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Summary

The flow physics for inviscid flow on an arbitrary two-dimensional streamsurface for application to turbomachines are re-examined without specialising to a particular type of surface. This re-examination introduces streamfunction as a function of three space dimensions which is two dimensional in the locality of the prescribed streamsurface and does not involve the concept of a force normal to the surface as introduced by Wu[1] and used by Marsh[2] and Smith[3]. A special principal equation of motion is developed (designated the $\bar{N}$-principal equation) which is applicable to any streamsurface. This $\bar{N}$-principal equation has several advantages over the principal equations formerly used, the most powerful being that rotation of axes as carried out by Marsh[2] is no longer necessary and solutions become possible for annular ducting of increased geometrical complexity including flow around a toroidal surface as in a fluid coupling. Numerical advantages of the $\bar{N}$-principal equation are a saving in computer store and improved numerical accuracy in certain regions of the solution.

The co-ordinate form of the $\bar{N}$-principal equation is developed for application to hub-to-shroud streamsurfaces and for blade-to-blade streamsurfaces of revolution. The $\bar{N}$-principal form of equation is compared to the principal forms of Marsh[2] and Smith[3] where it is shown that the $\bar{N}$-principal form gains its advantage chiefly by its freedom from indeterminacy which inevitably appears with other forms. The latter are shown to be the equation of motion resolved in certain prescribed planes whereas the $\bar{N}$-principal form is always resolved normal to the stream direction. However the $\bar{N}$-principal form can be recovered from other forms if the equations are further developed.

Other forms of indeterminacy and computational failure may arise from the choice of co-ordinate system used to express the derivatives and the manner in which the molecule grid points are chosen. Other failures or restrictions relate to the way in which the grid is set up. These are discussed and proposals made for ensuring their avoidance.

* Replaces A.R.C. 34 317

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1. Introduction

Computer programs for the solution of two-dimensional inviscid flow on arbitrary streamsurfaces using fixed grids have leaned heavily on the form and development of equations in Wu's original work[1]. Marsh[2] developed one of the earliest programs to solve for rotational flow on an arbitrary meridional streamsurface and Smith[3] developed a similar program for application to a blade-to-blade streamsurface. All previous programs appear to have employed the principal equation of motion expressed in a fixed, prescribed direction. This feature places certain limitations upon the choice of machine geometry for which a solution is admissible and although the device of rotating the co-ordinate system used in Refs. 2 and 3 can alleviate this difficulty it does not remove it. Also, because the principal equation of motion is expressed in a prescribed direction, the resulting equation does not take the simplest form and when solved numerically offers a lower degree of accuracy and may also require more computation.

2. The Basic Problem

The arbitrary streamsurface (S') on which the flow is to take place may be expressed in terms of the co-ordinates (r, θ, z) thus

$$S'(r, \theta, z) = 0, \tag{1}$$

so that the unit normal vector to the surface is given by

$$\bar{n} = \frac{\bar{n}'}{n'} \qquad \text{(3-components)} \tag{2}$$

where

$$\bar{n}' = \nabla S' \qquad \text{(3-components).} \tag{3}$$

In the absence of viscous stresses and heat transfer and for steady flow relative to a rotating blade row the physical laws may be expressed

$$\text{(continuity)} \qquad \oint_S \rho\, \overline{W}.\, d\bar{S} = 0, \tag{4}$$

$$\text{(energy)} \qquad \frac{Ds}{Dt} = 0 \tag{5}$$

and

$$\text{(motion)} \qquad \overline{W} \times (\nabla \times \overline{V}) = \nabla I - T\nabla s \qquad \text{(3-components),} \tag{6}$$

with equations of state for a perfect gas

$$p = \rho RT, \tag{7}$$

$$u = C_v T \tag{8a}$$

and

$$s\text{-}s_1 = C_p \log \frac{T}{T_1} - R \log \frac{p}{p_1}, \tag{9a}$$

and definitions

$$h \equiv u + \frac{p}{\rho}, \tag{8b}$$

$$I \equiv h_0 - \bar{U} \, . \, \bar{V} = h + \frac{W^2 - U^2}{2} \tag{9b}$$

and

$$\bar{V} \equiv \bar{W} + \bar{\omega} \times \bar{R} \qquad \text{(3-components)} \tag{10}$$

where

$$\bar{R} = \bar{R}(r, \theta, z) \qquad \text{(3-components).} \tag{11}$$

Equations (4) to (10) form a set of 13 equations which determine the 13 variables

$$p, \rho, T, s, u, h, I, \bar{W}, \bar{V},$$

when the values of

$$R, C_p, C_v, \bar{\omega}, \bar{R}$$

are given.

It may be shown that the continuity equation (equation (4)) (*see* Appendix) is satisfied for a streamsheet of normal thickness t by a streamfunction ψ satisfying

$$\nabla\psi = \rho t \, \bar{W} \times \bar{n} \tag{12}$$

This definition also implies that the flow is constrained to the surface i.e.

$$\bar{W} . \bar{n} = 0, \tag{13}$$

and that ψ is constant along streamlines i.e.

$$\bar{W} . \nabla\psi = 0. \tag{14}$$

When required, the velocity $\bar{W}$ may be recovered from the streamfunction ψ by equation (12) since

$$\frac{1}{\rho t}\bar{n} \times \nabla\psi = \bar{n} \times (\bar{W} \times \bar{n}) \equiv \bar{W}(\bar{n} \, . \, \bar{n}) - \bar{n}(\bar{W} \, . \, \bar{n}) = \bar{W}. \tag{15}$$

3. The Equations of Motion

The three equations of motion (equation (6)) must be satisfied by each particle in the flow, but since the flow is constrained to the surface, then the component of the equation of motion taken normal to the surface merely provides a value for the normal force necessary to achieve this constraint. This component does not determine the flow on the prescribed streamsurface and this equation is replaced by the geometrical constraint expressed by equation (13) and embodied in equation (12).

The flow on the streamsurface will be determined and must satisfy at every point, any two equations of motion taken in *non-coincident* directions and lying in the tangent plane to the surface. It is desired to choose two such directions which satisfy this condition and lead to the simplest forms of the equations. The simplest component equation is obtained when the direction of flow, $\bar{W}/W$, is chosen. In this case the scalar product with equation (6) gives

$$0 = \frac{\bar{W}}{W} . \nabla I - T\frac{\bar{W}}{W} . \nabla s \tag{16}$$

and since for steady flow

$$\bar{W} . \nabla = \frac{D}{Dt}, \tag{17}$$

(i.e. derivative with respect to time for a given particle) then equations (5) and (16) result in

$$\frac{1}{W}\frac{DI}{Dt} = 0, \tag{18a}$$

or simply

$$\frac{DI}{Dt} = 0 \tag{18b}$$

since $W \neq 0$. This is here regarded as an equation of motion in the streamwise direction as by Wu[1] although I is an energy term and Marsh[2] regarded this as the energy equation.

Since from equations (14) and (17)

$$\frac{D\psi}{Dt} = 0, \tag{19}$$

then the energy equation (5) and streamwise equation of motion (18) may be written

$$s = s(\psi) \tag{20}$$

and

$$I = I(\psi), \tag{21}$$

and in this form will be shown to offer increased numerical accuracy, reduction in computation time and reduction of computer store requirement.

The choice of the second (or principal) equation of motion in the tangent plane is now arbitrary, provided it is never allowed to coincide with the streamwise direction. Previous workers have always chosen a prescribed direction determined by the intersection of a prescribed plane with the tangent plane in order to express this last equation (*see* Comparison with Other Principal Equations, Section 9) and as a result there are some flows for which the computer program must in principle fail to give a solution (because the prescribed direction will coincide with the streamwise direction), although in practice it may result only in local inaccuracy.

Since the streamwise equation of motion is satisfied by equation (21), the only other component of the equation of motion completely independent of it (i.e. not containing a component of the streamwise equation) that need be satisfied is that taken in a direction normal to the stream and lying in the tangent plane. This direction varies from point to point of the flow and changes as the solution is approached by successive approximation; it is not known in advance, but an expression for it leads to a much simpler form than those previously used for prescribed directions. If this direction is denoted by unit vector $\bar{N}$, then

$$\bar{N} \equiv \frac{\bar{W}}{W} \times \bar{n} \tag{22}$$

hence

$$\bar{W} \times \bar{N} = \bar{W} \times \left(\frac{\bar{W}}{W} \times \bar{n}\right) \equiv W\bar{n}. \tag{23}$$

If now the $\bar{N}$ component of the equation of motion, equation (6), is taken (hereafter called the $\bar{N}$-principal equation) then

$$\bar{N} \,.\, \bar{W} \times (\nabla \times \bar{V}) = \bar{N} \,.\, (\nabla I - T\nabla s);$$

therefore

$$W\bar{n} \,.\, (\nabla \times \bar{V}) = \frac{\bar{W}}{W} \times \bar{n} \,.\, (\nabla I - T\nabla s), \tag{24}$$

and by equations (20) and (21),

$$\nabla I - T\nabla s = \nabla\psi\left(\frac{dI}{d\psi} - T\frac{ds}{d\psi}\right) \tag{25}$$

when by equation (12)

$$\nabla I - T\nabla s = \rho t W\left(\frac{\overline{W}}{W} \times \bar{n}\right)\left(\frac{dI}{d\psi} - T\frac{ds}{d\psi}\right),$$

which by equation (24) yields the $\overline{N}$-principal equation

$$\bar{n}\,.\,\nabla \times \overline{V} = \rho t\left(\frac{dI}{d\psi} - T\frac{ds}{d\psi}\right) \tag{26}$$

(cf. Ref. 4 for axi-symmetrical flow) since

$$\left(\frac{\overline{W}}{W} \times \bar{n}\right)^2 = 1.$$

Equation (26) is much simpler in form than components resolved in arbitrary planes.

4. General Co-ordinate Forms for the Operators

To obtain a principal equation in ψ it is necessary to substitute $\overline{W}$ from equation (15) into equation (26), observing equation (10). To give co-ordinate expression to the resulting equation it is then necessary to express the operators $\bar{n} \times \nabla$ and $\bar{n}.\ \nabla\times$ in an appropriate set of co-ordinates. To do this, these operators are first expressed in their known forms for the r-θ-z co-ordinate system, after which it is necessary only to transform the derivatives from this co-ordinate system to a new co-ordinate system, one co-ordinate of which is the prescribed streamsurface S, thereby ensuring that some derivatives lie on the streamsurface (Marsh's special derivatives[2]). Let the other two co-ordinates x and y be entirely arbitrary. The derivative transform from r-θ-z to x-y-S may now be written and the surface derivatives

$$\left(\frac{\partial}{\partial x}\right)_{y,S},\quad \left(\frac{\partial}{\partial y}\right)_{x,S}$$

separated from the off-surface derivative

$$\left(\frac{\partial}{\partial S}\right)_{x,y}$$

in the following manner.

$$\begin{bmatrix} \left(\frac{\partial}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial}{\partial z}\right)_{r,\theta} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial x}{\partial r}\right)_{\theta,z} & \left(\frac{\partial y}{\partial r}\right)_{\theta,z} & \left(\frac{\partial S}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_{r,z} & \frac{1}{r}\left(\frac{\partial y}{\partial \theta}\right)_{r,z} & \frac{1}{r}\left(\frac{\partial S}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial x}{\partial z}\right)_{r,\theta} & \left(\frac{\partial y}{\partial z}\right)_{r,\theta} & \left(\frac{\partial S}{\partial z}\right)_{r,\theta} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial}{\partial x}\right)_{y,S} \\ \left(\frac{\partial}{\partial y}\right)_{x,S} \\ \left(\frac{\partial}{\partial S}\right)_{x,y} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\partial x}{\partial r}\right)_{\theta,z} & \left(\frac{\partial y}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_{r,z} & \frac{1}{r}\left(\frac{\partial y}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial x}{\partial z}\right)_{r,\theta} & \left(\frac{\partial y}{\partial z}\right)_{r,\theta} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial}{\partial x}\right)_{y,S} \\ \left(\frac{\partial}{\partial y}\right)_{x,S} \end{bmatrix} + \bar{n}'\left(\frac{\partial}{\partial S}\right)_{x,y} \tag{27}$$

where (*see* equation (3))

$$\bar{n}' = \nabla S = \begin{bmatrix} \left(\frac{\partial S}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial S}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial S}{\partial z}\right)_{r,\theta} \end{bmatrix} \tag{28}$$

is the surface normal vector.

In r-θ-z co-ordinates

$$\nabla \equiv \begin{bmatrix} \left(\frac{\partial}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial}{\partial z}\right)_{r,\theta} \end{bmatrix} \tag{29}$$

and substituting from equation (27)

$$\nabla = \nabla_T + \bar{n}'\left(\frac{\partial}{\partial S}\right)_{x,y}, \tag{30}$$

where ∇_T contains only the surface derivatives. It may be verified that

$$\bar{n} \times \bar{n}' \equiv 0 \tag{31}$$

hence the operator

$$\bar{n} \times \nabla = \bar{n} \times \nabla_T. \tag{32}$$

In r-θ-z form

$$\nabla \times = \begin{bmatrix} 0 & -\frac{1}{r}\left(\frac{\partial (r)}{\partial z}\right)_{r,\theta} & \frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial}{\partial z}\right)_{r,\theta} & 0 & -\left(\frac{\partial}{\partial r}\right)_{\theta,z} \\ -\frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)_{r,z} & \frac{1}{r}\left(\frac{\partial (r)}{\partial r}\right)_{\theta,z} & 0 \end{bmatrix} \cdot \tag{33a}$$

$$= [\nabla \times] \tag{33b}$$

and when appropriate substitutions are made from equation (27) the above operator satisfies the transformation

$$[\nabla\times] = [\nabla\times]_T + [n']\left[\frac{\partial}{\partial S}\right], \tag{34}$$

where

$$[n']\left[\frac{\partial}{\partial S}\right] = \begin{bmatrix} 0 & -n'_z & n'_\theta \\ n'_z & 0 & -n'_r \\ -n'_\theta & n'_r & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{\partial}{\partial S}\right)_{x,y} & 0 & 0 \\ 0 & \left(\frac{\partial}{\partial S}\right)_{x,y} & 0 \\ 0 & 0 & \left(\frac{\partial}{\partial S}\right)_{x,y} \end{bmatrix}.$$

It may be verified that

$$\bar{n}\,.\,[n'] \equiv 0, \tag{35}$$

hence

$$\bar{n}\,.\,\nabla\times\; = \bar{n}\,.\,[\nabla\times]_T \tag{36}$$

and contains surface derivatives only.

5. $\bar{N}$-Principal Equation for a Hub-to-Shroud Streamsurface

For a meridional streamsurface which is proscribed from being multiple valued in θ (i.e. $n_\theta \neq 0$ everywhere) as would be the case for any real turbomachine, a suitable choice of co-ordinates free from multiple valued functions is

$$x = r, \quad y = z$$

thus forming an r-z-S co-ordinate system. The transform matrix (equation (27)) for this case reads

$$\begin{bmatrix} \left(\frac{\partial r}{\partial r}\right)_{\theta,z} & \left(\frac{\partial z}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial r}{\partial \theta}\right)_{r,z} & \frac{1}{r}\left(\frac{\partial z}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial r}{\partial z}\right)_{r,\theta} & \left(\frac{\partial z}{\partial z}\right)_{r,\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so that by equation (32)

$$\bar{n} \times \nabla = \bar{n} \times \begin{bmatrix} \left(\frac{\partial}{\partial r}\right)_{z,S} \\ 0 \\ \left(\frac{\partial}{\partial z}\right)_{r,S} \end{bmatrix}$$

and equation (15) reads

$$\begin{bmatrix} W_r \\ \\ W_\theta \\ \\ W_z \end{bmatrix} = \frac{1}{\rho t}\begin{bmatrix} n_\theta\left(\dfrac{\partial\psi}{\partial z}\right)_{r,S} \\ \\ n_z\left(\dfrac{\partial\psi}{\partial r}\right)_{z,S} \quad -n_r\left(\dfrac{\partial\psi}{\partial z}\right)_{r,S} \\ \\ -n_\theta\left(\dfrac{\partial\psi}{\partial r}\right)_{z,S} \end{bmatrix}$$

$$= \frac{1}{\rho t'}\begin{bmatrix} \left(\dfrac{\partial\psi}{\partial z}\right)_{r,S} \\ \\ \dfrac{1}{n_\theta}\left\{n_z\left(\dfrac{\partial\psi}{\partial r}\right)_{z,S} \quad -n_r\left(\dfrac{\partial\psi}{\partial z}\right)_{r,S}\right\} \\ \\ -\left(\dfrac{\partial\psi}{\partial r}\right)_{z,S} \end{bmatrix} \tag{37}$$

where

$$t' = \frac{t}{n_\theta} = \text{tangential thickness of streamsheet.}$$

By equations (10), (36) and (37)

$$\bar{n}\,.\,\nabla \times \bar{V} = \frac{n_\theta}{\rho t'}\left\{\left(\frac{\partial^2\psi}{\partial z^2}\right)_{r,S} + \left(\frac{\partial^2\psi}{\partial r^2}\right)_{z,S} + \left\{W_z\left(\frac{\partial(\rho t')}{\partial r}\right)_{z,S} - W_r\left(\frac{\partial(\rho t')}{\partial z}\right)_{r,S}\right\}\right\} + \frac{1}{r}\left\{n_z\left(\frac{\partial(rV_\theta)}{\partial r}\right)_{z,S} - n_r\left(\frac{\partial(rV_\theta)}{\partial z}\right)_{r,S}\right\},$$

so that the $\bar{N}$-principal equation (26) may be solved for ψ in the form

$$\left(\frac{\partial^2\psi}{\partial r^2}\right)_{z,S} + \left(\frac{\partial^2\psi}{\partial z^2}\right)_{r,S} = (\rho t')^2\left(\frac{dI}{d\psi} - T\frac{ds}{d\psi}\right) + \frac{\rho t'}{rn_\theta}\left\{n_r\left(\frac{\partial(rV_\theta)}{\partial z}\right)_{r,S} - n_z\left(\frac{\partial(rV_\theta)}{\partial r}\right)_{z,S}\right\}$$
$$+ \left\{W_r\left(\frac{\partial(\rho t')}{\partial z}\right)_{r,S} - W_z\left(\frac{\partial(\rho t')}{\partial r}\right)_{z,S}\right\} \tag{38}$$

and always expresses the equation of motion normal to the stream direction. It will be observed that the only divisor on the r.h.s. is the non-zero value (rn_θ); this feature makes the rotation of co-ordinates for solving flow with particular geometries, unnecessary.

6. $\bar{N}$-Principal Equation for a Blade-to-blade Streamsurface

Blade-to-blade streamsurfaces are usually surfaces of revolution (i.e. $n_\theta = 0$ everywhere) in order that simple cyclic boundary conditions may be realistically imposed on certain boundaries. The surface of revolution may be multiple valued in r and/or z making these co-ordinates unsuitable and the best choice of co-ordinates in this case is meridional distance $m = m(r, z)$ and tangential co-ordinate θ, forming an m-θ-S system.

The following simple geometrical properties of this streamsurface will be found useful

$$n_r^2 + n_z^2 = 1, \tag{39}$$

$$n_r = \left(\frac{\partial S}{\partial r}\right)_{\theta,z} = -\left(\frac{\partial m}{\partial x}\right)_{r,\theta}, \tag{40}$$

$$n_z = \left(\frac{\partial S}{\partial z}\right)_{r,\theta} = \left(\frac{\partial m}{\partial r}\right)_{\theta,z} \tag{41}$$

$$\left(\frac{\partial n_r}{\partial \theta}\right)_{r,z} = 0 \tag{42}$$

and

$$\left(\frac{\partial n_z}{\partial \theta}\right)_{r,z} = 0. \tag{43}$$

The transform matrix (equation (27)) for this case reads,

$$\begin{bmatrix} \left(\frac{\partial m}{\partial r}\right)_{\theta,z} & \left(\frac{\partial \theta}{\partial r}\right)_{\theta,z} \\ \frac{1}{r}\left(\frac{\partial m}{\partial \theta}\right)_{r,z} & \frac{1}{r}\left(\frac{\partial \theta}{\partial \theta}\right)_{r,z} \\ \left(\frac{\partial m}{\partial z}\right)_{r,\theta} & \left(\frac{\partial \theta}{\partial z}\right)_{r,\theta} \end{bmatrix} = \begin{bmatrix} n_z & 0 \\ 0 & \frac{1}{r} \\ -n_r & 0 \end{bmatrix}. \tag{44}$$

and by equation (32)

$$\bar{n} \times \nabla = \bar{n} \times \begin{bmatrix} n_z\left(\frac{\partial}{\partial m}\right)_{\theta,S} \\ \frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)_{m,S} \\ -n_r\left(\frac{\partial}{\partial m}\right)_{\theta,S} \end{bmatrix}, \tag{45}$$

so that by equation (10), equation (15) may be written,

$$\begin{bmatrix} W_r \\ W_\theta \\ W_z \end{bmatrix} = \begin{bmatrix} n_z \ W_m \\ W_\theta \\ -n_r \ W_m \end{bmatrix} = \begin{bmatrix} n_z \ V_m \\ V_\theta \ -\omega r \\ -n_r \ V_m \end{bmatrix} = \frac{1}{\rho t}\begin{bmatrix} -\frac{n_z}{r}\left(\frac{\partial \psi}{\partial \theta}\right)_{m,S} \\ -\left(\frac{\partial \psi}{\partial m}\right)_{\theta,S} \\ \frac{n_z}{r}\left(\frac{\partial \psi}{\partial \theta}\right)_{m,S} \end{bmatrix}. \tag{46}$$

By equations (36) and (46) and observing equations (39) to (43),

$$\begin{aligned} \bar{n}\,.\,\nabla \times \bar{V} &= \left(\frac{\partial W_\theta}{\partial m}\right)_{\theta,S} - \frac{1}{r}\left(\frac{\partial W_m}{\partial \theta}\right)_{m,S} - n_z\left(\omega + \frac{V_\theta}{r}\right) \\ &= \frac{1}{\rho t}\left\{\left(\frac{\partial^2 \psi}{\partial m^2}\right)_{\theta,S} + \frac{1}{r^2}\left(\frac{\partial^2 \psi}{\partial \theta^2}\right)_{m,S} - W_\theta\left(\frac{\partial(\rho t)}{\partial m}\right)_{\theta,S} \right. \\ &\qquad \left. + \frac{W_m}{r}\left(\frac{\partial(\rho t)}{\partial \theta}\right)_{m,S}\right\} - n_z\left(\omega + \frac{V_\theta}{r}\right), \end{aligned}$$

so that the $\bar{N}$-principal equation (26) may be solved for ψ in the form

$$\left(\frac{\partial^2 \psi}{\partial m^2}\right)_{\theta,S} + \frac{1}{r^2}\left(\frac{\partial^2 \psi}{\partial \theta^2}\right)_{m,S} = (\rho t)^2\left(\frac{dI}{d\psi} - T\frac{ds}{d\psi}\right) + \rho t\, n_z\left(\omega + \frac{V_\theta}{r}\right)$$

$$+\left\{W_\theta\left(\frac{\partial(\rho t)}{\partial m}\right)_{\theta,S}-\frac{W_m}{r}\left(\frac{\partial(\rho t)}{\partial\theta}\right)_{m,S}\right\}. \tag{47}$$

It is usual to consider the streamsheet normal thickness, t, to be constant in the θ-direction, so that

$$\left(\frac{\partial(\rho t)}{\partial\theta}\right)_{m,S}=t\left(\frac{\partial\rho}{\partial\theta}\right)_{m,S}, \tag{48}$$

and it will again be observed that the only divisor in equation (47) (i.e. r) is non-zero, thereby making it unnecessary to rotate the co-ordinate system as in Ref. 3.

It has also been customary to assume for axi-symmetric streamsurfaces that $dI/d\psi = 0$ and $ds/d\psi = 0$ so that equation (26) reduces to

$$\bar{n}\,.\,\nabla\times\bar{V}=0, \tag{49}$$

i.e. the component of absolute vorticity normal to the streamsurface is zero, but this does not imply that the three dimensional flow of which this streamsurface is part, is necessarily irrotational. It does imply that such vorticity as may be present lies in the surface tangent plane.

An equation for flow in the purely radial plane as analysed by Stanitz[5] is recovered from equation (47) in the form of equation (48) when

$$n_z = 1,$$

$$m = r \quad \text{and} \quad S = z$$

so that

$$\left(\frac{\partial^2\psi}{\partial r^2}\right)_{\theta,z}+\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2}\right)_{r,z}=\rho t\left(\omega+\frac{V_\theta}{r}\right)+W_\theta\left(\frac{\partial(\rho t)}{\partial r}\right)_{\theta,z}-\frac{tW_r}{r}\left(\frac{\partial\rho}{\partial\theta}\right)_{r,z}. \tag{50}$$

For an ideal axial flow machine for which the particles are constrained on a cylindrical surface

$$m = z \quad \text{and} \quad S = r$$

so that equation (47) reads

$$\left(\frac{\partial^2\psi}{\partial z^2}\right)_{r,\theta}+\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2}\right)_{r,z}=W_\theta\left(\frac{\partial(\rho t)}{\partial z}\right)_{\theta,r}-\frac{tW_z}{\rho}\left(\frac{\partial\rho}{\partial\theta}\right)_{z,r}. \tag{51}$$

7. Indeterminacy in the Principal Equation

It has been remarked that the principal equation may be any equation of motion resolved in the tangent plane but that the direction of resolution should not coincide with the stream direction. It is shown below that if this event does occur then the principal equation may become indeterminate and therefore that the analytical solution is not defined. Consider two mutually normal but otherwise arbitrary directions (x, y) lying in the tangent plane. Let the x-direction, defined by the unit vector $\bar{x}$, make an angle θ to the streamwise direction $\bar{W}/W$, then resolution of the equation of motion (equation (6)) in this direction leads to

$$\bar{x}\,.\,\bar{W}\times(\nabla\times\bar{V})=W\sin\theta\bar{n}\,.\,\nabla\times\bar{V}=\bar{x}\,.\,(\nabla I-T\nabla s),$$

and since

$$W\sin\theta = W_y$$

then

$$\bar{n}\,.\,\nabla \times \bar{V} = \frac{1}{W_y}\bar{x}\,.\,(\nabla I - T\nabla s) \tag{52a}$$

$$= \frac{1}{W_y}\left(\frac{\partial I}{\partial x} - T\frac{\partial S}{\partial x}\right). \tag{52b}$$

If $\bar{x}$ coincides with the stream direction $\bar{W}/W$, then as seen at equation (16), the numerator of the r.h.s. of equation (52) is zero but with this occurrence, W_y becomes the velocity component normal to the stream direction which is simultaneously zero. In this situation, then the r.h.s. of the principal equation expressed in the form of equation (52) becomes indeterminate and the equation fails to define the flow. Equation (52) is a generalised form of the usual expression for the principal equation as used by most workers who have analysed flow with a gradient of I and s (Refs. 2, 3 and 4).

The condition of indeterminacy of equation (52) will usually occur only on a given contour (or contours) in the flow field on which the direction of resolution $\bar{x}$ coincides with the flow direction. Whereas an analytical solution would fail completely in this event, numerical solutions usually do not. A numerical solution with such contours present will usually be successful but will be inaccurate in the neighbourhood of these contours and in subsonic flow the inaccuracy will remain local to the contours (unless of course W_y becomes zero to machine round-off at a grid point which would cause computation failure).

This indeterminacy can be avoided by rotating the axes of reference (Ref. 2) so that the direction $\bar{x}$ may be anticipated not to coincide with the direction of flow at any part of the solution. This device can be successfully applied (Ref. 2) then, if

(a) the direction of flow deflects through less than 180 degrees in the plane of reference and

(b) there are no eddies in the flow field (really a corollary of (a)).

With hub-to-shroud streamsurfaces, eddies do not usually appear in inviscid flow and in that part of the machine normally under investigation, the deflection in the r-z plane is frequently less than 180 degrees. With blade-to-blade streamsurfaces the situation is more restrictive in that inviscid eddies do appear on blade surfaces of mixed flow impellers when indeterminacy will be inevitable.

There are however other difficulties with eddies present in that the distribution of I and s within them, is not defined by the upstream boundary conditions.

It is usual in blade-to-blade flows on a surface of revolution to assume that ∇I and ∇S are zero and so to omit these terms from the formulation when the r.h.s. of equation (52) is identically zero and the indeterminacy will not occur.

It will be noticed that for the $\bar{N}$-principal form of the equation (26) the r.h.s. is always determinate, so that there are no limitations on its application and the rotation of axes is unnecessary. If equations (20) and (21) are observed it will be seen that the general form of the principal equation (52) may always be reduced to the $\bar{N}$-principal form since

$$W_y = \frac{1}{\rho t}\left(\frac{\partial\psi}{\partial x}\right)_{y,S}, \tag{53}$$

$$\frac{\partial I}{\partial x} = \frac{dI}{d\psi}\left(\frac{\partial\psi}{\partial x}\right)_{y,S} \tag{54}$$

and

$$\frac{\partial S}{\partial x} = \frac{ds}{d\psi}\left(\frac{\partial\psi}{\partial x}\right)_{y,S} \tag{55}$$

when

$$\bar{n}\,.\,\nabla \times \bar{V} = \rho t\left(\frac{dI}{d\psi} - T\frac{ds}{d\psi}\right)$$

(*see* equation (26)).

8. Numerical Accuracy and Computer Store Demand

In Section 7, attention is drawn to the numerical inaccuracy that will result when the direction of resolution of the principal equation is close to the stream direction. In practice of course this inaccuracy is always present in a numerical solution and reduces to a minimum when the direction of resolution is normal to the stream direction as is always the case with the $\bar{N}$-principal equation.

With the derivatives of I and s in the partial form of equation (52) it is necessary to store values of I and s at all grid points in order to compute these derivatives numerically. With the $\bar{N}$-principal form of the equation (26) it may be seen from equations (18b), (19) and (21) that

$$\frac{D}{Dt}\left(\frac{dI}{d\psi}\right) = \frac{d}{d\psi}\left(\frac{dI}{d\psi}\right)\frac{D\psi}{Dt} = 0, \tag{56}$$

and that therefore $(dI/d\psi)$ being constant along a streamline, may be determined from the upstream boundary values alone and only these values need be stored. Equations (5), (19) and (20) show that similar remarks apply to s. Where computer store is at a premium this can be a considerable saving.

9. Comparison with Other Principal Equations

Using the generalised transform equation (30), the equation of motion (26) may be written

$$\bar{W} \times (\nabla_T \times \bar{V}) = \nabla_T I - T\nabla_T s + \bar{n}\left(\frac{1}{\rho}\left(\frac{\partial p}{\partial S}\right)_{x,y} - \frac{\partial}{\partial S}\left(\frac{U^2}{2}\right)_{x,y}\right), \tag{57}$$

where it will be re-called that ∇_T contains only surface derivatives (*see* equation (27)). It may be seen clearly from equation (57) that only a component in the tangent plane will eliminate the vector

$$\bar{n}\left[\frac{1}{\rho}\left(\frac{\partial p}{\partial S}\right)_{x,y} - \frac{\partial}{\partial S}\left(\frac{U^2}{2}\right)_{x,y}\right] \tag{58}$$

which is normal to this plane and is not defined by properties on the surface. This vector has been regarded as a force $\bar{f}$ (Refs. 2 and 3) normal to the streamsurface and has been eliminated by resolving equation (57) into the tangent plane by using two only of its three r, θ, z components.

For a hub-to-shroud streamsurface, Marsh[2] offers a choice of resolution using either the r and θ or the z and θ components thus resolving the equation in the r-θ or z-θ planes respectively. When this resolution is carried out so as to eliminate $\bar{f}$, the resulting equation of motion is expressed in the direction of intersection of the r-θ and tangent planes or the z-θ and tangent planes, respectively. *See* Fig. 2. With this choice of co-ordinates ($x = r, y = z$)

$$\frac{\partial}{\partial S}\left(\frac{U^2}{2}\right)_{x,y} = \frac{\partial}{\partial S}\left(\frac{U^2}{2}\right)_{r,z} = 0$$

it is apparent that either of these directions will coincide with the stream direction at all points of the flow where $W_z = 0$ or $W_r = 0$, respectively, with consequent indeterminacy (previously mentioned) at these points and the *two* necessary equations of motion will there fail to be satisfied. The implications for turbomachine flow are that the first choice of principal equation will fail to define the flow in regions such as the outlet from a centrifugal impeller or the diffuser of such a machine or similarly through the nozzle ring or entry to the impeller of a centrifugal inflow turbine where $W_z = 0$. However for an axial-flow machine to which Marsh applied the program, the first choice of principal equation which he used seems to be entirely satisfactory.

By an analogous argument to that above, the application of the second choice of principal equation would fail to define the flow through an axial flow machine, at inlet to a typical centrifugal impeller or outlet from a typical radial inflow turbine impeller where $W_r = 0$.

If either of Marsh's principal equations (Ref. 2, equations (24) and (25)) *viz.*

$$\left(\frac{\partial^2\psi}{\partial r^2}\right)_{z,S} + \left(\frac{\partial^2\psi}{\partial z}\right)_{r,S} = \left(\frac{\partial\psi}{\partial r}\right)_{z,S}\frac{\partial}{\partial r}(\log(r\rho t))_{z,S} + \left(\frac{\partial\psi}{\partial z}\right)_{r,S}\frac{\partial}{\partial z}(\log(\rho r t))_{r,S}$$
$$+ \frac{r\rho t}{W_z}\left\{\left(\frac{\partial I}{\partial r}\right)_{z,S} - T\left(\frac{\partial S}{\partial r}\right)_{z,S} - \frac{W_\theta}{r}\frac{\partial}{\partial r}(rV_\theta)_{z,S} - f_r\right\} \tag{59a}$$

and

$$\left(\frac{\partial^2\psi}{\partial r^2}\right)_{z,S} + \left(\frac{\partial^2\psi}{\partial z}\right)_{r,S} = \left(\frac{\partial\psi}{\partial r}\right)_{z,S}\frac{\partial}{\partial r}(\log(r\rho t))_{z,S} + \left(\frac{\partial\psi}{\partial z}\right)_{r,S}\frac{\partial}{\partial z}(\log(r\rho t))_{r,S}$$

$$-\frac{r\rho t}{W_r}\left\{\left(\frac{\partial I}{\partial z}\right)_{r,S} - T\left(\frac{\partial S}{\partial z}\right)_{r,S} - \frac{W_\theta}{r}\frac{\partial}{\partial z}(rV_\theta)_{r,S} - f_z\right\} \tag{59b}$$

is further developed by substituting for f_r and f_z from the θ-component equation and cognisance is taken of equations (13), (53), (54) and (55) the $\bar{N}$-principal equation (38) results.

Smith[3] undertakes a similar analysis to that of Marsh[2], applied to a blade-to-blade surface of revolution. Smith offers a choice of principal equation resolved in either the r-θ or the r-z plane so that his equations (Ref. 3 equations (27.15a) and (27.15b) *viz.*

$$\frac{1}{r}\left(\frac{\partial^2\psi}{\partial\theta^2}\right)_{z,S} + \left(\frac{\partial^2\psi}{\partial z^2}\right)_{\theta,S} = \frac{1}{r^2}\frac{\partial}{\partial\theta}(\log(\rho t))_{z,S}\left(\frac{\partial\psi}{\partial\theta}\right)_{z,S} + \frac{\partial}{\partial z}(\log(\rho t))_{\theta,S}\left(\frac{\partial\psi}{\partial z}\right)_{\theta,S}$$

$$-\frac{\rho t}{W_z}\left\{-\frac{1}{r}\left(\frac{\partial I}{\partial\theta}\right)_{z,S} + \frac{T}{r}\left(\frac{\partial S}{\partial\theta}\right)_{z,S} + \frac{W_z\tan\lambda}{r}\frac{\partial}{\partial\theta}(W_z\tan\lambda)_{z,S} + W_z\tan\lambda\left(2+\frac{W_\theta}{r}\right)\right\} \tag{60a}$$

and

$$\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2}\right)_{z,S} + \left(\frac{\partial^2\psi}{\partial z^2}\right)_{\theta,S} = \frac{1}{r^2}\frac{\partial}{\partial\theta}(\log(\rho t))_{z,S}\left(\frac{\partial\psi}{\partial\theta}\right)_{z,S} + \frac{\partial}{\partial z}(\log(\rho t))_{\theta,S}\left(\frac{\partial\psi}{\partial z}\right)_{\theta,S}$$

$$+\frac{\rho t}{W_\theta}\left\{-\left(\frac{\partial I}{\partial z}\right)_{\theta,S} + T\left(\frac{\partial S}{\partial z}\right)_{\theta,S} + f_z + W_z\tan\lambda\frac{\partial}{\partial z}(W_z\tan\lambda)_{\theta,S}\right\} \tag{60b}$$

express the component of the equation of motion in the direction of the line of intersection of these planes with the surface tangent plane. *See* Fig. 3. Since in fact the θ-component already lies in the tangent plane to a surface of revolution then no component of $\bar{f}$ appears in this equation and the resolution is null. Resolution in the r-z plane is however necessary and results in the equation being expressed in the meridional direction when at all points of the flow for which $W_\theta = 0$, this direction coincides with the stream direction and indeterminacy ensues. Again, further development of equations (60) substituting for f_z from the r-component equation and observing equations (13), (53), (54) and (55) leads to the determinate, $\bar{N}$-principal form equation (47).

Equation (60a) which satisfies the equation of motion in the θ-direction will be indeterminate on contours in the flow field where W_m (and therefore W_z) is zero. Such will be the case everywhere on a certain contour joining the two stagnation points within an eddy on a blade surface. Smith's second principal equation (60b) will be indeterminate for contours on which W_θ is zero which would be particularly serious for many turbomachine impellers such as the top half of typical centrifugal compressor blades where the blades, being straight and radial, cause the streamwise direction to coincide with the meridional direction over a considerable part of the blade chord. Other types of blading suffering from this condition would be typical inlet guidevanes, outlet guidevanes and impulse blading.

As Smith appears ultimately to restrict application to flows having no gradient of I or s as is usual for streamsurfaces of revolution, then as pointed out (*see* Section 7) the indeterminacy vanishes as may be seen from equation (27.18) Ref. 3. Smith points out that his rotation of axes in this situation is to make his streamsurface S single valued in the space co-ordinate x and thereby maintain finite derivatives, and not to avoid indeterminacy of the equations (*see* Section 6).

10. Choice of Molecule

While computational failure may arise from a poor choice of principal equation, it may, as mentioned by Marsh[2], also arise in the process of determining the derivatives. The derivatives at a given grid point, referred to as 'nodal', are normally estimated as in Ref. 2 by assuming a local polynomial fit (in the form of a truncated Taylor series) to be made to a selected number of grid points, including the nodal one, the whole set of such points being referred to as a molecule. The number of points selected depends upon the order of derivative to be determined and the order of discretization error to which it is required, while the location of the points is to some extent arbitrary. Representing the function f at molecule point i by f_i and partial surface derivatives in arbitrary surface directions x and y at the nodal point by f_j as below, the truncated Taylor series may be expressed.

$$f_i = A_{ij} f_j \tag{61}$$

where

$$\begin{aligned}
f_{j=1} &= f, & A_{i1} &= 1, \\
f_{j=2} &= \frac{\partial f}{\partial x}, & A_{i2} &= x_i - x_0, \\
f_{j=3} &= \frac{\partial f}{\partial y}, & A_{i3} &= y_i - y_0, \\
f_{j=4} &= \frac{\partial^2 f}{\partial x^2}, & A_{i4} &= (x_i - x_0)^2/2!, \\
f_{j=5} &= \frac{\partial^2 f}{\partial x \partial y}, & A_{i5} &= (x_i - x_0)(y_i - y_0), \\
&\text{etc} & &\text{etc.}
\end{aligned}$$

and $i = 1$ refers to the nodal point.

The derivatives f_j are now determined from equation (61) thus

$$f_j = (A_{ij})^{-1} f_i \tag{62}$$

and are defined only in the case of A_{ij} being non-singular. It is important to establish what choice of molecule points will result in singularity of A_{ij} with consequent computational failure or inaccuracy (in practice).

The simplest cause of failure, not likely to arise in practice except by a programming error, arises if the molecule contains the same grid point chosen more than once. In this event two or more rows of A_{ij} and the corresponding f_i are identical and the equations are indeterminate.

Failure can occur in a less trivial and obvious manner. Consider the case of a three point molecule for which

$$|A_{ij}| \equiv \begin{vmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & (y_1 - y_0) \\ 1 & (x_2 - x_0) & (y_2 - y_0) \end{vmatrix}, \tag{63}$$

clearly if $x_0 = x_1 = x_2$ the second column is zero and if $y_0 = y_1 = y_2$ the third column is zero. The matrix is then singular if all three points lie on the same co-ordinate line i.e. either x or y, which is not surprising since then, one or other of the first derivatives which the matrix A defines, is not defined by the three functional values given. More generally the matrix is singular if all three points lie on any straight line, since then

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_2 - y_0}{x_2 - x_0} \tag{64}$$

$$|[A]| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{y_1 - y_0}{x_1 - x_0} \\ 0 & 1 & \dfrac{y_2 - y_0}{x_2 - x_0} \end{vmatrix} (x_1 - x_0)(x_2 - x_0),$$

by equation (64),

$$\|[A]\| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & \dfrac{y_2 - y_0}{x_2 - x_0} \end{vmatrix} (x_1 - x_0)(x_2 - x_0)$$

$$= 0. \tag{65}$$

For molecules with a greater number of grid points the argument and proof of failure becomes increasingly burdensome and tedious and so is not here pursued but the following simple rules should ensure $[A]$ does not become singular.

Fig. 4 illustrates a distribution of grid points from which the matrix A may be set up to ensure that a given order of discretisation error is achieved for derivatives of each successive order.

The form shown in Fig. 4 is a suitable distribution of points for derivatives to be determined at the lower left hand corner of a grid when the lower left hand grid point would be the nodal point and all derivatives with respect to x and y would of necessity be by forward differences. The purpose of the figure is more to illustrate the relative positioning of grid points for derivatives of different order and to indicate the total number of grid points required to determine that derivative by the basic procedure of equation (61). The total number of grid points required is indicated along the y-axis and is the number of grid points contained on and within the triangle of which there is one to each order of derivative. The triangle is formed by the hypotenuse (to which numbers relate) and the x and y axes respectively. The total number of points required for a derivative of given order is of course also the number of terms in the Taylor series up to and including all derivatives of that order.

Fig. 4 is intended to illustrate, not that the grid points must be arranged as shown but that for instance, if second order derivatives are required to third order discretisation error, then since all third order and fourth order derivatives will be required to be defined and a total of fifteen grid points will be needed then as indicated, there should be five points at one level of y, four grid points at a second level of y, three grid points at a third level of y, two grid points at a fourth level of y and one grid point at a fifth level of y. If the grid is regular, failure to arrange points in this way will lead to $[A]$ being singular. For irregular grids singularity of $[A]$ is likely only through mischance but it should be remembered that since irregular grids have grid points lying on defined contours (*i.e.* they are not random) then as the grid is refined it approaches regularity so that it is nontheless wise to observe the above rules. Moreover an irregular grid may, since it is defined anew for each application, become regular in certain domains and in particular applications.

The nodal point is chosen near the centre of the molecule if the magnitude of the discretisation error is to be kept small. The arrangement of six grid points for the second order Laplace operator (*i.e.* first order discretisation error) is accordingly best made as in Fig. 5a in which the dispositions make the nodal point fairly central except with respect to one point of the molecule. This unsymmetrical point serves strongly to define a value for $\partial^2 f/\partial x\, \partial y$, which value may possess a lower magnitude of discretisation error because of lack of centrality. Since the Laplace operator is determined entirely by the symmetrical points if the co-ordinates x and y are orthogonal (Fig. 5b) then for this case the coefficient corresponding to the unsymmetrical point will always be zero and the molecule effectively contains only five points. For convenient non-orthogonal grids (illustrated in Fig. 7) the six points are essential.

11. Choice of Co-ordinate System

In setting up the equations of motion to be solved, directly or indirectly, three co-ordinate systems are involved. These are (Fig. 6):

(a) the co-ordinates defined by the direction of resolution of the equations of motion *e.g.* S and N;

(b) the co-ordinates chosen to express the derivatives occurring in these equations *e.g.* r and z; and

(c) the co-ordinates defining the contours on which the grid points shall lie *e.g.* straight quasi-orthogonals and contours arbitrarily conformal with boundaries (Fig. 7).

There is no necessity, and it is usually numerically inconvenient as well as restricting the area of application, to choose these co-ordinate systems to coincide. It is shown in previous sections that the most convenient system for resolving the equations of motion to obtain the simplest expressions, guaranteed from indeterminacy, is that of the streamlines and their normals. By the preceding section the numerically determined streamsurface derivatives are in those co-ordinates used to locate the grid points. To avoid rotation of axes and the occurrence of infinite derivatives as in Ref. 3, it is necessary to choose this co-ordinate system in such a manner that the streamsurface is single valued in the co-ordinates chosen, while it is convenient in order to avoid mathematical transformations within the computer program, for these co-ordinates to be those in which the geometric data is readily to hand. In the hub-to-shroud solution these two requirements are compatible and (r, z) are suitable, but in the blade-to-blade problem (m, θ) satisfies the first requirement whereas the data is more likely to be available in (r, θ, z) co-ordinates. The choice of contours on which the grid points shall lie may be selected arbitrarily and is independent of the choice for (a) and (b) above, however because engineers are usually interested in hub, shroud or blade surface values of the solution, there is some merit in choosing one set of streamwise contours to conform with and include these surfaces. Such a choice avoids special extrapolation procedures to determine surface values of the solution and simplifies the programming necessary to define the grid. Such contours will not normally be expressible by analytic mathematical expressions, but this is of no consequence since the equations to be solved involve only the co-ordinates chosen for (b) above. The remaining set of cross-stream grid contours is again arbitrary, but it is necessary in order to avoid indeterminacy in evaluating the derivatives, to ensure that the two sets of grid contours will nowhere coincide (*i.e.* possess a common tangent). Additionally for this reason Marsh[2] found it necessary to rotate his co-ordinate system in which these contours are parallel straight lines but even so, the flow is always restricted to 180 degrees deflection in the (r, z) plane. The safest such contours are the true orthogonals to the streamlines which are not suitable due to the difficulty of their determination and their necessary re-determination at each iteration. It is sufficient to pre-define, fixed quasi-orthogonals for these contours as by Smith[3] for blade-to-blade flow or the author's program for hub-to-shroud flow (also suggested but not used by Marsh[2]), however it is still necessary to choose the molecule points as in the preceding section in order to avoid the indeterminacies there mentioned.

The lack of true orthogonality of the grid arising from the above proposals, does not alter the *order* of truncation error if the rules of the preceding section are observed. Nor does it necessarily imply a higher *magnitude* of this error, since this is a function of the local solution, and in every case vanishes in a region where all the derivatives in the error term approach zero. A numerical error due to round-off does however arise and is amplified by lack of orthogonality but in practice where some ten decimal places are being worked to, this will be quite insignificant.

12. Conclusions

The equations for inviscid flow on an arbitrarily prescribed streamsheet can be generalised both in respect of the equations of motion and of continuity. The principal equation of motion can be expressed for a direction $(\bar{N})$ of resolution which is always purely normal to the flow direction while lying on the prescribed streamsurface. Because the other equation of motion is always resolved in the stream direction, there is no possibility of indeterminacy arising from accidental coincidence of these equations. The $\bar{N}$-principal equation is both simpler in form and offers a higher numerical accuracy as well as a reduction of computer store demand, compared to previous forms of principal equation. The previous concepts of a force acting normal to the streamsurface and of the non-physical integrating factor in the continuity equation can be dispensed with.

The location co-ordinates (*e.g.* r, θ, z) and the choice of grid system may be chosen arbitrarily and independently of each other when giving expression to the $\bar{N}$-principal equation. Quasi-orthogonal grid systems offer the simplest means of avoiding accidental coincidence of grid co-ordinates while making possible calculation of flows with large deflections (greater than 360 degrees) without the need for rotation of reference axes. When setting up numerical expressions for the derivative operators with such grids, care must be taken of the way in which the operator molecule points are chosen, especially for higher orders of discretization error, if indeterminacy in calculating the numerical operator coefficients is to be avoided.

LIST OF SYMBOLS

f	A general function
h	Enthalpy
l	Contour
m	Meridional co-ordinate
n	Unit surface normal vector
n'	Surface normal vector
p	Static pressure
r	Radial co-ordinate
s	Entropy
t	Time, streamsheet normal thickness
t'	Streamsheet tangential thickness
u	Internal energy
x, y	Arbitrary co-ordinates
z	Axial co-ordinate
C_p	Specific heat at constant pressure
C_v	Specific heat at constant volume
I	Rothalpy (*see* equation 9)
$R, \bar{R}$	Gas constant, location vector
S, S'	Functions defining streamsurface
T	Temperature
U	Blade speed
V	Inertial velocity
W	Velocity relative to blade row
ρ	Density
ψ	Streamfunction
ω	Angular velocity of blades

Subscripts

0	Inertial stagnation value, nodal value
1	Datum value, matrix element
i, j	Matrix elements
m	Meridional vector component
r	Radial vector component
z	Axial vector component
θ	Tangential vector component

Operators

∇	Gradient
∇_T	Surface component of gradient
$\dfrac{D}{Dt}$	Time derivative for particle
$\left(\dfrac{\partial}{\partial x}\right)_{y,S}$ etc.	Space derivative
$\oint_S$	Integral over closed surface
$\oint_l$	Integral around closed contour
.	Scalar product
$\times$	Vector product

Symbols

$^{-}$	Vector quantity
[ ]	Matrix
| |	Determinant

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APPENDIX

Consider an arbitrary element of a streamsheet having a local normal thickness t and an edge contour l (Fig. 1). The equation of continuity (equation (4)) for the closed surface of this element may be written as the sum of the surface integrals taken over the three surfaces which form the element. Since however two of these surfaces are streamsurfaces by definition of streamsheet then integrals of mass flow

$$\int \rho \overline{W} . d\bar{S} \tag{A1}$$

on these surfaces are zero. The only remaining surface of the element is its edge defined by the contour l so that continuity may be expressed

$$\oint_S \rho \overline{W} . d\bar{S} = \int_l \rho \overline{W} . d\bar{S} = 0. \tag{A2}$$

An element of this edge surface $d\bar{S}$ is defined by

$$d\bar{S} = t\bar{n} \times d\bar{l}, \tag{A3}$$

hence by equation (A2)

$$\oint_S \rho \overline{W} . d\bar{S} = \int_{l \to 0} \rho \overline{W} . t\bar{n} \times d\bar{l} \equiv \int_{l \to 0} \rho t \overline{W} \times \bar{n} . d\bar{l} = 0$$

$$\equiv dS' \bar{n} . \nabla \times (\rho t \overline{W} \times \bar{n}) = 0 \tag{A3}$$

where dS' is the area of streamsurface contained by the element.
Since therefore

$$\nabla \times (\rho t \overline{W} \times \bar{n}) = 0 \tag{A4}$$

$$\rho t . \overline{W} \times \bar{n} \equiv \nabla \psi \tag{A5}$$

where ψ is a scalar function on the surface. Since by equation (A5)

$$\bar{n} . \nabla \psi = \rho t \overline{W} \times \bar{n} . \bar{n} \equiv 0,$$

then there is no variation of ψ normal to the surface and in this sense ψ is locally two dimensional having variation only on the surface.

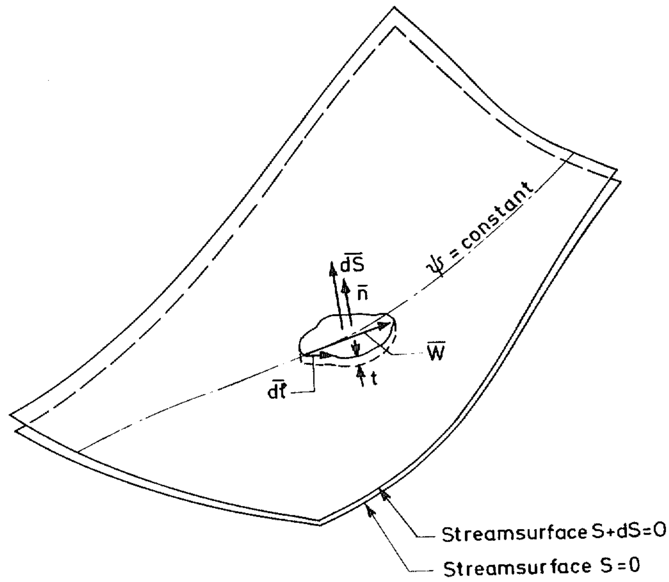

FIG. 1. Two neighbouring streamsurfaces S and $S + dS$ forming a thin streamsheet from which may be cut a small segment described by contour l.

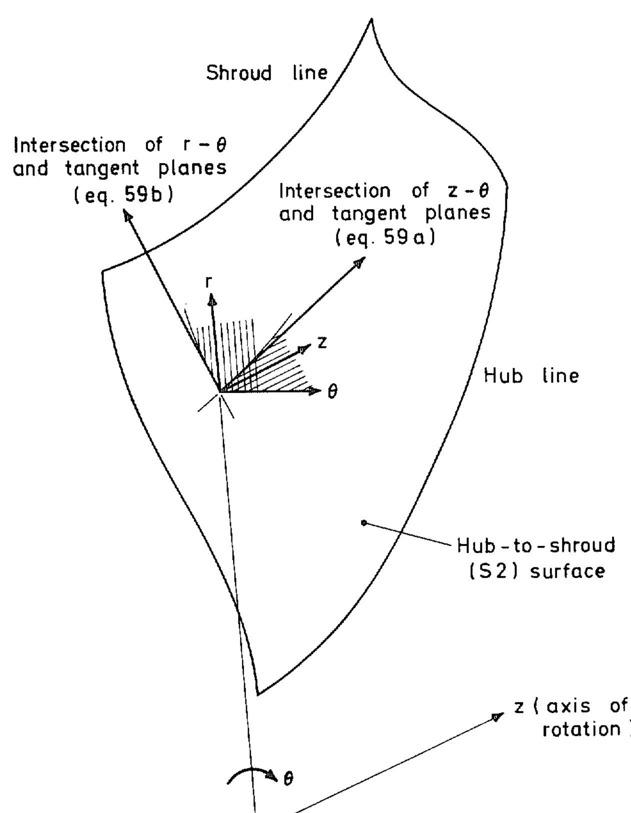

FIG. 2. The direction of resolution of Marsh's two principal equations.

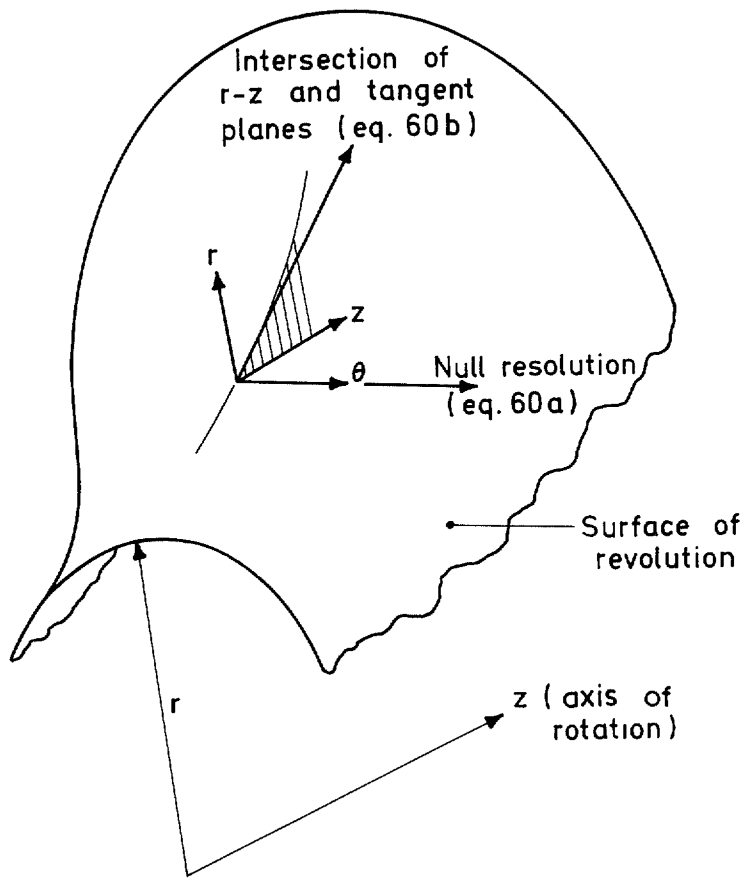

FIG. 3. The direction of resolution of Smith's two principal equations.

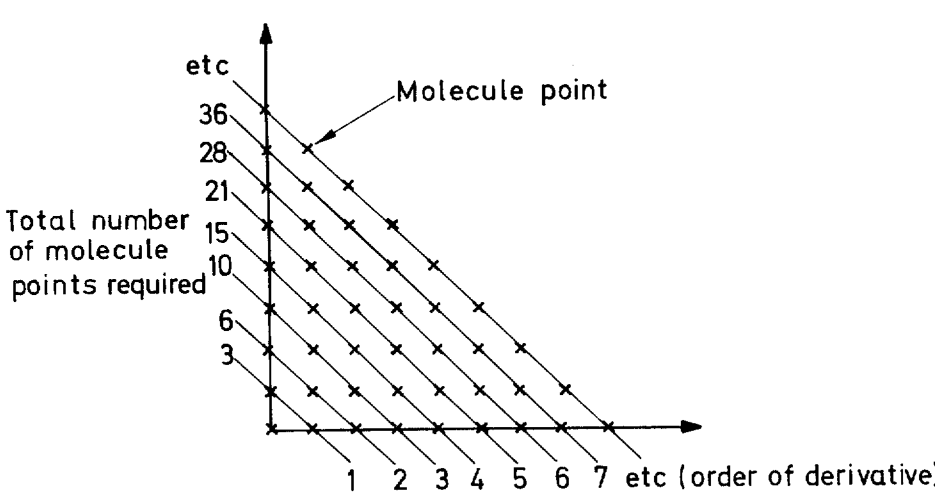

FIG. 4. Scheme providing the number and basic arrangement of grid points suitable to set up [A] for increasing orders of derivatives in two dimensions.

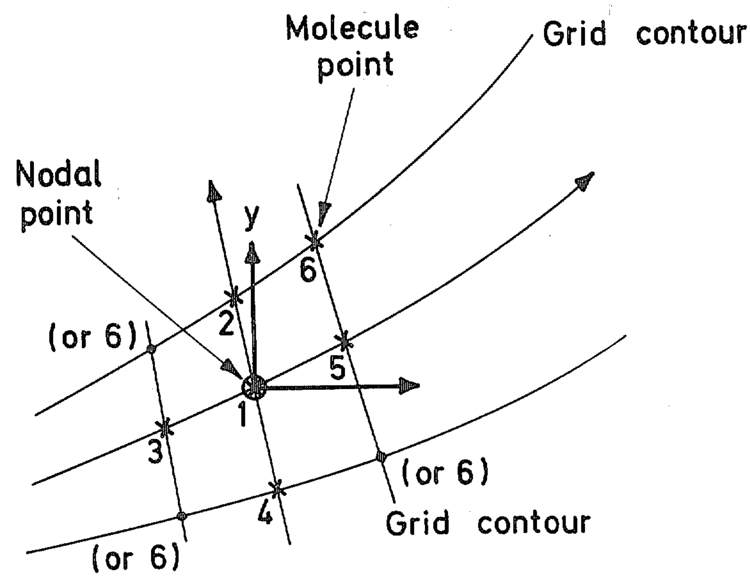

FIG. 5(a). Suitable arrangement for irregular grid showing necessary unsymmetrical point.

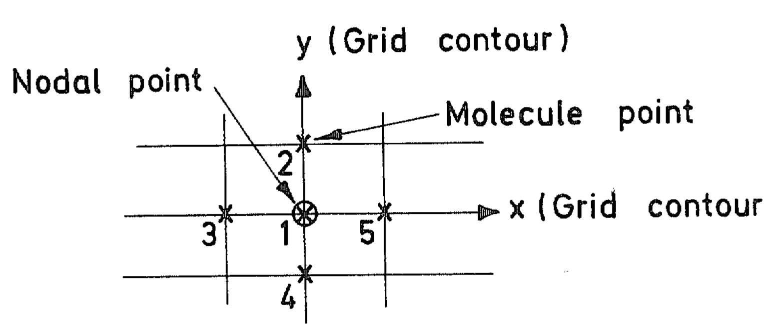

FIG. 5(b). Suitable arrangement for regular orthogonal grid. (Grid contours and location co-ordinates coincident).

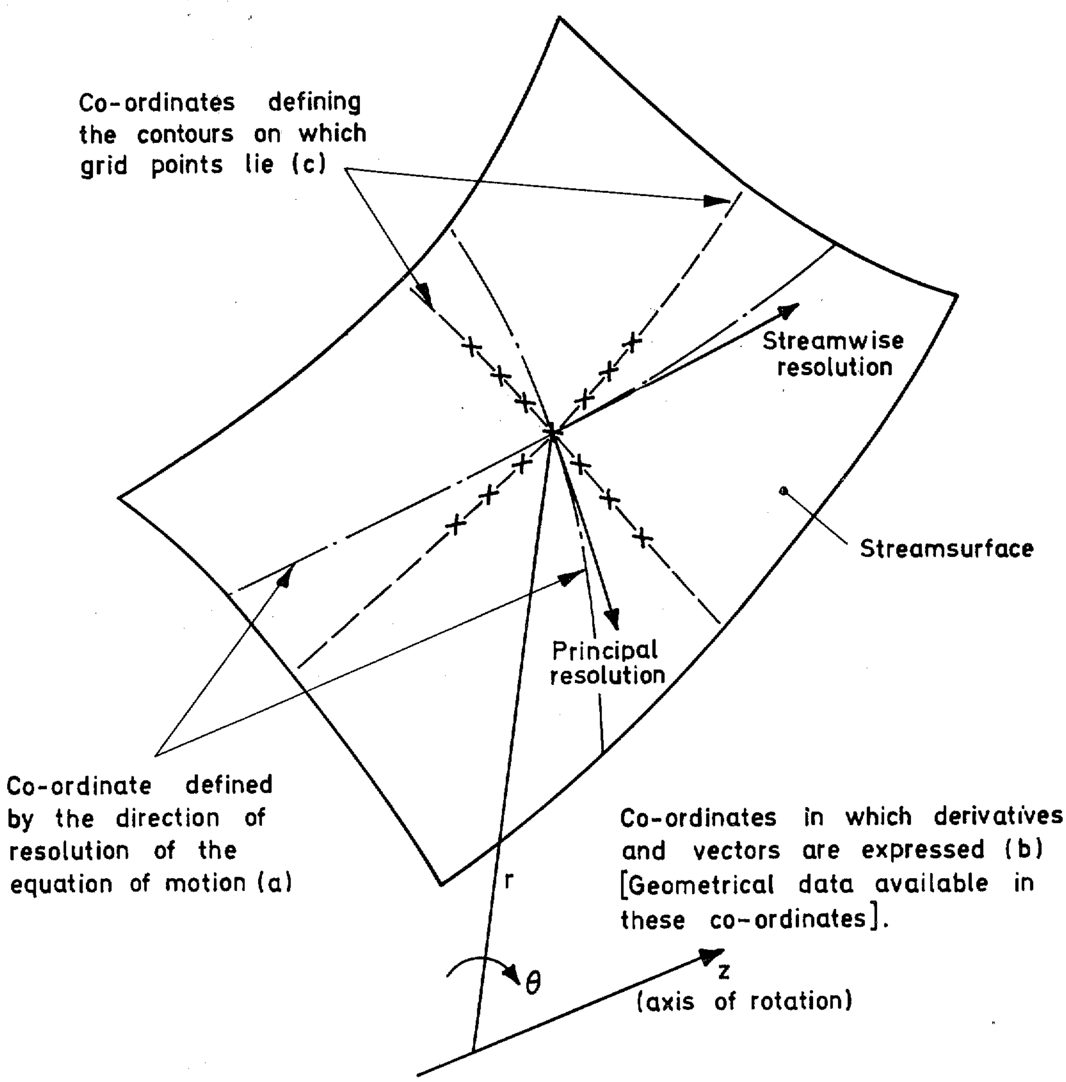

FIG. 6. The three co-ordinate systems implicitly involved in the problem.

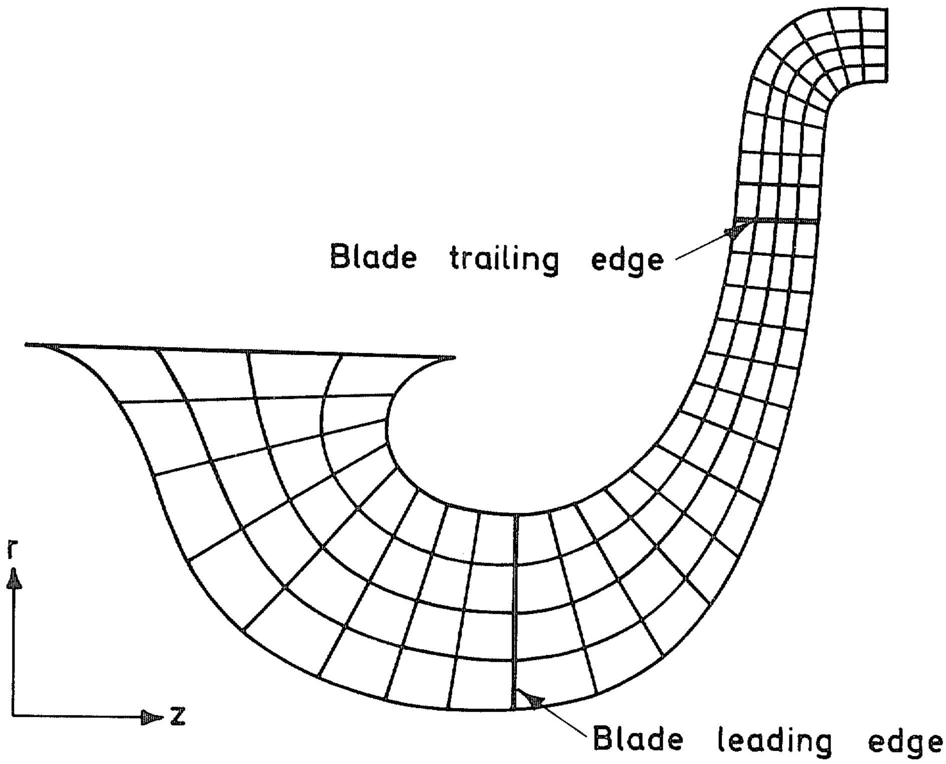

FIG. 7(a). Suitable quasi-orthogonal grid for computation on an $S2$ surface in a centrifugal turbomachine. This grid permits close definition where required and the positioning of quasi-orthogonals at blade L.E. and T.E.

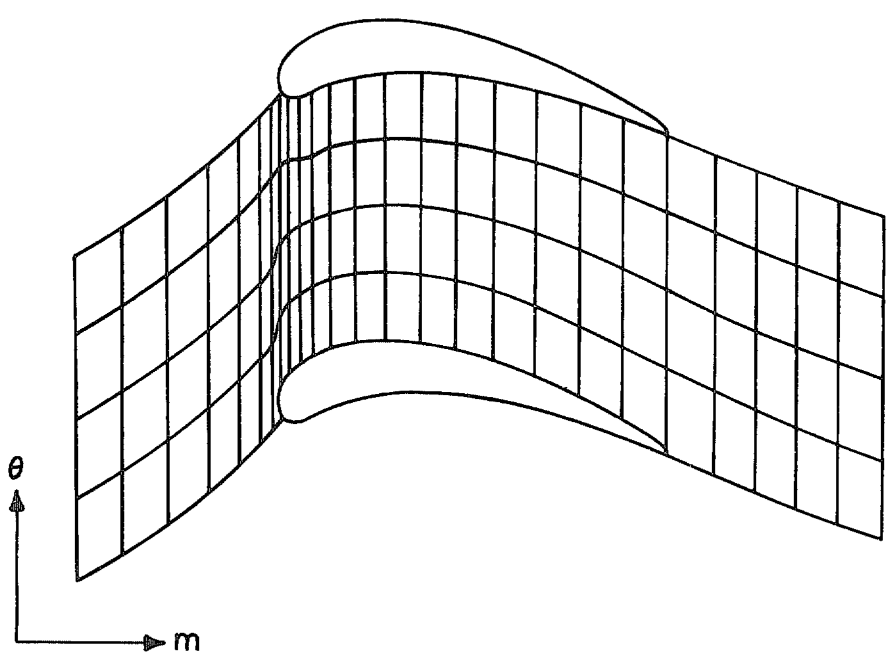

FIG. 7(b). Suitable quasi-orthogonal grid for computation on an $S1$ surface in a turbomachine (after Smith, D. J. L.). This grid permits easy imposition of boundary conditions with close profile definition where required.

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