\mu_1} \ldots & (\bar{X} > 0) \\ B_1 \operatorname{cosec} \mu_1 \pi |\bar{X}|^{\mu_1} \ldots & (\bar{X} < 0) \end{cases} \end{array}\right\} \tag{21}$$

where

$$P_1 = -A_1 \cot \lambda_1 \pi, \qquad Q_1 = B_1 \cot \mu_1 \pi.$$

If σ, n are measured along and normal to the sheet (see Fig. 2), with $\sigma = 0$ corresponding to the leading edge, then

$$d\zeta = dy + i\,dz = e^{i\psi}(d\sigma + i\,dn). \tag{22}$$

Thus from (16),

$$\frac{d\sigma_u}{dX} = e^h, \qquad \frac{dy_u}{dX} = e^h \cos \psi_u, \qquad \frac{dz_u}{dX} = e^h \sin \psi_u, \tag{23}$$

and integrating we can find σ_u, y_u, z_u. There are corresponding expressions in the lower variables and since they describe the same part of the (thin) sheet near the leading edge, the conditions

$$D(\psi) = \psi_u - \psi_l = 0, \tag{24}$$

$$D(\sigma) = \sigma_u - \sigma_l = 0, \tag{25}$$

enable us to relate the local expansions in X and $\bar{X}$.

Equation (21) gives, on the sheet,

$$h = H_0 + P_1 X^{\lambda_1} + H_1 X + \ldots - K \log X,$$

$$\bar{h} = \bar{H}_0 + Q_1 \bar{X}^{\mu_1} + \bar{H}_1 \bar{X} + \ldots ,$$

and so

$$\frac{d\sigma_u}{dX} = \mathrm{e}^h = \mathrm{e}^{H_0} X^{-K}[1 + P_1 X^{\lambda_1} + \ldots] \qquad (X > 0), \tag{26}$$

$$\frac{d\sigma_l}{d\bar{X}} = \mathrm{e}^{\bar{h}} = \mathrm{e}^{\bar{H}_0}[1 + Q_1 \bar{X}^{\mu_1} + \ldots] \qquad (\bar{X} > 0). \tag{27}$$

Hence, if $\alpha = 1 - K$,

$$\sigma_u = \frac{e^{H_0} X^\alpha}{\alpha}\left[1 + \frac{\alpha P_1}{\alpha + \lambda_1} X^{\lambda_1} + \ldots\right] \qquad (X > 0), \tag{28}$$

$$\sigma_l = e^{\bar{H}_0} \bar{X}\left[1 + \frac{Q_1}{1 + \mu_1} \bar{X}^{\mu_1} + \ldots\right] \qquad (\bar{X} > 0). \tag{29}$$

From equation (25) we find

$$\bar{X} = T X^\alpha \quad \text{with} \quad T = \frac{e^{H_0 - \bar{H}_0}}{\alpha} \cdot \tag{30}$$

In the same way from (19) and (24) we have

$$A_1 X^{\lambda_1} = B_1 \bar{X}^{\mu_1} + \ldots$$

and so from (30)

$$\lambda_1 = \alpha \mu_1, \qquad T^{\mu_1} = \frac{A_1}{B_1}. \tag{31}$$

Our original assumption that $\mu_1 < 1$ (see equation (19)) thus means that $\lambda_1 < \alpha$. On the wing surface, by (21),

$$e^h = e^{H_0} |X|^{-K} [1 - A_1 \operatorname{cosec} \lambda_1 \pi |X|^{\lambda_1} \ldots], \tag{32}$$

$$e^{\bar{h}} = e^{\bar{H}_0} [1 + B_1 \operatorname{cosec} \mu_1 \pi |\bar{X}|^{\mu_1} \ldots]. \tag{33}$$

Next we consider the complex potential $W = \Phi + i\Psi$ and find, along the sheet,

$$e^{i\psi^*} \frac{dW}{d\zeta} = \frac{\partial \Phi}{\partial \sigma} - i \frac{\partial \Phi}{\partial n}$$

$$= e^{-h^*} \frac{dW}{dZ^*} = e^{-h^*}\left(\frac{\partial \Phi}{\partial X^*} - i \frac{\partial \Phi}{\partial Y^*}\right),$$

which gives

$$\frac{\partial \Phi}{\partial \sigma} = e^{-h^*} \frac{\partial \Phi}{\partial X^*}, \qquad \frac{\partial \Phi}{\partial n} = e^{-h^*} \frac{\partial \Phi}{\partial Y^*}, \tag{34}$$

and so (14), (15) take the form

$$\frac{\partial \Phi}{\partial Y^*} = -e^{h^*} r \sin \phi, \tag{35}$$

$$D\Phi = D\left(e^{-h^*} \frac{\partial \Phi}{\partial X^*}\right)\left\{r \cos \phi - \left(e^{-h^*} \frac{\partial \Phi}{\partial X^*}\right)_m\right\}. \tag{36}$$

Let

$$\frac{dW}{dZ^*} = \frac{\partial \Phi}{\partial X^*} - i \frac{\partial \Phi}{\partial Y^*} = u^* + iq^*, \tag{37}$$

where q^* can be found from (35) both on the sheet surface and on the wing.

As in the case of h^*, for small X^* the function u^* consists of a singular part u_1^* and a regular part, and can be written

$$u^*(X^*) = u_1^*(X^*) + U_0^* + U_1^* X^* + \dots ,$$

so that in our local variables, close to the leading edge,

$$u(X) = u_1(X) + U_0 + U_1 X + \dots ,$$

$$\bar{u}(\bar{X}) = \bar{u}_1(\bar{X}) + \bar{U}_0 + \bar{U}_1 \bar{X} + \dots .$$

From equations (22), (26), (27), (32), (33), (35), (37), (B-11), (B-13), (B-14),

$$q(X) = \begin{cases} e^{H_0} X^{-K}\left[\sin\Omega_2 - \dfrac{A_1 \sin(\Omega_2 - \lambda_1 \pi)}{\sin \lambda_1 \pi} X^{\lambda_1} \dots\right] & (X>0) \\ e^{H_0}|X|^{-K}\left[-\sin\Omega_1 + \dfrac{A_1 \sin\Omega_1}{\sin\lambda_1 \pi}|X|^{\lambda_1} + \dots\right] & (X<0) \end{cases} \tag{38}$$

$$\bar{q}(\bar{X}) = \begin{cases} e^{\bar{H}_0}\left[\sin\Omega_2 + \dfrac{B_1 \sin(\Omega_2 + \mu_1 \pi)}{\sin \mu_1 \pi} \bar{X}^{\mu_1} \dots\right] & (\bar{X}>0) \\ e^{\bar{H}_0}\left[\sin\Omega_2 + \dfrac{B_1 \sin\Omega_2}{\sin\mu_1 \pi}|\bar{X}|^{\mu_1} \dots\right] & (\bar{X}<0). \end{cases} \tag{39}$$

In Appendix C we derive expressions for u_1, $\bar{u}_1$ from (38), (39) and including the regular terms, we obtain on the sheet

$$u(X) = e^{H_0} X^{-K}\left[\cos\Omega_2 + U_0\, e^{-H_0} X^K - \frac{A_1 \cos(\Omega_2 - \lambda_1 \pi)}{\sin\lambda_1 \pi} X^{\lambda_1} \dots\right] \tag{40}$$

$$\bar{u}(\bar{X}) = e^{\bar{H}_0}\left[\bar{U}_0\, e^{-\bar{H}_0} + \frac{B_1 \cos(\Omega_2 + \mu_1 \pi)}{\sin \mu_1 \pi} \bar{X}^{\mu_1} \dots\right]. \tag{41}$$

These expressions combined with (26) and (27) give, by (34),

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_u = e^{-h} u = \cos\Omega_2 + U_0\, e^{-H_0} X^K - A_1 \sin\Omega_2 X^{\lambda_1}, \tag{42}$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_l = e^{-\bar{h}} \bar{u} = \bar{U}_0\, e^{-\bar{H}_0} + B_1(d_0 \cot\mu_1\pi - \sin\Omega_2)\bar{X}^{\mu_1}, \tag{43}$$

where

$$d_0 = \cos\Omega_2 - \bar{U}_0\, e^{-\bar{H}_0}. \tag{44}$$

Using (30), (31) we express (43) in terms of the upper variables

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_l = \bar{U}_0\, e^{-\bar{H}_0} + A_1(d_0 \cot\mu_1\pi - \sin\Omega_2) X^{\lambda_1} \dots \tag{45}$$

and so from (42), (45) we deduce

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right) = d_0 + U_0\, e^{-H_0} X^K - A_1 d_0 \cot\mu_1\pi X^{\lambda_1} \dots \tag{46}$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_m = \frac{\cos\Omega_2 + \bar{U}_0\, e^{-\bar{H}_0}}{2} + \frac{U_0\, e^{-H_0}}{2} X^K + A_1\left(\frac{d_0 \cot\mu_1\pi}{2} - \sin\Omega_2\right) X^{\lambda_1} \dots . \tag{47}$$

Using equation (B-10) this leads to

$$r\cos\phi-\left(\frac{\partial\Phi}{\partial\sigma}\right)_m=\frac{d_0}{2}-\frac{U_0\,e^{-H_0}}{2}X^K-\frac{A_1d_0\cot\mu_1\pi}{2}X^{\lambda_1}\dots\,. \tag{48}$$

Equation (37) shows that $u=\partial\Phi/\partial X$ and hence integration of (40), (41) gives

$$\Phi_u=(\Phi_u)_0+\frac{e^{H_0}X^\alpha}{\alpha}[\cos\Omega_2+\dots], \tag{49}$$

$$\Phi_l=(\Phi_l)_0+\bar{U}_0\bar{X}\dots=(\Phi_l)_0+T\bar{U}_0X^\alpha\dots\,, \tag{50}$$

so that

$$D\Phi=D\Phi_0+\frac{d_0\,e^{H_0}}{\alpha}X^\alpha+\dots\,. \tag{51}$$

Substituting (46), (48) and (51) into the pressure condition (15) we obtain

$$D\Phi_0+\frac{d_0\,e^{H_0}}{\alpha}X^\alpha+\dots=\frac{d_0^2}{2}-A_1d_0^2\cot\mu_1\pi X^{\lambda_1}-\frac{(U_0\,e^{-H_0})^2}{2}X^{2K}. \tag{52}$$

Our original assumption that $\mu_1<1$ led to $\lambda_1<\alpha$ (equation (31)), and equation (52) therefore gives

$$D\Phi_0=\frac{d_0^2}{2}, \tag{53}$$

$$0=A_1d_0^2\cot\mu_1\pi X^{\lambda_1}+\frac{(U_0\,e^{-H_0})^2}{2}X^{2K}. \tag{54}$$

We examine equation (54) in detail in Appendix D and arrive at the conclusion that for all values of K, $\mu_1=\frac{1}{2}$ and $\lambda_1=\alpha/2$. Also, for $K<1/3$ we find that $U_0=0$.

We may express our leading terms as functions of the arc length:

$$\psi=\Omega_2+C\sigma^{\frac{1}{2}}+\dots\,,\qquad C=A_1(\alpha\,e^{-H_0})^{\frac{1}{2}}=B_1\,e^{\bar{H}_0/2},$$

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right)=d_0+o(\sigma^{\frac{1}{2}}).$$

Special case of a thin wing

For a thin wing $K=0$ and terms previously of order higher than λ_1 now become of the same order (*e.g.* λ_1+K). The expressions in the lower variables are largely unaltered apart from the simplifications introduced by having $\Omega_1=\Omega_2=0$. The bulk of the argument for the thick wing still applies but slight differences occur towards the end, necessitating a separate treatment.

Equations (26), (40), (42), (45) are replaced by:

$$e^h=e^{H_0}[1-A_1\cot\lambda_1\pi X^{\lambda_1}\dots],$$

$$U=e^{H_0}[1+U_0\,e^{-H_0}-A_1\cot\lambda_1\pi X^{\lambda_1}],$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_u=e^{-h}u=1+U_0\,e^{-H_0}+U_0\,e^{-H_0}A_1\cot\lambda_1\pi X^{\lambda_1}\dots,$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_l=e^{-\bar{h}}\bar{u}=\bar{U}_0\,e^{-\bar{H}_0}+A_1d_0\cot\mu_1\pi X^{\lambda_1},$$

where now $d_0=1-\bar{U}_0\,e^{-\bar{H}_0}$.

Relationships defined by (30), (31) become

$$\bar{X}=TX; \qquad T=e^{H_0-\bar{H}_0}; \qquad \lambda_1=\mu_1; \qquad T^{\mu_1}=\frac{A_1}{B_1}.$$

Let

$$b_1=d_0+U_0\,e^{-H_0}, \qquad b_2=d_0-U_0\,e^{-H_0}. \tag{55}$$

Equations (46), (48) to (51) then become:

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right)=b_1-b_2A_1\cot\mu_1\pi X^{\lambda_1}\dots, \tag{56}$$

$$r\cos\phi-\left(\frac{\partial\Phi}{\partial\sigma}\right)_m=\frac{b_2}{2}-\frac{b_1A_1\cot\mu_1\pi}{2}X^{\lambda_1}\dots, \tag{57}$$

$$\Phi_u=(\Phi_u)_0+e^{H_0}X[1+\dots],$$
$$\Phi_l=(\Phi_l)_0+T\bar{U}_0X=(\Phi_l)_0+\bar{U}_0\,e^{H_0-\bar{H}_0}X+\dots,$$

therefore

$$D\Phi=D\Phi_0+e^{H_0}\,d_0X+\dots. \tag{58}$$

Substituting (56) to (58) into the pressure condition (36) we obtain

$$D\Phi_0+d_0\,e^{H_0}X\dots=\frac{b_1b_2}{2}-\frac{(b_1^2+b_2^2)}{2}A_1\cot\mu_1\pi X^{\lambda_1}.$$

Hence,

$$D\Phi_0=\frac{b_1b_2}{2}, \tag{59}$$

$$\left(\frac{b_1^2+b_2^2}{2}\right)A_1\cot\mu_1\pi X^{\lambda_1}=0. \tag{60}$$

For physical reasons $D\Phi_0>0$, $D(\partial\Phi/\partial\sigma)_0<0$ so that $b_1<0$, $b_2<0$. Moreover $(b_1^2+b_2^2)>0$ and hence we may deduce from (60)

$$\cot\mu_1\pi=0, \qquad \mu_1=\lambda_1=\tfrac{1}{2}.$$

Here we are unable to say anything about the value of U_0.
Also,

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right)=b_1+o(\sigma^{\frac{1}{2}})$$

which reduces to the expression for the thick wing if $U_0=0$.

4. Second-Order Solution

We now introduce a second term into the expressions for ψ_u, ψ_l and apply the same analysis as in the previous section.

$$\psi_u=\begin{cases}\Omega_2+A_1X^{\alpha/2}+A_2X^{\lambda_2} & (X>0)\\ -\Omega_1 & (X<0)\end{cases}; \qquad \psi_l=\begin{cases}\Omega_2+B_1\bar{X}^{\frac{1}{2}}+B_2\bar{X}^{\mu_2} & (\bar{X}>0)\\ \Omega_2 & (\bar{X}<0)\end{cases}. \tag{61}$$

Equations (28), (29) become ($X, \bar{X}>0$)

$$\sigma_u = \frac{e^{H_0} X^{\alpha}}{\alpha}\left[1 - \frac{2A_1 t}{3} X^{\alpha/2} \ldots\right], \tag{62}$$

$$\sigma_l = e^{\bar{H}_0} \bar{X}[1 + o(\bar{X}^{\frac{1}{2}})], \tag{63}$$

where $t = \cot \alpha\pi/2$.

Let the second-order expression for $\bar{X}$ be

$$\bar{X} = T X^{\alpha}(1 + S_1 X^{\beta_1}), \tag{64}$$

then (24) and (25) lead to

$$A_2 X^{\lambda_2} = \frac{A_1 S_1}{2} X^{\beta_1 + \alpha/2} + B_2 T^{\mu_2} X^{\alpha\mu_2}, \tag{65}$$

$$S_1 X^{\beta_1} = -\frac{2A_1 t}{3} X^{\alpha/2}. \tag{66}$$

Equation (66) indicates that

$$\beta_1 = \frac{\alpha}{2}, \qquad S_1 = -\frac{2A_1 t}{3}, \tag{67}$$

and so (65) becomes

$$A_2 X^{\lambda_2} = -\frac{A_1^2 t}{3} X^{\alpha} + B_2 T^{\mu_2} X^{\alpha\mu_2}. \tag{68}$$

We obtain expressions for u, $\bar{u}$, cf. (40), (41)

$$u(X) = e^{H_0} X^{-K}\left[\cos\Omega_2 + U_0 e^{-H_0} X^{K} - \frac{A_1 \cos(\Omega_2 - \alpha\pi/2)}{\sin\alpha\pi/2} X^{\alpha/2} - \frac{A_2 \cos(\Omega_2 - \lambda_2\pi)}{\sin\lambda_2\pi} X^{\lambda_2} + \frac{A_1^2 \cos(\Omega_2 - \alpha\pi)}{2\sin^2\alpha\pi/2} X^{\alpha} \ldots\right], \tag{69}$$

$$\bar{u}(\bar{X}) = e^{\bar{H}_0}\left[\bar{U}_0 e^{-\bar{H}_0} - B_1 \sin\Omega_2 \bar{X}^{\frac{1}{2}} + \frac{B_2 \cos(\Omega_2 + \mu_2\pi)}{\sin\mu_2\pi} \bar{X}^{\mu_2} + \left(\bar{U}_1 e^{-\bar{H}_0} - \frac{B_1^2 \cos\Omega_2}{2}\right)\bar{X} \ldots\right]. \tag{70}$$

In the coefficient of $\bar{X}$ in equation (70) the second term is the coefficient of $\bar{X}^{2\mu_1}$ which is not absorbed into the global regular term, for clarity.

Expressions for $\partial\Phi/\partial\sigma$ then become, cf. (42), (43)

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_u = \cos\Omega_2 + U_0 e^{-H_0} X^{K} - A_1 \sin\Omega_2 X^{\alpha/2} - A_2 \sin\Omega_2 X^{\lambda_2} - \frac{A_1^2 \cos\Omega_2}{2} X^{\alpha} + U_0 e^{-\bar{H}_0} A_1 t X^{K+\alpha/2}, \tag{71}$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_l = \bar{U}_0 e^{-\bar{H}_0} - B_1 \sin\Omega_2 \bar{X}^{\frac{1}{2}} + B_2(d_0 \cot\mu_2\pi - \sin\Omega_2)\bar{X}^{\mu_2} + \left[(\bar{U}_1 - \bar{U}_0 \bar{H}_1) e^{-\bar{H}_0} - \frac{B_1^2 \cos\Omega_2}{2}\right]\bar{X} \tag{72}$$

$$= \bar{U}_0 e^{-\bar{H}_0} - A_1 \sin\Omega_2 X^{\alpha/2} + B_2 T^{\mu_2}(d_0 \cot\mu_2\pi - \sin\Omega_2) X^{\alpha\mu_2} + \left[T(\bar{U}_1 - \bar{U}_0 \bar{H}_1) e^{-\bar{H}_0} - \frac{A_1^2 \cos\Omega_2}{2} + \frac{A_1^2 t \sin\Omega_2}{3}\right] X^{\alpha} \tag{73}$$

where we have used expressions (67).

To second order we have

$$r\cos\phi=\cos\Omega_2-A_1\sin\Omega_2X^{\alpha/2}-A_2\sin\Omega_2X^{\lambda_2}+\left(\frac{e^{H_0}}{\alpha}-\frac{A_1^2\cos\Omega_2}{2}\right)X^{\alpha}\dots$$

and so we find that if

$$C_1=T(\bar{U}_1-\bar{U}_0\bar{H}_1)\,e^{-\bar{H}_0}+\frac{A_1^2t\sin\Omega_2}{3}, \tag{74}$$

then

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right)=d_0+U_0\,e^{-H_0}X^K-B_2T^{\mu_2}(d_0\cot\mu_2\pi-\sin\Omega_2)X^{\alpha\mu_2}-A_2\sin\Omega_2X^{\lambda_2}$$
$$-C_1X^{\alpha}+U_0\,e^{-H_0}A_1tX^{\frac{1}{2}(1+K)}, \tag{75}$$

and

$$r\cos\phi-\left(\frac{\partial\Phi}{\partial\sigma}\right)_m=\frac{d_0}{2}-\frac{U_0\,e^{-H_0}X^K}{2}-\frac{B_2T^{\mu_2}}{2}(d_0\cot\mu_2\pi-\sin\Omega_2)X^{\alpha\mu_2}-\frac{A_2\sin\Omega_2}{2}X^{\lambda_2}$$
$$+\left(\frac{e^{H_0}}{\alpha}-\frac{C_1}{2}\right)X^{\alpha}-\frac{U_0\,e^{-H_0}A_1t}{2}X^{\frac{1}{2}(1+K)}. \tag{76}$$

From (50), (75) and (76) to order X^{α}, (36) gives

$$D\Phi_0+\frac{d_0\,e^{H_0}}{\alpha}X^{\alpha}+\dots$$
$$=d_0^2-\frac{(U_0\,e^{-H_0})^2}{2}X^{2K}-d_0A_2\sin\Omega_2X^{\lambda_2}+\left(\frac{e^{H_0}}{\alpha}-C_1\right)d_0X^{\alpha}$$
$$-d_0B_2T^{\mu_2}(d_0\cot\mu_2\pi-\sin\Omega_2)X^{\alpha\mu_2}-(U_0\,e^{-H_0})^2A_1tX^{\frac{1}{2}(1+3K)}+\dots. \tag{77}$$

This equation is discussed in Appendix E and we find that for all thickness angles

$$\lambda_2=\alpha,\qquad \mu_2>1,\qquad A_2=-\frac{A_1^2t}{3}. \tag{78}$$

Putting these results back into the pressure condition we obtain relationships required in Section 5.

(i) $K\neq\frac{1}{3}$

Equations (E-2), (78) give

$$(C_1+A_2\sin\Omega_2)X^{\alpha}=0,$$

i.e. (from (74))

$$T(\bar{U}_1-\bar{U}_0\bar{H}_1)\,e^{-\bar{H}_0}=0. \tag{79}$$

(ii) $K=\frac{1}{3}$

$$T(\bar{U}_1-\bar{U}_0\bar{H}_1)\,e^{-\bar{H}_0}+\frac{U_0^2\,e^{-2H_0}}{2d_0}=0. \tag{80}$$

5. Third-Order Solution

We now introduce a third term in the expression for ψ_u, but no further terms in ψ_l, since μ_2 is still unknown, though of higher order than λ_2.

$$\psi_u = \begin{cases} \Omega_2 + A_1 X^{\alpha/2} - \dfrac{A_1^2 t}{3} X^{\alpha} + A_3 X^{\lambda 3} & (X>0) \\ -\Omega_1 & (X<0) \end{cases} \tag{81}$$

$$\psi_l = \begin{cases} \Omega_2 + B_1 \bar{X}^{\frac{1}{2}} + B_2 \bar{X}^{\mu_2} & (\bar{X}>0) \\ \Omega_2 & (\bar{X}<0) \end{cases} \tag{82}$$

and we obtain, as previously,

$$\sigma_u = \frac{e^{H_0} X^{\alpha}}{\alpha}\left[1 - \frac{2A_1 t}{3} X^{\alpha/2} + \frac{A_1^2(4t^2-1)}{12} X^{\alpha} \dots\right], \tag{83}$$

$$\sigma_l = e^{\bar{H}_0}\bar{X}\left[1 + \frac{\bar{H}_1 \bar{X}}{2} + \dots\right]. \tag{84}$$

We shall later find for this case that the solution is more easily obtained by working in the lower variable $\bar{X}$. We therefore invert (64), using the result found in (67) and including a further term:

$$X^{\alpha} = R\bar{X}\left[1 + \frac{2B_1 t}{3}\bar{X}^{\frac{1}{2}} + S_2 \bar{X}^{\beta_2}\right], \tag{85}$$

where

$$R = T^{-1} = \alpha\, e^{\bar{H}_0 - H_0}. \tag{86}$$

Hence, (24) and (25) give

$$B_2 \bar{X}^{\mu_2} = \frac{B_1 S_2}{2}\bar{X}^{\beta_2 + \frac{1}{2}} + A_3 {R_3}^{\lambda_3/\alpha}\bar{X}^{\lambda_3/\alpha} - \frac{5B_1^3 t^2}{18}\bar{X}^{\frac{3}{2}}, \tag{87}$$

$$S_2 \bar{X}^{\beta_2} = \left(\frac{\bar{H}_1}{2} + \frac{B_1^2(4t^2+1)}{12}\right)\bar{X}. \tag{88}$$

We find, using (79) or (80), (85), (86),

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_l = \bar{U}_0\, e^{-\bar{H}_0} - B_1 \sin\Omega_2 \bar{X}^{\frac{1}{2}} - \frac{B_1^2 \cos\Omega_2}{2}\bar{X} + B_2(d_0 \cot\mu_2\pi - \sin\Omega_2)\bar{X}^{\mu_2} + \frac{B_1^3 \sin\Omega_2}{6}\bar{X}^{\frac{3}{2}} \tag{89}$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_u = \cos\Omega_2 - B_1 \sin\Omega_2 \bar{X}^{\frac{1}{2}} - \frac{B_1^2\cos\Omega_2}{2}\bar{X} + \left[B_1^3 \sin\Omega_2\left(\frac{5t^2+3}{18}\right) - \frac{B_1\, e^{\bar{H}_0}}{3}\cot\left(\frac{5\alpha\pi}{2}\right)\right]\bar{X}^{\frac{3}{2}} + E(\bar{X}), \tag{90}$$

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right) = d_0 + \left[B_1^3 \sin\Omega_2 \frac{5t^2}{18} - \frac{B_1\, e^{\bar{H}_0}}{3}\cot\left(\frac{5\alpha\pi}{2}\right)\right]\bar{X}^{\frac{3}{2}} - B_2(d_0\cot\mu_2\pi - \sin\Omega_2)\bar{X}^{\mu_2} + E(\bar{X}), \tag{91}$$

and

$$r\cos\phi - \left(\frac{\partial\Phi}{\partial\sigma}\right)_m = \frac{d_0}{2} + e^{\bar{H}_0}\bar{X} - \left[B_1^3\sin\Omega_2\frac{5t^2}{36} - \frac{B_1\, e^{\bar{H}_0}}{3}\cot\left(\frac{5\alpha\pi}{2}\right)\right]\bar{X}^{\frac{3}{2}} - \frac{B_2}{2}(d_0\cot\mu_2\pi + \sin\Omega_2)\bar{X}^{\mu_2} - \frac{E(\bar{X})}{2}, \tag{92}$$

where

$$E(\bar{X}) = U_0 \, e^{-H_0} R^{K/\alpha} \bar{X}^{K/\alpha} + U_0 \, e^{-H_0} B_1 t R^{K/\alpha}\left(1+\frac{2K}{3\alpha}\right)\bar{X}^{K/\alpha+\frac{1}{2}}$$
$$-A_3 R^{\lambda_3/\alpha} \sin \Omega_2 \bar{X}^{\lambda_3/\alpha} - \frac{B_1 S_2}{2} \sin \Omega_2 \bar{X}^{\beta_2+\frac{1}{2}}$$
$$+\frac{U_0 A_1^2(2t^2+1)}{6} e^{-H_0} R^{1/\alpha} \bar{X}^{1/\alpha} + (U_1 - U_0 H_1) \, e^{-H_0} R^{(1+K)/\alpha} \bar{X}^{(1+K)/\alpha} \ldots \tag{93}$$

We find that

$$\Phi_l = (\Phi_l)_0 + \bar{U}_0 \bar{X} - \frac{2B_1 \, e^{\bar{H}_0} \sin \Omega_2}{3} \bar{X}^{\frac{3}{2}}$$

$$\Phi_u = (\Phi_u)_0 + \frac{e^{H_0} \cos \Omega_2}{\alpha} X^\alpha + U_0 X - \frac{2A_1 \, e^{H_0}}{3\alpha}(t \cos \Omega_2 + \sin \Omega_2) X^{3\alpha/2}$$
$$= (\Phi_u)_0 + e^{\bar{H}_0} \cos \Omega_2 \bar{X} + U_0 R^{1/\alpha} \bar{X}^{1/\alpha} - \frac{2B_1 \, e^{\bar{H}_0} \sin \Omega_2}{3} \bar{X}^{\frac{3}{2}}.$$

Therefore

$$D\Phi = D\Phi_0 + d_0 \, e^{H_0} \bar{X} + U_0 R^{1/\alpha} \bar{X}^{1/\alpha} + \ldots . \tag{94}$$

Hence to order $\bar{X}^{\frac{3}{2}}$ the pressure condition (36) is

$$D\Phi_0 + d_0 \, e^{\bar{H}_0} \bar{X} + U_0 R^{1/\alpha} \bar{X}^{1/\alpha} \ldots$$
$$= \frac{d_0^2}{2} + d_0 \, e^{\bar{H}_0} \bar{X} - \frac{(U_0 \, e^{-H_0})^2}{2} R^{2K/\alpha} \bar{X}^{2K/\alpha} - d_0^2 B_2 \cot \mu_2 \pi \bar{X}^{\mu_2} + \ldots . \tag{95}$$

If we ignore the constant terms, this leads to,

$$U_0 R^{1/\alpha} \bar{X}^{1/\alpha} + \frac{(U_0 \, e^{-H_0})^2}{2} R^{2K/\alpha} \bar{X}^{2K/\alpha} + d_0^2 B_2 \cot \mu_2 \pi \bar{X}^{\mu_2} = 0. \tag{96}$$

This equation is discussed in Appendix F, where we conclude that $U_0 = 0$ for $K < 3/7$ and that $\mu_2 \geq \frac{3}{2}$ for all K, though it appears likely from the form of the equations that $\mu_2 = \frac{3}{2}$.

Equation (88) shows that $\beta_2 \geq 1$, and using the above result for μ_2 in (87), it can be seen that $\lambda_3 \geq 3\alpha/2$. Unfortunately we are unable to place a greater restriction on the values from the available information and, although it seems likely that $\lambda_3 = 3\alpha/2$ and $\beta_2 = 1$, we can satisfy all the equations without requiring either of these equalities.

6. Extension to Wing Surfaces

(a) *Upper surface*

Let $l = \operatorname{cosec} \alpha\pi/2$, then we have seen previously that

$$\psi_u = -\Omega_1 \tag{20}$$

$$(r \sin \phi)_u = -\sin \Omega_1, \tag{B-14}$$

and extending (32),

$$e^{h}=e^{H_0}|X|^{-K}\left[1-A_1 l|X|^{\alpha/2}+\frac{2A_1^2 l^2}{3}|X|^{\alpha}-H_1|X|-\frac{A_1^3 l^3}{3}|X|^{3\alpha/2}-\frac{A_3}{\sin\lambda_3\pi}|X|^{\lambda_3}\right].$$

Hence,

$$\sigma=\frac{e^{H_0}|X|^{\alpha}}{\alpha}\left[1-\frac{2A_1 l}{3}|X|^{\alpha/2}+\frac{A_1^2 l^2}{3}|X|^{\alpha}-\frac{\alpha H_1}{1+\alpha}|X|-\frac{2A_1^3 l^3}{15}|X|^{3\alpha/2}-\frac{\alpha A_3}{(\alpha+\lambda_3)\sin\lambda_3\pi}|X|^{\lambda_3+\alpha}\right]. \quad (97)$$

We find u_1 on the wing in an analogous manner to its evaluation on the sheet, using Appendix A:

$$u_1=e^{H_0}|X|^{-K}\cos\Omega_1\left[1-A_1 l|X|^{\alpha/2}+\frac{2A_1^2 l^2}{3}|X|^{\alpha}-H_1|X|\right.$$
$$\left.-\frac{A_1^3 l^3}{3}|X|^{3\alpha/2}-\frac{A_3}{\sin\lambda_3\pi}|X|^{\lambda_3}\ldots\right]+\frac{e^{2H_0}A_1}{3\alpha\sin\left(\frac{5\alpha\pi}{2}\right)}|X|^{3\alpha/2-K},$$

for $\Omega\neq\pi/5$.
We find that

$$e^{-h}u_1=\cos\Omega_1+\frac{e^{H_0}A_1}{3\alpha\sin\frac{5\alpha\pi}{2}}|X|^{3\alpha/2}\ldots,$$

and so

$$e^{-h}u=\cos\Omega_1-U_1 e^{-H_0}|X|^{1+K}+\frac{e^{H_0}A_1}{3\alpha\sin\left(\frac{5\alpha\pi}{2}\right)}|X|^{3\alpha/2}\ldots. \quad (98)$$

Also,

$$\Phi_u=(\Phi_u)_0-\int_0^{|X|}u\,d|X|$$
$$=(\Phi_u)_0-\frac{e^{H_0}|X|^{\alpha}}{\alpha}\cos\Omega_2\left[1-\frac{2A_1 l}{3}|X|^{\alpha/2}+\frac{A_1^2 l^2}{3}|X|^{\alpha}-\frac{\alpha H_1}{1+\alpha}|X|\right.$$
$$\left.-\frac{2A_1^3 l^3}{15}|X|^{3\alpha/2}-\frac{\alpha A_3}{(\alpha+\lambda_3)\sin\lambda_3\pi}|X|^{\lambda_3}\right]$$
$$+\frac{U_1}{2}|X|^2-\frac{2A_1 e^{2H_0}}{15\alpha^2\sin\left(\frac{5\alpha\pi}{2}\right)}|X|^{5\alpha/2}\ldots. \quad (99)$$

(b) *Lower surface*
Previously we had

$$\psi_l=\Omega_2,$$

$$(r\sin\phi)_l=\sin\Omega_2,$$

and extending (33) we obtain

$$e^{\bar{h}}=e^{\bar{H}_0}\left[1+B_1|\bar{X}|^{\frac{1}{2}}+\left(\frac{B_1^2}{2}-\bar{H}_1\right)|\bar{X}|+\left(\frac{B_1^3}{6}-B_1\bar{H}_1-B_2\right)|\bar{X}|^{\frac{3}{2}}\ldots\right].$$

Hence, by integration,

$$\sigma = e^{\bar{H}_0}|\bar{X}|\left[1+\frac{2B_1}{3}|\bar{X}|^{\frac{1}{2}}+\left(\frac{B_1^2}{4}-\frac{\bar{H}_1}{2}\right)|\bar{X}|+\frac{2}{5}\left(\frac{B_1^3}{6}-B_1\bar{H}_1-B_2\right)|\bar{X}|^{\frac{3}{2}}\ldots\right]. \tag{100}$$

Then, as for the upper surface,

$$\bar{u} = e^{\bar{H}_0}\left[\bar{U}_0\, e^{-\bar{H}_0}+B_1 \cos\Omega_2|\bar{X}|^{\frac{1}{2}}+\left(\frac{B_1^2\cos\Omega_2}{2}+\bar{U}_1\, e^{-\bar{H}_0}\right)|\bar{X}|+\left(\frac{B_1^3}{6}-B_1\bar{H}_1-B_2\right)\cos\Omega_2|\bar{X}|^{\frac{3}{2}}\ldots\right].$$

Using equation (79) we obtain

$$e^{-\bar{h}}\bar{u} = \bar{U}_0\, e^{-\bar{H}_0}+d_0B_1|\bar{X}|^{\frac{1}{2}}-\frac{d_0B_1^2}{2}|\bar{X}|+d_0\left(\frac{B_1^3}{6}-B_2\right)|\bar{X}|^{\frac{3}{2}}+\ldots, \tag{101}$$

and integrating $\bar{u}$ we find

$$\Phi_l = (\Phi_l)_0-\bar{U}_0|\bar{X}|-\frac{2B_1\, e^{\bar{H}_0}}{3}\cos\Omega_2|\bar{X}|^{\frac{3}{2}}+\frac{\bar{U}_1}{2}|\bar{X}|^2-\frac{2}{5}\left(\frac{B_1^3}{6}-B_1\bar{H}_1-B_2\right)e^{\bar{H}_0}\cos\Omega_2|\bar{X}|^{5/2}\ldots. \tag{102}$$

7. Summary of Results

In previous Sections quantities have been expressed in terms of X, $\bar{X}$, the variables in the transformed plane. They become more indicative of the physical behaviour if expressed in terms of σ, the arc length from the leading edge. We consider only the case $U_0=0$, corresponding to $K<3/7$ (i.e. $\Omega<77°$), and the most likely third-order solution, namely

$$\mu_2=\frac{3}{2}, \qquad \lambda_3=\frac{3\alpha}{2}.$$

Define the constants,

$$C=B_1\, e^{-\bar{H}_0/2}=A_1(\alpha\, e^{-H_0})^{\frac{1}{2}}, \tag{103}$$

$$C^*=\left(\frac{C}{B_1}\right)^3\left[B_2-\frac{B_1\bar{H}_1}{4}\right]=\left(\frac{C}{A_1}\right)^3\left[A_3-A_1^3\left(\frac{t^2}{9}-\frac{1}{24}\right)\right]. \tag{104}$$

Equations (44), (46) and (53) give

$$-\sqrt{2D\Phi_0}=d_0=\cos\Omega_2-\bar{U}_0\, e^{-\bar{H}_0}=(D\Phi_\sigma)_0<0 \tag{105}$$

and we summarise the other results as follows.

7.1. Sheet

Shape

$$\psi=\Omega_2+C\sigma^{\frac{1}{2}}+C^*\sigma^{\frac{3}{2}}+\ldots. \tag{106}$$

Normal velocity

$$\frac{\partial\Phi}{\partial n}=-\sin\Omega_2+C\cos\Omega_2\sigma^{\frac{1}{2}}+\frac{C^2\sin\Omega_2}{2}\sigma-\left[\frac{C}{3}+\left(C^*-\frac{C^3}{6}\right)\cos\Omega_2\right]\sigma^{\frac{3}{2}}+\ldots. \tag{107}$$

Tangential velocities

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_u = \cos\Omega_2 - C\sin\Omega_2\sigma^{\frac{1}{2}} - \frac{C^2\cos\Omega_2\sigma}{2} + \frac{U_1}{\alpha}\left(\frac{C}{A_1}\right)^{4/\alpha}\sigma^{2/\alpha-1}$$

$$-\left[\left(C^* - \frac{C^3}{6}\right)\sin\Omega_2 + \frac{C}{3}\tan\left(\frac{5\Omega}{2}\right)\right]\sigma^{3/2}\ldots, \tag{108}$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_l = (\cos\Omega_2 - d_0) - C\sin\Omega_2\sigma^{\frac{1}{2}} - \frac{C^2\cos\Omega_2\sigma}{2} - \left(C^* - \frac{C^3}{6}\right)\sin\Omega_2\sigma^{3/2}\ldots, \tag{109}$$

$$D\left(\frac{\partial\Phi}{\partial\sigma}\right) = d_0 + \frac{U_1}{\alpha}\left(\frac{C}{A_1}\right)^{4/\alpha}\sigma^{2/\alpha-1} - \frac{C}{3}\tan\left(\frac{5\Omega}{2}\right)\sigma^{3/2}\ldots, \tag{110}$$

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_m = (\cos\Omega_2 - \tfrac{1}{2}d_0) - C\sin\Omega_2\sigma^{\frac{1}{2}} - \frac{C^2\cos\Omega_2\sigma}{2} + \frac{U_1}{2\alpha}\left(\frac{C}{A_1}\right)^{4/\alpha}\sigma^{2/\alpha-1}$$

$$-\left[\left(C^* - \frac{C^3}{6}\right)\sin\Omega_2 + \frac{C}{6}\tan\left(\frac{5\Omega}{2}\right)\right]\sigma^{3/2}\ldots. \tag{111}$$

The potentials may be obtained by integrating (108) and (109). The case $\Omega = \pi/5$ introduces logarithmic terms into the expansions and is not dealt with here.

Note that we have used (79) to express $\bar{U}_1$ in terms of the other constants.

7.2. Wing

We measure σ from the leading edge once again and quote the results for $-(\partial\Phi/\partial\sigma)$, the tangential velocity towards the leading edge:

$$-\left(\frac{\partial\Phi}{\partial\sigma}\right)_u = \cos\Omega_1 - \frac{U_1}{\alpha}\left(\frac{C}{A_1}\right)^{4/\alpha}\sigma^{2/\alpha-1} + \frac{1}{3}C\sec\left(\frac{5\Omega}{2}\right)\sigma^{\frac{3}{2}}\ldots, \tag{112}$$

$$-\left(\frac{\partial\Phi}{\partial\sigma}\right)_l = (\cos\Omega_2 - d_0) + d_0C\sigma^{\frac{1}{2}} - \frac{5}{6}d_0C^2\sigma + \left(\frac{47}{72}d_0C^3 - C^*\right)\sigma^{\frac{3}{2}}\ldots. \tag{113}$$

$(d_0 < 0)$

Normal velocities

$$\left(\frac{\partial\Phi}{\partial n}\right)_u = \sin\Omega_1 \tag{114}$$

$$\left(\frac{\partial\Phi}{\partial n}\right)_l = -\sin\Omega_2. \tag{115}$$

Here, on the lower surface, n is in the direction indicated in Fig. 2, *i.e.* into the wing.

7.3. Concluding Notes

(i) In Section 2, the potential was non-dimensionalised by $\Phi^* = \Phi/kUs$ and the results we quote are in terms of this Φ^*. For the physical velocities and potentials the expressions given are multiplied by $k = \tan\gamma$.

(ii) All the results are true for the special case of the thin wing, with the exception that U_0 is not necessarily zero which means (105) does not hold in this form. For the thin wing the constants have values:

$$\Omega = \Omega_1 = \Omega_2 = K = t = 0; \qquad \alpha = l = 1.$$

(iii) In the results we notice the intrusion of a 'rogue' term in $\sigma^{2/\alpha-1}$. For $K > 1/5$, $2/\alpha - 1 > 3/2$ and this term lies beyond the range of our investigations. For $K \leqslant 1/5$, $1 < 2/\alpha - 1 \leqslant 3/2$ and this term dominates the terms from the third-order expansion. It is possible, as in the case of U_0, that U_1 is zero for this range, but we are unable to confirm this without going to higher-order expansions.

LIST OF SYMBOLS

A	Area of cross-section (Section 2)
A_i, B_i	Coefficients of X^{λ_i}, $\bar{X}^{\mu_i}$ in ψ
c	$\cos \Omega_2$
C	Coefficient of $\sigma^{\frac{1}{2}}$ in ψ Arbitrary constant (Appendix A)
C^*	Coefficient of $\sigma^{\frac{3}{2}}$ in ψ
C_p	Pressure coefficient
d_0	$= \cos \Omega_2 - \bar{U}_0 e^{-\bar{H}_0}$
D	Difference operator across sheet
F	Analytic function of Z^*
g_1, g_2	*See* equation (B-9)
$G_i, \bar{G}_i$	Coefficients in u_1, $\bar{u}_1$ (Appendix C)
$h, \bar{h}$	Real parts of $\ln\left(\frac{d\zeta}{dZ}\right)$, $\ln\left(\frac{d\bar{\zeta}}{d\bar{Z}}\right)$
$h_1, \bar{h}_1$	Singular parts of h, $\bar{h}$
$H_i, \bar{H}_i$	Coefficients of X^i, $\bar{X}^i$ in regular parts of h, $\bar{h}$
k	$= s/x = \tan \gamma$ (Section 2)
K	$= \Omega/\pi$
$\bar{K}$	Arbitrary constant (Appendix A)
l	$= \operatorname{cosec}\left(\frac{\alpha\pi}{2}\right) = \sec \frac{\Omega}{2}$
L, M	Arbitrary coefficients (Appendix A)
n	Normal to sheet or wing
P_i	$= -A_i \cot \lambda_i \pi$
Q_i	$= B_i \cot \mu_i \pi$
$q, \bar{q}$	$= \left(-\frac{\partial \Phi}{\partial Y^*}\right)_u, \left(-\frac{\partial \Phi}{\partial Y^*}\right)_l$
r	*See* Fig. 2
R	Coefficient of $\bar{X}$ in X^{α} (equation (85)) $= \lvert Z \rvert$ (Appendix A)
$\bar{R}$	$= \lvert \bar{Z} \rvert$ (Appendix A)
s	Semi-span (Section 2) $\sin \Omega_2$
S	$S(x, r, \theta) = 0$ is equation of sheet
S_i	Coefficients of X^{β_i} in $\bar{X}$ or $\bar{X}^{\beta_i}$ in X^{α}
t	$= \cot \frac{\alpha\pi}{2} = \tan \frac{\Omega}{2}$
T	Coefficient of X^{α} in $\bar{X}$

LIST OF SYMBOLS (concluded)

$u, \bar{u}$	$=\left(\frac{\partial \Phi}{\partial X^*}\right)_u, \left(\frac{\partial \Phi}{\partial X^*}\right)_l$
U	Free stream velocity
$U_i, \bar{U}_i$	Coefficients of X^i, $\bar{X}^i$ in regular parts of u, $\bar{u}$
$\mathbf{V}$	Velocity vector
W	Complex potential
x, y, z	Cartesian co-ordinates in original frame
X^*, Y^*	Co-ordinates in transformed plane
X, Y	Upper local co-ordinates in transformed plane
$\bar{X}, \bar{Y}$	Lower local co-ordinates in transformed plane
Z^*	$=X^*+iY^*$
$Z, \bar{Z}$	$=X+iY, \bar{X}+i\bar{Y}$

Greek

α'	= angle of incidence
α	$=1-K$
β_i	Powers of X in $\bar{X}$ or of $\bar{X}$ in X^α
γ	Semi-apex angle of wing
ζ	$=y+iz$
θ	*See* Fig. 2; $=\arg(Z)$ (Appendix A)
$\bar{\theta}$	$=\arg(\bar{Z})$ (Appendix A)
λ_i, μ_i	Powers of X, $\bar{X}$ in ψ
ν	Arbitrary power (Appendix A)
σ	Arc length along sheet
Φ	Velocity potential
$\phi, \psi, \chi, \Omega, \Omega_1, \Omega_2$	See Fig. 2

Subscripts

i	Imaginary part
l	Lower surface
m	Mean value
r	Real part
u	Upper surface

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APPENDIX A

Real and Imaginary Parts of Analytic Functions

If, in the transformed plane, $Z^* = X^* + iY^*$, we know the imaginary part F_i of an analytic function, $F(Z^*) = F_r + iF_i$, on the boundary (in Fig. 3c, the real axis) then we are able to calculate the real part, F_r, on the boundary to within an arbitrary constant. The means used is Poisson's integral which involves an integration along the boundary (*see* Moretti[8]).

In the main text we are concerned with two such functions,

$$\ln\left(\frac{d\zeta}{dZ^*}\right) \quad \text{and} \quad \frac{dW}{dZ^*},$$

where

$$\ln\left(\frac{d\zeta}{dZ^*}\right) = h^* + i\psi^*, \qquad \frac{dW}{dZ^*} = u^* + iq^*.$$

In both cases we know the singular behaviour of the imaginary part in the neighbourhood of the leading edge, but the behaviour at more distant points depends upon the global properties of the flow and cannot be ascertained from our local analysis.

The real part can then be calculated in terms of a local contribution, probably singular, and a regular contribution from the distant parts of the integral. The arbitrary constant may, for our purposes, be absorbed into the regular function.

In this Appendix we seek an analytic function whose imaginary part behaves locally on the boundary in the same way as F_i. The real part of this function will then give the local behaviour of F_r.

A.1. Upper Leading Edge

From Fig. 4 we see that $dZ = dZ^*$ and so

$$\ln\left(\frac{d\zeta}{dZ}\right) = h + i\psi; \qquad \frac{dW}{dZ} = u + iq.$$

If $Z = R\,e^{i\theta}$, then on the sheet $R = X$ and $\theta = 0$, while on the wing $R = |X|$ and $\theta = \pi$. We choose a function, $F(Z)$, to try to match the known behaviour of F_i on the boundary.

(i) If $F(Z) = C\,e^{i\alpha} Z^\nu$ then,

$$F_r = C\cos(\nu\theta + \alpha)R^\nu, \qquad F_i = C\sin(\nu\theta + \alpha)R^\nu. \tag{A-1}$$

Taking the case of q in equation (38), we know that

$$F_i = \begin{cases} LX^\nu & \text{on} \quad \theta = 0 \text{ (sheet)}, \\ M|X|^\nu & \text{on } \theta = \pi \text{ (wing)}, \end{cases} \qquad [\nu = -K, \lambda_1 - K, \ldots]$$

giving, from (A-1),

$$L = C\sin\alpha; \qquad M = C\sin(\alpha + \nu\pi).$$

We may now calculate $F_r(=u)$ on the boundary in terms of L and M and find that

$$F_r = \begin{cases} \dfrac{M - L\cos\nu\pi}{\sin\nu\pi} X^\nu & \text{on} \quad \theta = 0, \\[2ex] \dfrac{M\cos\nu\pi - L}{\sin\nu\pi} |X|^\nu & \text{on} \quad \theta = \pi. \end{cases}$$

We obtain the expression for μ_1 by treating each term successively as above—*see* Appendix C.
(ii) If $F(Z) = C\,e^{i\alpha}Z^\nu - K\log Z + i\Omega_2$ then

$$F_r = C\cos(\alpha+\nu\theta)R^\nu - K\log R,$$

$$F_i = C\sin(\alpha+\nu\theta)R^\nu - K\theta + \Omega_2.$$

Taking the case of ψ in equations (19), (20), we know that

$$F_i = \begin{cases} \Omega_2 + A_1 X^{\lambda_1} & (\theta = 0), \\ -\Omega_1 & (\theta = \pi). \end{cases}$$

If $\nu = \lambda_1$, $A_1 = C\sin\alpha$, $\alpha = -\nu\pi$, $K = (\Omega_1+\Omega_2)/\pi$, then F_i has the required behaviour, giving

$$F_r = \begin{cases} -K\log X - A_1\cot\lambda_1\pi X^{\lambda_1} & (\theta = 0), \\ -K\log|X| - A_1\operatorname{cosec}\lambda_1\pi|X|^{\lambda_1} & (\theta = \pi). \end{cases} \tag{A-2}$$

This deals with the discontinuity in ψ and the first term in the expansion. In Sections 4 and 5 the further terms in the expansion are treated as in (i) above.

A.2. Lower Leading Edge

From Fig. 4 we see that $d\bar{Z} = -dZ^*$ and so

$$\ln\left(\frac{d\zeta}{d\bar{Z}}\right) = \bar{h} + i(\psi_l - \pi); \qquad -\frac{dW}{d\bar{Z}} = \bar{u} + i\bar{q}. \tag{A-3}$$

If $\bar{Z} = \bar{R}\,e^{i\bar{\theta}}$, then on the sheet $\bar{R} = \bar{X}$ and $\bar{\theta} = 0$ while on the wing $\bar{R} = |\bar{X}|$ and $\bar{\theta} = -\pi$. We proceed as in Section A.1.

If $F(\bar{Z}) = C\,e^{i\alpha}\bar{Z}^\nu + i\bar{K}$ then

$$F_r = C\cos(\nu\theta+\alpha)R^\nu, \qquad F_i = C\sin(\nu\theta+\alpha)R^\nu + \bar{K}. \tag{A-4}$$

Taking the case of $\bar{q}$ in equation (39),

$$F_i = \begin{cases} L\bar{X}^\nu + \bar{K} & \text{on } \theta = 0 \quad \text{(sheet)} \\ M|\bar{X}|^\nu + \bar{K} & \text{on } \theta = -\pi \quad \text{(wing)} \end{cases} \quad [\nu = \mu_1, \ldots]$$

giving, from (A-4),

$$L = C\sin\alpha, \qquad M = C\sin(\alpha - \nu\pi)$$

leading to, as in Section A.1,

$$F_r = \begin{cases} \dfrac{L\cos\nu\pi - M}{\sin\nu\pi}\bar{X}^\nu & \text{on } \theta = 0 \quad \text{(sheet)} \\ \dfrac{L - M\cos\nu\pi}{\sin\nu\pi}|\bar{X}|^\nu & \text{on } \theta = -\pi \quad \text{(wing)} \end{cases}$$

In the case of ψ_l, from equations (19), (20) and (A-3) we see that $\bar{K} = \Omega_2 - \pi$, $M = 0$, $L = B_1$, $\nu = \mu_1$ and so

$$\bar{h}_1 = \begin{cases} B_1\cot\mu_1\pi\bar{X}^{\mu_1} & \text{on } \theta = 0 \quad \text{(sheet)} \\ B_1\operatorname{cosec}\mu_1\pi|\bar{X}|^{\mu_1} & \text{on } \theta = -\pi \quad \text{(wing)} \end{cases}. \tag{A-5}$$

APPENDIX B

Series Expansions for the Geometrical Expressions

In Section 3 we require expressions for $r \cos \phi$ and $r \sin \phi$ in terms of $X, \bar{X}$. The derivations involve simple but tedious manipulation which we have included in this Appendix for convenience. This work is for the surface of the sheet, i.e. $X, \bar{X} > 0$. Note that $\lambda_1 < \alpha$.

Let

$$\chi = \psi - \Omega_2, \tag{B-1}$$

then from equations (19),

$$(\cos \chi)_u = 1 - \dots, \qquad (\sin \chi)_u = A_1 X^{\lambda_1} - \dots \ , \tag{B-2}$$

and thus using (26),

$$(\mathrm{e}^h \cos \chi)_u = \mathrm{e}^{H_0} X^{-K}[1 + \dots], \tag{B-3}$$

$$(\mathrm{e}^h \sin \chi)_u = \mathrm{e}^{H_0} X^{-K}[A_1 X^{\lambda_1} + \dots]. \tag{B-4}$$

Let

$$\cos \Omega_2 = c, \qquad \sin \Omega_2 = s,$$

then (B-1) gives,

$$\cos \psi = c \cos \chi - s \sin \chi, \qquad \sin \psi = s \cos \chi + c \sin \chi, \tag{B-5}$$

and hence, using equations (23), (B-3) and (B-4) we obtain,

$$\frac{dy_u}{dX} = \mathrm{e}^{H_0} X^{-K}[c + \dots], \qquad \frac{dz_u}{dX} = \mathrm{e}^{H_0} X^{-K}[s + \dots].$$

Integrated these become,

$$y_u = 1 + \frac{\mathrm{e}^{H_0} X^{\alpha}}{\alpha}[c + \dots], \qquad z_u = \frac{\mathrm{e}^{H_0} X^{\alpha}}{\alpha}[s + \dots], \tag{B-6}$$

since $y_u = 1$, $z_u = 0$ when $X = 0$.

From Fig. 2 we see that

$$\phi = \psi - \theta, \qquad r \cos \theta = y; \qquad r \sin \theta = z.$$

Thus,

$$r \cos \phi = r \cos (\psi - \theta) = y \cos \psi + z \sin \psi, \tag{B-7}$$

$$r \sin \phi = y \sin \psi - z \cos \psi. \tag{B-8}$$

Let

$$g_1 = y \cos \chi + z \sin \chi, \qquad g_2 = y \sin \chi - z \cos \chi, \tag{B-9}$$

then it can be seen from (B-1), (B-7) and (B-8) that

$$r \cos \phi = c g_1 - s g_2, \qquad r \sin \phi = s g_1 + c g_2.$$

Equations (B-2), (B-6) and (B-9) give

$$(g_1)_u = 1 - \ldots; \qquad (g_2)_u = A_1 X^{\lambda_1} \ldots .$$

Thus,

$$(r \cos \phi)_u = c - sA_1 X^{\lambda_1} \ldots \tag{B-10}$$

$$(r \sin \phi)_u = s + cA_1 X^{\lambda_1} \ldots . \tag{B-11}$$

Similar analysis for the lower variables gives

$$(r \cos \phi)_l = c - sB_1 \bar{X}^{\mu_1} \ldots , \tag{B-12}$$

$$(r \sin \phi)_l = s + cB_1 \bar{X}^{\mu_1} \ldots . \tag{B-13}$$

If we consider the wing surface, a similar analysis would give $r \sin \phi$. However, simple geometry gives, since the span is unity,

$$(r \sin \phi)_u = -\sin \Omega_1; \qquad (r \sin \phi)_l = \sin \Omega_2. \tag{B-14}$$

APPENDIX C

Calculation of u, $\bar{u}$

From equations (38), (39) we evaluate u_1, $\bar{u}_1$ using Appendix A Sections A.1 and A.2 so that u_1, $\bar{u}_1$ are of the form

$$u_1(X) = e^{H_0} X^{-K}[G_0 + G_1 X^{\lambda_1} + \ldots]$$

$$\bar{u}_1(\bar{X}) = e^{\bar{H}_0}[\bar{G}_0 + \bar{G}_1 \bar{X}^{\mu_1} + \ldots].$$

In fact the expression $e^{\bar{H}_0}\bar{G}_0$ will be absorbed into the regular part of $\bar{u}$ and may be omitted from $\bar{u}_1$.

C.1. Upper Surface (on the sheet)

Using the notation of Appendix A,

$$G_i = \frac{M_i - L_i \cos \nu_i \pi}{\sin \nu_i \pi}.$$

Note that $\Omega_1 = \Omega - \Omega_2$, $\Omega = K\pi$.

(i) Taking $\nu_0 = -K$

$$G_0 = \frac{-\sin \Omega_1 - \sin \Omega_2 \cos K\pi}{-\sin K\pi} = \frac{-\sin \Omega \cos \Omega_2}{-\sin \Omega} = \cos \Omega_2.$$

(ii) Taking $\nu_1 = \lambda_1 - K$

$$G_1 = \frac{A_1}{\sin \lambda_1 \pi} \frac{\sin \Omega_1 + \sin(\Omega_2 - \lambda_1 \pi)\cos(\Omega - \lambda_1 \pi)}{-\sin(\Omega - \lambda_1 \pi)} = -\frac{A_1 \cos(\Omega_2 - \lambda_1 \pi)}{\sin \lambda_1 \pi}.$$

C.2. Lower Surface (on the sheet)

$$\bar{G}_i = \frac{\bar{L}_i \cos \bar{\nu}_i \pi - \bar{M}_i}{\sin \bar{\nu}_i \pi}.$$

Taking $\bar{\nu}_i = \mu_1$

$$\bar{G}_1 = \frac{B_1}{\sin \mu_1 \pi} \frac{\sin(\Omega_2 + \mu_1 \pi)\cos \mu_1 \pi - \sin \Omega_2}{\sin \mu_1 \pi} = \frac{B_1 \cos(\Omega_2 + \mu_1 \pi)}{\sin \mu_1 \pi}.$$

APPENDIX D

The Several Possibilities of Equation (54)

Equation (54) is

$$A_1 d_0^2 \cot \mu_1 \pi X^{\lambda_1} + \frac{(U_0\,e^{-H_0})^2}{2} X^{2K} = 0$$

and the possibilities arise from the relative values of the powers λ_1 and $2K$, where $K = \Omega/\pi$.

(a) $K \geq \frac{1}{3} (\alpha = 1 - K \leq \frac{2}{3})$

We have assumed that $\mu_1 < 1$ and so, by (31), $\lambda_1 < \alpha \leq 2K$. Thus (54) reduces to

$$A_1 d_0^2 \cot \mu_1 \pi X^{\lambda_1} = 0. \quad \text{(D-1)}$$

Since $D\Phi_0 \neq 0$, equation (53) shows $d_0 \neq 0$ and (D-1) leads to the conclusion that $\cot \mu_1 \pi = 0$, i.e. $\mu_1 = \frac{1}{2}$ and $\lambda_1 = \alpha/2$.

(b) $K < \frac{1}{3} (\alpha > \frac{2}{3})$

(i) $\lambda_1 < 2K$

Equation (54) then gives immediately $\mu_1 = \frac{1}{2}$. This leads to the further conclusion that $U_0 = 0$, since for $K < \frac{1}{3}$, $2K < \alpha$.

(ii) $\lambda_1 > 2K$

We again obtain $U_0 = 0$, $\mu_1 = \frac{1}{2}$.

(iii) $\lambda_1 = 2K$

$$\mu_1 = \frac{\lambda_1}{\alpha} = \frac{2K}{1-K}$$

$$\cot \mu_1 \pi \geq 0 \quad \text{for} \quad \mu_1 \leq \tfrac{1}{2},$$

(i.e. for $K \leq \frac{1}{5}$).

Equation (54) gives

$$A_1 d_0^2 \cot \mu_1 \pi + \frac{(U_0\,e^{-H_0})^2}{2} = 0. \quad \text{(D-2)}$$

Thus for $K \leq \frac{1}{5}$, since $A_1 > 0$, (D-2) contains two positive terms which must separately be zero, giving $U_0 = 0$ and $\mu_1 = \frac{1}{2}$.

There remains a possibility that $\lambda_1 = 2K$, $\frac{1}{5} \leq K < \frac{1}{3}$. However, this does give rise to a discontinuity in λ_1 at $K = \frac{1}{3}$ (*see* case (a)).

Therefore we conclude that $\lambda_1 = \alpha/2$ and $\mu_1 = \frac{1}{2}$ throughout and that $U_0 = 0$ for $K < \frac{1}{3}$. Later it is shown (Appendix F) that $U_0 = 0$ for a wider range of K.

APPENDIX E

The Several Possibilities of Equation (77)

Ignoring the constant terms, equation (77) becomes

$$\frac{(U_0\,e^{-H_0})^2}{2}X^{2K}+d_0A_2\sin\Omega_2X^{\lambda_2}+d_0C_1X^{\alpha}+d_0B_2T^{\mu_2}(d_0\cot\mu_2\pi-\sin\Omega_2)X^{\alpha\mu_2}+(U_0\,e^{-H_0})^2A_1tX^{\frac{1}{2}(1+3K)}=0. \tag{E-1}$$

(a) $K<\frac{1}{3}$

In Section 3 we saw that from (54), $U_0=0$ for $K<\frac{1}{3}$. Hence (E-1) becomes, since $d_0\neq 0$,

$$A_2\sin\Omega_2X^{\lambda_2}+C_1X^{\alpha}+B_2T^{\mu_2}(\cot\mu_2\pi-\sin\Omega_2)X^{\alpha\mu_2}=0. \tag{E-2}$$

If we assume that $\mu_2\neq 1$ (this can be justified by a lengthy argument) then (E-2) allows two possibilities:

(i) $\lambda_2=\alpha\mu_2<\alpha$, (i.e. $\mu_2<1$) and so from (68), $A_2=B_2T^{\mu_2}$,

(ii) $\lambda_2=\alpha<\alpha\mu_2$, (i.e. $\mu_2>1$) and so from (68), $A_2=-\dfrac{A_1^2\cot\alpha\pi/2}{3}$.

Putting (i) into equation (E-2) we obtain

$$B_2T^{\mu_2}d_0\cot\mu_2\pi X^{\alpha\mu_2}=0,$$

which is impossible with $B_2\neq 0$ and $\frac{1}{2}=\mu_1<\mu_2<1$. Hence we are left with condition (ii) and

$$\lambda_2=\alpha,\qquad \mu_2>1,\qquad A_2=-\frac{A_1^2t}{3}. \tag{E-3}$$

(b) $K>\frac{1}{3}$ $(\alpha<\frac{2}{3})$

All terms involving U_0 are of higher order than those considered in (E-2) and the results of (a) still hold.

(c) $K=\frac{1}{3}$ $(\alpha=\frac{2}{3}=2K)$

(E-1) now becomes

$$d_0A_2\sin\Omega_2X^{\lambda_2}+\left[\frac{(U_0\,e^{-H_0})^2}{2}+d_0C_1\right]X^{\alpha}+d_0B_2T^{\mu_2}(d_0\cot\mu_2\pi-\sin\Omega_2)X^{\alpha\mu_2}=0.$$

The argument in Section (a) is unaltered and (E-3) is again valid.
Thus we see that (E-3) holds for all values of K (i.e. all thickness angles).

APPENDIX F

The Several Possibilities of Equation (96)

Equation (96) is

$$U_0 R^{1/\alpha} \bar{X}^{1/\alpha} + \frac{(U_0 \, e^{-H_0})^2}{2} R^{2K/\alpha} \bar{X}^{2K/\alpha} + d_0^2 B_2 \cot \mu_2 \pi \bar{X}^{\mu_2} = 0,$$

and again we look at the different ranges of K.

(a) $K < \frac{1}{3}$

Here $U_0 = 0$ (*see* Appendix D) and (96) becomes (to order $\bar{X}^{\frac{3}{2}}$)

$$d_0^2 B_2 \cot \mu_2 \pi \bar{X}^{\mu_2} = 0.$$

Thus since $d_0 \neq 0$ we see that $\mu_2 \geq \frac{3}{2}$.

(b) $\frac{1}{3} < K < \frac{3}{7}$ $(1/\alpha > \frac{3}{2}, 1 < 2K/\alpha < \frac{3}{2})$

Equation (96) now gives

$$\frac{(U_0 \, e^{-H_0})^2}{2} R^{2K/\alpha} \bar{X}^{2K/\alpha} + d_0^2 B_2 \cot \mu_2 \pi \bar{X}^{\mu_2} = 0.$$

There are three possibilities:

(i) $\mu_2 < 2K/\alpha < \frac{3}{2}$, which leads to $\cot \mu_2 \pi = 0$ with $\frac{1}{2} = \mu_1 < \mu_2 < \frac{3}{2}$, giving a contradiction.

(ii) $\mu_2 = 2K/\alpha < \frac{3}{2}$, which means that $\mu_2 \to 1$ as $K \to \frac{1}{3}+, \ldots$. Since we wish to avoid sharp changes in μ_2 for increasing K, having regard to case (a) above we rule this possibility out.

(iii) $2K/\alpha < \mu_2$, which leads to $U_0 = 0$ which subsequently gives (96) the same form as in case (a).

Thus for $\frac{1}{3} < K < \frac{3}{7}$ we have, as in case (a), $\mu_2 \geq \frac{3}{2}$ and $U_0 = 0$.

(c) $K = \frac{1}{3}$ $(2K/\alpha = 1, 1/\alpha = \frac{3}{2})$

In this case the terms in $\bar{X}^{2K/\alpha}$ will have been previously absorbed into those of order $\bar{X}$ by the use of equation (80) in place of (79) after equation (88). Thus (96) becomes

$$d_0^2 B_2 \cot \mu_2 \pi \bar{X}^{\mu_2} + 0(\bar{X}^{\frac{3}{2}}) = 0,$$

and as in case (a) $\mu_2 \geq \frac{3}{2}$.

(d) $K \geq \frac{3}{7}$ $(2K/\alpha \geq \frac{3}{2}, 1/\alpha \geq \frac{7}{4})$

Equation (96) again assumes the form of case (a) and we can conclude that $\mu \geq \frac{3}{2}$.

Therefore we may conclude that $U_0 = 0$ for $K < \frac{3}{7}$ and $\mu_2 \geq \frac{3}{2}$ for all K.

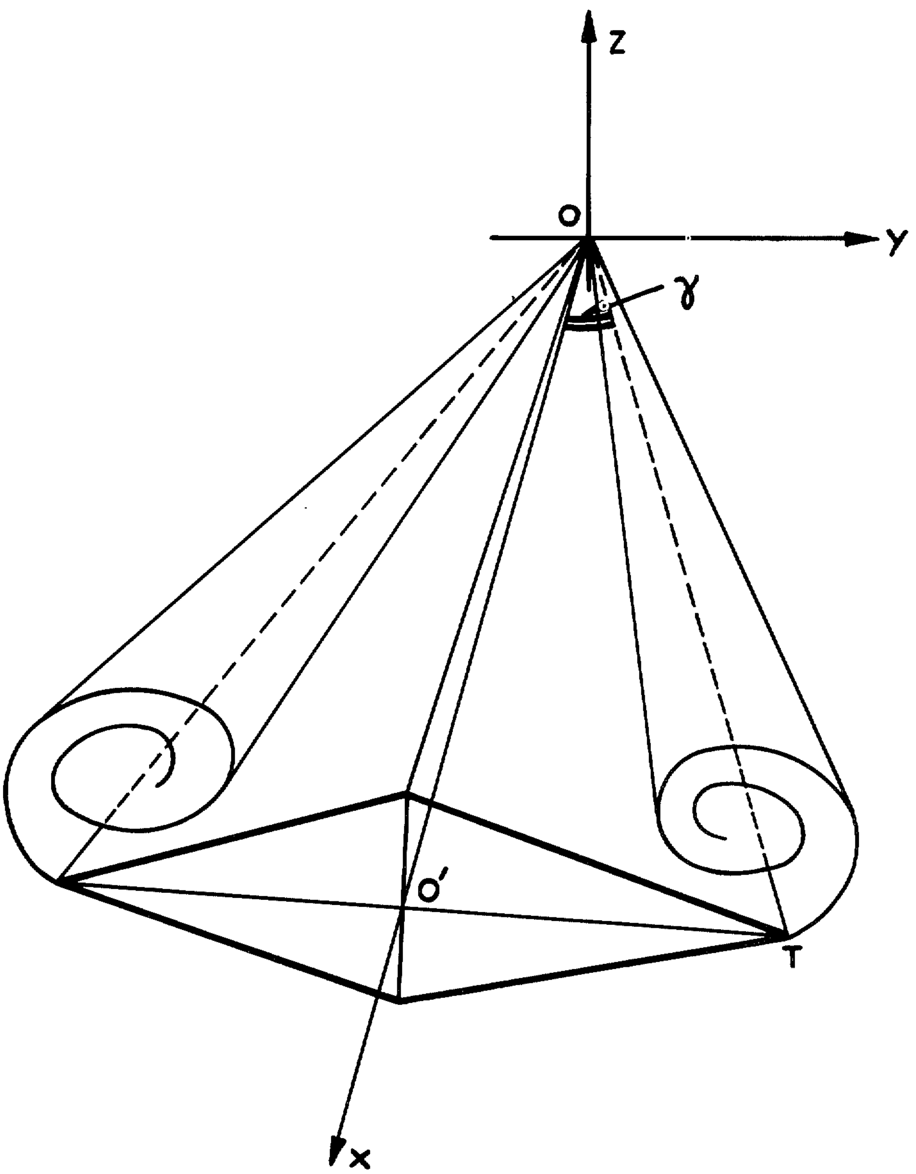

FIG. 1. Wing and coordinate system.

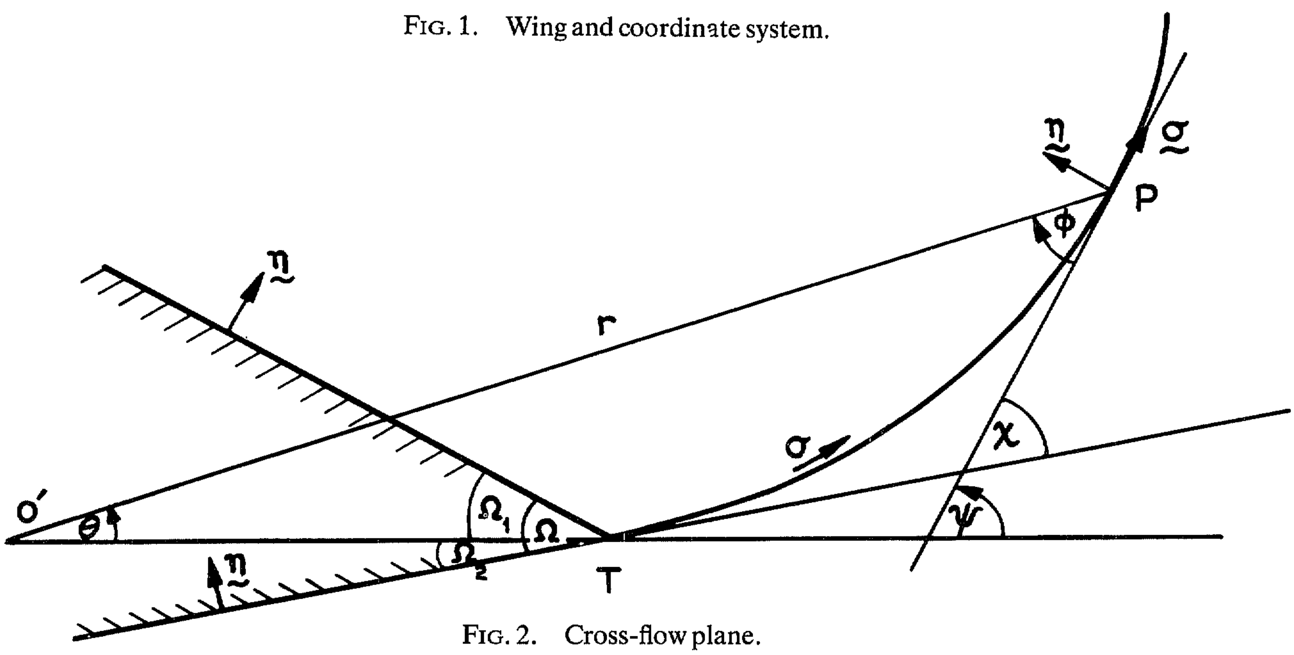

FIG. 2. Cross-flow plane.

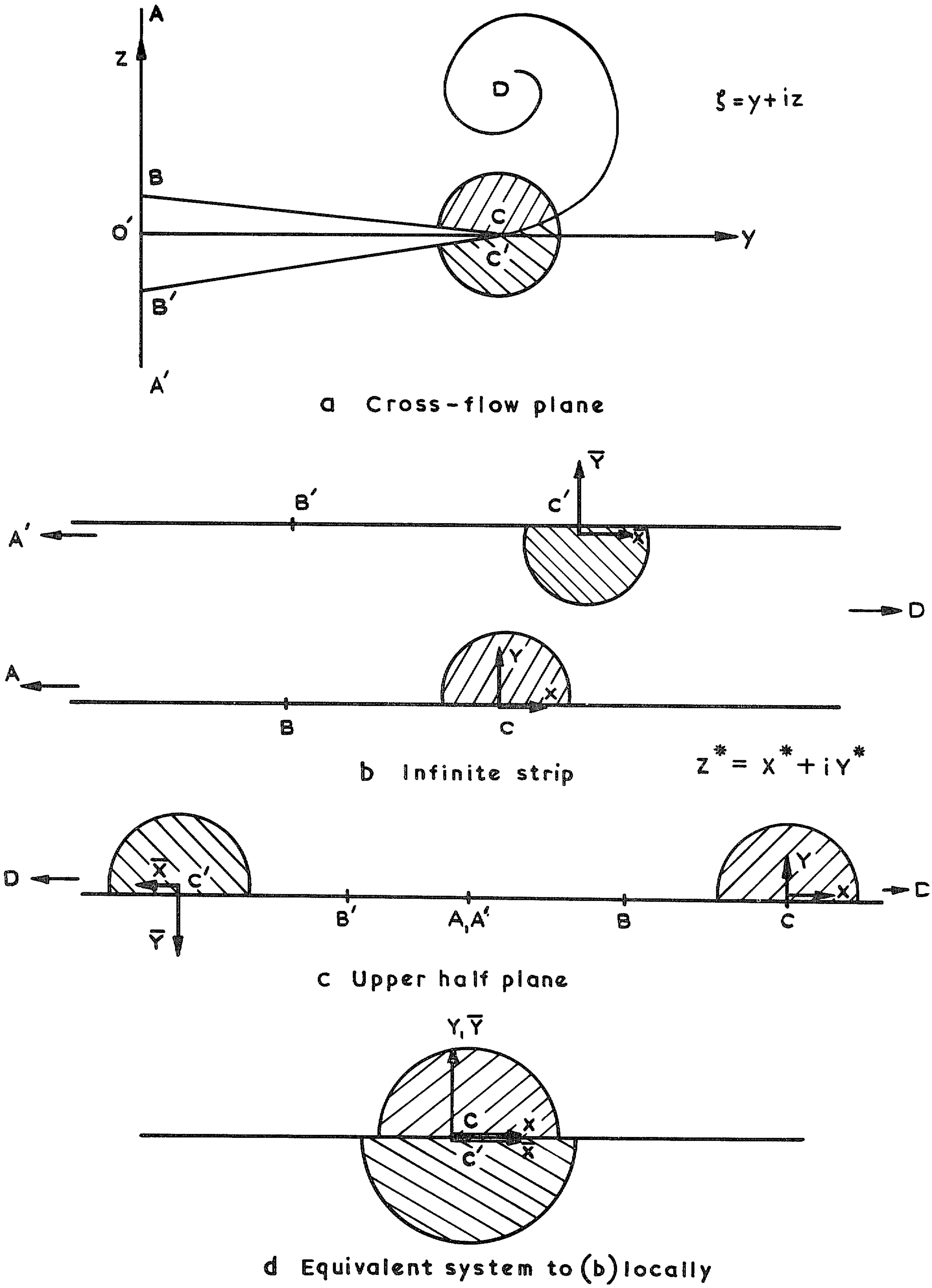

FIG. 3a–d. Several representations of the region of the starboard leading edge.

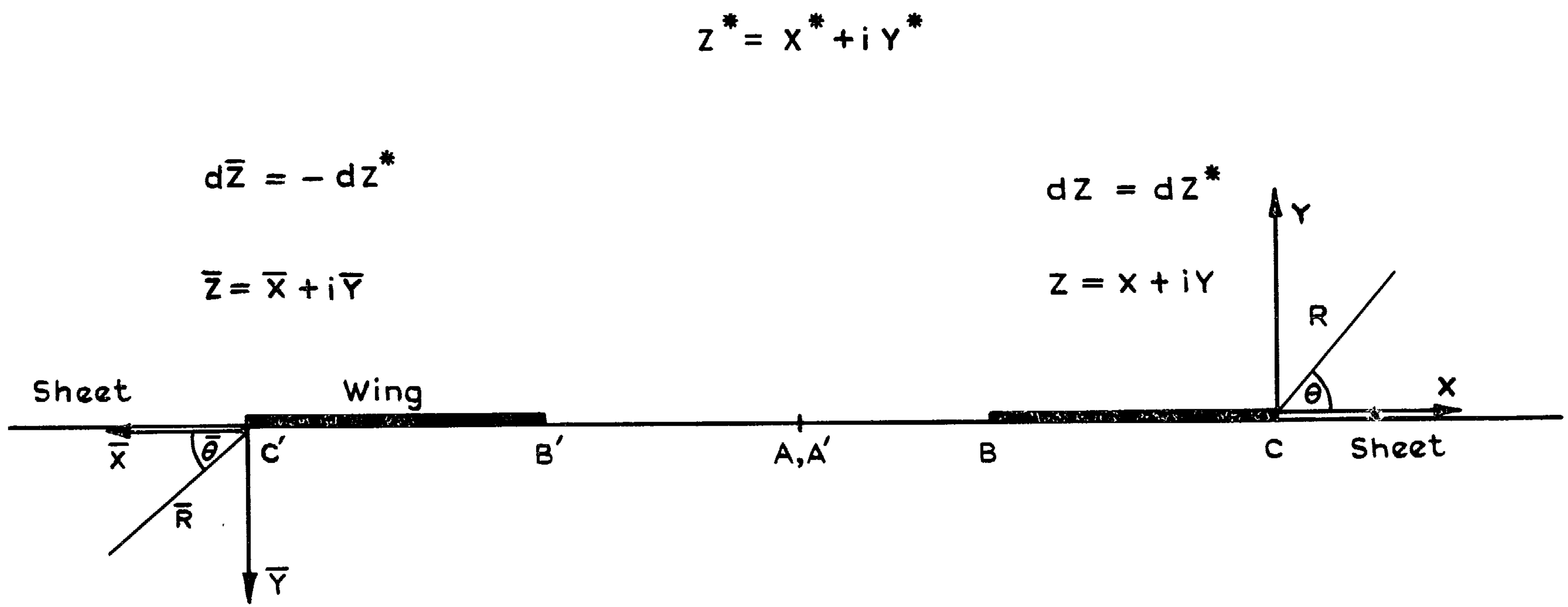

FIG. 4. Enlargement of Fig. 3c.

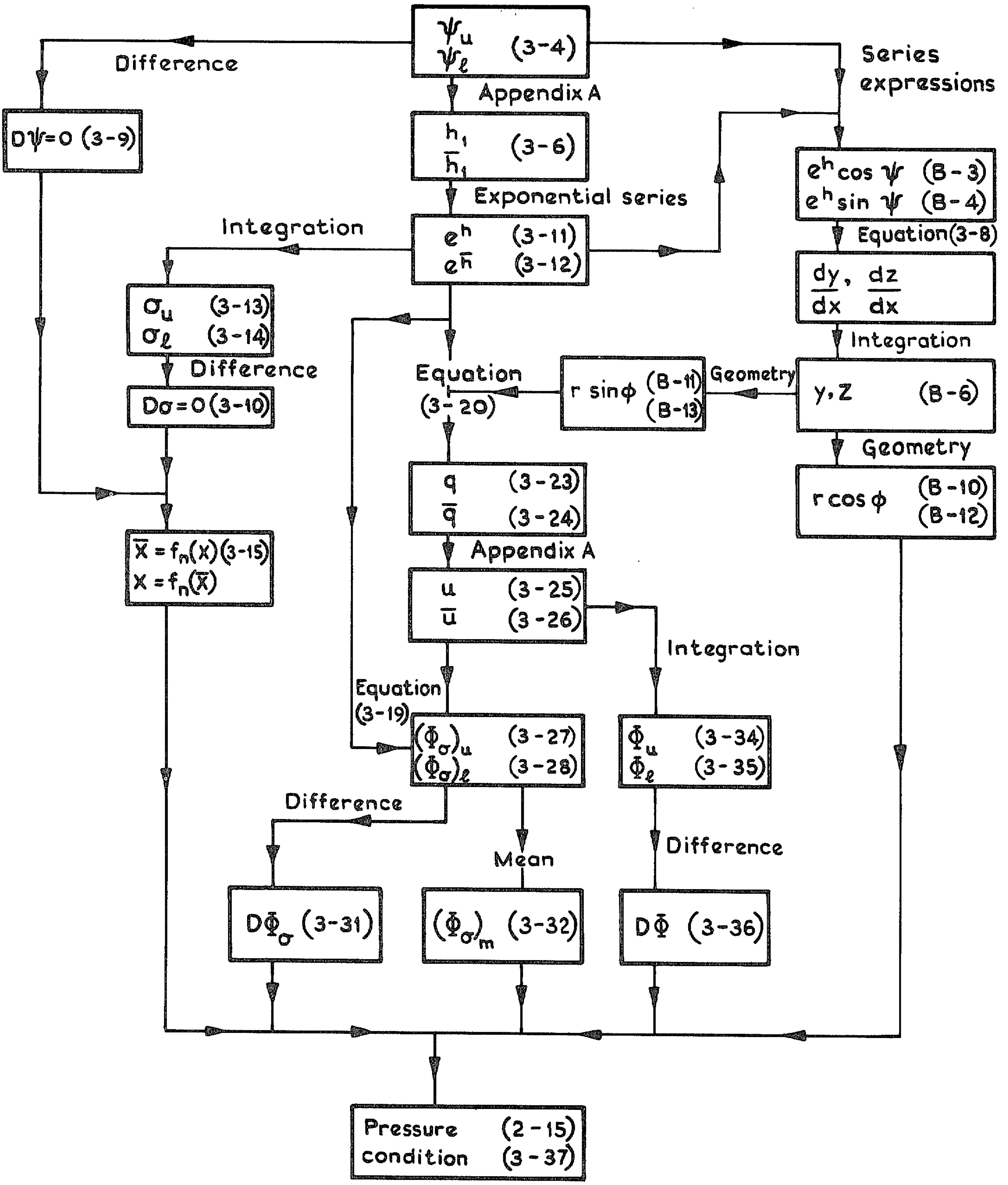

FIG. 5. Flow diagram for first-order solution.

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