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An Introduction to Helicopter Air Resonance

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An Introduction to Helicopter Air Resonance

By A. R. S. BRAMWELL

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Summary

The phenomenon known as 'air resonance', peculiar to helicopters with hingeless rotors, has been analysed and presented in as simple a manner as possible.

In the air resonance case the 'body mode' with frequency close to the whirling frequency of the rotor C.G. is much more heavily damped than in ground resonance and the damping of the modes of motion is apparently unaffected by coincidence of the body and whirl frequences.

The slower of the two C.G. whirl modes is unstable if only the aerodynamic lag damping is present and it appears that at least 4 per cent of critical damping is required for all the modes to be stable.

A resonant situation may occur if the lag stiffness is so high that the blade lag frequency coincides with the flapping frequency. With low damping large lag amplitudes may be excited by Coriolis forces when blade flapping occurs.

* Replaces A.R.C. 33 886.

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Detachable Abstract Cards

1. Introduction

In recent years several experimental and production helicopters have appeared employing hingeless rotor blades. Flight experience and theoretical investigations show that these helicopters may be subject to a phenomenon known as 'air resonance' in which instability or large lag plane amplitudes may occur due to coupling of blade lagging and flapping with the modes of body motion.

Some papers, e.g. Refs. 1 and 2, have given a qualitative explanation of air resonance, whilst Ref. 6, although giving the outline of a theoretical method, explains the phenomenon mainly in terms of the numerical solutions of certain flight cases. Reference 7, which is not easily available, gives a detailed treatment of the problem but most of the results have been obtained by the drastic simplification of assuming that the pitching and rolling moments of inertia are the same, whereas they are usually in the ratio of about five to one.

The object of this report is to give a simple theoretical treatment of the phenomenon regarding it as a logical extension of the more familiar classical helicopter stability problem. In particular, the so-called 'pendulum mode' is discussed in some detail. The analysis throughout has been kept as simple as possible in order not to lose sight of the physical nature of the problem.

It is emphasised that in keeping the treatment simple not all of the parameters which could effect the motion have been included or fully investigated, but it is hoped that they are either relatively unimportant or, for practical reasons, are capable of only small variations.

2. The Gyroscopic Modes

Before obtaining the complete set of equations, it will be interesting to examine the motion of the helicopter when unsteady flapping motion is taken into account.

For the moment we consider only blade flapping motion and associated fuselage pitching and rolling. Blade flapping in this report is defined as blade motion in a plane containing the blade in question and the rotor-hub axis.

If $z = RS_1(x)P_1(\psi)$ is the displacement of a point of the blade in its first flapping mode, the azimuth coordinate is given by, Ref. 4,

$$\frac{d^2P_1}{d\psi^2}+\lambda_1^2P_1(\psi)=\frac{1}{\Omega^2R\int_0^1 mS_1^2(x)\,dx}\int_0^1\frac{\partial F}{\partial x}S_1(x)\,dx. \tag{1}$$

Considering only pitching and rolling in hovering flight, and in the notation of Ref. 3, the blade loading distribution $\partial F/\partial x$ is given by

$$\frac{\partial F}{\partial x}=\frac{1}{2}\rho ac\Omega^2R^3\left[-xS_1\frac{dP_1}{d\psi}+x\hat{p}\sin\psi+x\hat{q}\cos\psi\right]+$$
$$+2m\Omega^2R^2(x\hat{p}\cos\psi-x\hat{q}\sin\psi)+m\Omega^2R^2\left(x\frac{d\hat{p}}{d\psi}\sin\psi+x\frac{d\hat{q}}{d\psi}\cos\psi\right). \tag{2}$$

In calculating $\partial F/\partial x$ the steady blade incidence due to collective pitch and induced velocity has been omitted since it makes no contribution to the disturbed motion. We then have, again in the notation of Ref. 3,

$$\frac{d^2P_1}{d\psi^2}+\frac{\gamma E_1}{2}\frac{dP_1}{d\psi}+\lambda_1^2P_1=\frac{\gamma}{2}[F_1\hat{p}\sin\psi+F_1\hat{q}\cos\psi]+$$
$$+\frac{\gamma}{\gamma_1}\left[2\hat{p}\cos\psi-2\hat{q}\sin\psi+\frac{d\hat{p}}{d\psi}\sin\psi+\frac{d\hat{q}}{d\psi}\cos\psi\right]. \tag{3}$$

Now, in disturbed motion, we cannot assume steady blade flapping and the motion of each blade must, at first, be considered separately. Since only first mode flapping is being considered, the suffix 1 can be omitted from the terms in equation (3) and we can then write, for the kth blade,

$$P_k'' + \frac{\gamma E}{2} P_k' + \lambda_1^2 P_k = \frac{\gamma}{2}[F\hat{p} \sin \psi_k + F\hat{q} \cos \psi_k] +$$
$$+ \bar{F}\left[2\hat{p} \cos \psi_k - 2\hat{q} \sin \psi_k + \frac{d\hat{p}}{d\psi} \sin \psi_k + \frac{d\hat{q}}{d\psi} \cos \psi_k\right] \tag{4}$$

where P_k' denotes $dP_k/d\psi_k$, etc. and $\bar{F} = \gamma/\gamma_1$.

Let us define

$$\bar{a}_1 = -\frac{2}{b} \sum_{k=1}^{b} P_k \cos \psi_k; \qquad \bar{b}_1 = -\frac{2}{b} \sum_{k=1}^{b} P_k \sin \psi_k \tag{5}$$

where ψ_k takes the values $\psi + 2\pi(k-1)/b$, $k = 1, 2 \ldots b$.

The relationships, equations (5), correspond to the classical Coleman transformation which convert the rotating blade modes to fixed body modes. It follows from (5) that

$$\sum_{k=1}^{b} P_k' \sin \psi_k = \frac{b}{2}(\bar{a}_1 - \bar{b}_1'); \qquad \sum_{k=1}^{b} P_k' \cos \psi_k = -\frac{b}{2}(\bar{a}_1' + \bar{b}_1);$$

$$\sum_{k=1}^{b} P_k'' \sin \psi_k = -\frac{b}{2}(\bar{b}_1'' - 2\bar{a}_1' - \bar{b}_1); \qquad \sum_{k=1}^{b} P_k'' \cos \psi_k = -\frac{b}{2}(\bar{a}_1' + 2\bar{b}_1' - \bar{a}_1),$$

and it can easily be shown that

$$\sum_{k=1}^{b} \sin^2 \psi_k = \sum_{k=1}^{b} \cos^2 \psi_k = \frac{b}{2}$$

and

$$\sum_{k=1}^{b} \sin \psi_k \cos \psi_k = 0.$$

Then multiplying equation (4) by $\cos \psi_k$ and $\sin \psi_k$ summing over the b blades of the rotor and using the above relations, gives

$$\bar{a}_1'' + \nu\bar{a}_1' + (\lambda_1^2 - 1)\bar{a}_1 + 2\bar{b}_1' + \nu\bar{b}_1 = -\frac{\gamma F}{2}\hat{q} - \bar{F}\left(2\hat{p} + \frac{d\hat{q}}{d\psi}\right) \tag{6}$$

and

$$\bar{b}_1'' + \nu\bar{b}_1' + (\lambda_1^2 - 1)\bar{b}_1 - 2\bar{a}_1' - \nu\bar{a}_1 = -\frac{\gamma F}{2}\hat{p} + \bar{F}\left(2\hat{q} - \frac{d\hat{p}}{d\psi}\right) \tag{7}$$

where $\nu = (\gamma/2)E_1$.

It is shown in Ref. 4, that the moment exerted on the helicopter by the kth blade is

$$M_k = \Omega^2 R^3 (\lambda_1^2 - 1) P_k \int_0^1 m x S_1(x)\, dx, \tag{8}$$

and the rolling and pitching moments are, respectively, $-M_k \sin \psi_k$ and $-M_k \cos \psi_k$. The total rolling and pitching moments are, therefore,

$$L = -\sum_{k=1}^{b} M_k \sin \psi_k = \tfrac{1}{2} b \Omega^2 R^3 (\lambda_1^2 - 1)\bar{b}_1 \int_0^1 m x S_1(x)\, dx = \frac{b\rho a c \Omega^2 R^4 (\lambda_1^2 - 1)\bar{b}_1}{2\gamma_1},$$

and

$$M = -\sum_{k=1}^{b} M_k \cos \psi_k = \frac{b\rho ac\Omega^2 R^4(\lambda_1^2 - 1)\bar{a}_1}{2\gamma_1}.$$

The equations of rolling and pitching motion of the helicopter are

$$A\frac{dp}{dt} = L = \frac{b\rho ac\Omega^2 R^4(\lambda_1^2 - 1)\bar{b}_1}{2\gamma_1} \tag{9}$$

and

$$B\frac{dq}{dt} = M = \frac{b\rho ac\Omega^2 R^4(\lambda_1^2 - 1)\bar{a}_1}{2\gamma_1}. \tag{10}$$

Introducing the non-dimensional inertias

$$i_A = \frac{A}{W/gR^2} \quad \text{and} \quad i_B = \frac{B}{W/gR^2}$$

and the relative density parameter

$$\mu^* = \frac{W/g}{\rho SAR}, \quad \text{where the solidity } s = \frac{bc}{\pi R}$$

enable (9) and (10) to be written in the non-dimensional form

$$\frac{d\hat{p}}{d\psi} = \frac{a(\lambda_1^2 - 1)}{2\mu^* i_A \gamma_1}\bar{b}_1 \tag{11}$$

$$\frac{d\hat{q}}{d\psi} = \frac{a(\lambda_1^2 - 1)}{2\mu^* i_B \gamma_1}\bar{a}_1. \tag{12}$$

The complete motion is described by the four equations

$$\bar{a}_1'' + \nu\bar{a}_1' + \eta\bar{a}_1 + 2\bar{b}_1' + \nu\bar{b}_1 + 2\bar{F}\hat{p} + \bar{F}\frac{d\hat{q}}{d\psi} + \kappa\hat{q} = 0, \tag{6}$$

$$-2\bar{a}_1' - \nu\bar{a}_1 + \bar{b}_1'' + \nu\bar{b}_1' + \eta\bar{b}_1 + \bar{F}\frac{d\hat{p}}{d\psi} + \kappa\hat{p} - 2F\hat{q} = 0, \tag{7}$$

$$k_A\bar{b}_1 - \frac{d\hat{p}}{d\psi} = 0 \tag{13}$$

and

$$k_B\bar{a}_1 - \frac{d\hat{q}}{d\psi} = 0, \tag{14}$$

where

$$k_A = \frac{a(\lambda_1^2 - 1)}{2\mu^* i_A \gamma_1}; \qquad k_B = \frac{a(\lambda_1^2 - 1)}{2\mu^* i_B \gamma_1};$$

$$\kappa = \frac{\gamma}{2}F_1; \qquad \eta = \lambda_1^2 - 1.$$

Assuming, in the usual way, that typical solutions of the equations are of the form $\hat{p} = \hat{p}_0 \, e^{\lambda\psi}$, etc., leads to a characteristic equation of the form

$$a\lambda^6 + b\lambda^5 + c\lambda^4 + d\lambda^3 + e\lambda^2 + f\lambda + g = 0 \tag{15}$$

where

$$\begin{aligned}
a &= 1, \\
b &= 2\nu, \\
c &= 4 + 2\eta + \nu^2 + \bar{F}(k_A + k_B), \\
d &= 2\nu(2+\eta) + (k_A + k_B)(\kappa + \bar{F}\nu), \\
e &= \eta^2 + \nu^2 + (k_A + k_B)\{\bar{F}(4+\eta) + \kappa\nu\} + k_A k_B \bar{F}^2, \\
f &= (k_A + k_B)(\eta\kappa + 2\bar{F}\nu) + 2\bar{F}\kappa k_A k_B, \\
g &= k_A k_B(\kappa^2 + 4\bar{F}^2).
\end{aligned}$$

We take the following typical values

$$\nu = 0{\cdot}836, \qquad \eta = 0{\cdot}245, \qquad \bar{F} = 1{\cdot}08, \qquad \kappa = 1{\cdot}146, \qquad k_A = 0{\cdot}102, \qquad k_B = 0{\cdot}0204.$$

Equation (15) then becomes

$$\lambda^6 + 1{\cdot}672\lambda^5 + 5{\cdot}32\lambda^4 + 4\lambda^3 + 1{\cdot}44\lambda^2 + 0{\cdot}26\lambda + 0{\cdot}0124 = 0$$

the roots of which are,

$$\begin{aligned}
\lambda_{1,2} &= -0{\cdot}408 \pm 2{\cdot}03i, \\
\lambda_{3,4} &= -0{\cdot}215 \pm 0{\cdot}246i, \\
\lambda_5 &= -0{\cdot}356 \quad \text{and} \quad \lambda_6 = -0{\cdot}0698.
\end{aligned}$$

It is interesting to investigate the meaning of the two oscillatory roots. To do so we consider the simpler case of a rotor having centrally hinged blades rotating *in vacuo* and with the shaft axis fixed. The equation of free motion of a blade is

$$\frac{d^2\beta_k}{d\psi^2} + \beta_k = 0, \tag{16}$$

with the familiar solution

$$\beta_k = -a_{1_k} \cos\psi - b_{1_k} \sin\psi, \tag{17}$$

a_{1_k} and b_{1_k} being constants determined by the initial conditions of the kth blade.

Applying the transformations of equation (5) gives a characteristic equation of the form

$$\lambda^2(\lambda^2 + 4) = 0 \tag{18}$$

whose roots are

$$\lambda_{1,2} = \pm 2i \quad \text{and} \quad \lambda_{3,4} = 0.$$

The first pair of roots imply that $\bar{a}_1$ and $\bar{b}_1$ oscillate with realtime frequency 2Ω; the second pair of roots imply that $\bar{a}_1$ and $\bar{b}_1$ are constants.

Now $\bar{a}_1$ and $\bar{b}_1$ represent, respectively, the blade flapping displacements resolved in the longitudinal and lateral directions, summed over all the blades. If the blade motions are not identical but if each, nevertheless, sweeps out a plane in accordance with equation (16), it follows that $\bar{a}_1$ and $\bar{b}_1$ have periodic components of frequency 2Ω. This is quite easy to see if we imagine a four-bladed rotor with one blade having different flapping from the other three. It also follows that the amplitudes of $\bar{a}_1$ and $\bar{b}_1$ must be identical; in fact, the eigenvectors corresponding to equation (17), for $\lambda = \pm 2i$, confirm that the ratio of $\bar{a}_1$ to $\bar{b}_1$ is unity. A similar sort of phenomenon is present with the two-bladed helicopter, even when the flapping is sinusoidal, because of the asymmetry of the rotor. It is easy to see physically that the longitudinal and lateral components of flapping have 2Ω fluctuations with exactly equal amplitudes.

If the flapping of the blades are identical, the blade tips trace out the same plane. Substitution of this motion into the summations, equation (5), give constant values for $\bar{a}_1$ and $\bar{b}_1$; further, $\bar{a}_1$ and $\bar{b}_1$ have precisely the same values as a_1 and b_1 of the individual blades—indeed, it was for this reason that $\bar{a}_1$ and $\bar{b}_1$ were chosen as the notation for the summations, the bars denoting the flapping of the hingeless blade as explained in Ref. 4.

Thus, the two sets of roots, $\lambda_{1,2}$ and $\lambda_{3,4}$ correspond to a 2Ω fluctuation of total blade flapping superimposed upon a steady tip path plane.

Returning to the original problem, the presence of elastic root restraint, aerodynamic damping and coupling with the rolling and pitching motions, changes the values of the roots considerably. The pair of roots $\lambda_{1,2}$ is still very close to the 2Ω of the simple case chosen for illustration, but $\lambda_{3,4}$ no longer corresponds to a stationary tip path plane but one which precesses with frequency $0{\cdot}264\Omega$, the blade a natural frequency being no longer identical with the shaft rotational frequency.

If the shaft is fixed these roots then become

$$\lambda_{1,2} = -0{\cdot}419 \pm 2{\cdot}025i$$

and

$$\lambda_{3,4} = -0{\cdot}417 \pm 0{\cdot}212i$$

It can be seen that the coupling of the blade motion with pitch and roll of the helicopter has little effect on the 'twice per rev' oscillation, as might be expected from its relatively high frequency, but has a considerable effect on the other mode, particularly the damping.

The general form of the characteristic equation in this latter case, if the damping is made zero, is simply

$$(\lambda^2 + \eta)^2 + 4\lambda^2 = 0, \tag{19}$$

from which one can easily see that the corresponding frequencies are $\Omega(1+\lambda_1)$ and $\Omega(1-\lambda_1)$ since $\eta = \lambda_1^2 - 1$.

Now gyroscopic theory gives the steady precession rates ω of a gyro as the solution of

$$A\omega^2 \cos\alpha - Cn\omega - Wl = 0, \tag{20}$$

where equation (20) is expressed in the standard notation of gyroscope theory, Ref. 5.

Regarding the rotor as a disc gives $C = 2A$; also the angular velocity of the disc is the shaft angular velocity, i.e. $n = \Omega$, the weight 'stiffness' Wl corresponds to the elastic hinge stiffness if $\alpha < \pi/2$. The elastic stiffness is represented by the coefficient of P_k in equation (18) averaged in a given direction over all the blades i.e. $\frac{1}{2}(\lambda_1^2 - 1)b\Omega^2R^3\int_0^1 mxS_1(x)\,dx$. But $bR^3\int_0^1 mxS_1(x)\,dx$ is almost exactly equal to the polar moment of inertia C, and since the axis of the rotor is only slightly inclined to the shaft, $\cos\alpha \simeq 1$. On substituting these values into equation (20) we get the same solution as that of equation (19); that is, the oscillations represented by the roots $\lambda_{1,2}$ and $\lambda_{3,4}$ correspond to the fast and slow (retrograde) precessions of a spring restrained gyroscope.

The frequency of the slow mode is likely to be very close to that of the in-plane oscillations and we might expect the coupling to be important. This latter mode has often been referred to as the 'pendulum' mode, but this is clearly a misnomer since the motion is independent of gravity and, as has been shown above, the mode is still present when the shaft is fixed. From the above discussion a more appropriate term would be 'gyroscopic' mode.

Both the oscillations discussed above appear in the analysis as a result of including unsteady flapping terms in the equations of motion. Neglecting these terms is equivalent to assuming that the motion of the rotor does not depend on its displacement relative to the shaft, but only on such 'external' influences as pitching, rolling, forward speed, etc.; it also corresponds to the 'quasi-steady' assumption of classical stability analysis which has been based on examination of the frequency response of a blade to given shaft oscillations. The analysis of this section further justifies the 'quasi-steady' assumption in conventional stability calculations, since the modes corresponding to $\lambda_{1,2}$ and $\lambda_{3,4}$ can be seen to have frequencies well above those of the usual stability modes.

3. Approximations to the Modes

It has been found that a very good approximation to the fast 'gyroscopic' mode can be obtained by assuming the shaft to be fixed, i.e. by solving equation (15) with k_A and k_B put to zero.

If, now, the $\bar{a}_1''$ and $\bar{b}_1''$ terms are omitted from equations (6) and (7), the characteristic equation reduces to

$$\lambda^4+0{\cdot}823\lambda^3+0{\cdot}306\lambda^2+0{\cdot}0554\lambda+0{\cdot}00264=0$$

with roots

$$\lambda_{3,4}=-0{\cdot}21\pm 0{\cdot}263i$$

$$\lambda_5=-0{\cdot}334 \quad \text{and} \quad \lambda_6=-0{\cdot}0698.$$

The 'twice per rev' fast gyroscopic oscillation has disappeared and, apart from a slight change in λ_5, the remaining roots are extremely close to those of the original sextic.

If, in addition, the rolling freedom is omitted, the characteristic equation reduces to the cubic

$$\lambda^3+0{\cdot}803\lambda^2+0{\cdot}186\lambda+0{\cdot}00906=0$$

with roots

$$\lambda_{3,4}=-0{\cdot}369\pm 0{\cdot}038i$$

and

$$\lambda_6=-0{\cdot}0659.$$

This gives a poor approximation to the slow gyroscopic mode.

If rolling in included but pitching omitted, the characteristic equation is

$$\lambda^3+0{\cdot}819\lambda^2+0{\cdot}282\lambda+0{\cdot}0452=0,$$

with roots

$$\lambda_{3,4}=-0{\cdot}212\pm 0{\cdot}264i$$

and

$$\lambda_5=-0{\cdot}395.$$

These roots are very good approximations to the original roots. Thus, the slow gyroscopic mode represents a strong coupling between the slow precession mode of the rotor and the rolling of the fuselage. The approximate equations of this motion are therefore

$$\nu\frac{d\bar{a}_1}{d\psi}+\eta\bar{a}_1+2\frac{d\bar{b}_1}{d\psi}+\nu\bar{b}_1+2\bar{F}_1\hat{p}=0, \tag{21}$$

$$-2\frac{d\bar{a}_1}{d\psi}-\nu\bar{a}_1+\nu\frac{d\bar{b}_1}{d\psi}+\eta\bar{b}_1+\bar{F}\frac{d\hat{p}}{d\psi}+\kappa\hat{p}=0 \tag{22}$$

and

$$k_A\bar{b}_1 - \frac{d\hat{p}}{d\psi} = 0. \tag{23}$$

The characteristic equation is

$$a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1 = 0$$

where

$$\begin{aligned}
a_1 &= 4 + \nu^2, \\
b_1 &= 2\nu(2+\eta) + k_A\bar{F}\nu, \\
c_1 &= \eta^2 + \nu^2 + k_A\{\bar{F}(4+\eta) + \kappa\nu\} \\
d_1 &= k_A(\kappa\eta + 2\bar{F}\nu).
\end{aligned}$$

It remains to be seen if the approximations discussed above apply also when lagging motion is introduced.

4. Blade Lagging Motion

By analogy with the flapping motion the first mode lagging displacement of the blade will be represented by

$$Y = RT_1(x)Q_1(\psi) \tag{24}$$

$T_1(x)$ being the mode shape and $Q_1(\psi)$ the time or azimuth coordinate. If $\partial G/\partial r$ is the blade lagwise loading, we shall have

$$\frac{d^2Q_1}{d\psi^2} + \kappa_1^2 Q_1 = \frac{1}{\Omega^2 R \int_0^1 mT_1^2(x)\,dx} \int_0^1 \frac{\partial G}{\partial x} T_1(x)\,dx, \tag{25}$$

where $k_1\Omega$ is the natural undamped frequency of the first lagging mode. $\partial G/\partial r$ will consist of inertia terms due to blade flapping, helicopter translation and angular motion, and of changes in local blade drag.

In order to keep the analysis as simple as possible, structural coupling between the flapping and lagging modes has been deliberately left out of account even though it may have a significant effect on the lagging motion. This is because it would probably arise from blade twist and collective pitch which are determined by the aerodynamic design and flight condition and are not really at the designer's disposal for controlling the lagging motion. In a full calculation structural coupling would be included. Other parameters will also be seen to be practically fixed in advance. Reference 6 considers the effect of an autostabiliser on the coupling between the airframe and the blade modes.

4.1. Inertia Terms

In order to be able to calculate the Coriolis acceleration due to blade flapping, it is convenient to regard the blade as rigid and moving about a hinge. For this purpose we can use the offset hinge representation of Ref. 4 and illustrated in Fig. 1.

Take a set of unit vectors with **i** along the span of the undeflected blade, **k** upwards along the hub axis and **j** completing the right hand set, Fig. 2.

The position vector **r** of a point P on the blade is

$$\mathbf{r} = (eR + r\cos\beta)\mathbf{i} + (r\sin\beta + hR)\hat{\mathbf{k}}$$

and the angular velocity $\mathbf{A}$ of the axes system is

$$\mathbf{A}=(-p\cos\psi+q\sin\psi)\mathbf{i}+(p\sin\psi+q\cos\psi)\mathbf{j}+\Omega\mathbf{k}$$

Calculating the velocity and acceleration of P by operating on $\mathbf{r}$ twice with $(\partial/\partial t+\mathbf{A}\times)$ leads to rather lengthy expressions, but neglecting squares and products of p and q, and making the usual small angle assumption for β, gives for the component of acceleration in the $\mathbf{j}$-direction

$$a_j=\dot{p}hR\cos\psi-\dot{q}hR\sin\psi-2\Omega r\beta\dot{\beta}+2rp\dot{\beta}\cos\psi-2rq\dot{\beta}\sin\psi.$$

Now, in disturbed motion, the amplitudes of p and $\dot{\beta}$ can be expected to be proportional to one another, whereas β will contain the coning angle which is independent of the amplitude of the motion. Thus, the terms $2rp\dot{\beta}\cos\psi$ and $2rq\dot{\beta}\sin\psi$ can be regarded as second order compared with the others. If now we add the translational acceleration of the origin to complete the motion, we can write for small disturbances

$$a_j=\dot{p}hR\cos\psi-\dot{q}hR\sin\psi-2\Omega r\beta\dot{\beta}+\dot{u}\sin\psi+\dot{v}\cos\psi \tag{26}$$

4.2. Aerodynamic Loading

It can be seen from Fig. 3 that, due to flapping and lagging motion, the component of the change of aerodynamic force Y_a in the $\mathbf{j}$ direction, in an element of blade, is approximately

$$\begin{aligned}dG_a=&-\tfrac{1}{2}\rho ac\Omega^2r^2\,dr\left[\left(\theta-\varphi_i-\frac{1}{x}\frac{\partial z}{\partial\psi}\right)\left(\varphi_i+\frac{1}{x}\frac{\partial z}{\partial\psi}\right)-(\theta-\varphi_i)\varphi_i\right]-\\&-\tfrac{1}{2}C_D\rho ac\,dr\left[\left(\Omega r+\frac{\partial Y}{\partial t}\right)^2-\Omega^2r^2\right].\end{aligned} \tag{27}$$

The term $\varphi_i+(1/x)(\partial z/\partial\psi)$ represents the local inflow angle due to the induced velocity and blade flapping; the terms in the square brackets represent the force difference between disturbed and steady motion.

For small disturbances, equation (27) can be linearised to

$$dG_a=-\tfrac{1}{2}\rho ac\Omega^2rR\,dr\left[(\theta-2\varphi_i)\frac{\partial z}{\partial\psi}+2C_D\frac{\partial y}{\partial\psi}\right]. \tag{28}$$

Letting the blade-lag displacement be expressed as $Y=RT_1(x)Q_1(\psi)$, the total aerodynamic and inertia loading is

$$\begin{aligned}\frac{\partial G}{\partial r}=&-m\dot{p}hR\cos\psi+m\dot{q}hR\sin\psi-m\dot{u}\sin\psi-m\dot{v}\cos\psi+2m\Omega^2RS_1(x)S_1'(x)P_1(\psi)P_1'(\psi)-\\&-\tfrac{1}{2}\rho ac\Omega^2R^2x[(\theta-2\varphi_i)S_1(x)P_1'(\psi)+2C_DT_1(x)Q_1'(\psi)].\end{aligned}$$

Substituting in equation (25) gives

$$\begin{aligned}\frac{d^2Q_1}{d\psi^2}+\delta\frac{dQ_1}{d\psi}+\kappa_1^2Q_1=&-\frac{1}{\int_0^1 mT_1^2(x)\,dx}\left[\left(\frac{d\hat{p}}{d\psi}h\cos\psi-\frac{d\hat{q}}{d\psi}h\sin\psi\right)\int_0^1 mT_1(x)\,dx\right]+\\&+\left(\frac{d\hat{u}}{d\psi}\sin\psi+\frac{d\hat{v}}{d\psi}\cos\psi\right)\int_0^1 mT_1(x)\,dx-\\&-2P_1(\psi)P_1'(\psi)\int_0^1 mS_1(x)S_1'(x)T_1(x)\,dx+\\&+\tfrac{1}{2}\rho acR\left\{P_1'(\psi)\int_0^1 x(\theta-2\varphi_i)S_1(x)T_1(x)\,dx\right\}\end{aligned} \tag{29}$$

where

$$\delta = \tfrac{1}{2}\gamma' \int_0^1 xC_D T_1^2(x)\,dx \quad \text{and} \quad \gamma' = \frac{\rho acR}{\int_0^1 mT_1^2(x)\,dx}.$$

The δ term could include any artificial damping which may be added to augment the low value of the natural air damping.

Equation (29) can be written more shortly as

$$\frac{d^2Q_1}{d\psi^2} + \delta\frac{dQ_1}{d\psi} + \kappa_1^2 Q_1 = -\Big[\bar{H}\frac{d\hat{p}}{d\psi}h\cos\psi - \bar{H}\frac{d\hat{q}}{d\psi}h\sin\psi + \bar{H}\frac{d\hat{u}}{d\psi}\sin\psi + \bar{H}\frac{d\hat{v}}{d\psi}\cos\omega - \\ -2\bar{a}_0 P_1'(\psi)\bar{J} + \tfrac{1}{2}\gamma' CP_1'(\psi)\Big] \tag{30}$$

where

$$\bar{H} = \frac{\int_0^1 mT_1(x)\,dx}{\int_0^1 mT_1^2(x)\,dx}; \qquad \bar{J} = \frac{\int_0^1 mS_1S_1'T_1\,dx}{\int_0^1 mT_1^2\,dx}; \qquad C = \int_0^1 (\theta - 2\varphi_i)xS_1T_1\,dx$$

and $\bar{a}_0P_1'(\psi)$ is the linearized form of $P_1(\psi)P_1'(\psi)$. Note that the last two terms on the right-hand side of equation (30) represent respectively the Coriolis inertia force and the aerodynamic force component in the lag plane due to blade-flapping velocity. Numerical calculations show that the former is much larger than the latter.

We now define

$$e_1 = -\frac{2}{b}\sum_{k=1}^{b} Q_k\cos\psi_k \quad \text{and} \quad f_1 = -\frac{2}{b}\sum_{k=1}^{b} Q_k\sin\psi_k. \tag{31}$$

Substituting these relations into equation (30) and summing over all the blades gives

$$e_1'' + \delta e_1' + \xi e_1 + 2f_1' + \delta f_1 = \bar{H}h\frac{d\hat{p}}{d\psi} + \bar{H}\frac{dv}{d\psi} + \Big(2\bar{a}_0\bar{J} - \frac{\gamma' C}{2}\Big)(\bar{a}_1' + \bar{b}_1) \tag{32}$$

and

$$f_1'' + \delta f_1' + \xi f_1 - 2e_1' - \delta e_1 = -\bar{H}h\frac{d\hat{q}}{d\psi} + \bar{H}h\frac{d\hat{u}}{d\psi} + \Big(2\bar{a}_0\bar{J} - \frac{\gamma' C}{2}\Big)(\bar{b}_1' - \bar{a}_1) \tag{33}$$

where

$$\xi = \kappa_1^2 - 1.$$

The variables e_1 and f_1 can be interpreted as the lateral and longitudinal displacements of the centre of gravity of the rotor when the blades move out of phase with one another. The right-hand sides represent the inertial and aerodynamic excitation of the lagging motion.

Putting these terms and the damping term δ to zero gives the free-lagging motion, whose characteristic equation is

$$\lambda^4 + (4 + 2\xi)\lambda^2 + \xi^2 = 0. \tag{34}$$

With $\xi = \kappa_1^2 - 1$ the roots of equation (32) can be written

$$\lambda = \pm(\kappa_1 - 1)i \quad \text{and} \quad \lambda = \pm(\kappa_1 + 1)i. \tag{35}$$

Thus, the displacement of the rotor C.G. has two components; one rotates round the hub in the same direction as the rotor with angular velocity $(\kappa_1 + 1)\Omega$, the other rotates round the hub with angular velocity

$(\kappa_1 - 1)\Omega$ in the same, or in the opposite direction to the rotor, according as κ_1 is less than or greater than unity.

Omitting the e_1'' and f_1'' terms gives a characteristic equation

$$4\lambda^2 + \xi^2 = 0$$

with roots

$$\lambda = \pm\tfrac{1}{2}i(\kappa_1^2 - 1). \tag{36}$$

Taking $\kappa_1 = 0{\cdot}5$ as a typical value, it can be seen by comparing equation (33) with equation (34), that omitting e_1'' and f_1'' leads to a poor approximation to the roots, unlike the corresponding flapping case where the neglect of these terms was seen to be not very important. The reason for this is that the flap damping is very large and tends to dominate the motion whereas the lag damping (here assumed to be zero) will probably be small even with artifical damping. The natural lag damping is typically less than 2 per cent of critical. Hence e_1'' and f_1'' should be included in the analysis.

4.3. Motion of the Fuselage due to Lag Damping

We have now to calculate the effect of the lagging motion on the helicopter airframes. It has already been assumed that the flapping motion is unaffected by the blade lagging but the displacement of the rotor C.G., mentioned above, will produce inertia forces and pitching and rolling moments. To calculate them consider the displacement of a point P of the blade when deflected by lag bending, Fig. 4.

The displacement of a point P of the blade in the Y' direction is

$$Y' = Y_0' + xR \sin \psi_k + RT_1(x)Q_k(\psi) \cos \psi_k.$$

The acceleration of P in this direction is

$$\frac{d^2Y'}{dt^2} = \ddot{Y}_0' - x\Omega^2 R \sin \psi_k + \Omega^2 RT_1(x)\frac{d^2}{d\psi^2}\{Q_k \cos \psi_k\}.$$

The inertia force on the helicopter corresponding to this acceleration is $-m\ddot{Y}'\,dr$ and the total force due to one blade is therefore

$$-\ddot{Y}_0' \int_0^1 m\,dr + \Omega^2 R \sin \psi_k \int_0^1 mxR\,dx - \Omega^2 R\frac{d^2}{d\psi^2}\{Q_k \cos \psi_k\} \int_0^1 mRT_1(x)\,dx.$$

Using definitions (31), the force due to the whole rotor is

$$-M_b\ddot{Y}_0' + \tfrac{1}{2}b\Omega^2 R^2 e_1'' \int_0^1 mT_1(x)\,dx \tag{37}$$

where M_b is the mass of all the blades.

The term $-M_b\ddot{Y}_0'$ denotes the blade inertia due to the motion of the fuselage and can therefore be absorbed into the total inertia of the helicopter.

Similarly, the inertia force in the X' direction is

$$\tfrac{1}{2}b\Omega^2 R^2 f_1'' \int_0^1 mT_1(x)\,dx. \tag{38}$$

The rolling and pitching moments are respectively $\tfrac{1}{2}bh\Omega^2 R^3 e_1'' \int_0^1 mT_1(x)\,dx$ and $-\tfrac{1}{2}bh\Omega^2 R^3 f_1'' \int_0^1 mT_1(x)\,dx$.

Finally, the only other contributions to the translational motion are the force components due to the tilt of the disc, Fig. 5, which shows rolling combined with a sideways tilt of the disc. We need not consider the effect of roll and pitch since the inertia force on a blade element due to the acceleration is exactly balanced by the gravitational component in the lag plane, i.e. there would be no resultant motion in the lag plane. This is not so when the rotor tilts relative to the shaft, however, because the acceleration thus produced has no compensating gravitational component in the lag plane.

From Ref. 4 we find that the component of force in the X' (forward) direction, due to a tilt of the disc, is $-T\bar{a}_1$ and in the Y' direction it is $T\bar{b}_1$. Putting thrust equal to weight, the translational equations of motion are

$$\frac{W}{g}\frac{du}{dt} = -W\bar{a}_1 + \tfrac{1}{2}b\Omega^2R^2f_1''\int_0^1 mT_1(x)\,dx \tag{39}$$

and

$$\frac{W}{g}\frac{dv}{dt} = +W\bar{b}_1 + \tfrac{1}{2}b\Omega^2R^2e_1''\int_0^1 mT_1(x)\,dx. \tag{40}$$

In non-dimensional form these equations become

$$\frac{d\hat{u}}{d\psi} = -\hat{g}\bar{a}_1 + \hat{H}f_1'' \tag{41}$$

and

$$\frac{d\hat{v}}{d\psi} = +\hat{g}\bar{b}_1 + \hat{H}e_1'' \tag{42}$$

where

$$\hat{g} = \frac{g}{\Omega^2R} \quad \text{and} \quad \hat{H} = \frac{bgR}{2W}\int_0^1 mT_1(x)\,dx.$$

It must be appreciated that, in equations (39) to (42), u and v are the velocities due to the rotor tilt relative to the shaft and do not include the contributions due to tilt of the whole helicopter, as explained earlier.

The inertia moments calculated above must now be added to complete the equations (13) and (14), which now become

$$\frac{d\hat{p}}{d\psi} = k_A\bar{b}_1 + \frac{h\hat{H}}{i_A}e_1'' \tag{43}$$

and

$$\frac{d\hat{q}}{d\psi} = k_B\bar{a}_1 - \frac{h\hat{H}}{i_B}f_1''. \tag{44}$$

4.4. Final Coupled Equations

It can be seen that $du/d\psi$ and $dv/d\psi$ can easily be eliminated from equations (32) and (33) by means of (41) and (42). Thus, the final equations can be written

$$\bar{a}_1'' + \nu\bar{a}_1' + \eta\bar{a}_1 + 2\bar{b}_1' + \nu\bar{b}_1 + 2\bar{F}\hat{p} + \bar{F}\frac{d\hat{q}}{d\psi} + \kappa\hat{q} = 0, \tag{6}$$

$$-2\bar{a}_1' - \nu\bar{a}_1 + \bar{b}_1'' + \nu\bar{b}_1' + \eta\bar{b}_1 + \bar{F}\frac{d\hat{p}}{d\psi} + \kappa\hat{p} - 2\bar{F}\hat{q} = 0, \tag{7}$$

$$\frac{h\hat{H}}{i_A}e_1'' + k_A\bar{b}_1 - \frac{d\hat{p}}{d\psi} = 0, \tag{43}$$

$$-\frac{h\hat{H}}{i_B}f_1'' + k_A\bar{a}_1 - \frac{d\hat{q}}{d\psi} = 0, \tag{44}$$

$$(1-\bar{H}\hat{H})e_1'' + \delta e_1' + \xi e_1 + 2f_1' + \delta f_1 - \bar{H}h\frac{d\hat{p}}{d\psi} - L'\bar{a}_1' - L\bar{b}_1 = 0 \tag{45}$$

and

$$(1-\bar{H}\hat{H})f_1'' + \delta f_1'' + \xi f_1 - 2e_1' - \delta e_1 + \bar{H}h\frac{d\hat{q}}{d\psi} + L\bar{a}_1 - L'\bar{b}_1' = 0, \tag{46}$$

where

$$L = 2\bar{a}_0\bar{J} - \frac{\gamma'}{2}C + \bar{H}\hat{g} \quad \text{and} \quad L' = 2\bar{a}_0\bar{J} - \frac{\gamma'}{2}C.$$

It is interesting to examine equations (6), (7), (43) to (46) to see which terms can be regarded as variables in the coupled motion.

In equations (6) and (7), the constants ν, F, κ, η are determined by the aerodynamics and blade mode shape in flapping motion and these can be varied over only a narrow range. Only η, which represents the blade flapping stiffness, is capable of being varied somewhat more readily. Nevertheless, in the present context they may be regarded as essentially constant. Similarly, in equations (43) and (44), the constants depend upon quantities, such as the rotor height and fuselage inertias, which are more or less fixed by overall design considerations and, again, cannot be arbitrarily varied except between narrow limits. In the last pair of equations, (45) and (46), the coefficients L and L' represent the combined effects of the Coriolis force and blade drag and, in the case of L' an extra inertia term due to horizontal acceleration. It can be seen from equations (29) and (30) that the Coriolis force depends on the coning angle $\bar{a}_0$ and the drag term on the collective pitch angle θ and downwash angle φ_i. These terms are governed by the aerodynamic design and flight condition and, in practice, cannot be varied to affect the lagging oscillation. The terms $\bar{J}$ and C depend a little on blade lag mode shape which is related to lag frequency.

Thus, the only parameters which can be varied arbitrarily are the damping δ and frequency ratio κ_1 of the blade lag motion.

5. Calculations

5.1. Solution of the Equations

The numerical values of some of the parameters have already been given in Section 2.1. The blade mass distribution and shape of the first lagging mode lead to the following additional constants:

$$\bar{H} = 1{\cdot}5, \qquad \hat{H} = 0{\cdot}0117, \qquad \bar{J} = 1{\cdot}22 \quad \text{and} \quad \bar{a}_0 = 4{\cdot}5 \text{ degs.}$$

these quantities having been defined in Section 4.2.

The blade was assumed to have 8 degrees twist and the collective pitch was chosen to lift an 8000 lb helicopter at normal operating rotor speed. In the calculations the collective pitch (actually 8·2 degrees at 0·75 blade radius) was kept constant although, strictly speaking, it should have varied with rotor r.p.m. to maintain constant thrust. This point will be commented upon later.

Since interest is confined to the airborne case the range of rotor speed considered is from 0·8 to 1·2 times the normal operating value. Over this range the flapping frequency ratio can be assumed to be constant but the lagging frequency is assumed to conform to a Southwell formula

$$\kappa_1^2\Omega^2 = \kappa_0^2\Omega_0^2 + 0{\cdot}23\Omega^2 \tag{47}$$

where $\kappa_0\Omega_0$ is the lagging frequency of the blade when not rotating and Ω_0 is the normal operating rotor angular velocity. Equation (47) can also be written as

$$\kappa_1^2 = \frac{\kappa_0^2}{\hat{\Omega}^2} + 0{\cdot}23 \tag{48}$$

where

$$\hat{\Omega} = \frac{\Omega}{\Omega_0}.$$

In the calculations κ_0 has been given the values 0·2, 0·4, 0·6, 0·8.

The damping coefficient δ is given as a percentage of the critical damping of the free lag oscillation, the values chosen being 0, 2, 5, 10.

Some extra cases were considered to investigate the effect of blade torsional motion. To keep the calculations simple, torsional damping was excluded and the blade was assumed to twist linearly. The centre of gravity was assumed to be 1 per cent ahead of the flexural axis. It was further assumed that blade twist increased the lift only and gave no extra drag i.e. the twist gave rise to no lagwise force component.

The equations of torsional-flapping motion are taken as

$$\frac{d^2\beta_k}{d\psi^2} + 0{\cdot}836\frac{d\beta_k}{d\psi} + 1{\cdot}245\beta_k = 0{\cdot}836\theta_k \tag{49}$$

and

$$-5\left(\frac{d^2\beta_k}{d\psi^2} + \beta_k\right) = \frac{d^2\theta_k}{d\psi^2} + \tau^2\theta_k \tag{50}$$

where θ_k is the torsional frequency ratio and took the values 2, 4, 6, 8, 10.

The pitch angle, θ_k, was transformed by a pair of Coleman coordinates and the appropriate terms on the right-hand side of equation (49) were added to equations (6) and (7). The two transformed equations corresponding to equation (50) were added to the set of coupled equations of the previous section.

5.2. Results and Discussion of the Calculations

The frequencies of the two blade lag modes are shown in Fig. 6. These frequencies, which represent the whirling motion of the rotor C.G., are almost completely independent of the lag damping so that the results for a given non-rotating stiffness κ_0 can be shown in each case as a single curve. The broken lines show the two gyroscopic mode frequencies and these are found to be practically independent of the lag stiffness and damping.

Air resonance might be expected, as with ground resonance, if the slow frequency of the C.G. whirling mode coincides with a 'body' mode, i.e. one of the two frequencies of the motion which we have called 'gyroscopic' modes. For self-excited oscillations a body mode must also coincide with the lag frequency of an individual blade. For these conditions to be satisfied simultaneously we must have

$$|\kappa_1\Omega - \Omega| = \Omega_b \tag{51}$$

where Ω_b is a body frequency.

This relationship can be written

$$|\sqrt{\kappa_0^2 + 0{\cdot}23\hat{\Omega}^2} - \hat{\Omega}| = \hat{\Omega}_b$$

where

$$\hat{\Omega}_b = \frac{\Omega_b}{\Omega}.$$

The left-hand side of equation (51) corresponds to the frequency of the slow-lag-whirl mode (Section 4.2) and equation (51) is therefore satisfied when the slow-lag mode of Fig. 6 intersects the slow-gyro curve. For the cases considered, this can be seen to occur when $\hat{\Omega} = 1{\cdot}04$ for $\kappa_0 = 0{\cdot}6$. But we see from Figs. 7 to 10 that the damping of both the lag modes show no particular changes in this region and no 'resonance' is apparent. The reason for this is that the damping of the slow gyro mode is very high, as we have seen in Section 2; in fact it is over 60 per cent of critical, most of it being derived from the blade flapping motion. The system's natural damping, therefore, is much higher than is the case with ground resonance. This is confirmed by the work of Ref. 6 where it is shown, by means of a numerical example, that when the aerodynamic forces are absent resonance occurs when the slow lag mode coincides with the slow gyro mode and also when it coincides with the mode in which the pitching and rolling motions are coupled. When the aerodynamic terms are included these local instabilities are replaced by broad unstable regions as indicated in Figs. 7 to 10 of this report. There is the further difference that, whereas, in the latter case, the body motion can be regarded as a simple second-order mass-spring-damper system, the slow gyro mode is the result of coupling between the blade flapping and fuselage angular motion and is governed by a cubic characteristic equation.

Nevertheless, as Figs. 7 to 9 show, the slow lag mode is unstable for all the lag stiffness and rotor speeds considered unless the lag damping is at least 4 per cent of critical. The fast lag mode appears to be stable for all these cases. Fig. 11 also shows that, unless the blade is very soft torsionally, twisting motion has little effect on the damping of the modes.

Since the natural lag damping is very small (about 2 per cent of critical), one might expect that when the total system damping is redistributed by coupling of the modes, a small shift of damping away from one of the lag modes would have a large relative effect. It is not surprising, therefore, that at least one of the lag modes should become unstable and that more than the natural blade damping may be necessary to ensure that both lag modes are stable. Similar conclusions are presented in Refs. 6 and 7.

There may be a number of parameters which appear to have quite a large effect on the stability of the lag motion, but as emphasised above, this is only because the damping is so low that these effects are only large relatively. One such parameter is the collective pitch which has been assumed constant in the calculations. Varying the collective pitch changes the values of the last two terms on the right hand side of equation (30). These terms represent the lag moments due to the Coriolis and aerodynamic forces arising from the flapping velocity; the first is proportional to the coning angle, which itself is roughly proportional to the collective pitch, whilst the second, and smaller term, is roughly porportional to the collective pitch also. It may well be that a particular range of collective pitch angles, together with certain values of other parameters fortuitously causes both lag modes to be stable but, as discussed in Section 4.4, these combinations could not be chosen arbitrarily and, in any case could not be expected to hold over the complete flight range. The effects of the individual collective pitch terms are discussed in Ref. 6.

A genuine lag resonance, but not connected with self-excited oscillations, may arise when the flapping frequency of an individual blade is the same as the lag frequency so that large response to Coriolis excitation can occur. This flapping frequency can be found by subtracting unity from the frequency ratio of the fast gyro mode (i.e. to convert the Coleman mode back to rotating axes). Similarly, the lag-frequency ratio can be found by adding unity to the slow lag motion. Coincidence of these frequencies is shown graphically in Fig. 12 and can be seen to occur when $\kappa_0 = 0{\cdot}8$ and $\hat{\Omega} = 0{\cdot}89$. The lower diagram of Fig. 12 shows the ratio of lagging to flapping amplitude ratio and we see that these ratios reach a peak when the frequencies coincide. The ratios reach very large values when only the natural aerodynamic damping is present, indicating that if blade flapping occurs due, say, to a gust or a manoeuvre, the lag motion may reach excessively lag amplitudes.

Although the motion is more stable in this region the possibility that very large lag amplitudes might occur may be more serious.

6. Conclusions

6.1. When unsteady flapping and body angular motion are considered two oscillations appear; one is heavily damped with a frequency of about twice per rotor revolution; the other a slower motion, but also heavily damped, whose frequency depends on the flapping stiffness. For the stiffness typical of hingeless rotors, the slower oscillation has a frequency of about one-quarter rotor speed.

In these modes the rotor can be shown to behave like a spring-restrained gyroscope, the 'spring' being the blade-flapping stiffness. The term 'pendulum mode' often used to describe the slower oscillation, is inappropriate and 'slow gyroscopic mode' is preferred. More generally, these modes are called 'body modes' to indicate that fuselage movement is involved.

6.2. The phenomenon called 'air resonance' is different in character from 'ground resonance'. In air resonance the slower body mode, mentioned above, may have a frequency similar to that of the slow C.G. whirling mode due to the blade-lag motion. But the high damping of the flapping motion seems to suppress

self-excited oscillations since little change of damping of the modes occur when there is coincidence of the flap and lag frequencies.
6.3. It appears that when the blade lagging motion is coupled with the gyroscopic modes one of the two lag modes (the slow one) becomes unstable when only the low aerodynamic damping (2 per cent of critical) is present. Damping of at least 4 per cent of critical is required to ensure that all the modes are stable. The only parameters which appear, in practice, to have a considerable effect on the lagging motion are the lag damping and, to a lesser extent, the lag stiffness.
6.4. A resonant situation may occur when the lag stiffness is high enough to bring the blade frequency (not the whirling frequency) of an individual blade into coincidence with the flapping frequency. In this situation the Coriolis forces excite the lagging motion which could reach large amplitudes when the lag damping has the low natural aerodynamic value.

Acknowledgement

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LIST OF SYMBOLS

a	Blade section lift slope
A	Moment of inertia about roll axis
$\left.\begin{matrix}\bar{a}_1\\ \bar{b}_1\end{matrix}\right\}$	Coleman coordinates of blade flapping motion
b	Number of blades
B	Moment of inertia about pitch axis
c	Blade chord
C	Aerodynamic integral defined in Section 4.2.
E_1	$\int_0^1 mS_1^2(x)\,dx$
F_1	$\int_0^1 mS_1(x)\,dx$
$\bar{F}_1$	$\dfrac{\gamma}{\gamma_1}$
$\left.\begin{matrix}e_1\\ f_1\end{matrix}\right\}$	Coleman coordinates of blade lag motion
hR	Height of rotor hub above helicopter C.G.
$\left.\begin{matrix}\bar{H}\\ \bar{J}\end{matrix}\right\}$	Inertia integrals defined in Section 4.2.
i_A	$\dfrac{Ag}{WR^2}$
i_B	$\dfrac{Bg}{WR^2}$
κ	$\dfrac{\gamma F}{2}$
k_A	$\dfrac{a(\lambda_1^2-1)}{2\mu^* i_A \gamma_1}$
k_B	$\dfrac{a(\lambda_1^2-1)}{2\mu^* i_B \gamma_1}$
L	Rolling moment
m	Mass per unit length of blade
M	Pitch moment
p	Roll angular velocity
$P(\psi)$	Azimuth coordinate of blade flapping motion
q	Pitch angular velocity
$Q(\psi)$	Azimuth coordinate of blade lagging motion
r	Radial distance of point on blade
R	Blade radius

$S(x)$	Flapping mode shape
$T(x)$	Lagging mode shape
$\left.\begin{matrix}X\\Y\\Z\end{matrix}\right\}$	Coordinates of point on blade
x	$\dfrac{X}{R}$
y	$\dfrac{Y}{R}$
z	$\dfrac{Z}{R}$
α	Incidence of blade section
β	Flapping angle of blade
γ	$\dfrac{\rho a c R}{\int_0^1 m S_1^2\,dx}$
γ_1	$\dfrac{\rho a c R}{\int_0^1 m x S_1\,dx}$
δ	Damping coefficient of lag motion
η	$\lambda_1^2 - 1$
θ	Blade section pitch angle
κ_1	Frequency ratio of lag motion
λ_1	Frequency ratio of flapping motion
ν	$\dfrac{\gamma E_1}{2}$
ξ	$\kappa_1^2 - 1$
ρ	Air density
τ	Undamped torsional frequency ratio
φ	Inflow angle at blade section
ψ	Azimuth angle of blade
Ω	Angular velocity of blade

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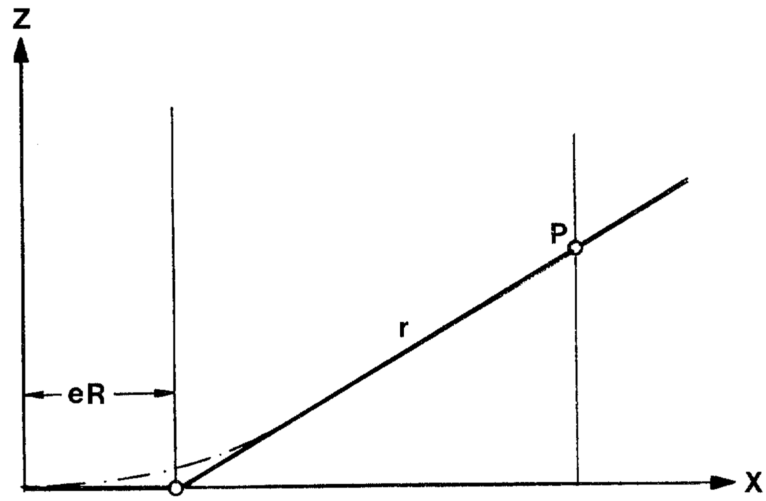


FIG. 1 Representation of hingeless blade.

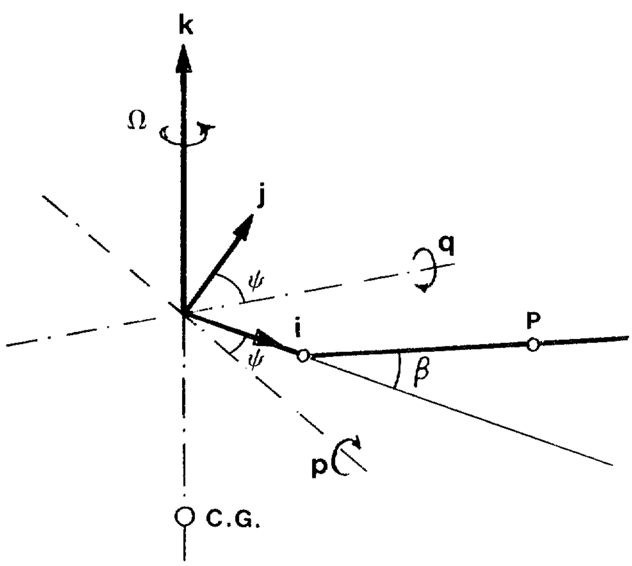


FIG. 2. Rotor coordinates.

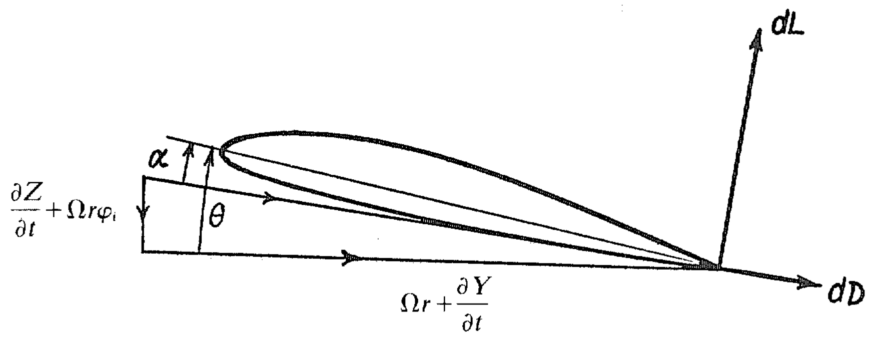


FIG. 3. Blade forces and velocities.

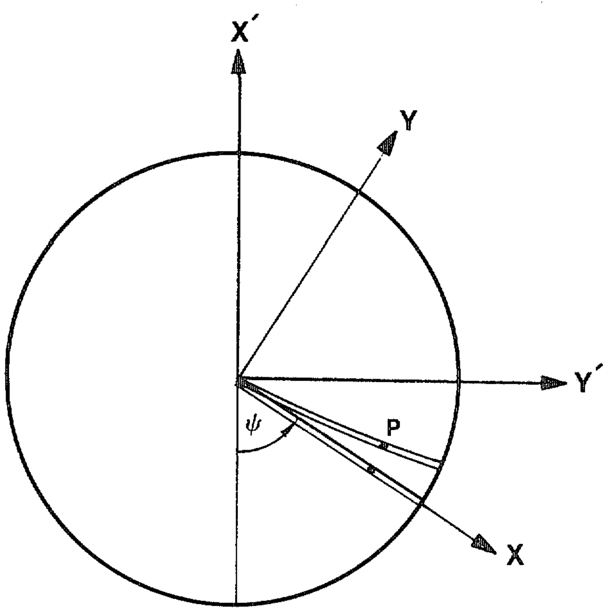


FIG. 4. Lag coordinates.

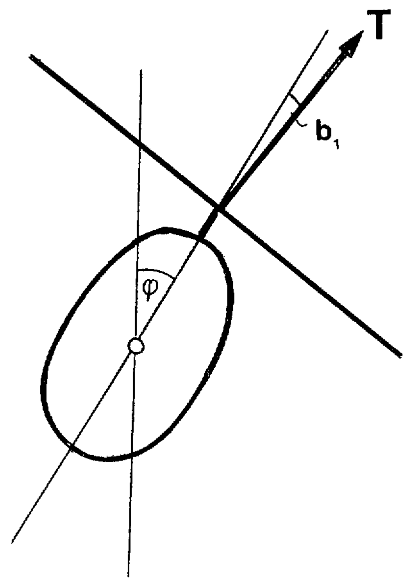


FIG. 5. Roll coordinates.

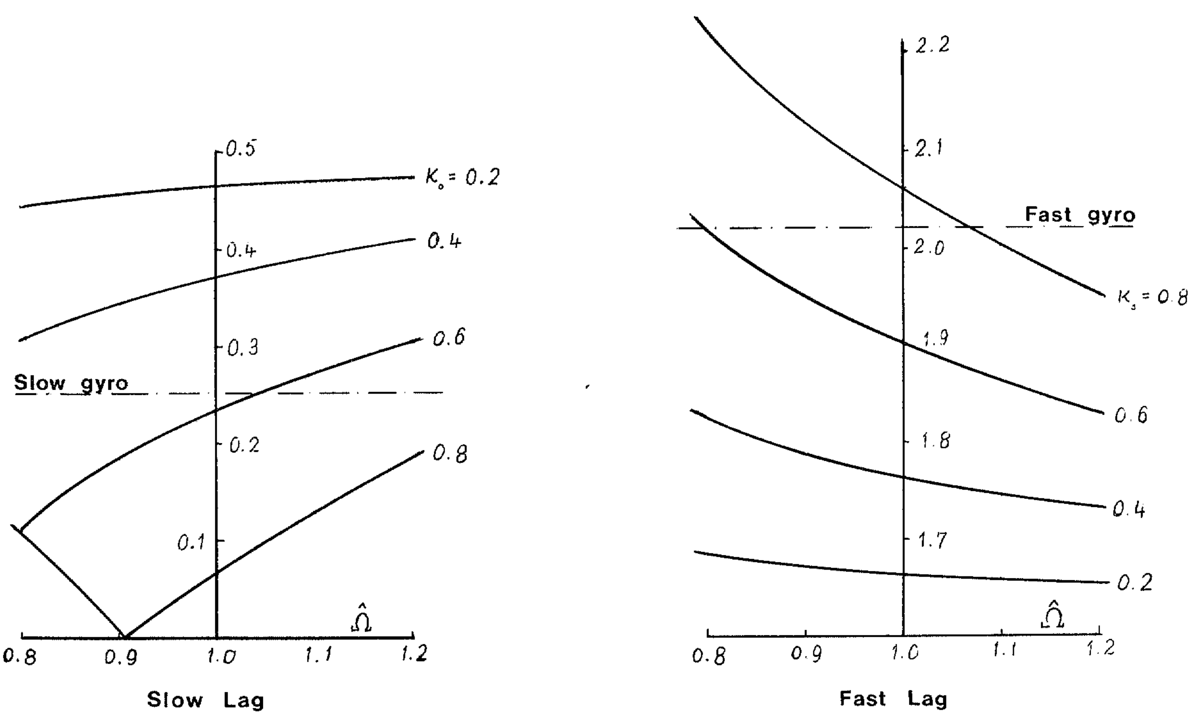


FIG. 6. Frequency ratios of lag modes.

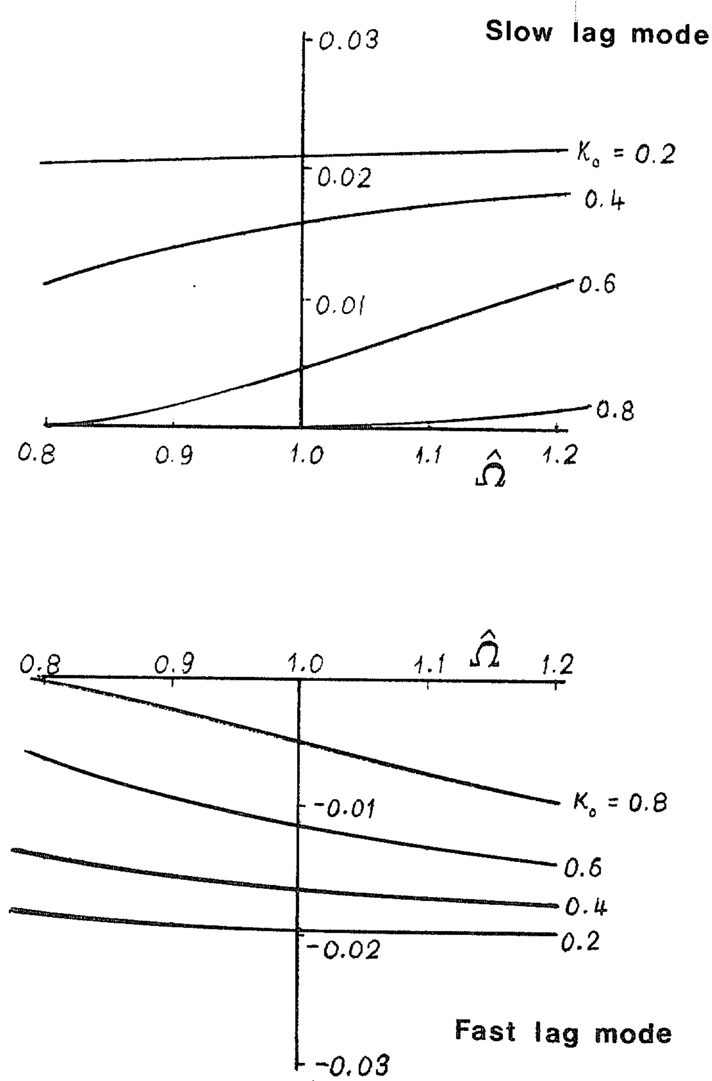
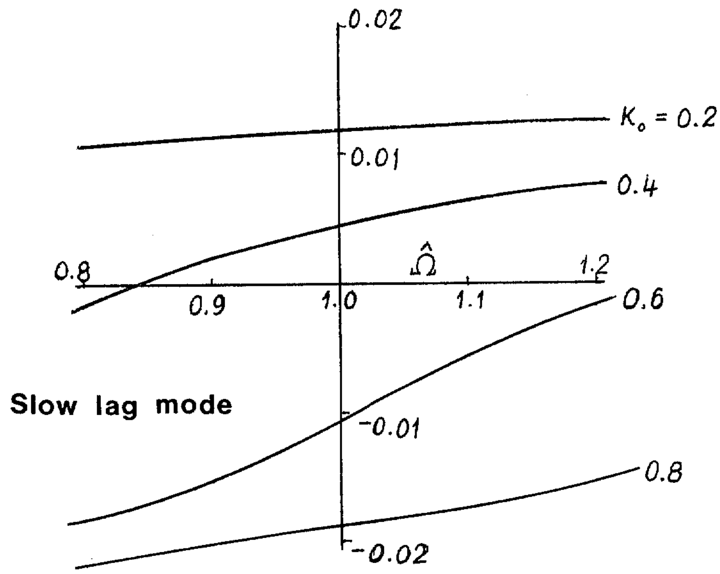


FIG. 7. Real part of lag mode roots, $\delta = 0$.



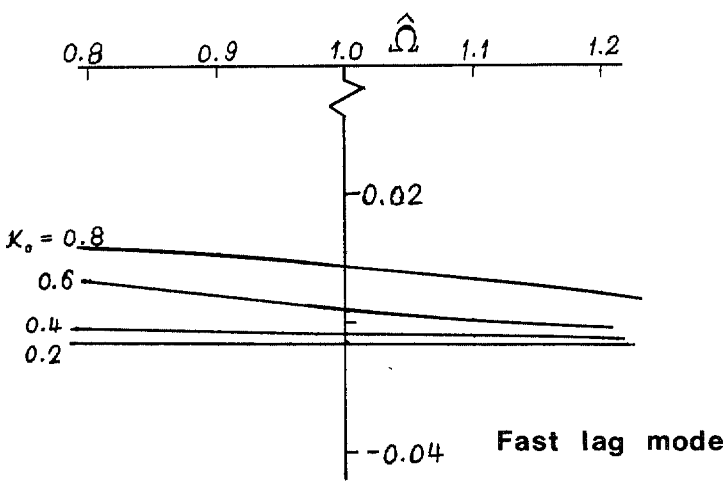


FIG. 8. Real parts of lag mode roots, $\delta = 0{\cdot}02$.

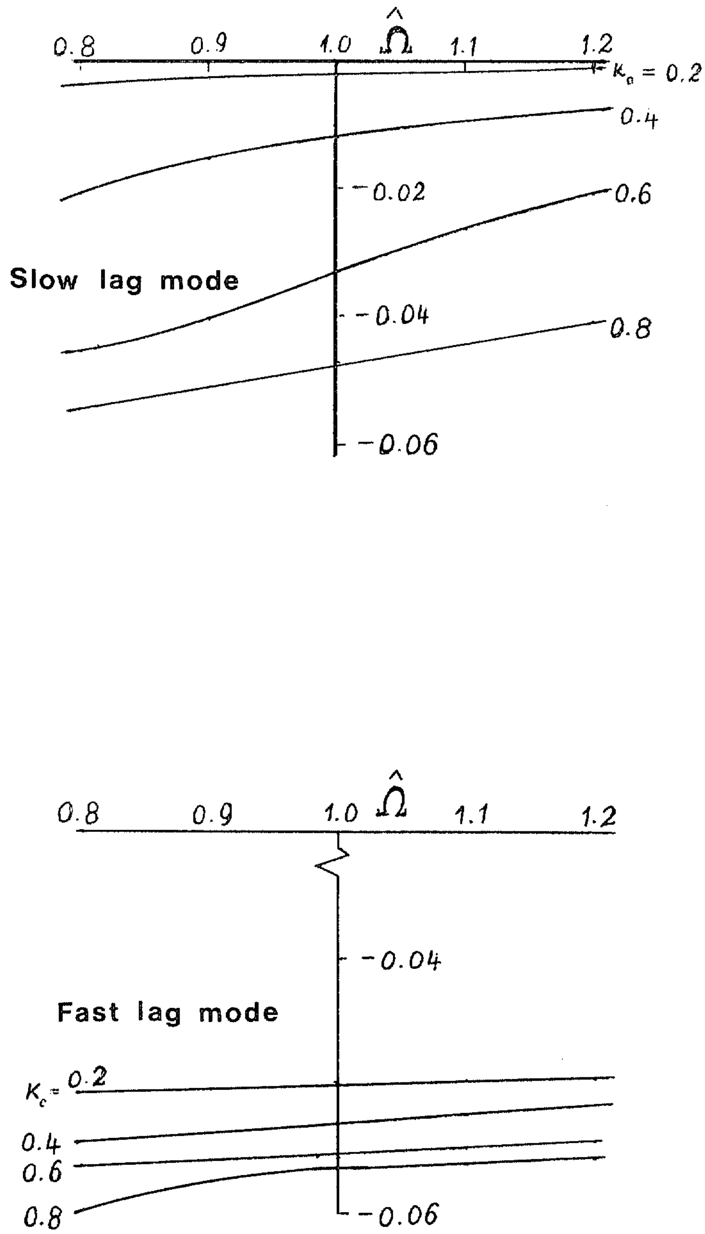
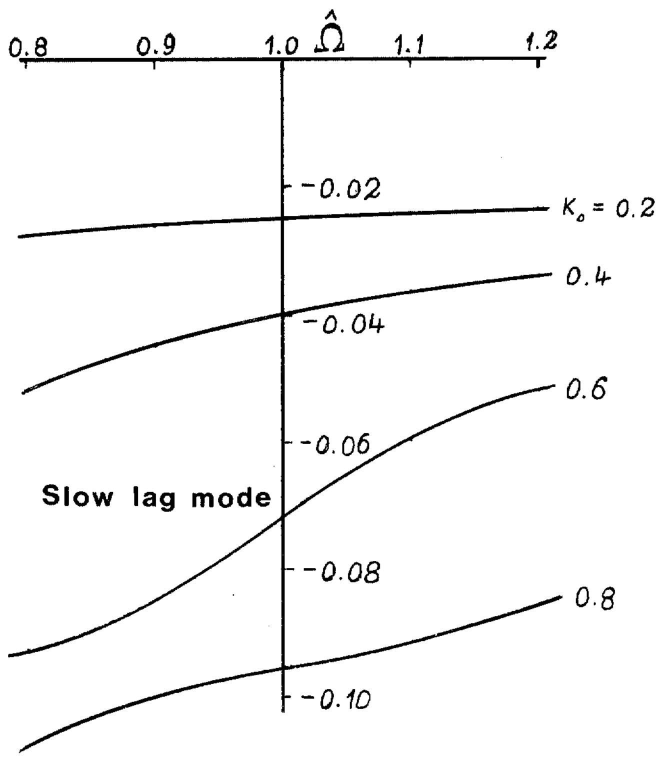


FIG. 9. Real parts of lag mode roots, $\delta = 0{\cdot}05$.



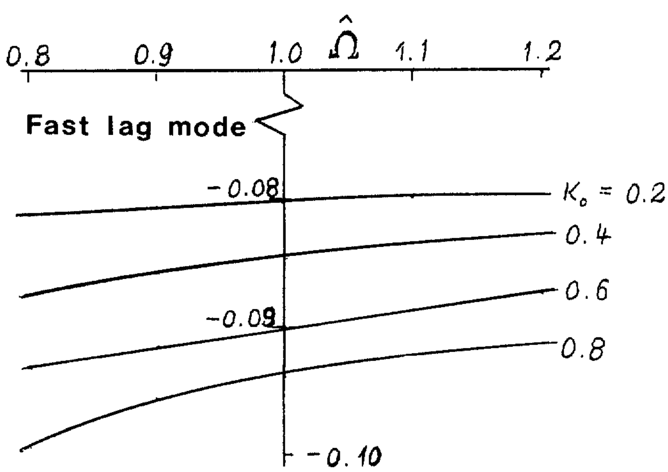


FIG. 10. Real parts of lag mode roots, $\delta = 0{\cdot}10$.

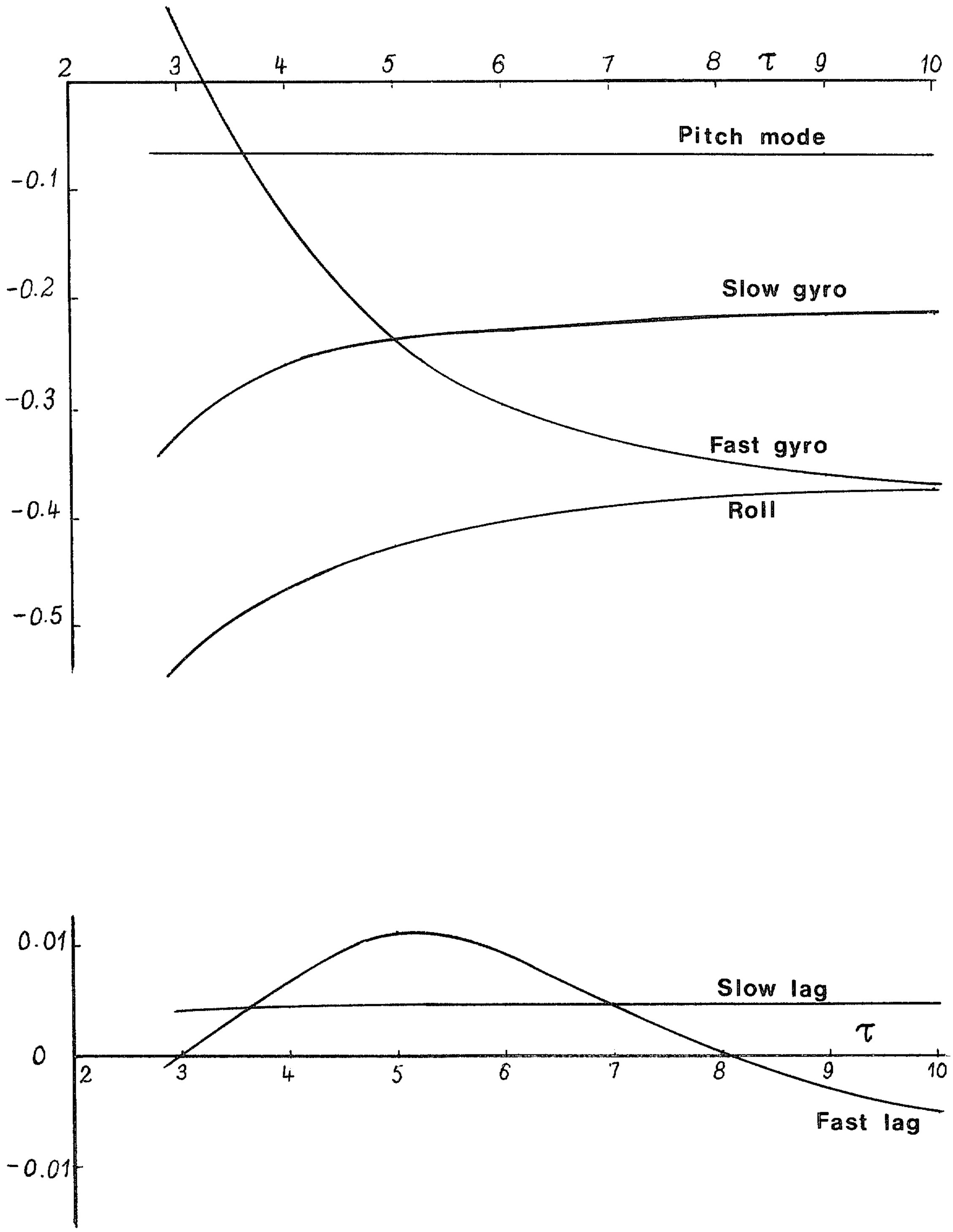


FIG. 11. Effect of torsional stiffness on mode damping.

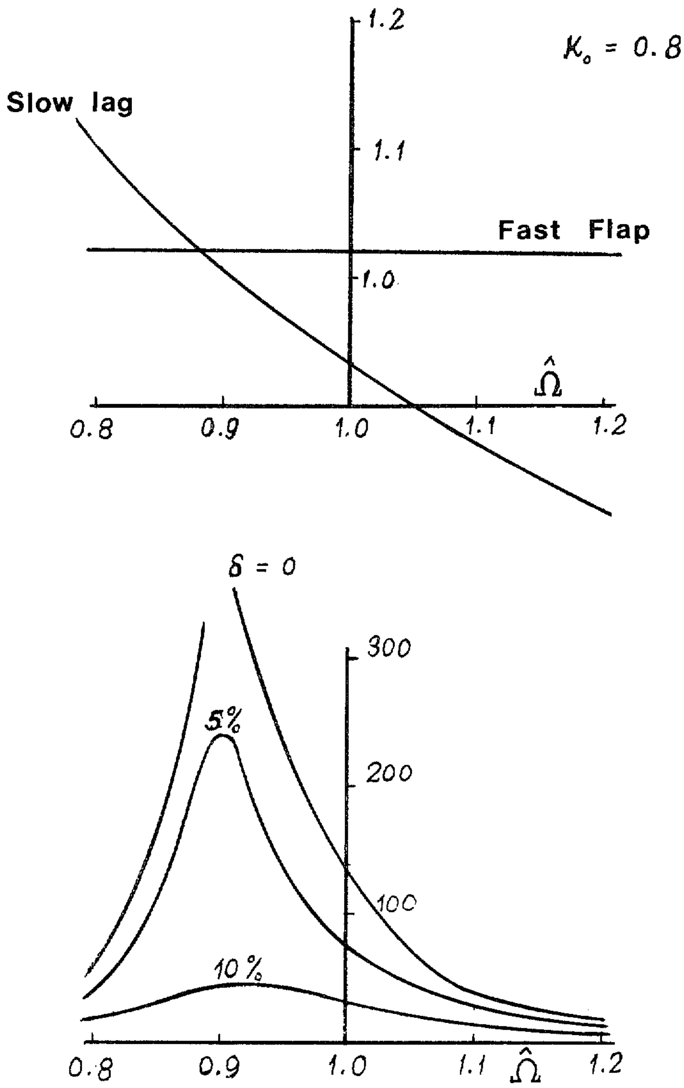


FIG. 12. Lag/flap ratios near resonance.

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