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Approximate Analytic Methods for the Solution of a Class of Strongly Non-Linear Differential Equations—a Comparison

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Summary

A review is made of certain approximate analytic methods which are capable of solving a wide class of strongly nonlinear, ordinary, differential equations. Contained in this class are many equations which arise from the analysis of physical situations, e.g. the equation of Van der Pol and the unforced Duffing equation. The relative accuracy of these methods is assessed by applying them to a particular strongly nonlinear equation and comparing the solutions with numerical solutions obtained on a digital computer. The results show the R.A.E. and C.J. methods to be a useful and accurate means of solving this particular equation over a wide range of the coefficients involved.

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1. Introduction

The differential equation

$$\ddot{x} + F(x, \dot{x}) = 0, \tag{1}$$

has been the subject of many studies. When $F(x, \dot{x})$ has the form

$$F(x, \dot{x}) = c_1 x + \varepsilon f(x, \dot{x}), \tag{2}$$

it is referred to as 'weakly nonlinear' with a small parameter ε and may be solved, to a high degree of accuracy, by techniques such as that of Kryloff and Bogoliuboff, hereafter referred to as K and B. The method of K and B, described in Ref. 1, is based on small perturbations of the harmonic oscillator equation

$$\ddot{x} + c_1 x = 0, \qquad c_1 > 0, \tag{3}$$

and makes use of the 'principle of averaging' which, as presented in Ref. 1, appears to be a crude approximation but has since been put on a rigorous mathematical basis. (*See* Ref. 2, Chapter 15.) A convenient summary of small perturbation techniques, which use the solutions of equation (3) as 'generating solutions', may be found in Ref. 3, pp. 211–415, together with the original references. These include the formal perturbation procedure of Poincaré, the developments of the K and B procedure by Bogoliuboff and Mitropolsky and the 'stroboscopic method' of Minorsky.

When $F(x, \dot{x})$ does not have a weakly nonlinear form, equation (1) is said to be 'strongly nonlinear' and the application of small perturbation techniques will, in most cases, give inaccurate results. In this situation, recourse may be made to qualitative methods associated with the concept of the 'phase plane', by which means existence and (in)stability of solutions may be demonstrated and the solution curves constructed by various graphical means. (*See* for example Ref. 4, Chapter 2 or Ref 5, Chapters 1 and 2.) Such methods do not, however, give an explicit algebraic form for the solution.

In cases where equation (1) may be written

$$\ddot{x} + b\dot{x} + c_1 x + \varepsilon f(x, \dot{x}) = 0, \tag{4}$$

with $b, c_1 > 0$, the characteristic roots of the degenerate linear equation have negative real parts. If $f(x, \dot{x})$ is analytic in $x, \dot{x}$ then the solution of equation (4) may be expressed in series form by means of Lyapunov's Expansion Theorem, (*see* Ref. 6, pp. 95–106), or by the topological method of Ref. 7. For instance in Refs. 7 and 8 it has been shown that when the characteristic roots of

$$\ddot{x} + b\dot{x} + c_1 x = 0 \tag{5}$$

are complex, the general solution of

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = 0 \tag{6}$$

is

$$x = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} e^{-rkt} \{ a_r + b_{rn} \sin nmt + c_{rn} \cos nmt \}, \tag{7}$$

where

$$k = \tfrac{1}{2}b \quad \text{and} \quad m^2 = c_1 - k^2. \tag{8}$$

It is, of course, re-assuring to know that such a solution exists, but in strongly nonlinear cases the usefulness of equation (7) is questionable, since, to obtain adequate accuracy, it is necessary to evaluate a large number of terms in the summation.

For many physical applications it is desirable to have more manageable, though perhaps approximate, forms for the solution of equations (1) or (4) and several workers have obtained these. Probably the first is that given in Ref. 1, Chapters 3 and 4 and the extensions to this technique by Bogoliuboff and Mitropolsky.

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More recently a modified form of the K and B method was proposed by Beecham and Titchener in Ref. 9. The form of the solution,

$$x = a \cos \psi, \tag{9}$$

is retained but the nonlinear 'damping', λ, and 'frequency', ω, are defined as $\lambda = \dot{a}/a$ and $\omega = \dot{\psi}$, where these are obtained by a simple iterative procedure as functions of the amplitude, $a(t)$. Further extensions and examples of this technique have been discussed by Ross, by Titchener and by Simpson in Refs. 10, 11, 12 and 13. These include applications to the interpretation and presentation of nonlinear characteristics of flight vehicles, from experimental data, and the prediction of their dynamic response. This method, which has, in Refs. 10 and 13, been extended to systems of higher order and several degrees of freedom, will be referred to as the 'R.A.E. method'.

A different approach to the problem stems from the fact that the equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0 \tag{10}$$

has an exact general solution in terms of Jacobian elliptic functions, (see Ref. 14, Chapter 7). Using this as a basis Barkham and Soudack, in a series of papers,[15-18] have developed and refined an approximate solution for the class of equations

$$\ddot{x} + m^2(x + \gamma x^3) + \varepsilon f(x, \dot{x}, t) = 0, \tag{11}$$

with $\gamma > 0, m^2 > 0$ and ε a small constant. This solution has the form

$$x = a(t) Cn\{\psi(t), \mu(t)\}, \tag{12}$$

where $Cn\{ \ \}$ is the Jacobian elliptic function of argument $\psi(t)$ and modulus μ, (see Ref. 14, Chapter 6).

In cases such as

$$\varepsilon f(x, \dot{x}, t) \equiv b\dot{x},$$

where the transformation

$$x = v \exp\left\{-\frac{1}{2}\int_0^t b \, dt\right\} = v \exp\left(-\tfrac{1}{2}bt\right) \tag{13}$$

reduces equation (11) to the form

$$\ddot{v} + \alpha v + \beta(t)v^3 = 0, \tag{14}$$

it is possible to apply the work of Kuzmak,[20] to obtain an approximate solution.

An approach, based on perturbations of the elliptic equation (10), has also been used by Christopher in Ref. 19, where attention has been restricted to the case $\varepsilon f(x, \dot{x}, t) \equiv b\dot{x}$. Subsequent development of this work has given rise to a method which is capable of being applied to the more general class of equation (11). This technique will, with apologies, be referred to as the CJ method (Christopher-Jacobian method).

The purpose of the present report is first to review briefly the R.A.E. and CJ methods and then to apply them to a particular strongly nonlinear equation for comparison with exact numerical solutions.

2. The R.A.E. Method

2.1. Description of Method

The R.A.E. method as described in Ref. 9 was originally developed to give approximate analytic solutions for the damping and frequency as functions of the amplitude of the envelope, but it may be readily extended to give the complete solution, with the independent variable and the phase angle as parametric functions of the amplitude.

The solution is assumed to be of the form

$$x = a \cos \psi, \tag{15}$$

where 'damping', λ, and 'frequency', ω, are defined as

$$\lambda = \frac{\dot{a}}{a} \quad \text{and} \quad \omega = \dot{\psi}. \tag{16}$$

Thus

$$\dot{x} = a(\lambda \cos\psi - \omega \sin\psi)$$

and

$$\ddot{x} = -a\{(\omega^2 - \lambda^2 - \dot{\lambda})\cos\psi + (\dot{\omega} + 2\lambda\omega)\sin\psi\}.$$

As in the Kryloff and Bogoliuboff method, equation (1) is multiplied by $\sin\psi$ and $\cos\psi$ respectively, and the average value over one cycle of ψ is assumed to be zero. Here, the damping and frequency, λ and ω, are assumed to be equal to their values at mid-cycle, but their derivatives are retained. Then Ref. 9 gives that

$$\lambda = I_1/2\omega\left[1 + \frac{a}{4\omega^2}\frac{d(\omega)^2}{da}\right] \tag{17}$$

and

$$\omega^2 = \lambda^2 + I_2 + \lambda a\frac{d\lambda}{da} \tag{18}$$

where

$$I_1 = \frac{1}{\pi a}\int_0^{2\pi} F\sin\psi \, d\psi, \qquad I_2 = \frac{1}{\pi a}\int_0^{2\pi} F\cos\psi \, d\psi$$

and d/dt has been replaced by $a\lambda(d/da)$.

Equations (17) and (18) may be solved iteratively, to give $\lambda(a)$ and $\omega(a)$. The extension to the original report is to integrate combinations of these functions, to give

$$t = \int_0^t dt = \int_{a(0)}^a \frac{dt}{da}da = \int_{a(0)}^a \frac{da}{\lambda a}, \tag{19}$$

and

$$\psi = \int_0^t \omega \, dt = \int_{a(0)}^a \frac{\omega}{\lambda a}da. \tag{20}$$

The complete solution, $x(t) = a\cos\psi$, is then readily obtained in the form $x(a)$, $t(a)$.

2.2. Application to Equation (6)

With $F(x, \dot{x}) = b\dot{x} + c_1 x + c_3 x^3$, then

$$I_1 = -b\omega$$

and

$$I_2 = b\lambda + c_1 + \tfrac{3}{4}c_3 a^2.$$

It is shown in Ref. 9 that the second approximation, obtained after one iteration, gives good agreement for the variation of λ and ω with amplitude when compared with numerical solution of the equation. Thus

$$\lambda = \frac{-b\omega^2}{2\omega^2 + \tfrac{3}{4}c_3 a^2} \tag{21}$$

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and

$$\omega^2 = c_1 - \frac{b^2}{4} + \tfrac{3}{4}c_3 a^2 ; \tag{22}$$

i.e.

$$\lambda = \frac{-b(m^2 + \tfrac{3}{4}c_3 a^2)}{2m^2 + \tfrac{9}{4}c_3 a^2},$$

where $m^2 = c_1 - b^2/4$.

On substituting into equation (19),

$$t = \frac{1}{b} \log_e \left\{ \frac{a^2(0)[m^2 + \tfrac{3}{4}c_3 a^2(0)]^{\frac{1}{2}}}{a^2[m^2 + \tfrac{3}{4}c_3 a^2]^{\frac{1}{2}}} \right\}, \tag{23}$$

and equation (20) gives that

$$\psi = -\frac{1}{b} \int_{a(0)}^{a} \frac{(2m^2 + \tfrac{9}{4}c_3 a^2)}{a(m^2 + \tfrac{3}{4}c_3 a^2)^{\frac{1}{2}}} \, da. \tag{24}$$

The form of the solution of equation (24) depends on the sign of m^2. For positive m^2, with $4m^2/3c_3 = \sigma^2$, then

$$\psi = \frac{3\sqrt{3c_3}}{2b} \left\{ [\sigma^2 + a^2(0)]^{\frac{1}{2}} - [\sigma^2 + a^2]^{\frac{1}{2}} + \frac{2\sigma}{3} \log_e \left(\frac{a(0)[(\sigma^2 + a^2)^{\frac{1}{2}} + \sigma]}{a[[\sigma^2 + a^2(0)]^{\frac{1}{2}} + \sigma]} \right) \right\} \tag{25}$$

and

$$t = \frac{1}{b} \log_e \left\{ \frac{a^2(0)[\sigma^2 + a^2(0)]^{\frac{1}{2}}}{a^2[\sigma^2 + a^2]^{\frac{1}{2}}} \right\}. \tag{26}$$

For $m^2 = 0$, then

$$\psi = \frac{3\sqrt{3c_3}}{2b}[a(0) - a] \tag{27}$$

and

$$t = \frac{3}{b} \log_e \left\{ \frac{a(0)}{a} \right\}. \tag{28}$$

With negative m^2, for which the degenerate linear problem is more than critically damped,

$$\psi = \frac{3\sqrt{3c_3}}{2b} \left\{ [a^2(0) - \delta^2]^{\frac{1}{2}} - [a^2 - \delta^2]^{\frac{1}{2}} + \frac{2\delta}{3} \left(\sin^{-1} \left[\frac{\delta}{a(0)} \right] - \sin^{-1} \left[\frac{\delta}{a} \right] \right) \right\} \tag{29}$$

$$t = \frac{1}{b} \log_e \left\{ \frac{a^2(0)[a^2(0) - \delta^2]^{\frac{1}{2}}}{a^2[a^2 - \delta^2]^{\frac{1}{2}}} \right\},$$

where $\delta^2 = -4m^2/3c_3$.

This latter solution exists only for $a > \delta$, which from equation (22), in turn implies that the frequency is real.

3. The CJ Method

The method was developed in Ref. 19, in relation to the equation

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = 0 \quad \text{for} \quad b, c_1, c_3 > 0 \tag{30}$$

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and has subsequently been generalised in relation to equation (11). Because of certain differences in the formulation of the solutions in Ref. 19 and in the more general case, and because the computations of Section 4 are based on the formulation of Ref. 19, it is convenient to restrict the present section to a summary of the method of Ref. 19.

Using the transformation equation (13) and normalizing with respect to $x(0) = v(0)$ gives

$$\ddot{u} + m^2\{u + \gamma(t)u^3\} = 0, \tag{31}$$

where

$$u = v/x(0),$$

$$\dot{u} = du/dt,$$

$$k = b/2,$$

$$m^2 = c_1 - k^2,$$

$$\gamma(0) = c_3 x^2(0)/m^2$$

and

$$\gamma(t) = \gamma(0)\,e^{-2kt}.$$

It is assumed, on the basis of arguments which are developed in Ref. 19, Section 3, that the solution of equation (31) may be written as

$$u(t) = C(t)Cn\{\psi(t), \mu(t)\}, \tag{32}$$

where

$$\psi(t) = \int_0^t \omega(t)\,dt - \phi(t) \tag{33}$$

is the argument of the Jacobian elliptic function $Cn\{\ \}$, and $\mu(t)$ its modulus. All four of the functions $C(t)$, $\omega(t)$, $\phi(t)$ and $\mu(t)$ are required to be determined.

Differentiating equation (32) with respect to t then gives

$$\dot{u} = -C\omega SnDn + \{\dot{C}Cn + CSnDn\dot{\phi} + C(1 - \mu^2)^{-1}\mu\dot{\mu}Sn(DnI - SnCn)\} \tag{34}$$

where

$$I(\psi, \mu) = \int_0^\psi Cn^2\,d\psi = \mu^{-2}\{E(\psi, \mu) - (1 - \mu^2)\psi\} \tag{35}$$

and

$$E(\psi, \mu) = \int_0^\psi Dn^2\,d\psi \tag{36}$$

is the incomplete elliptic integral of the second kind. A second differentiation then gives

$$\ddot{u} = -C\omega^2 Cn(Dn^2 - \mu^2 Sn^2) - \omega SnDn\dot{C} - CSnDn\dot{\omega} + \omega CCn(1 - 2Dn^2)\dot{\phi} -$$

$$-\omega C\mu(1 - \mu^2)^{-1}\{ICn(1 - 2Dn^2) + SnDn(1 - 2Sn^2)\}\dot{\mu} +$$

$$+\frac{d}{dt}\{\dot{C}Cn + CSnDn\dot{\phi} + C(1 - \mu^2)^{-1}\mu\dot{\mu}Sn(DnI - SnCn)\}. \tag{37}$$

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Because equation (31) is an equation of second order, it requires only two distinct arbitrary functions in the formulation of its general solution; it is, therefore, possible to impose two constraints upon the form of $\dot{u}$ and $\ddot{u}$ in such a way as to produce simultaneous relationships of dependence between $C(t)$, $\omega(t)$, $\phi(t)$ and $\mu(t)$. One way of doing this is to require that

$$\dot{u} = -C\omega SnDn$$

and

$$\ddot{u} = -C\omega^2 Cn(Dn^2 - \mu^2 Sn^2)$$

$$\left.\right\} \quad (38)$$

which is the form these derivatives would have if $b = 0$. For equation (38) to be true, it follows that the constraint relations

$$\dot{C}Cn + CSnDn\dot{\phi} + C(1 - \mu^2)^{-1}\mu Sn(DnI - SnCn)\dot{\mu} = 0 \quad (39)$$

and

$$\omega SnDn\dot{C} + CSnDn\dot{\omega} + \omega CCn(1 - 2Dn^2)\dot{\phi} + \omega C\mu(1 - \mu^2)^{-1}\{ICn(1 - 2Dn^2) + SnDn(1 - 2Sn^2)\}\dot{\mu} = 0 \quad (40)$$

must be satisfied.

It is the approximate satisfaction of equations (39) and (40), and, thereby, the determination of $C(t)$, ...etc., that constitutes the basis of the CJ method. The imposition of constraints in this manner is, of course, not new and closely parallels the approach of K and B in Ref. 1 when dealing with perturbations of equation (3).

An added advantage of the imposition of equation (38) is that the solution satisfies the identity

$$\frac{d^2}{d\psi^2}Cn\{\psi, \mu\} = (2\mu^2 - 1)Cn\{\psi, \mu\} - 2\mu^2 Cn^3\{\psi, \mu\} \quad (41)$$

and, thereby, gives rise to the expressions

$$\omega^2 = m^2(1 + \gamma C^2), \quad (42)$$

$$2\mu^2 = \gamma C^2/(1 + \gamma C^2), \quad (43)$$

$$2\omega^2\mu^2 = m^2\gamma C^2 \quad (44)$$

and

$$\gamma C^2 = \omega^2/m^2 - 1 = \mu^2/(\tfrac{1}{2} - \mu^2). \quad (45)$$

It will be noted that for $\gamma C^2 \geqslant 0$, $0 \leqslant \mu^2 \leqslant \tfrac{1}{2}$; the value $\gamma = 0$, $\mu = 0$ corresponding to the degeneration of equation (30) to the equation of the perturbed harmonic oscillator.

The initial values, which will be known, are

$$u(0) = \xi_1 = C(0)Cn\{-\phi(0), \mu(0)\} = 1$$

and

$$\dot{u}(0) = \xi_2 = -C(0)\omega(0)Sn\{-\phi(0), \mu(0)\}Dn\{-\phi(0), \mu(0)\},$$

from which it follows that

$$C(0) = 1/Cn\{-\phi(0), \mu(0)\}$$

or

$$\phi(0) = \arccos\{1/C(0)\}; \quad (46)$$

and that

$$C^2(0) = \gamma^{-1}(0) \left[-1 \pm \left\{ \{1 + \gamma(0)\}^2 + \frac{2\gamma(0)}{m^2} \left(\frac{\xi_2}{\xi_1}\right)^2 \right\}^{\frac{1}{2}} \right].$$ (47)

For real C only the positive branch of the root is taken.

The qualitative nature of the solution may readily be obtained by resort to 'phase plane' techniques and will be of a 'damped' form which may, or may not, be oscillatory in character. Both the frequency and degree of damping will be dependent on u and t. Provided $m^2 > 0$, γC^2 will be finite, therefore $\mu^2(0) < \frac{1}{2}$ and $\mu^2(t) \to 0$ as $t \to \infty$.

The solutions $C(t)$, $\omega(t)$, $\phi(t)$ and $\mu(t)$ to the simultaneous equations (39), (40), (42) and (43) are not apparent. However, the coefficients of $\dot{C}$, $\dot{\omega}$, $\dot{\phi}$ and $\dot{\mu}$ in equations (39) and (40) are capable of being expressed as the sums of integral powers of Sn. It follows that these coefficients have a period of $4K(\mu)$, where $K(\mu)$ is the complete elliptic integral of the first kind. Provided then that C, ω, ϕ and μ are varying slowly with t, compared to the variation of the periodic coefficients, it is then appropriate to consider an approximate solution for C, ω, ϕ and μ, based on the principle of averaging. This procedure again closely parallels that of K and B in Ref. 1. The resulting averaged equations constitute an approximation to equations (39) and (40) whose accuracy will be greatest when the instantaneous frequency, $\psi = \omega - \dot{\phi}$, is large in comparison with $\dot{C}$, $\dot{\omega}$, $\dot{\mu}$ and $\dot{\phi}$. See Ref. 2, Chapters 14, 15 and 16 for a rigorous presentation of the averaging theorems.

3.1. First Approximation, $\dot{\mu} = 0$

Earlier work has demonstrated that a reasonable first approximation is achieved by taking $\dot{\mu} = 0$ in equations (39) and (40). This does not, of course, mean that μ is constant, but that μ is varying sufficiently slowly with t for the terms containing $\dot{\mu}$ to be ignored. Thus equations (39) and (40) become

$$\dot{C}Cn + CSnDn\dot{\phi} = 0$$ (48)

and

$$(\omega\dot{C} + C\dot{\omega})SnDn + \omega C(1 - 2Dn^2)Cn\dot{\phi} = 0,$$ (49)

respectively. Eliminating $\dot{\phi}$ between these equations gives, after division by ωC an expression in terms of Sn,

$$\frac{\dot{C}}{C}(1 - 2\mu^2 Sn^2 + \mu^2 Sn^4) + \frac{\dot{\omega}}{\omega}(Sn^2 - \mu^2 Sn^4) = 0.$$ (50)

The coefficients of $\dot{C}/C$ and $\dot{\omega}/\omega$ are periodic of period $4K(\mu)$ and, as explained above, an approximation to equation (50) may be obtained by averaging the periodic coefficients over the period 0 to $4K(\mu)$. This gives

$$\frac{\dot{C}}{C}(1 - 2\mu^2 Q_1 + \mu^2 Q_2) + \frac{\dot{\omega}}{\omega}(Q_1 - \mu^2 Q_2) = 0$$ (51)

where, as shown in the appendix of Ref. 19,

$$Q_1(\mu) = \frac{1}{4K} \int_0^{4K} Sn^2 \, d\phi = \mu^{-2}\{1 - E(\mu)/K(\mu)\}$$

$$= \frac{1}{2} + \frac{1}{16}\mu^2 + \frac{1}{32}\mu^4 + \dots$$ (52)

and

$$Q_2(\mu) = \frac{1}{4K} \int_0^{4K} Sn^4 \, d\phi = \frac{3}{8} + \frac{1}{16}\mu^2 + \frac{35}{1024}\mu^4 + \dots,$$ (53)

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these series being rapidly convergent. If terms up to an including μ^4 are retained, then upon substitution for Q_1 and Q_2 in equation (51), and division throughout by the coefficient of $\dot{C}/C$, there emerges the equation

$$\frac{\dot{C}}{C} + \tfrac{1}{2}\frac{\dot{\omega}}{\omega} = 0. \tag{54}$$

Integration of equation (54) with respect to t then gives

$$\frac{C(t)}{C(0)} = \left[\frac{\omega(t)}{\omega(0)}\right]^{-\frac{1}{2}} = \left[\frac{1 + \gamma(t)C^2(t)}{1 + \gamma(0)C^2(0)}\right]^{-\frac{1}{4}} = \left[\frac{\tfrac{1}{2} - \mu^2(t)}{\tfrac{1}{2} - \mu^2(0)}\right]^{\frac{1}{4}} \tag{55}$$

Eliminating $\dot{C}$ between equations (48) and (49) gives, after multiplication throughout by Cn,

$$\omega C \dot{\phi} \{ Cn^2(1 - 2Dn^2) - Sn^2 Dn^2 \} + CSnCnDn\dot{\omega} = 0,$$

which upon averaging over the period 0 to $4K(\mu)$ becomes

$$(-1 + 2\mu^2 Q_1 - \mu^2 Q_2)\dot{\phi} = -\left(\frac{\dot{\omega}}{\omega}\right)\frac{1}{4K}\int_0^{4K} SnCnDn\, d\psi = 0$$

or

$$\dot{\phi} = 0$$

and

$$\phi(t) = \phi(0). \tag{56}$$

Now

$$\int_0^t \omega\, dt = m\int_0^t \{1 + \gamma C^2\}^{\frac{1}{2}}\, dt. \tag{57}$$

Write

$$y^2 = \gamma(t)C^2(t)$$

then upon differentiation

$$\frac{\dot{y}}{y} = \frac{\dot{C}}{C} - k = -k - \tfrac{1}{2}\frac{\dot{\omega}}{\omega},$$

from equation (54). Differentiating equation (42) gives

$$\frac{\dot{\omega}}{\omega} = y\dot{y}/(1 + y^2);$$

hence

$$\frac{\dot{y}}{y} = -k - \tfrac{1}{2}y\dot{y}/(1 + y^2)$$

or

$$\dot{y} = -ky(1 + y^2)/(1 + \tfrac{1}{2}y^2).$$

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Expressing equation (57) in terms of y then gives

$$\int_0^t \omega\,dt = -\frac{m}{k}\int_{y(0)}^{y(t)}(1 + \tfrac{3}{2}y^2)y^{-1}(1 + y^2)^{-\frac{1}{2}}\,dy$$

$$= -\frac{m}{2k}\left[3\{\Omega(t) - \Omega(0)\} + \log_e\left\{\frac{\{\Omega(t) - 1\}\{\Omega(0) + 1\}}{\{\Omega(t) + 1\}\{\Omega(0) - 1\}}\right\}\right],$$ (58)

where
$$\Omega(t) = \omega(t)/m = (1 + y^2)^{\frac{1}{2}}.$$ (59)

It will be seen that equations (55), (58) and (59) correspond to those of the second approximation of Ref. 19.

3.2. Second Approximation

In this approximation the terms containing $\dot{\mu}$, omitted from the first approximation, are now included. From equation (39)

$$C\dot{\phi} = -\{C(1 - \mu^2)^{-1}\mu\dot{\mu}Sn(DnI - SnCn) + \dot{C}Cn\}/(SnDn),$$

which upon substitution into equation (40) gives, after multiplication by $SnDn$,

$$(\omega\dot{C} + C\dot{\omega})Sn^2Dn^2 - \omega\dot{C}Cn^2(1 - 2Dn^2) + \omega C(1 - \mu^2)^{-1}\mu\dot{\mu}\{Sn^2Dn^2(1 - 2Sn^2) + Sn^2Cn^2(1 - 2Dn^2)\} = 0.$$ (60)

From equations (44) and (45)

$$2\mu^2\omega^2 = \omega^2 - m^2$$

which upon differentiation gives

$$\mu\dot{\mu} = \dot{\omega}(\tfrac{1}{2} - \mu^2)/\omega.$$ (61)

Substituting this result into equation (60) and dividing by ωC yields, after expression in terms of Sn,

$$\frac{\dot{C}}{C}(1 - 2\mu^2Sn^2 + \mu^2Sn^4) + \frac{\dot{\omega}}{\omega}(Sn^2 - \tfrac{1}{2}Sn^4) = 0.$$ (62)

Averaging this equation over the period 0 to $4K(\mu)$, and dividing throughout by the resulting coefficient of $\dot{C}/C$, gives

$$\frac{\dot{C}}{C} + R(\mu^2)\frac{\dot{\omega}}{\omega} = 0,$$ (63)

where terms up to and including μ^4 have been retained and

$$R(\mu^2) = \tfrac{5}{16} + \tfrac{29}{128}\mu^2 + \tfrac{359}{2048}\mu^4 + \ldots.$$ (64)

From equations (61) and (63)

$$\frac{\dot{C}}{C} = -\frac{R(\mu^2)\mu\dot{\mu}}{(\tfrac{1}{2} - \mu^2)}$$

thus

$$\int_0^t \frac{\dot{C}}{C}\,dt = -\int_{\mu(0)}^{\mu(t)} R(\mu^2)\mu(\tfrac{1}{2} - \mu^2)^{-1}\,d\mu$$

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or

$$C(t) = C(0)\left[\frac{\frac{1}{2} - \mu^2(t)}{\frac{1}{2} - \mu^2(0)}\right]^{0.2348} \exp\left[G\{\mu^2(t)\} - G\{\mu^2(0)\}\right], \tag{65}$$

where

$$G(\mu^2) = 0.1572\mu^2 + 0.0438\mu^4 + \dots. \tag{66}$$

Now

$$\frac{\dot{y}}{y} = \frac{\dot{C}}{C} - k,$$

which upon substitution for $\dot{C}/C$ from equation (63), and some further manipulation, gives

$$\dot{y} = -y\{k + R(\mu^2)y\dot{y}/(1 + y^2)\}. \tag{67}$$

Upon substitution from equation (67) into equation (57), and expression of $R(\mu^2)$ in terms of y, gives

$$\int_0^t \omega \, dt = -\frac{m}{k}\left[\frac{1}{2}\log_e\left\{\frac{\{\Omega(t) - 1\}\{\Omega(0) + 1\}}{\{\Omega(t) + 1\}\{\Omega(0) - 1\}}\right\} + 1.469\{\Omega(t) - \Omega(0)\} + \right.$$

$$\left. +0.201\{\Omega^{-1}(t) - \Omega^{-1}(0)\} - 0.015\{\Omega^{-3}(t) - \Omega^{-3}(0)\} + \dots\right]. \tag{68}$$

From equation (39)

$$SnDn\{\dot{\phi} + (1 - \mu^2)^{-1}I\mu\dot{\mu}\} = (1 - \mu^2)^{-1}Sn^2Cn\mu\dot{\mu} - Cn\frac{\dot{C}}{C},$$

which upon multiplication by $SnDn$ and averaging over 0 to $4K(\mu)$ becomes

$$(Q_1 - \mu^2 Q_2)\dot{\phi} = \frac{1}{4K}\left\{(1 - \mu^2)^{-1}\mu\dot{\mu}\int_0^{4K} Sn^3CnDn \, d\psi - \frac{\dot{C}}{C}\int_0^{4K} SnCnDn \, d\psi\right\} -$$

$$-(1 - \mu^2)^{-1}\mu\dot{\mu}\{Sn^2Dn^2I\}_{average}. \tag{69}$$

The first and second of these integrals are zero, whilst

$$\{Sn^2Dn^2I\}_{average} = \mu^{-2}[Sn^2Dn^2\{E(\psi, \mu) - (1 - \mu^2)\psi\}]_{average}.$$

Now, as shown in Ref. 14, p. 136, equation (7), $E(\psi, \mu)$ may be expanded in the form

$$E(\psi, \mu) = \frac{2}{\pi}E\left(\frac{\pi}{2}, \mu\right)\psi + SnCn\{\alpha_0(\mu^2) + \alpha_2(\mu^2)Sn^2 + \dots + \alpha_{2m}(\mu^2)Sn^{2m} + \dots\} \tag{70}$$

thus

$$(Q_1 - \mu^2 Q_2)\dot{\phi} = -\frac{1}{4}K^{-1}\mu^{-1}(1 - \mu^2)^{-1}\dot{\mu}\left\{\left\{\frac{2}{\pi}E\left(\frac{\pi}{2}, \mu\right) - (1 - \mu^2)\right\}\psi\int_0^{4K} Sn^2Dn^2 \, d\psi + \right.$$

$$\left. + \sum_{m=0}^{\infty} \alpha_{2m}(\mu^2)\int_0^{4K} Sn^{2m+3}CnDn^2 \, d\psi\right\}.$$

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Since

$$\int_0^{4K} Sn^{2m+3} Cn Dn^2 \, d\psi = 0, \quad \text{for } m = 0, 1, 2, \ldots,$$

then

$$\dot{\phi} = -(1 - \mu^2)^{-1} \mu^{-1} \dot{\mu} \left\{ \frac{2}{\pi} E\left(\frac{\pi}{2}, \mu\right) - (1 - \mu^2) \right\} \psi$$

$$= -\tfrac{3}{4}(1 - \mu^2)^{-1} \mu \dot{\mu}(1 - \tfrac{1}{16}\mu^2 - \ldots)\psi.$$

In the first approximation $\dot{\phi} = 0$. It might reasonably be anticipated that an adequate second approximation for $\dot{\phi}$ will be obtained by replacing $(1 - \tfrac{1}{16}\mu^2 - \ldots)$ by unity; hence

$$\dot{\phi} - \tfrac{3}{4}(1 - \mu^2)^{-1}\mu\dot{\mu}\phi = -\tfrac{3}{4}(1 - \mu^2)^{-1}\mu\dot{\mu} \int_0^t \omega \, dt. \tag{71}$$

Integration of this equation then gives

$$\phi(t) = \phi(0) + (1 - \mu^2)^{-\frac{3}{8}}\{H(t) - H(0)\}, \tag{72}$$

where

$$H(t) = \frac{3m}{4K} \times 2^{-\frac{3}{8}}\{-1.469\Omega^{-1} - 0.5\Psi\Omega^{-2} + 0.572\Omega^{-3} + 0.156\Psi\Omega^{-4} - 0.177\Omega^{-5}$$

$$-0.083\Psi\Omega^{-6} + 0.146\Omega^{-7} + 0.055\Psi\Omega^{-8} - 0.033\Omega^{-9}\ldots\} \tag{73}$$

and

$$\Psi = -1.469\Omega(0) + 0.799\Omega^{-1}(0) + 0.348\Omega^{-3}(0) + 0.2\Omega^{-5}(0) + \ldots. \tag{74}$$

4. A Numerical Comparison and Conclusions

The equation chosen as the basis for the numerical comparison between the R.A.E. and CJ methods was

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = 0, \tag{75}$$

with $b, c_1, c_3 > 0$. By writing

$$\tau = vt \tag{76}$$

equation (75) can be transformed into

$$v^2 x'' + vbx' + c_1 x + c_3 x^3 = 0,$$

where $x' = dx/d\tau$. Choosing $v^2 \equiv c_1$ then permits this equation to be reduced to the normalized form

$$x'' + qx' + x + px^3 = 0 \tag{77}$$

where $q = b/v$ and $p = c_3/v^2 = c_3/c_1$. For the purpose of the comparison it was assumed that $c_1 = 1.0$, throughout, and, thereby, that equation (75) was already in normalized form.

The comparison was made between solutions obtained by the following methods:

(a) numerical integration using a standard integration technique on a digital computer,

(b) the R.A.E. method,

(c) the CJ method.

(d) the 'refined approximation' of Barkham and Soudack as described in Refs. 17 and 18.

The technique of (d) gives the solution of equation (75) in the form

$$x = a(t)Cn\{\psi(t), \mu\}, \qquad \qquad (78)$$

where

$$a(t) = \exp\left[\frac{-bt}{2(1 + \mu^2)}\right], \qquad (79)$$

$$\psi(t) = \left[1 + p\exp\left\{\frac{-bt}{1 + \mu^2}\right\}\right]^{\frac{1}{4}} vt - \phi(0), \qquad (80)$$

$$\mu^2 = \frac{pa^2(0)}{2\{1 + pa^2(0)\}}, \qquad (81)$$

$$v^2 = c_1$$

and

$$p = c_3/c_1. \qquad (82)$$

For the purpose of this comparison the calculations using methods (b) and (c) were programmed for digital machines at the R.A.E. Farnborough and C.I.T., Cranfield, respectively. In many cases it would be anticipated that these approximate methods might be employed in association with small electronic calculators or slide rules. Some such calculations were done and, although not presented here, gave accurate comparisons with their programmed counterparts. The values of Cn required in the methods (c) and (d) were obtained either, from the tables of Ref. 21, or, in the case of the programmed solutions, by means of a sub-routine based on the descending Landen transformation (see Ref. 22, p. 573).

Computations were made for the fourteen cases:

$$b = 0{\cdot}1, \qquad c_3 = 1, 2, 10, 100$$

$$b = 0{\cdot}5, \qquad c_3 = 1, 2, 10, 100$$

$$b = 1{\cdot}0, \qquad c_3 = 1, 2, 10, 100$$

$$b = 2{\cdot}0, \qquad c_3 = \quad 2, 10$$

with $c_1 = 1{\cdot}0$, $x(0) = 1{\cdot}0$, $\dot{x}(0) = 0$ throughout. The results are shown in graphical form in Figs. 1 to 14. It can be seen from these that the method of Barkham and Soudack (d) is, in general, of a lower order of accuracy than the other methods. The R.A.E. and CJ methods give solutions which are, for most of these cases, close to each other and have a high degree of accuracy. Both of these methods accurately predict the envelopes of maximum and minimum, and both tend to become less accurate in cases where the ratio b/c_3 is relatively large, as does the method of Barkham and Soudack. This inaccuracy is particularly marked at lower values of t. With b fixed there is a tendency for the R.A.E. method to develop a 'phase' error with increasing c_3 and t. This effect is much smaller in the CJ method. A shortcoming of the CJ method, as formulated in Section 3 is that it is limited to cases where $m^2 = c_1 - b^2/4 > 0$ and, therefore, cannot be used when $b^2 > 4c_1$ or, in the present example, when $b \geqslant 2$. This restriction does not, however, apply to the other techniques and a limited comparison for $b = 2$ is shown in Figs. 13 and 14. Both these figures demonstrate the inaccuracy associated with relatively large values of b/c_3 already observed in Figs. 1 to 12.

It is clear that both the R.A.E. and CJ methods offer a useful means of solving equation (75) to a satisfactory degree of accuracy for many purposes. In order to be able to draw similar conclusions about the application of these techniques to a wider class of equations such as equation (11) or equation (1) further extensive numerical comparisons are required.

LIST OF SYMBOLS

$\left.\begin{array}{c} b \\ c_1 \\ c_3 \end{array}\right\}$ Constant coefficients, equation (6)

k $b/2$

m $(c_1 - k^2)^{\frac{1}{2}}$

p c_3/c_1

q b/v

r *See* equation (7)

t Independent variable, usually time in applications to dynamics

u $v/x(0)$

v $x \exp (kt)$

x Dependent variable

$\dot{x}$ dx/dt

x' $dx/d\tau$

y $\{\gamma(t)C^2(t)\}^{\frac{1}{2}}$

$a(t)$ Amplitude function, equation (15)

$C(t)$ Amplitude function, equation (32)

$\left.\begin{array}{c} F(x, \dot{x}) \\ f(x, \dot{x}) \end{array}\right\}$ Nonlinear functions, equations (1) and (2)

$H(t)$ *See* equations (72) and (73)

I_1 $\dfrac{1}{\pi a} \displaystyle\int_0^{2\pi} F \sin \psi \, d\psi$

I_2 $\dfrac{1}{\pi a} \displaystyle\int_0^{2\pi} F \cos \psi \, d\psi$

$K(\mu)$ Complete elliptic integral of the first kind of modulus μ

$Q_1(\mu^2)$ $\dfrac{1}{4K} \displaystyle\int_0^{4K} Sn^2 \, d\psi$

$Q_2(\mu^2)$ $\dfrac{1}{4K} \displaystyle\int_0^{4K} Sn^4 \, d\psi$

$R(\mu^2)$ *See* equation (64)

$\left.\begin{array}{c} Cn(\psi, \mu) \\ Sn(\psi, \mu) \\ Dn(\psi, \mu) \end{array}\right\}$ Jacobian elliptic functions of argument ψ and modulus μ

$I(\psi, \mu)$ $\displaystyle\int_0^{\psi} Cn^2 \, d\psi$

$E(\psi, \mu)$ $\displaystyle\int_0^{\psi} Dn^2 \, d\psi$

$\left.\begin{array}{c} \alpha \\ \beta(t) \end{array}\right\}$ Coefficients in equation (14)

$\gamma(t)$ $\gamma(0) \exp (-2kt)$

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$\gamma(0)$ $c_3 x^2(0)/m^2$

δ $\{-4m^2/(3c_3)\}^{\frac{1}{2}}$

ε A small parameter, equation (2)

λ $\dot{a}/a$

$\mu(t)$ Modulus of Jacobian elliptic functions; *see* equations (32) and (45)

ν $c_1^{\frac{1}{2}}$

σ $\{4m^2/(3c_3)\}^{\frac{1}{2}}$

τ νt

$\phi(t)$ Instantaneous phase angle, *see* equation (33)

$\psi(t)$ Angle function, equation (15), argument of Jacobian elliptic function, equation (32)

$\omega(t)$ Instantaneous frequency, equations (16), (33) and (42)

$\Psi(\Omega)$ *See* equation (74)

Ω $\omega(t)/m$, non-dimensional frequency

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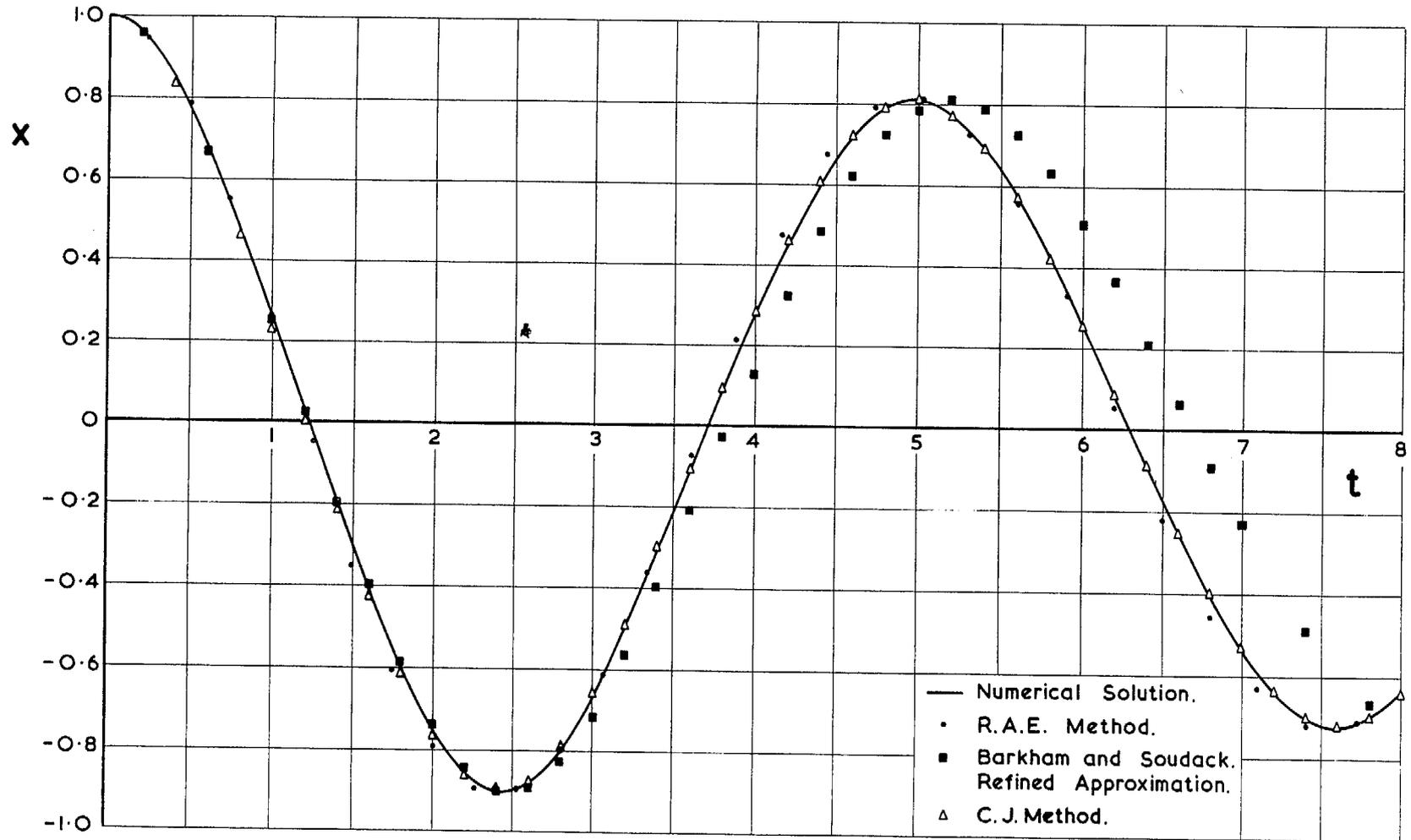
FIG. 1.　$b = 0.1, c_1 = 1.0, c_3 = 1.0.$
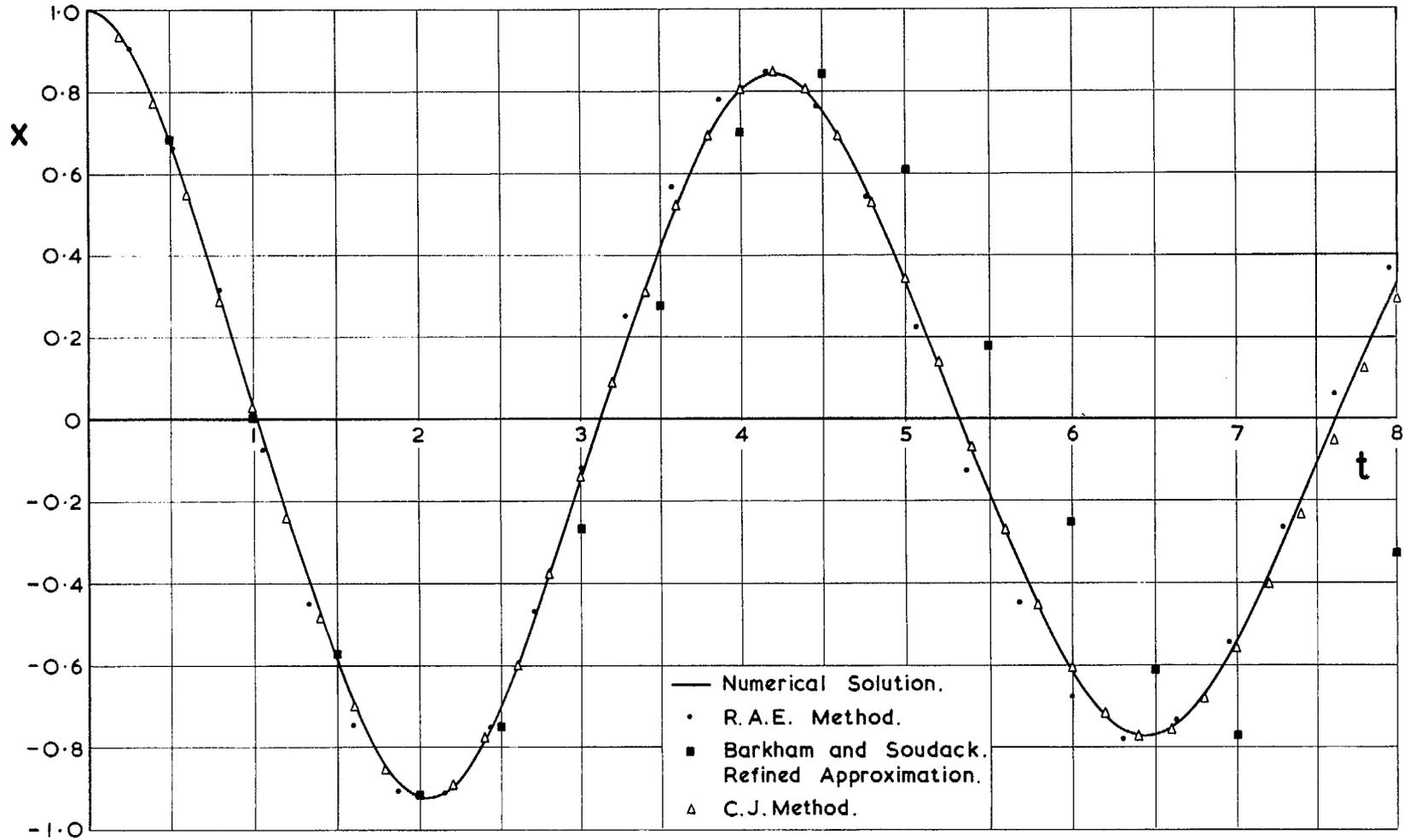
FIG. 2. $b = 0{\cdot}1, c_1 = 1{\cdot}0, c_3 = 2{\cdot}0.$
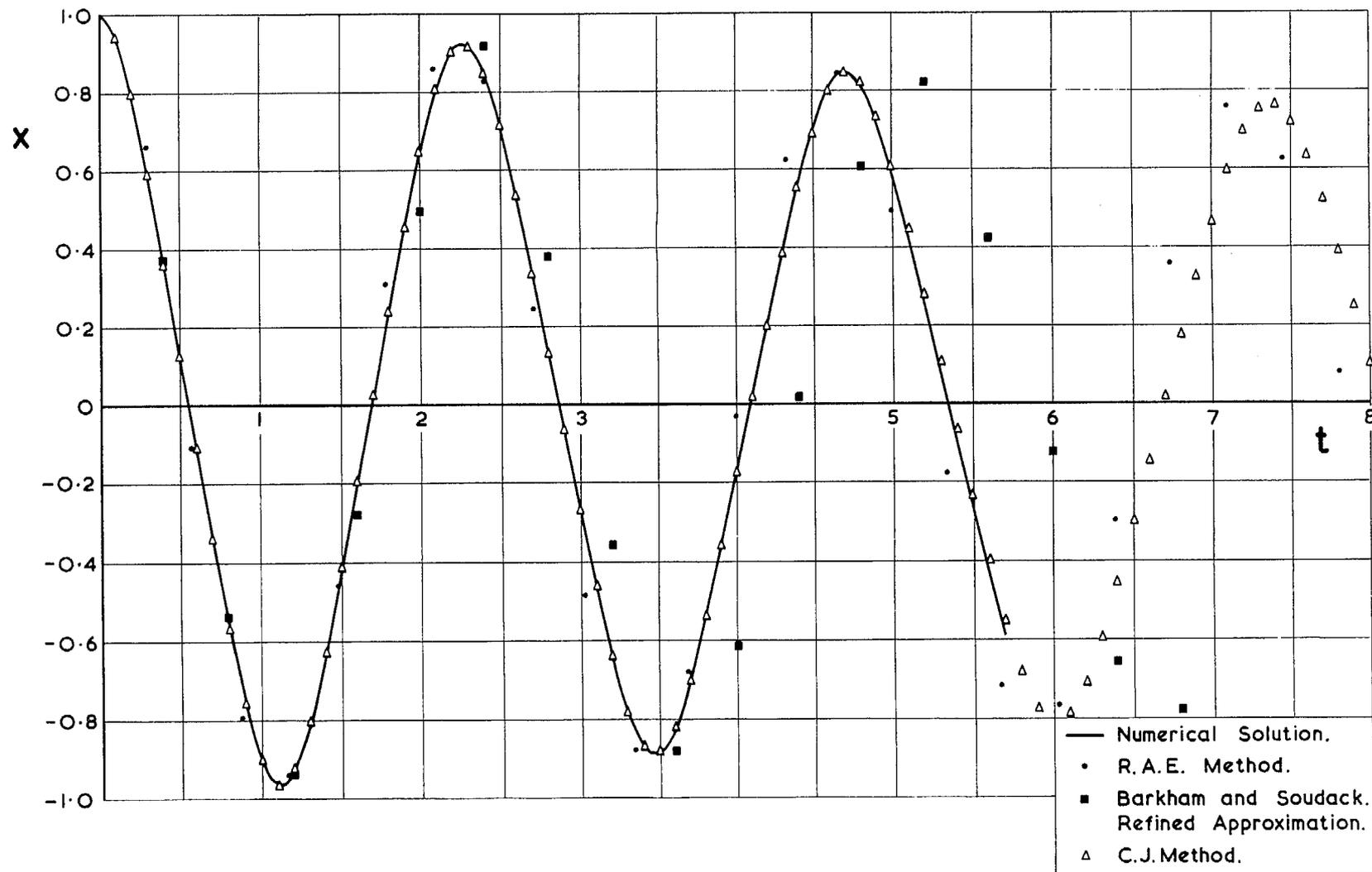
FIG. 3. $b = 0.1, c_1 = 1.0, c_3 = 10.0.$
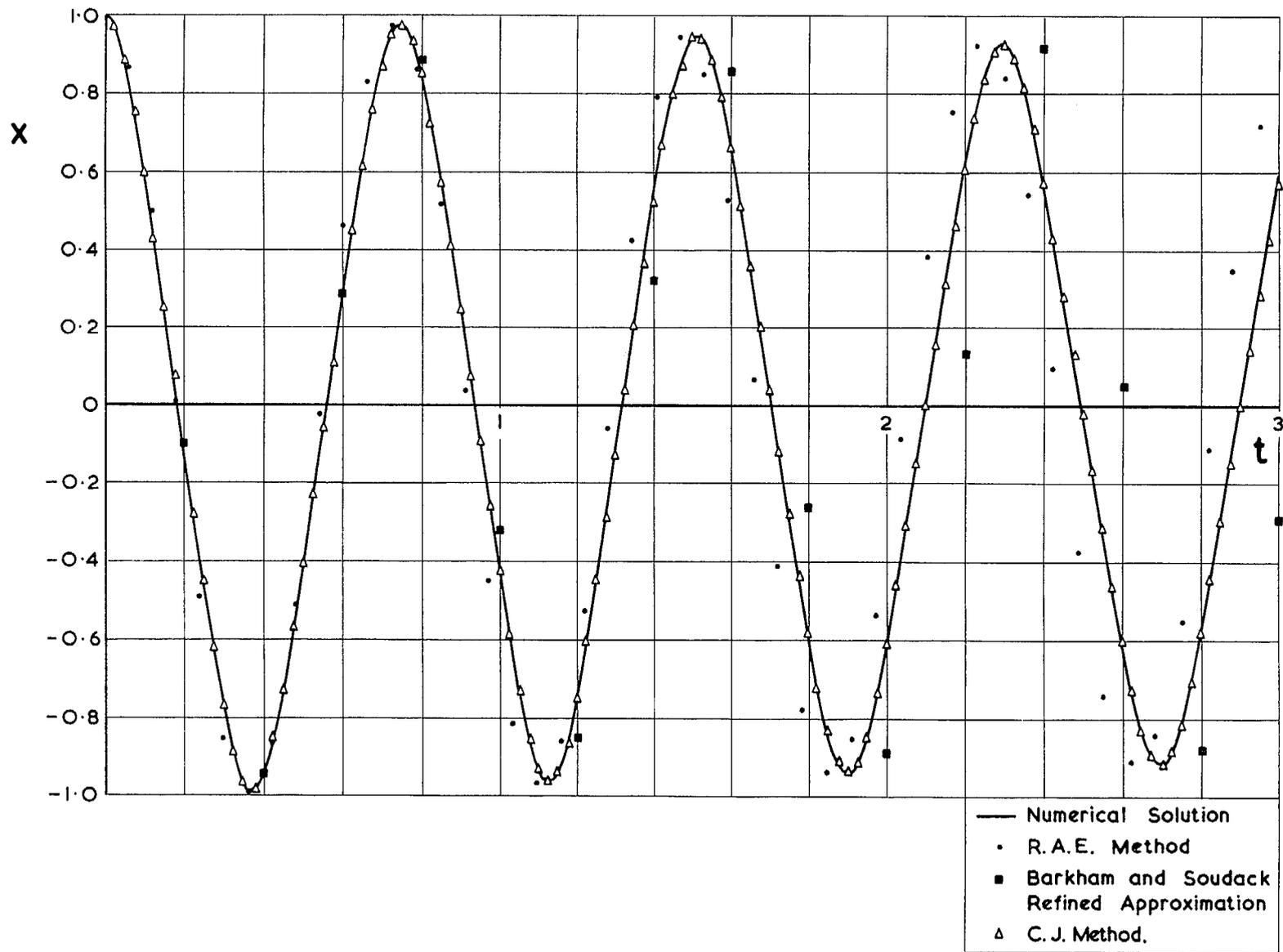
FIG. 4. $b = 0.1, c_1 = 1.0, c_3 = 100.$
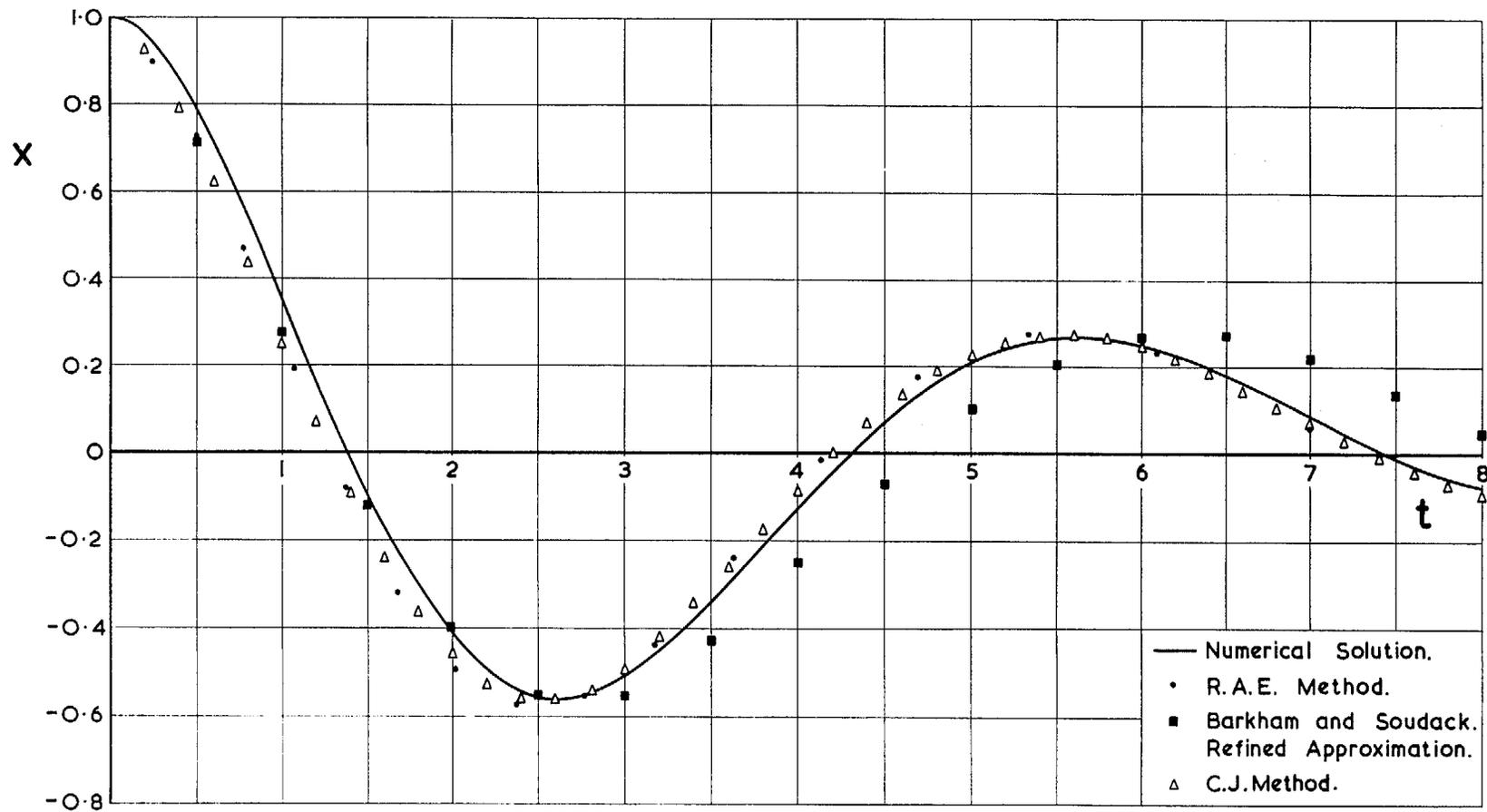
FIG. 5. $b = 0.5, c_1 = 1.0, c_3 = 1.0$.
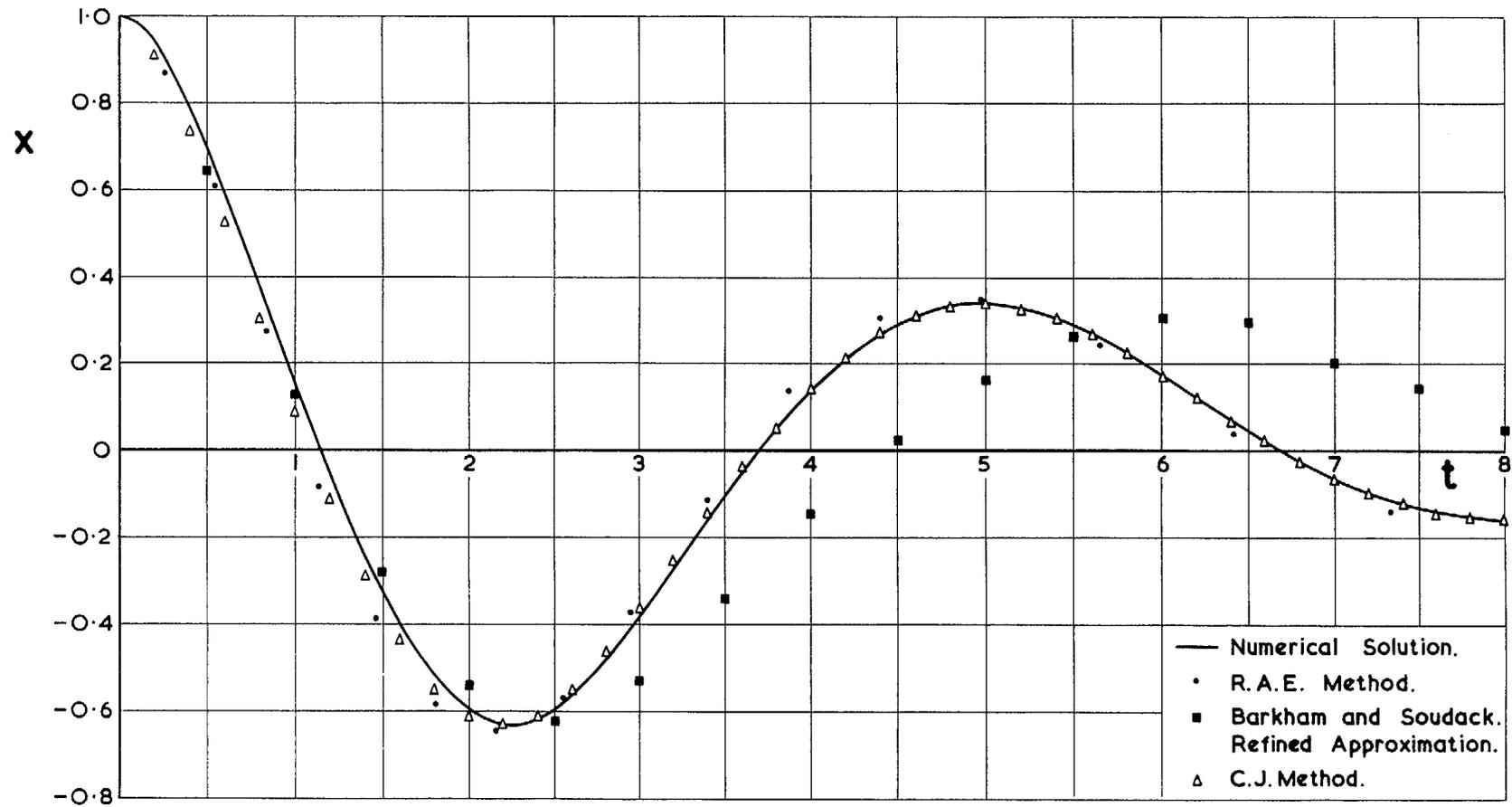
FIG. 6. $b = 0.5, c_1 = 1.0, c_3 = 2.0$.
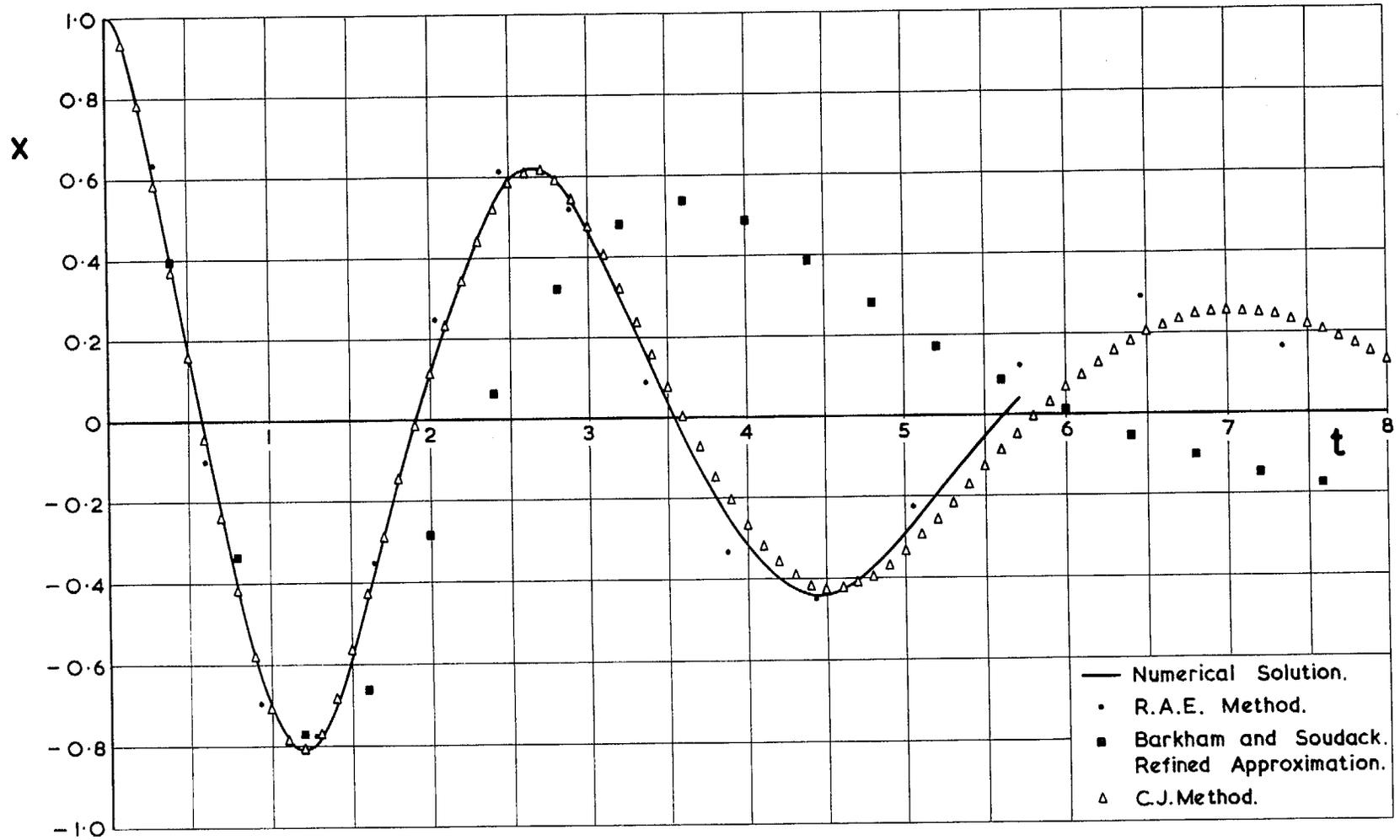
FIG. 7. $b = 0.5, c_1 = 1.0, c_3 = 10.0$.
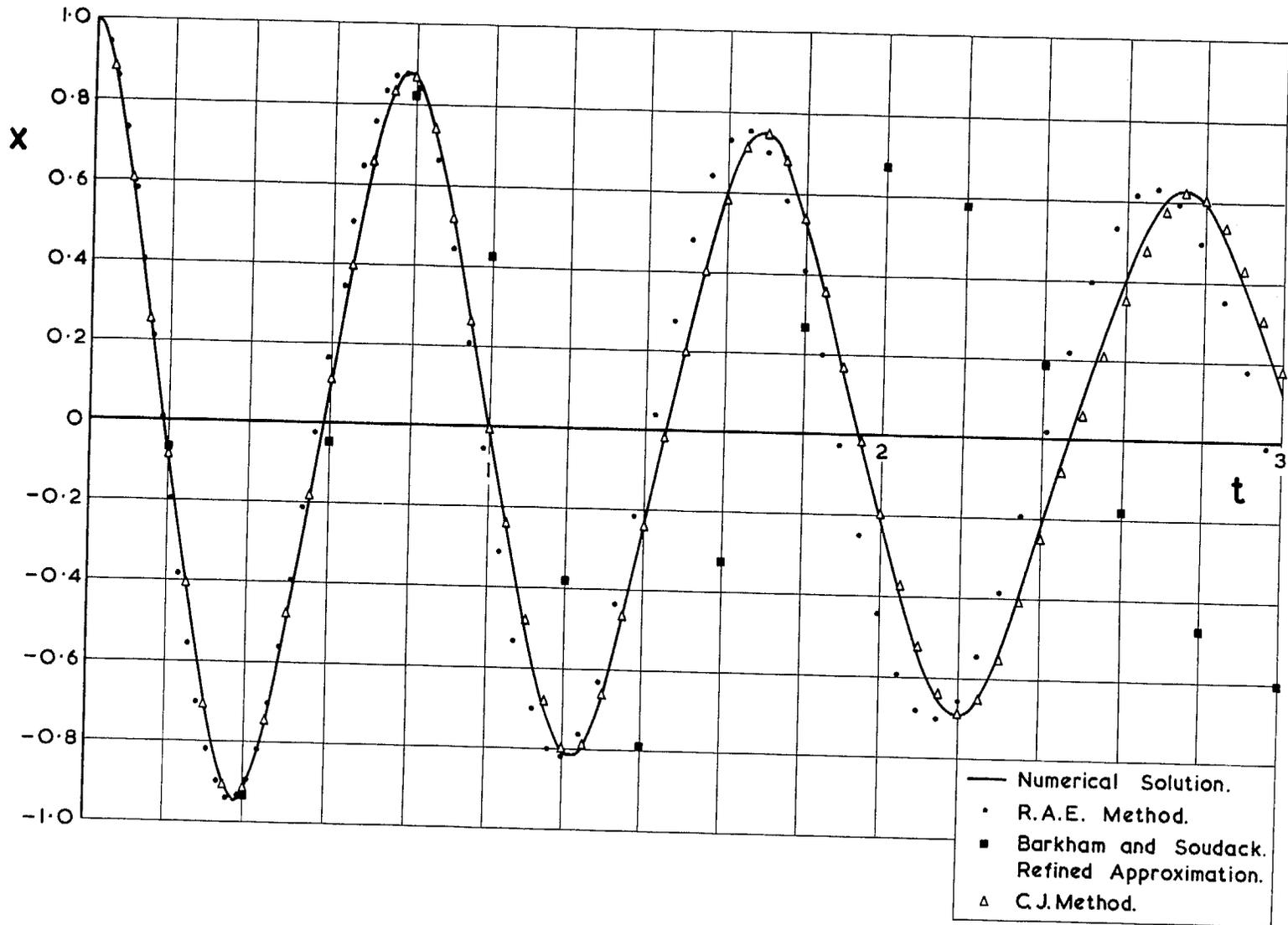
FIG. 8. $b = 0.5, c_1 = 1.0, c_3 = 100.$
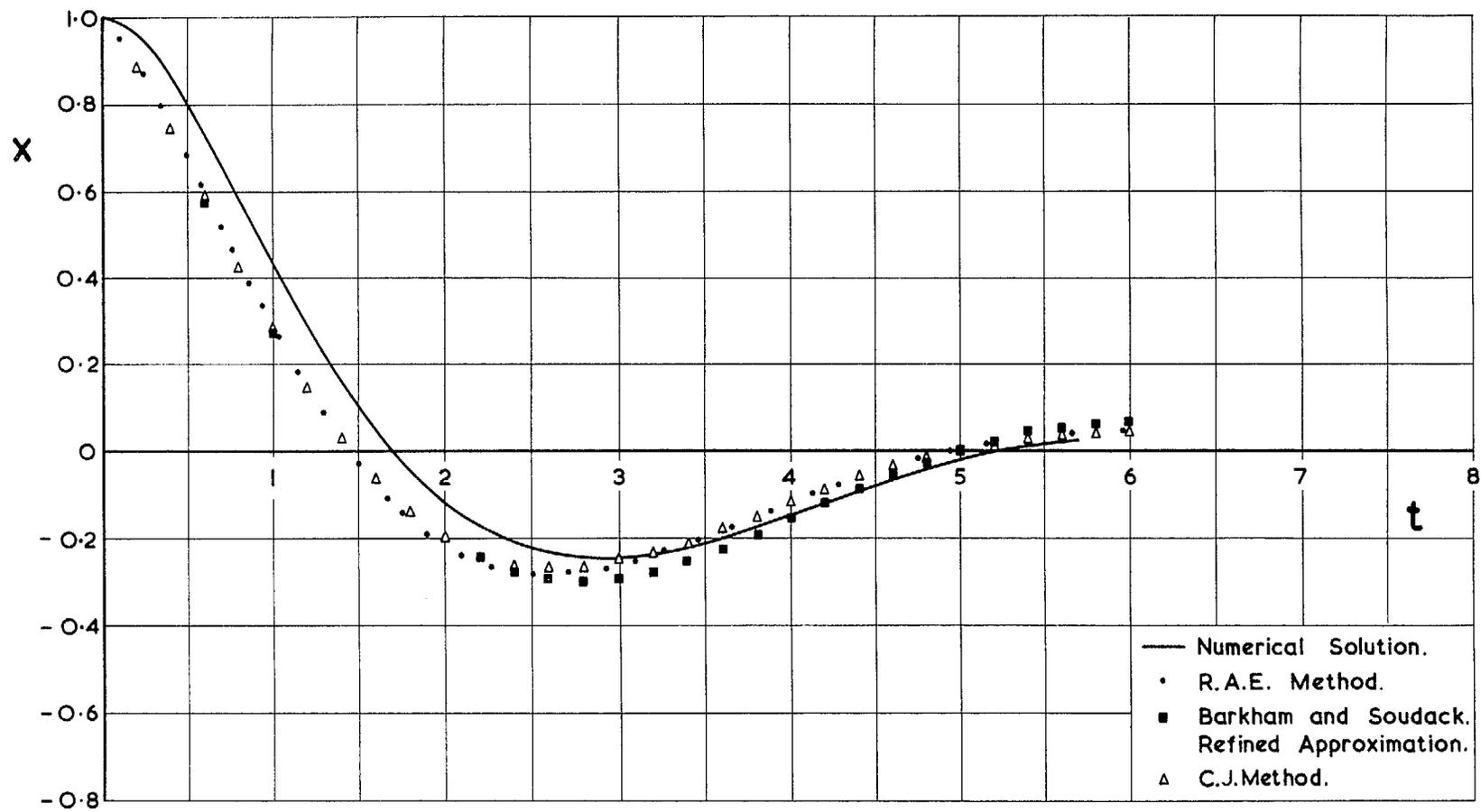
FIG. 9.　$b = 1\cdot0, c_1 = 1\cdot0, c_3 = 1\cdot0.$
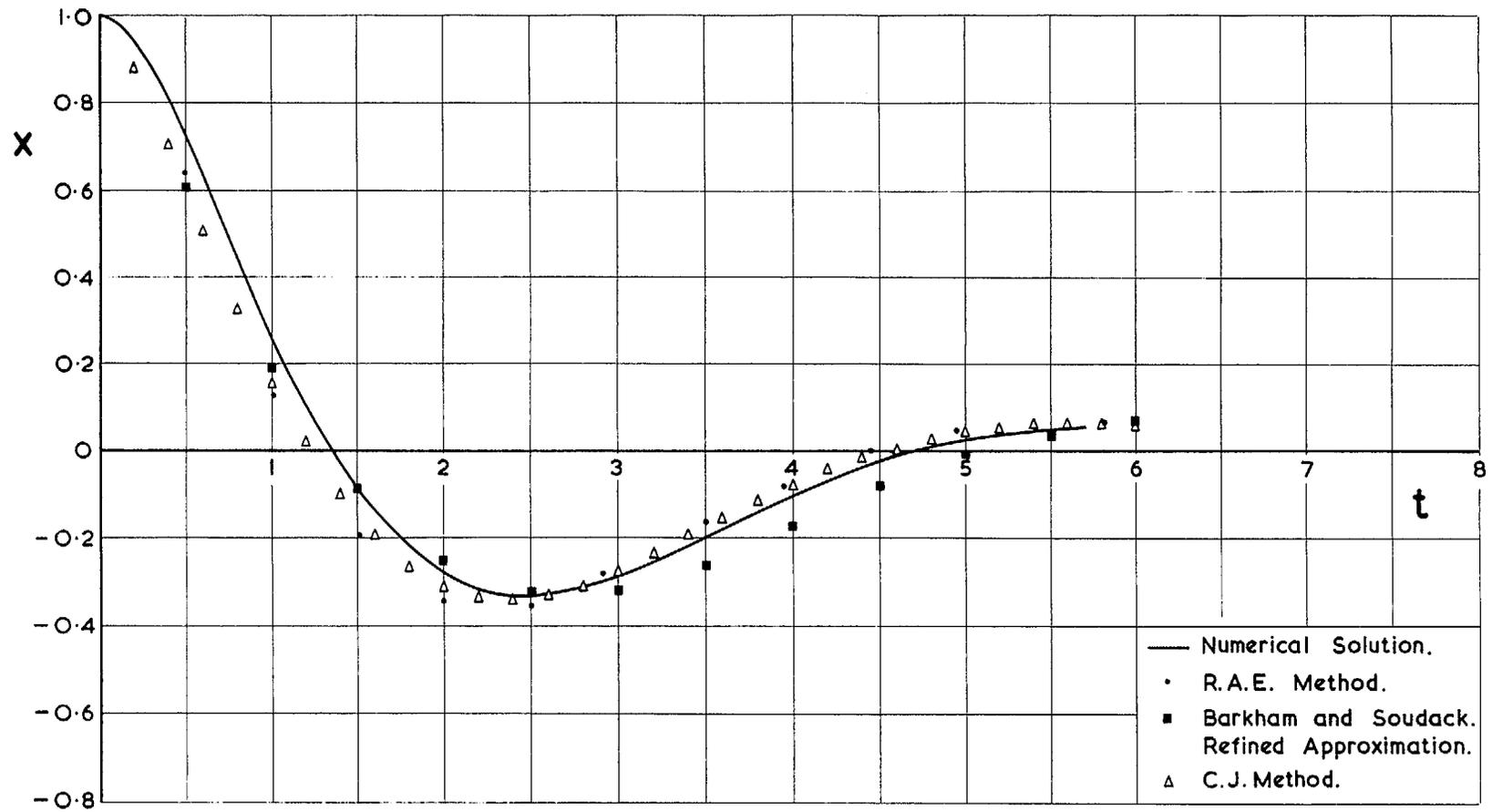
FIG. 10. $b = 1.0, c_1 = 1.0, c_3 = 2.0.$

FIG. 11. $b = 1 \cdot 0, c_1 = 1 \cdot 0, c_3 = 10 \cdot 0.$

The legend within the figure:

—— Numerical Solution.

• R.A.E. Method.

■ Barkham and Soudack.
  Refined Approximation.

△ C.J. Method.
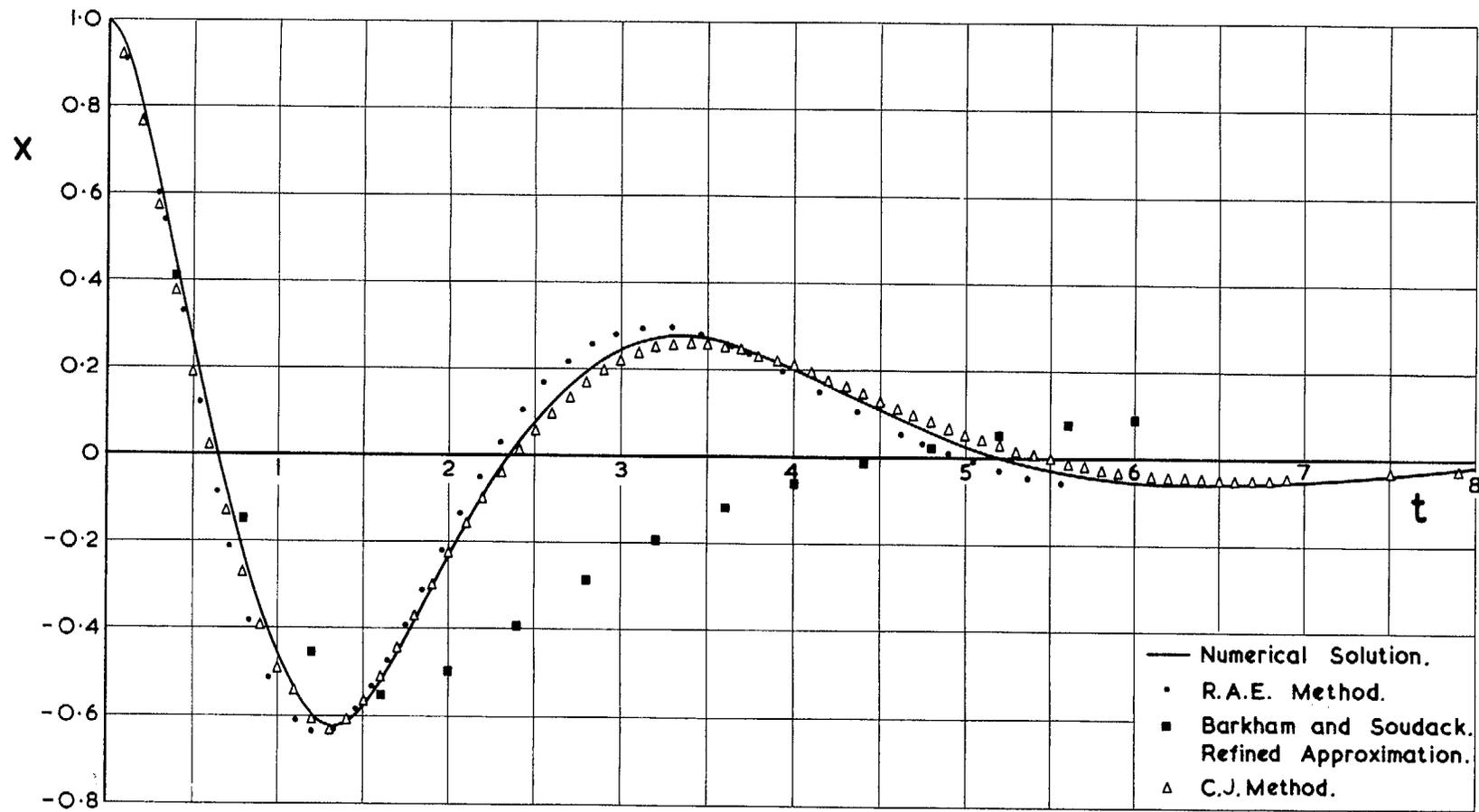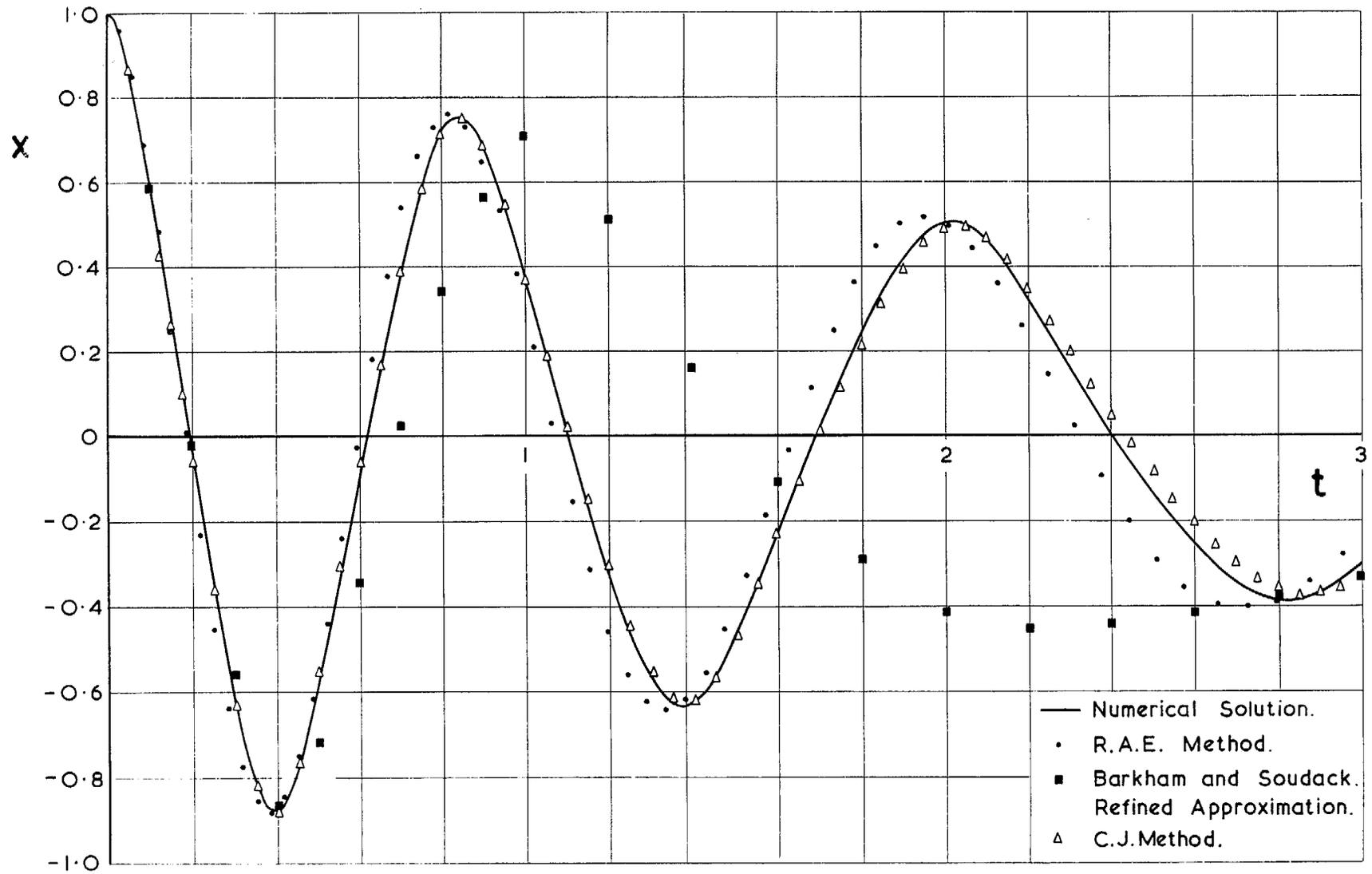
FIG. 12. $b = 1.0, c_1 = 1.0, c_3 = 100.$

FIG. 13. $b = 2{\cdot}0, c_1 = 1{\cdot}0, c_3 = 2{\cdot}0.$

X

1·0

0·8

0·6

0·4

0·2

0

−0·2

−0·4

2   3   4   5   6   7   8

t

— Numerical Solution

· R.A.E. Method

■ Barkham and Soudack Refined Approximation

FIG. 14. $b = 2 \cdot 0, c_1 = 1 \cdot 0, c_3 = 10 \cdot 0.$