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Kron's Method: An Algorithm for the Eigenvalue Analysis of Large-Scale Structural Systems

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Summary

In this report, the Kron eigenvalue procedure is established by the application of Hamilton's principle to a constrained primitive Lagrangian, comprising the characteristics of the various sub-systems into which a composite system is 'torn' to facilitate analysis. The computational merits of the method are outlined—particularly in relation to a scanning algorithm derived from a procedure developed by Wittrick and Williams.

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1. Introduction

When venturing to assess the history of piecewise analysis of large-scale engineering systems, many modern authors seem not to be aware of the profound contributions of G. Kron[1] in laying the foundations in this vast and expanding field. In particular, Kron's work on the solution of large network problems, involving the inversion of very large matrices at a time when only 'zeroth generation' digital computing facilities existed, has now been largely forgotten; probably because the modern breed of computers can cope with systems of considerable dimensionality on standard package programmes.

However, there comes a time when the (n + 1) th − generation problems are to be solved on nth generation machines. When such problems have arisen in the past, one has only had to wait for the next generation of machines to arrive: in a nutshell, the problems have been solved by building bigger and better computers! One might rightly ask, 'Where does this stop?' In certain areas, there can be no doubt whatsoever that Kron's philosophy offers a means for the solution of this problem. It is a philosophy which enables *existing facilities* to be used to their utmost.[7]

In this report, the Kron method of solution of large-scale natural frequency eigenvalue problems is reconsidered via a Lagrangian approach.[6] This method involves the tearing assunder of the composite system to form independent sub-systems—the size of the largest of which is, ideally, dictated by the standard capacity of the largest available computer. From the equations of sub-system solutions, the equations of composite system solutions are deduced by transformation, thus yielding a non-linear eigenvalue problem of relatively small order. The resulting determinantal equation is solved by frequency scanning—a procedure aided greatly by the canonical decomposition of the sub-systems.

In recent years, the development of a scanning procedure incorporating Newton's method[2,3] has removed the hit and miss element from the conventional frequency scan. More recently, the powerful counting algorithm of Wittrick and Williams,[4] developed to deal with the scanning problem in the dynamic stiffness formulation, has been adapted for use with Kron's determinant[5] for systems with real eigenvalues. Undoubtedly the embellishment of Kron's method with this algorithm renders it most attractive from the viewpoint of general structural usage.

It is hoped that the method will come into its own in the aeronautical context as the adoption of finite-element solutions for the various 'branches' increases greatly the dimensionality of the eigenvalue problems. Indeed, if full 'branch' solutions are available, the use of the method should enable the full normal modes solution to be undertaken in systems of very large order without the need for 'throwing away' branch modes.

2. The Kron Determinant

Consider a composite system which requires so many state variables to describe adequately its dynamical behaviour that the natural frequency eigenvalue problem exceeds the capacity of the largest computer available. Now tear the composite system assunder to yield 'u' independent, non-interacting, sub-systems—the largest of which, ideally, can *just* be coped with by use of standard programmes. Let the stiffness and inertia matrices of the ith sub-system be E_i (symmetric, positive semi-definite) and A_i (symmetric, positive definite) respectively relative to the sub-system state vector, X_i. The *primitive* matrices may now be defined:

$$\left.\begin{aligned} \mathbf{E}_p &= \text{Comp Diag}\,[E_1 E_2 \ldots E_u], \\ \text{and} \qquad \mathbf{A}_p &= \text{Comp Diag}\,[A_1 A_2 \ldots A_u] \\ \mathbf{X}_p &= \{X_1 X_2 \ldots X_u\} \end{aligned}\right\}. \tag{1}$$

Hence, the primitive Lagrangian* is

$$L_p = T_p^* - U_p \equiv \tfrac{1}{2}\dot{\mathbf{X}}_p' \mathbf{A}_p \dot{\mathbf{X}}_p - \tfrac{1}{2}\mathbf{X}_p' \mathbf{E}_p \mathbf{X}_p \tag{2}$$

T_p^* being the primitive kinetic co-energy and U_p the primitive potential energy. Use of Hamilton's principle now leads to the sub-system equations of motion,

$$A_i \ddot{X}_i + E_i X_i = 0 \text{ and} \qquad i = 1, 2, \ldots, u. \tag{3}$$

*A dual approach based on the *co-Lagrangian* is presented in Reference 6.

Now from $\mathbf{X}_p$ select all those co-ordinates involved in connexion relations. These are known as 'tie' co-ordinates and may be stacked in vector $\mathbf{X}_t$ such that

$$\mathbf{X}_t = \mathbf{T}\mathbf{X}_p \tag{4}$$

where $\mathbf{T}$ is a 'selection' matrix which isolates the 'tie' co-ordinates. The actual connexions (assumed holonomic) which re-constitute the composite system may be expressed as

$$\mathbf{C}\mathbf{X}_t = \mathbf{C}\mathbf{T}\mathbf{X}_p = \mathbf{0}, \tag{5}$$

each *row* of this matrix relationship representing a single holonomic constraint upon the elements of $\mathbf{X}_p$. Each constraint may be assigned a Lagrange multiplier, β_j, and if $\boldsymbol{\beta} = \{\beta_1\beta_2 \ldots \beta_n\}$ the Lagrangian for the composite system may be written

$$L_c = L_p + \boldsymbol{\beta}'\mathbf{C}\mathbf{T}\mathbf{X}_p. \tag{6}$$

Note that as L_c has the units of work and $\mathbf{X}_p$ the units of generalised displacement, $\boldsymbol{\beta}$ has units of *generalised force*. Indeed, the β_j are the forces within the married 'tie' co-ordinates.

By Hamilton's principle

$$\delta\int_{t_1}^{t_2} L_c\,dt = 0, \quad \delta\mathbf{X}_p\Big]_{t_1}^{t_2} = 0 \quad \text{and} \quad \delta\boldsymbol{\beta}\Big]_{t_1}^{t_2} = 0 \tag{7}$$

the equations of motion of the composite system are found to be

$$\left.\begin{aligned} \mathbf{A}_p\ddot{\mathbf{X}}_p + \mathbf{E}_p\mathbf{X}_p - \mathbf{T}'\mathbf{C}'\boldsymbol{\beta} = \mathbf{0} \\ \mathbf{C}\mathbf{T}\mathbf{X}_p = \mathbf{0} \end{aligned}\right\} \tag{8}$$

and

With $D \equiv d/dt$, these become

$$\begin{bmatrix} \mathbf{A}_pD^2 + \mathbf{E}_p, & -\mathbf{T}'\mathbf{C}' \\ -\mathbf{C}\mathbf{T}, & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{X}_p \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{8A}$$

While it is formally possible to treat equation (8A) as a matrix eigenvalue problem in the usual way, the order of the leading sub-matrix will in general be too large for this to be practically possible. (*See* Appendix I).

Now the system matrix of the ith sub-system, $A_i^{-1}E_i$, has the eigenvalues $\omega_{i1}^2, \omega_{i2}^2, \ldots, \omega_{ir}^2$ and eigenvectors $X_{(1)}^i X_{(2)}^i \ldots X_{(r)}^i$ and hence the spectral and modal matrices are,

$$\left.\begin{aligned} \Lambda_i = \text{Diag}\,[\omega_{i1}\omega_{i2}\ldots\omega_{ir}] \\ M_i = [X_{(1)}^i \ldots X_{(r)}^i]; \qquad |M_i| \neq 0 \end{aligned}\right\} \tag{9}$$

and

Although some of the ω_{ij} might be zero (rigid body modes of the ith sub-system), the matrix M_i can always be chosen such that

$$M_i'A_iM_i = I \quad \text{and hence } M_i'E_iM_i = \Lambda_i^2. \tag{10}$$

The primitive spectral and modal matrices

$$\left.\begin{aligned} \boldsymbol{\Lambda}_p = \text{Comp Diag}\,[\Lambda_1 \ldots \Lambda_u] \\ \mathbf{M}_p = \text{Comp Diag}\,[M_1 \ldots M_u] \end{aligned}\right\} \tag{11}$$

and

can now be defined and from equation (10)

$$\mathbf{M}_p'\mathbf{A}_p\mathbf{M}_p = \mathbf{I} \quad \text{and} \quad \mathbf{M}_p'\mathbf{E}_p\mathbf{M}_p = \mathbf{\Lambda}_p^2. \tag{12}$$

Thus, writing $\mathbf{X}_p = \mathbf{M}_p\mathbf{Q}_p$ in equation (8A) and completing the transformation, one obtains (with $\mathbf{\Gamma}' = \mathbf{CTM}_p$),

$$\begin{bmatrix} \mathbf{\Lambda}_p^2 + \mathbf{I}D^2, & -\mathbf{\Gamma} \\ -\mathbf{\Gamma}', & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_p \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{13}$$

Then with D^2 replaced by $-\omega^2$, it is seen that

$$\mathbf{Q}_p = [\mathbf{\Lambda}_p^2 - \omega^2\mathbf{I}]^{-1}\mathbf{\Gamma}\boldsymbol{\beta} \tag{14}$$

and

$$\mathbf{\Gamma}'[\mathbf{\Lambda}_p^2 - \omega^2\mathbf{I}]^{-1}\mathbf{\Gamma}\boldsymbol{\beta} = \mathbf{0}. \tag{14A}$$

For non-trivial $\boldsymbol{\beta}$

$$|\mathbf{R}_4| \equiv |\mathbf{\Gamma}'\mathbf{D}\mathbf{\Gamma}| = 0, \tag{15}$$

where $\mathbf{D} = (\mathbf{\Lambda}_p^2 - \omega^2\mathbf{I})^{-1}$ = diagonal matrix. This is Kron's determinantal frequency equation. The composite system eigenvalues are the zeros of $|\mathbf{R}_4|$ while the sub-system eigenvalues are clearly the poles.

It should be noted that the case $\boldsymbol{\beta} = \mathbf{0}$ is not necessarily a trivial solution of the composite system eigenvalue problem. In general, $\boldsymbol{\beta} = \mathbf{0}$ will be associated with a composite system eigenvalue in those cases where the 'tie' co-ordinates experience no forces upon re-connexion, implying a matching of sub-system and composite system eigenvalues as well, in many cases, as a geometrical symmetry requirement on the sub-systems. (*See* equation (8) where $\boldsymbol{\beta} = \mathbf{0}$ implies $\mathbf{A}_p\ddot{\mathbf{X}}_p + \mathbf{E}_p\mathbf{X}_p = \mathbf{0}$, which is the sub-system result, along with $\mathbf{CT}\,\mathbf{X}_p = \mathbf{0}$, which in this context is a free compatibility requirement on the sub-system eigenvectors.). Under these conditions, it is clear that equation (15) will not yield the composite system eigenvalues associated with $\boldsymbol{\beta} = \mathbf{0}$. Such cases are discussed briefly in Ref. 2 in particular relation to the problem of composite system rigid body modes, while Ref. 5 presents a method (*see* Section 3.4 below) which locates the composite system eigenvalues regardless of whether or not $\boldsymbol{\beta}$ is null.

A salient feature of the above formulation is the central diagonal matrix, **D**. This feature is not shared by the stiffness formulation (Appendix II). It might also be noted that while the above formulation is in terms of stiffness, the ultimate Kron equation, (14A), is a *force* equation. (*See* Ref. 2 for a complete derivation of Kron's method via receptances).

3. Numerical Methods

The various published methods for obtaining solutions of equation (15) are now discussed:—

3.1. Frequency Scanning

This is a 'brute force' method wherein a value of ω^2 is assigned and $|\mathbf{R}_4|$ evaluated. The zeros of the $|\mathbf{R}_4|/\omega^2$ plot are then obtained by interpolation.

3.2. Scalar Frequency Scanning

In equation (14A) partition $\boldsymbol{\beta} = \{\beta_1\boldsymbol{\beta}_R\}$ and $\mathbf{\Gamma} = [\mathbf{g}_1\mathbf{\Gamma}_R]$, where $\beta_1 = 1$, say, is scalar and $\mathbf{g}_1$ is a column vector. Equation (14A) then gives

$$\boldsymbol{\beta}_R = -[\mathbf{\Gamma}_R'\mathbf{D}\mathbf{\Gamma}_R]^{-1}\mathbf{\Gamma}_R\mathbf{D}\mathbf{g}_1 \tag{16}$$

and

$$f = \mathbf{g}_1'[\mathbf{D} - \mathbf{D}\mathbf{\Gamma}_R(\mathbf{\Gamma}_R'\mathbf{D}\mathbf{\Gamma}_R)^{-1}\mathbf{\Gamma}_R'\mathbf{D}]\mathbf{g}_1 \equiv \mathbf{g}_1'\mathbf{L}\mathbf{g}_1 = 0 \tag{16A}$$

The zeros of $|\mathbf{R}_4|$ and f are identical; hence f may be used as the basis of a frequency scan. The advantage over the straightforward determinantal scan is that the poles and zeros of f *alternate* along the ω^2 axis of the f/ω^2 plot. (*See* Ref. 2).

3.3. Scalar Scanning Allied with Newton Iteration

This technique was developed in Ref. 2. It provides a powerfully convergent substitute for 'hit and miss' scanning. It is easy to show that

$$\frac{df}{d\omega^2} = \mathbf{g}_1'\mathbf{L}^2\mathbf{g}_1 \tag{17}$$

and hence that if ω_a^2 is an approximation to a zero of f, a better approximation will be

$$\omega_b = \omega_a^2 - \frac{\mathbf{g}_1'\mathbf{L}(\omega_a^2)\mathbf{g}_1}{\mathbf{g}_1'\mathbf{L}^2(\omega_a^2)\mathbf{g}_1}. \tag{18}$$

This technique is extended to the complex eigenvalue problem in Ref. 3.

3.4. Adaptation of the Wittrick–Williams Counting Algorithm

This powerful algorithm[4] has recently been adapted[5] to deal with equation (15). In the present context, this method represents successive applications, to the primitive system, of the connexion constraints joining the sub-systems. Partition $\mathbf{\Gamma} = [\mathbf{g}_1\mathbf{g}_2 \ldots \mathbf{g}_n]$, so that each column represents one constraint. Also let $\mathbf{\Gamma}_{1j} = [\mathbf{g}_1\mathbf{g}_2 \ldots \mathbf{g}_j]$ and

$$\mathbf{m}_j = \mathbf{\Gamma}_{1j}'\mathbf{D}\mathbf{\Gamma}_{1j} \tag{19}$$

so that $|\mathbf{m}_j| = m_j$ is the jth principal minor of $\mathbf{R}_4$ (equation (15)).

Clearly

$$m_{j+1} = m_j\mathbf{g}_{j+1}'[\mathbf{D} - \mathbf{D}\mathbf{\Gamma}_{1j}\mathbf{m}_j^{-1}\mathbf{\Gamma}_{1j}'\mathbf{D}]\mathbf{g}_{j+1} \equiv m_j\mathbf{g}_{j+1}'\mathbf{L}_j\mathbf{g}_{j+1}, \tag{20}$$

defining a Sturm sequence for the matrix $\mathbf{R}_4$ in which one may place $m_0 = 1$. Defining $l_j = m_j/m_{j-1}$, the number of negative elements in the sequence $l_1, l_2, \ldots, l_n$ defines the number of negative eigenvalues[4,5] of $\mathbf{R}_4$ at the particular value of ω for which the l_j are calculated. This number is called the *sign count*[4] of $\mathbf{R}_4$ and is written $s[\mathbf{R}_4]$. The sign count does not have to be calculated as indicated in equations (19) and (20); indeed it is pointed out in Ref. 4 that the elements $l_1, l_2, \ldots, l_n$ are the diagonal elements of the upper triangular matrix which results from $\mathbf{R}_4$ on Gaussian triangulation without row interchanges. This provides a rapid procedure for determining $s[\mathbf{R}_4]$.

Differentiating l_{j+1} w.r.t. ω^2 one obtains from equation (20)

$$dl_{j+1}/d\omega^2 = \mathbf{g}_{j+1}'\mathbf{L}_j^2\mathbf{g}_{j+1} = (\mathbf{L}_j\mathbf{g}_{j+1})'(\mathbf{L}_j\mathbf{g}_{j+1}) \geqslant 0 \tag{21}$$

so that the poles and zeros of l_{j+1} alternate along the ω^2 axis. Also, the *zeros* of l_j are the *poles* of l_{j+1} since $\mathbf{m}_j^{-1}$ appears in the $\mathbf{l}_{j+1}$ expression. The zeros of l_j and l_{j+1} are therefore interlaced[5] (*See* Fig. 1) and one may state

$$\begin{bmatrix}\text{Number of zeros of} \\ l_{j+1} \text{ exceeded by } \omega = \bar{\omega}\end{bmatrix} = \begin{bmatrix}\text{No. of poles, } J_j, \text{ of } l_j \\ \text{thus exceeded}\end{bmatrix} \quad \text{if } l_j > 0, l_{j+1} > 0$$

$$= [J_j - 1] \quad \text{if } l_j l_{j+1} < 0$$

$$= [J_j - 2] \quad \text{if } l_j < 0, l_{j+1} < 0. \tag{22}$$

Thus, by induction from entire sequence

$$K(\bar{\omega}) = s[\mathbf{D}(\bar{\omega})] - s[\mathbf{R}_4(\bar{\omega})], \tag{23}$$

where $K(\bar{\omega})$ is the number of composite system eigenvalues exceeded by $\bar{\omega}$ and $s[\mathbf{D}(\bar{\omega})]$ gives the number of sub-system eigenvalues thus exceeded.

While the proof of equation (23) requires modification in cases where equal eigenvalues occur, equation (23) may be shown to hold good.[5] This equation gives the basis for the scanning algorithm. Using it, it is quite impossible to 'miss' a composite system eigenvalue. There is no difficulty in combining this algorithm with the Newton method of Section 3.3 since

$$l_n = m_n/m_{n-1} = \mathbf{g}_n'\mathbf{L}_{n-1}\mathbf{g}_n \tag{24}$$

which is the last term on the diagonal of the triangulated $\mathbf{R}_4$ matrix and which is equivalent to equation (16A) except that elimination has been performed along the *last* row and column instead of the *first*.

From equation (21)

$$dl_n/d\omega^2 = \mathbf{g}_n'\mathbf{L}_{n-1}^2\mathbf{g}_n \tag{25}$$

and the application of Newton's method to $l_n = 0$ defines the composite system eigenvalues.

3.5. Comments

As a general eigenvalue algorithm, the use of equation (23), along with equations (24) and (25) to give quadratic convergence to the zeros of $|\mathbf{R}_4|$, could hardly be bettered. However, evaluation of $dl_n/d\omega^2$ requires a matrix inversion of order $n-1$, while l_n can be obtained from triangulation of $\mathbf{R}_4$. This might be construed as a difficulty, and if n is large it might be advisable to use equation (23) without the Newton refinement. One might note that while it is possible to use Newton iteration in the stiffness method,[4] the expressions for the derivatives are lengthy to the point of being unmanageable.[2]

4. An Example

The simplest example suffices to illustrate the principles of the method; hence we consider the mass/spring system of Fig. 2.

For sub-systems 1 and 3:— $E = 2, A = 1, M = 1, \Lambda^2 = 2$.

For sub-system 2:

$$E_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A_2 = \text{Diag}\,[1, 2, 1]$$

$$M_2 = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}, \qquad \Lambda_2 = \text{Diag}\,[0, 1, 2].$$

Thus from equation (11)

$$\Lambda_p^2 = \text{Diag}\,[2, 0, 1, 2, 2]$$

and

$$\mathbf{D} = \text{Diag}\left[\frac{1}{2-\lambda}, \frac{-1}{\lambda}, \frac{1}{1-\lambda}, \frac{1}{2-\lambda}, \frac{1}{2-\lambda}\right]; \qquad \lambda \equiv \omega^2.$$

Also

$$\mathbf{M}_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/\sqrt{2} & 1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & -1/\sqrt{2} & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and from equations (4), (5):

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Hence from equation (13),

$$\mathbf{\Gamma}' = \left[\begin{array}{cccc|c} 1 & -1/2 & -1/\sqrt{2} & -1/2 & 0 \\ 0 & 1/2 & -1/\sqrt{2} & 1/2 & -1 \end{array}\right] \equiv \begin{bmatrix} \mathbf{g}_1' \\ \mathbf{g}_2' \end{bmatrix}$$

From equation (15),

$$\mathbf{R}_4 = \frac{1}{4}\begin{bmatrix} \dfrac{5}{2-\lambda} + \dfrac{2}{1-\lambda} - \dfrac{1}{\lambda}, & -\dfrac{1}{2-\lambda} + \dfrac{2}{1-\lambda} + \dfrac{1}{\lambda} \\ -\dfrac{1}{2-\lambda} + \dfrac{2}{1-\lambda} + \dfrac{1}{\lambda}, & \dfrac{5}{2-\lambda} + \dfrac{2}{1-\lambda} - \dfrac{1}{\lambda} \end{bmatrix}.$$

Forgetting the factor 1/4, we have from equation (20)

$$l_1 = \frac{5}{2-\lambda} + \frac{2}{1-\lambda} - \frac{1}{\lambda}$$

and

$$l_2 = \frac{5}{2-\lambda} + \frac{2}{1-\lambda} - \frac{1}{\lambda} - \frac{\left[-\dfrac{1}{2-\lambda} + \dfrac{2}{1-\lambda} + \dfrac{1}{\lambda}\right]^2}{\dfrac{5}{2-\lambda} + \dfrac{2}{1-\lambda} - \dfrac{1}{\lambda}}.$$

It is seen that l_1 has zeros at $\lambda = (3 \pm \sqrt{5})/4$ and poles at 0, 1, 2; while l_2 has zeros at $\lambda = \frac{1}{2}, \frac{3}{2}$, and 2 and poles at $(3 \pm \sqrt{5})/4$ and 2 (twice). Fig. 3 illustrates the implementation of equation (23) at seven frequency points.

5. Kron's Method versus 'Branch Modes'

The branch mode method is restricted to that topologically-special class of systems with branch-like connectivity. It can, of course, be 'bent' to fit special circumstances, though such 'distortion' will usually be reflected in the proliferation of coupling terms in the characteristic matrices defining normal-mode frequencies. In its unadulterated form, the branch mode method will give a diagonal stiffness matrix and a relatively lightly coupled mass matrix. The 'branches' are normally of the fixed-free type and must, in effect, be 'released' after solution. It is the process of releasing the branches which leads to mass coupling terms.

The ultimate stiffness and inertia matrices in the branch mode formulation will usually be of large order. In order to reduce the dimensionality, it is common practice to 'throw-out' certain of the high frequency branch modes. While this is done in the light of practical experience, it is a procedure which must be regarded

as generally unsatisfactory. The sparse nature of the ultimate dynamical matrix renders it amenable to solution by the standard 'rotation' procedures.

On the other hand, Kron's method is unrestricted in the topological sense and, indeed, is *topologically* superior to the stiffness method.[4,5] Sub-systems are usually of the free-free type (the exceptions being those with natural earthing) and must be analysed as such. Hence, the sub-systems of Kron's method may be larger than the corresponding branch-mode 'branches' to the extent of six rigid body freedoms. The order of the Kron determinant is usually quite small, being equal to the number of connexion relations, n. This number might be as low as six for, say, the symmetric modes of a conventional aeroplane, but the number of degrees of freedom might well be 1000 plus.

The computationally efficient form[2] of the ultimate determinantal equation, (15), renders it unnecessary to 'prune' the number of degrees of freedom at any stage. As a result, extremely accurate eigenvalue estimates can be made. When the composite eigenvalues have been found, the associated vectors can be obtained from the relation given in equation (14) since

$$\mathbf{X}_p = \mathbf{M}_p\mathbf{Q}_p = \mathbf{M}_p\mathbf{D}\Gamma\boldsymbol{\beta} \tag{26}$$

wherein $\mathbf{D}$ is evaluated at the eigenvalue in question, and $\boldsymbol{\beta}$ is found from equation (16). Note that since $\mathbf{M}_p$ is 'composite diagonal', the eigenvector calculation can be performed piecewise too.

It is clear that use of the algorithm in equation (23) enables eigenvalues to be calculated within any specified range. The number of resonance frequencies in the range may be deduced immediately by calculating $\mathrm{K}(\omega_1)$ and $\mathrm{K}(\omega_2)$ for the extreme frequencies, ω_1 and ω_2, of the range and subtracting to give

$$\nu = \begin{bmatrix}\text{Number of composite system natural} \\ \text{frequencies between } \omega_1 \text{ and } \omega_2\end{bmatrix} = \mathrm{K}(\omega_2) - \mathrm{K}(\omega_1).$$

This device was first noted by Wittrick and Williams[4] in relation to the stiffness formulation. It provides an advantage not shared by the standard computational methods.

6. Conclusions

An outline of the Kron eigenvalue procedure has been given with a view to airing its potential with regard to the vibration problems of aeronautical engineering. The method is seen to present an attractive alternative to branch modes calculations—being able to cope with substantially larger problems.

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APPENDIX I

Standard Eigenvalue Treatment of the Constrained Equations

Rather than look at equation (8A) we choose to examine the 'canonical' form given in equation (13). With $D^2 = -\omega^2$, this equation may be written

$$\begin{bmatrix} \Lambda_p^2 & -\Gamma \\ -\Gamma' & 0 \end{bmatrix} \begin{bmatrix} Q_p \\ \beta \end{bmatrix} = \omega^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_p \\ \beta \end{bmatrix}. \tag{I.1}$$

Because of the sub-system singularity, we may partition $\Lambda_p = \text{Comp Diag}\,[\Omega_p\ 0]$ and conformably partition

$$\Gamma = \begin{bmatrix} \Gamma_e \\ \Gamma_b \end{bmatrix}, \qquad Q_p = \begin{bmatrix} Q_e \\ Q_b \end{bmatrix}.$$

The 'stiffness' matrix may now be written

$$\begin{bmatrix} \Omega_p^2 & R \\ R' & B \end{bmatrix} = \bar{E} \tag{I.2}$$

wherein

$$R = [0 \quad -\Gamma_e], \qquad B = \begin{bmatrix} 0 & -\Gamma_b \\ -\Gamma_b' & 0 \end{bmatrix} = B' \tag{I.3}$$

Provided $|B - R'\Omega_p^{-2}R| \neq 0$, the 'flexibility' matrix is

$$\bar{E}^{-1} = \begin{bmatrix} \Omega_p^{-2} - \Omega_p^{-2}R\delta R'\Omega_p^{-2}, & -\Omega_p^{-2}R\delta \\ -\delta R'\Omega_p^{-2} & \delta \end{bmatrix} \tag{I.4}$$

with

$$\delta = [B - R'\Omega_p^{-2}R]^{-1} = \delta'. \tag{I.5}$$

The 'mass matrix' $\bar{A} = \text{Comp Diag}\,[I\alpha]$, wherein $\alpha = \text{Comp Diag}\,[I\,0]$, the number of unit terms being equal to the number of zero elements in Λ_p. Accordingly, the 'dynamical' matrix is

$$\Phi \equiv \bar{E}^{-1}\bar{A} = \begin{bmatrix} (\Omega_p^{-2} - \Omega_p^{-2}R\delta R'\Omega_p^{-2}), & -\Omega_p^{-2}R\delta\alpha \\ -\delta R'\Omega_p^{-2}, & \delta\alpha \end{bmatrix}. \tag{I.6}$$

The degeneracy of Φ is n-fold corresponding to the 'n' null columns of α. This procedure can be modified to cope with cases where $\bar{E}$ is singular.

APPENDIX II

The Stiffness Method

The dynamic stiffness matrix of the ith sub-system is

$$S_i = E_i - \omega^2 A_i \equiv \begin{bmatrix} S_i^t & S_i^{tn} \\ S_i^{nt} & S_i^n \end{bmatrix} \tag{II.1}$$

so that the 'tie' stiffness is

$$S_{it} = S_i^t - S_i^{tn}(S_i^n)^{-1} S_i^{nt}. \tag{II.2}$$

In the stiffness method, S_i^n has to be inverted for each ω value specified. It is therefore logical to use its canonical form. The eigenvalue problem for $(A_i^n)^{-1} E_i^n$ is the one which results when *the 'tie' coordinates are locked.* Let the spectral matrix be ρ_i and the normalised modal matrix N_i so that (II.2) can be written

$$S_{it} = S_i^t - S_i^{tn} N_i [\rho_i^2 - \omega^2 I]^{-1} N_i' S_i^{nt}. \tag{II.3}$$

Defining the primitive matrices

$$\mathbf{S}_{tp} = \text{Comp Diag } [S_{1t} \ldots S_{ut}],$$

$$\mathbf{S}_p^t = \text{Comp Diag } [S_1^t \ldots S_u^t] \text{etc. etc.}$$

and the connexion matrix $\boldsymbol{\gamma}$, the frequency determinant[5] is then

$$|\boldsymbol{\gamma}' \mathbf{S}_{tp} \boldsymbol{\gamma}| \equiv |\boldsymbol{\gamma}' \mathbf{S}_p^t \boldsymbol{\gamma} - \boldsymbol{\gamma}' \mathbf{S}_p^{tn} \mathbf{N}_p (\boldsymbol{\rho}_p^2 - \omega^2 \mathbf{I})^{-1} \mathbf{N}_p' \mathbf{S}_p^{nt} \boldsymbol{\gamma}| = 0. \tag{II.4}$$

While the inversion bottleneck is avoided by use of the canonical form, this determinal equation is much more complex than the Kron equation (15). In general, however, the order of the above determinant will be smaller than that of Kron's determinant. The Williams-Wittrick method[5] may be applied *directly* to (II.4) with $\mathbf{J}_0$ determined from $s[\boldsymbol{\rho}_p^2 - \omega^2 \mathbf{I}]$.

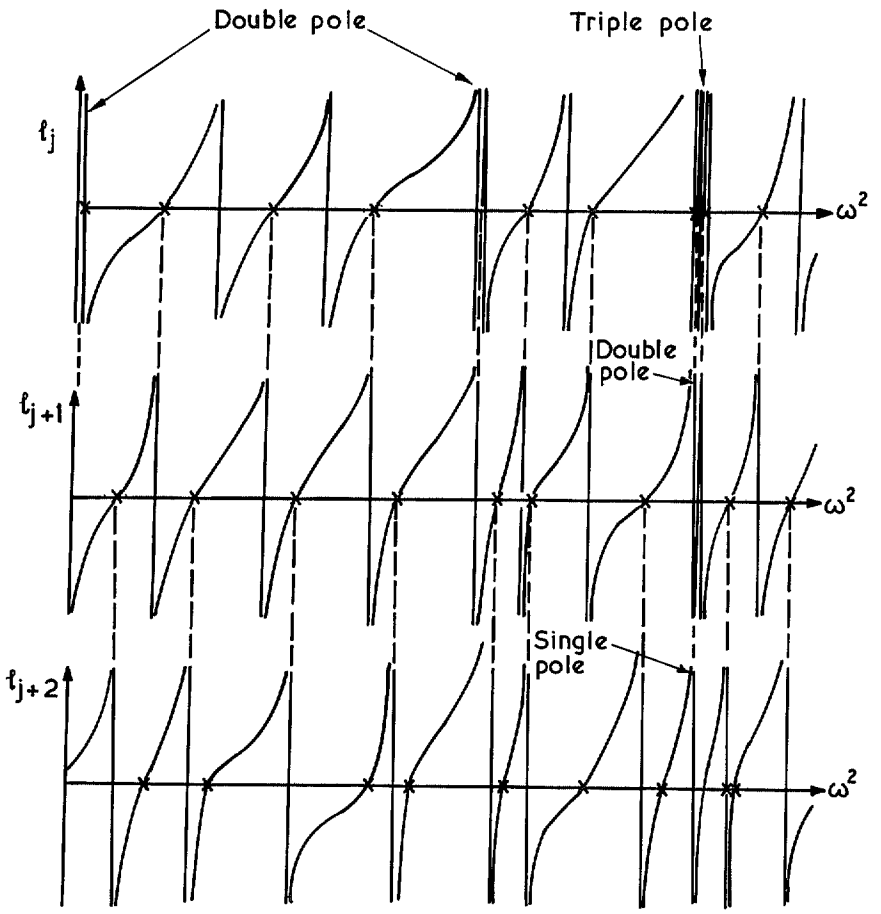


FIG. 1. Diagram illustrating 'interlacedness' of the zeros of the Sturm sequence $l_1 \ldots l_j \ldots l_n$.

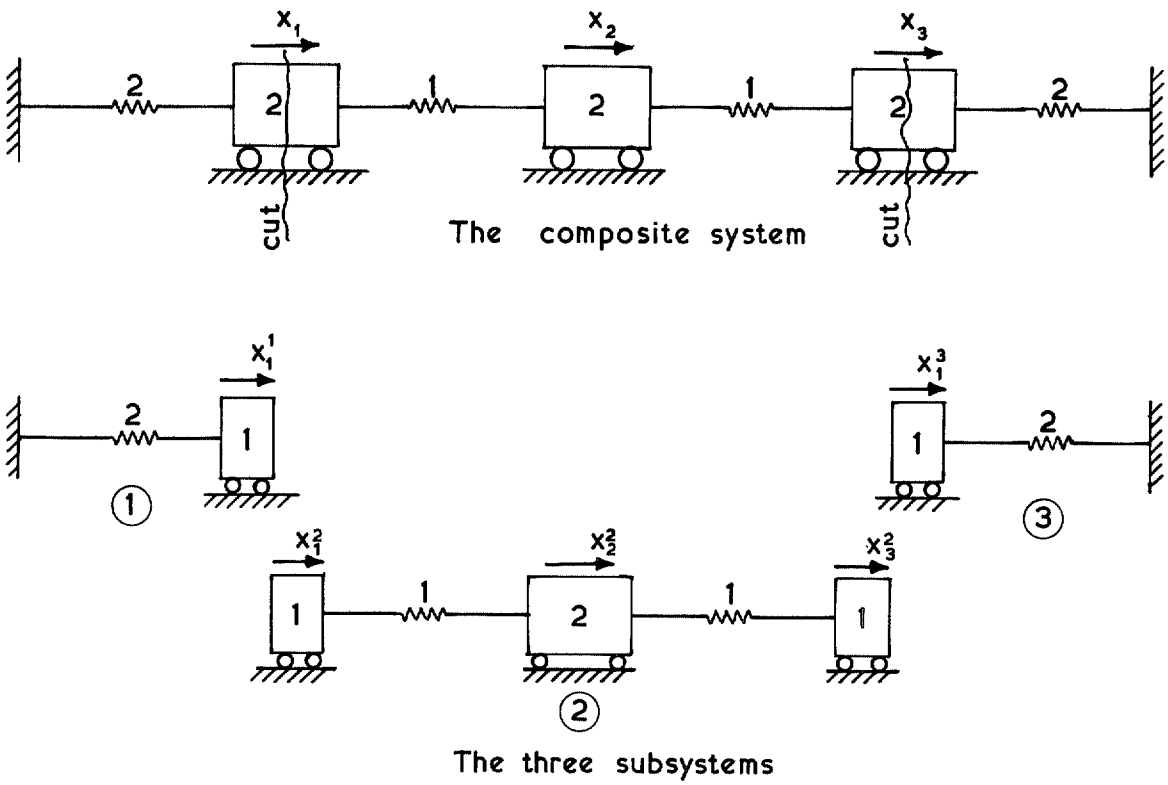
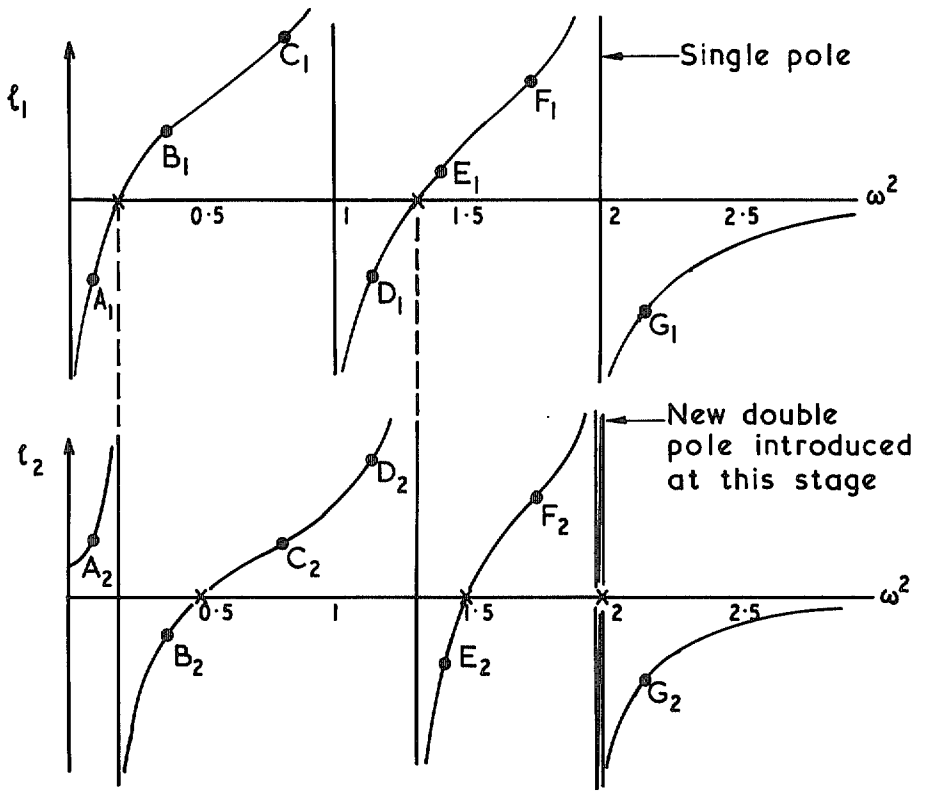


FIG. 2. Kron analysis of a mass-spring system.



ω^2 station	A_1 A_2	B_1 B_2	C_1 C_2	D_1 D_2	E_1 E_2	F_1 F_2	G_1 G_2
s [$\underset{\sim}{D}$]	1	1	1	2	2	2	5
s [$\underset{\sim}{R}_4$]	1	1	0	1	1	0	2
K	0	0	1	1	1	2	3

FIG. 3. Sturm sequence, 1_1 1_a, for mass-spring system showing validity of equation (23).

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