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The Prediction of Instabilities of Linear Differential Equations with Periodic Coefficients

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Summary

The stability of the solutions of a system of differential equations with periodic coefficients has been examined using Floquet's theorem and a general method of solution has been programmed in ICL 1900 Fortran. The application of the method is illustrated by the solution of two dynamical systems both of which are unsymmetrical rigid rotors in unsymmetrical bearings and the program has been used to obtain solutions for up to six simultaneous second-order differential equations with periodic coefficients. The relevance of the solutions for an associated system of equations having constant coefficients, to the solutions of the periodic system is discussed and used to propound a method for predicting the unstable regions of periodic systems of particular form. For Lagrangian systems without viscous damping a method is presented for calculating the limits of an unstable region.

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1. Introduction

The present need of a method for testing the stability of a set of second-order differential equations with periodic coefficients arose from an investigation into the dynamic behaviour of a flexible aircraft with two unsymmetric rigid rotors mounted on it. The method for testing the stability of such a system followed that given by Williams in Ref. 1. A general form of this was programmed in Mercury Autocode for the Atlas computer and subsequently re-programmed in ICL 1900 Fortran.

The availability of this program was a big advance, since it was now possible to test the stability of large systems of equations with periodic coefficients (solutions for a system of six equations have been obtained), but it looked as though its usefulness would be restricted because apparently the program gave the answer 'stable' or 'unstable' and little more. Thus without further information, the assessment of the stability of a system would involve solving the periodic equations for equal increments of a parameter, say rotor rotational speed, over the required range of that parameter. This would involve considerable computation and depending on the increment size may or may not show the unstable regions. However, since it was necessary to find the rotor speeds defining the unstable regions of the flexible-aircraft rotor system, several binary systems were solved for a large number of rotor speeds. At this stage it was noticed that the principal value solutions (*see* Section 3) for the periodic system followed very closely the principal value solutions of an associated constant coefficient system (obtained by omitting the time dependent terms of the periodic system) and that the results differed locally at an instability. Thus by examining solutions of this constant coefficient system it was possible to locate regions which were likely to have unstable solutions when the complete periodic equations were solved. These empirical observations combined with the solutions obtained for the periodic systems of Gladwell and Stammers[2] led to the method, detailed in Section 4, for predicting the unstable regions of a system of periodic equations of a particular form, but not restricted to two simultaneous equations. In essence the prediction method is purely a guide since the stability of the system can only be assessed by solving the periodic equations, but its value lies in the fact that it indicates values of the parameter for which solutions should be obtained initially. Also as more solutions of the periodic system are obtained, the plotting of these solutions on the 'principal value' graphs continually reduces the interpolations required and confirms with increasing certainty that all important regions are being investigated. Thus the necessity for solving the periodic equations at equal increments of a parameter has been removed together with the uncertainties inherent in such a process. When applied to larger systems of equations the method proved successful and it was of considerable help in locating the instabilities of a system with large periodic coefficients, i.e. when the solutions of the periodic system differ greatly from those of the constant coefficient system.

It is shown in Section 6 that for a system without structural damping it is possible to calculate the range of an instability from three values of the unstable solution within the region.

2. Analysis

2.1. System Equations

The equations for the system are assumed to have the form

$$(A_1 + A_2 \sin 2\omega t + A_3 \cos 2\omega t)\ddot{q} + D_1\dot{q} + \omega(D_2 + D_3 \sin 2\omega t + D_4 \cos 2\omega t)\dot{q} + Eq = 0, \qquad (1)$$

where A_1, A_2 etc. are square matrices of order m and the equations are periodic of period τ/ω. The form of the periodic terms is appropriate for the rotor systems considered later, ω then being the angular frequency of rotation of the rotor and t the real time. Not included in the form of equation (1) are Hill's equation and Mathieu's equation, both of which have periodic coefficients in the q term, but the program of Section 3 is easily modified to cope with such terms.

2.2. Application of Floquet's Theorem

Equation (1) can be written in the form,

$$I\ddot{q} + A^{-1}D\dot{q} + A^{-1}Eq = 0 \qquad (2)$$

where A and D are periodic matrices and I is the unit matrix. This set of second order differential equations is reduced to a first order set by writing $p = \dot{q}$ and rearranging

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -A^{-1}D & -A^{-1}E \\ I & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{3}$$

or

$$\dot{y} = Z(t)y \tag{4}$$

where $Z(t)$ is the square matrix of order $n(= 2m)$ on the right hand side of equation (3). Since $Z(t)$ is periodic of period $\tau(= \pi/\omega)$,

$$Z(t + \tau) = Z(t). \tag{5}$$

If $\phi_i(i = 1, 2, \ldots n)$ is a set of n linearly independent solutions of equation (4) these are said to form a basis or a fundamental set[4] of solutions. Let $\Phi(t)$ be the matrix whose n columns are n linearly independent solutions of equation (4), then $\Phi(t)$ is known as a fundamental matrix and is non-singular (*see* Ref. 4). It follows, therefore, that any fundamental matrix satisfies equation (4), i.e.

$$\dot{\Phi} = Z(t)\Phi. \tag{6}$$

Let Φ_1, Φ_2 be any two fundamental matrices, then

$$\begin{aligned} Z(t)\Phi_1 = \dot{\Phi}_1 &= \frac{d}{dt}\{\Phi_2\Phi_2^{-1}\Phi_1\} \\ &= \dot{\Phi}_2\Phi_2^{-1}\Phi_1 + \Phi_2\frac{d}{dt}\{\Phi_2^{-1}\Phi_1\} \\ &= Z(t)\Phi_1 + \Phi_2\frac{d}{dt}\{\Phi_2^{-1}\Phi_1\}. \end{aligned} \tag{7}$$

Consequently

$$\frac{d}{dt}\{\Phi_2^{-1}\Phi_1\} = 0 \tag{8}$$

or

$$\Phi_1 = \Phi_2 C \tag{9}$$

where C is a constant non-singular matrix.
Since $Z(t)$ is periodic we have,

$$\begin{aligned} \dot{\Phi}_1(t + \tau) &= Z(t + \tau)\Phi_1(t + \tau) \\ &= Z(t)\Phi_1(t + \tau) \end{aligned}$$

and hence $\Phi_1(t + \tau)$ is a fundamental matrix. From equation (9)

$$\begin{aligned} \Phi_1(t + \tau) &= \Phi_1(t)C_1 \\ &= \Phi_1(t)\,e^{R_1\tau}, \end{aligned} \tag{10}$$

where R_1 is a non-unique matrix[4,5,6] such that

$$C_1 = e^{R_1\tau}. \tag{11}$$

We then have,

$$\begin{aligned}\Phi_1(t+\tau)\,e^{-R_1(t+\tau)} &= \Phi_1(t)\,e^{R_1\tau}\,e^{-R_1(t+\tau)} \\ &= \Phi_1(t)\,e^{-R_1 t}\end{aligned}$$

and so $\Phi_1(t)\,e^{-R_1 t}(\equiv P(t))$ is periodic of period τ.
Consequently any fundamental matrix of equation (4) has the form

$$\Phi_1(t) = P(t)\,e^{R_1 t} \tag{12}$$

and this is Floquet's theorem. $P(t)$ is determined by equation (12) over a period, and, since it is periodic, it is determined over $(-\infty, \infty)$. The significance of Floquet's theorem is that the determination of a fundamental matrix $\Phi(t)$ over a period of $P(t)$ leads, at once, to the determination of $\Phi(t)$ over $(-\infty, \infty)$.

From equation (10) taking $t = 0$ and $\Phi_1(0) = I$, as we do later, then

$$\Phi_1(\tau) = e^{R_1\tau}. \tag{13}$$

The fundamental matrix $\Phi_1(\tau)$ can be calculated using a numerical step-by-step integration procedure over the period.

From equations (9) and (10) we have,

$$\begin{aligned}\Phi_2(t+\tau) &= \Phi_1(t+\tau)C^{-1} \\ &= \Phi_1(t)\,e^{R_1\tau}C^{-1} \\ &= \Phi_2(t)C\,e^{R_1\tau}C^{-1}.\end{aligned} \tag{14}$$

Thus every fundamental matrix determines a matrix $C\,e^{R_1\tau}C^{-1}$ which is similar to $e^{R_1\tau}$. Such a similarity transformation does not affect a number of the quantities associated with $e^{R_1\tau}$. In particular the eigenvalues of $e^{R_1\tau}$ are the same as those of $C\,e^{R_1\tau}C^{-1}$.

If K is the matrix of right hand eigenvectors of $e^{R\tau}$ and $\Lambda = \operatorname{diag}\lambda_j$ where all the λ_j are assumed distinct, then

$$e^{R_1\tau} = K\Lambda K^{-1}$$

and therefore[5,6]

$$\log(e^{R_1\tau}) = R_1\tau = K\log\Lambda K^{-1}$$

giving

$$R_1 = K(1/\tau\log\Lambda)K^{-1}.$$

So the eigenvalues of R_1, $\sigma_j(=\alpha_j + i\beta_j)$, are given by $1/\tau\log\Lambda$ and are determined modulo $2\pi i/\tau$, the eigenvectors combined in K are the same as those of $e^{R_1\tau}$ and are called the Floquet modes[1,7]. The eigenvalues of $e^{R_1\tau}$, $\lambda_j(=\mu_j + i\nu_j)$, are called the characteristic roots or multipliers and the σ_j are called the characteristic exponents.

If we write

$$\Sigma = \operatorname{diag}(\sigma_j)$$

where the σ_j are assumed to be distinct, then

$$K^{-1}R_1K = \Sigma.$$

Putting $\Phi = \Phi_1 K$ and $P_1 = PK$ in equation (12) gives

$$\Phi(t) = P_1(t)\,e^{\Sigma t} \quad \text{and} \quad P_1(t+\tau) = P_1(t).$$

It follows that the columns $\phi_1, \ldots, \phi_n$ of the matrix Φ, which forms a set of n linearly independent solutions of equation (4), are of the form,

$$\phi_j(t) = e^{\sigma_j t} p_j(t) \quad j = 1, 2, \ldots, n$$

where the $p_1, \ldots, p_n$ are periodic column vectors of P_1. Thus if $\mathscr{R}(\sigma_j) < 0$ or equivalently

$$|\lambda_j| < 1 \quad \text{then as} \quad t \to \infty, \phi_j(t) \to 0 \tag{15}$$

exponentially fast.

2.3. Reciprocal Systems

A reciprocal system is one for which any characteristic root is either unity or one of a reciprocal pair and possible configurations of roots are given in Ref. 3. Thus if λ_i is a non-unity eigenvalue of the fundamental matrix Φ then so is $1/\lambda_i$, and the characteristic exponents are then either zero or $\pm\sigma_i$.

Consider the system of n equations given by equation (4) for n independent initial conditions, i.e.

$$\frac{dY}{dt} = Z(t)Y. \tag{16}$$

Now, if Φ is a fundamental matrix of equation (16), we know that

$$\frac{d\Phi}{dt} = Z(t)\Phi \tag{17}$$

and if the system is reciprocal the eigenvalues of Φ will be the same as those of Φ^{-1}. Assuming that the eigenvalues of Φ are distinct we have,

$$\Phi = P(\Phi')^{-1}P^{-1}.$$

Substituting for Φ in equation (17) gives

$$P\frac{d}{dt}(\Phi')^{-1}P^{-1} = ZP(\Phi')^{-1}P^{-1}$$

therefore,

$$\frac{d}{dt}(\Phi')^{-1} = P^{-1}ZP(\Phi')^{-1}. \tag{18}$$

But,

$$\Phi'(\Phi')^{-1} = I$$

which on differentiation gives,

$$\Phi' \frac{d}{dt}(\Phi')^{-1} + \frac{d}{dt}(\Phi')(\Phi')^{-1} = 0$$

and so

$$\frac{d}{dt}(\Phi')^{-1} = -(\Phi')^{-1}\frac{d}{dt}(\Phi')(\Phi')^{-1}.$$

Using equation (18) we have,

$$P^{-1}ZP = -(\Phi')^{-1}\frac{d}{dt}(\Phi')$$

giving

$$\frac{d}{dt}(\Phi') = -\Phi' P^{-1}ZP. \tag{19}$$

Equation (17) gives, on transposition

$$\frac{d}{dt}(\Phi') = \Phi' Z'$$

and so

$$Z' = -P^{-1}ZP. \tag{20}$$

Thus if we can find a constant matrix P which satisfies equation (20) then the system given by equation (16) is reciprocal.

The form of the characteristic equation for a reciprocal system is,

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda + 1 = 0$$

where $a_k = a_{n-k}$ for $k = 1, 2, \ldots, n-1$. The constant term is given by the determinant of the fundamental matrix, Φ, and so $|\Phi| = |\Phi^{-1}| = 1$.

Consider the equation,

$$A\ddot{q} + D\dot{q} + Eq = 0$$

which when written in the form of equation (4), i.e. by making the substitution $p = A\dot{q}$, gives,

$$Z = \left[\begin{array}{c|c} -(\dot{A} - D)A^{-1} & -E \\ \hline A^{-1} & 0 \end{array}\right] \tag{21}$$

This system is reciprocal if the matrices A and E are symmetric and the matrix $(\dot{A} - D)$ is constant and skew symmetric. Then the constant matrix P is given by,

$$P = \left[\begin{array}{c|c} -(\dot{A} - D) & -I \\ \hline I & 0 \end{array}\right].$$

3. Calculation of Characteristic Exponents

The method for calculating the characteristic exponents, and thereby testing the stability, of a set of equations with periodic coefficients has been programmed in ICL 1900 Fortran. The form of the equations programmed is,

$$(A_1 + A_2\varepsilon \sin 2\omega t + A_3\varepsilon \cos 2\omega t)\ddot{q} + D_1\dot{q} + \frac{\omega}{\omega_0}(\bar{D}_2 + \bar{D}_3\varepsilon \sin 2\omega t + \bar{D}_4\varepsilon \cos 2\omega t)\dot{q} + Eq = 0 \tag{22}$$

where ε is a factor on the time dependent coefficients and ω_0 is a standard frequency associated with the matrices $\bar{D}_2$, $\bar{D}_3$ and $\bar{D}_4$. These differential equations are reduced to a first-order set as in equation (3) and then integrated numerically over a period, with $\Phi(0) = I$, using a Runge–Kutta variable step length procedure[8]. The characteristic roots (λ_j) and exponents (σ_j) of the fundamental matrix are calculated together with the Floquet modes, and the application of the conditions given in equation (15) determines the stability of the system.

Provision is made for varying ω, the matrices A_1, A_2 etc., the factor ε on the periodic coefficients and the accuracy of the numerical integration. When ω is zero or the periodic terms are zero ($\varepsilon = 0$) then the solution of the system of equations is merely an eigenvalue problem and numerical integration is not used. During the integration over a period, the step length is varied automatically to maintain the required accuracy and relaxing the accuracy of the integration results in a considerable saving in computation time. The form of the input and details of the output of the program are given in Appendix I. An important detail concerning the calculation of the characteristic exponents from the characteristic roots is the value taken by the imaginary part of $\log \lambda_j$. The principal value of $\mathscr{I}(\log \lambda_j)$ i.e. $-\pi \leqslant \mathscr{I}(\log \lambda_j) \leqslant \pi$ is always used and since $(\sigma_j) = 1/\tau \mathscr{I}(\log \lambda_j)$ and

$$\tau = \tau/\omega, \text{ the range of } \mathscr{I}(\sigma_j) \text{ will be } -\omega \leqslant \mathscr{I}(\sigma_j) \leqslant \omega.$$

4. Prediction of Unstable Regions

4.1. Properties of Constant Coefficient Equations

Let Z be an $n \times n$ constant coefficient matrix obtained by omitting the periodic terms of equation (1) and reducing the system so obtained to a set of first-order differential equations. Thus following equations (3) and (4) we have,

$$\dot{y} = Zy. \tag{23}$$

Let Φ be a fundamental matrix of equation (23), it being obtained by integrating equation (23) over an interval of time. Then the fundamental matrix[4] is given by

$$\Phi(t) = e^{tZ} \quad \text{for} \quad |t| < \infty \quad \text{and} \quad \Phi(0) = I$$

and so

$$\frac{d\Phi}{dt} = Z\Phi.$$

Thus the use of the program determines the characteristic exponents which are $1/t \log(\lambda_i)$, where $\lambda_i (i = 1, 2, \ldots n)$ are the eigenvalues of e^{tZ}.

Now if σ_i is an eigenvalue of Z then

$$e^{t\sigma_i} \text{ is an eigenvalue of } e^{tZ},$$

therefore $\lambda_i = e^{t\sigma_i}$ and so the characteristic exponents determined by the program are $1/t \log(e^{t\sigma_i}) = \sigma_i + 2\pi ni/t$, i.e. they are the eigenvalues of Z determined modulo $2\pi i/t$.

Taking $t = \tau = \pi/\omega$ the uncertainty in the imaginary part of σ_i becomes $2n\omega$ where n is an integer. Again the computer program calculates principal value solutions for the characteristic exponents and so we have $-\omega \leqslant \mathscr{I}(\sigma_i) \leqslant \omega$. Thus the form of the solutions obtained for the constant coefficient system is the same as that of the periodic system.

However, there is no need to resort to the method of Section 2 to solve equation (23) since we know that $y = e^{\sigma_i t}k_i$ is a typical solution giving on substitution,

$$\sigma_i k_i = Zk_i.$$

This is now an eigenvalue problem and as such is not restricted to principal value solutions for the σ_i. Thus if we wish to compare the principal value solutions obtained for the periodic system with the eigenvalue solutions obtained for the constant coefficient system, we may either eliminate the 'uncertainty' for the periodic system[1], which would entail a considerable amount of computation, or 'condense' the solutions of the constant coefficient system to 'principal β_j^*' (*see* below). The latter is obviously preferable and can be achieved in the following manner.

For the constant coefficient system, the pair of values $\pm\beta_j^*$ (where β_j^* is the positive imaginary part of the characteristic exponent) are 'condensed' to a value between the lines $\beta = 0$ and $\beta = \omega$ (Fig. 1) by the addition of $2n\omega$, where n is an appropriate integer. We shall call these condensed values the 'principal β_j^*'. The condensation can be made graphically or using

$$\text{principal } \beta_j^* = 2\omega r - \beta_j^* \, sgn\, r$$

where $0 \leqslant$ principal $\beta_j^* \leqslant \omega$, r has the values $-0, 1, -1, 2, -2$ etc. and $sgn(-0) = -1$. The values of r and the range over which they apply can be determined from the intersection of the lines $\beta = s\omega (s = 1, 2, 3, \ldots)$ with the curve of β_j^* versus ω, Fig. 1. The modulus of r is the same either side of $\beta = s\omega$ for even s, $|r| = s/2$ and the sign of r is negative to the left of the line of even s. The condensation of a typical line to principal β_j^* is shown in Fig. 1. If we had solved the constant coefficient system in the same manner as the periodic system, i.e. integrating over an interval of time then the characteristic exponents would have been principal value solutions and would have been equivalent to the principal β_j^*.

4.2. Relevance of Constant Coefficient Solutions

The output of the program of Section 3 yields the valuable answer to the question 'Is the system stable or not?'. However, the output would be even more useful if there were indications from a stable solution showing for what values of the variable an instability may result. The computer solutions for some simple reciprocal periodic two degree of freedom rotor systems were obtained for a large number of rotor speeds and various presentations of the results were tried to this end. The presentation finally established is shown in Figs. 3a and 3b, and from this came the empirical observation that instabilitites seemed to result from parameter values in the region of the intersection of the constant coefficient principal value solutions, as indicated in Fig. 2.

Gladwell and Stammers in Ref. 2 have given the conditions for limiting stability of a simple reciprocal two degree of freedom periodic system. If σ_1, σ_2 are two characteristic exponents of the system then the condition is,

$$\left.\begin{aligned} \sigma_1 - \sigma_2 &= 2ik\omega \\ \text{for} \qquad k &= 0, \pm 1, \pm 2, \text{etc.} \end{aligned}\right\} \qquad (24)$$

In general the characteristic exponents, σ_i, will be complex ($= \alpha_i + i\beta_i$) and occur in conjugate pairs, but for a stable reciprocal system the real part of σ_i will be zero, and the characteristic exponents will then have the form $\pm i\beta_1$, $\pm i\beta_2$. Using these roots in equation (24) gives the following conditions for limiting stability

$$\beta_i = k\omega \tag{25}$$

for

$$k = 0, \pm 1, \pm 2 \text{ etc.} \quad \text{and} \quad i = 1, 2$$

and

$$\left.\begin{aligned} \beta_i - \beta_j &= 2k\omega \\ \beta_i + \beta_j &= 2k\omega \end{aligned}\right\} \tag{26}$$

for

$$k = 0, \pm 1, \pm 2, \text{etc.} \qquad i, j = 1, 2 \quad \text{and} \quad i \neq j.$$

These conditions are then applied to the solutions of a constant coefficient system (obtained by neglecting the periodic terms) in order to deduce the regions in which instability is likely to occur when the periodic terms are included.

As outlined previously, in order to minimise the amount of computation it is preferable to condense the constant coefficient solutions to principal values. Then it is a simple exercise to show that the Gladwell and Stammers conditions $\beta(\omega) = k\omega$ of equation (25) correspond to intersections of Type 1 (Fig. 2) for the principal value solutions and those given by equation (26) correspond to intersections of Types 2 and 3 (Fig. 2). Type 3 has been distinguished from Type 2 because it has been found to give rise to a major instability for systems with distinct β_j^* at $\omega = 0$. In the majority of cases examined, the Type 2 intersection does not develop into an unstable region with the inclusion of the periodic terms and can be thought of as a line of limiting stability, i.e. the line is mathematically significant it being a stability boundary, (*see* Ref. 2). The degree of instability developing from a Type 1 intersection depends very much on the system but the unstable range of the periodic system decreases as ω tends to zero. For all types of instability of the periodic system, the degree of instability can be assessed from the real part of the characteristic exponent.

The empirical observations with regard to the relevance of the constant coefficient solutions have been extended to more complicated systems. A four degree of freedom case of different form (but still a reciprocal system when $D1 = 0$) is shown in Fig. 4a and it may be seen how clearly the critical regions are shown. It is emphasised that the prediction method is only a guide to the practical detail of using the computer program. If types of equations exist for which the constant coefficient solutions are irrelevant, then this will be apparent as soon as the initial solutions of the periodic system are plotted. The comparatively trivial effort used to obtain the constant coefficient solutions will then be wasted, but the user of the program will still be able to check the stability of the periodic system. The number of cases to be solved in order to check the stability over a range of parameter values will, however, have to be greater than for the cases considered above, where interpolations may soon be made with confidence.

4.3. Summary of the Method

To summarise the procedure to be adopted for the prediction of instabilities. First solve the associated constant coefficient system obtained by omitting the periodic terms from the equations. Then condense these eigenvalue solutions to principal values, this is easily effected graphically, and the type of intersection, Fig. 2, will be apparent immediately. Next solve the periodic system, using the computer program, in the regions indicated by the constant coefficient solutions. The real part of the characteristic exponents for the periodic system determines the stability and deviations in the variation of the imaginary parts of the characteristic exponents from that of the constant coefficient solutions indicate further regions to be examined. It is important to investigate any deviations in the trends of the solutions since this usually means that an unstable region is in the vicinity.

The procedure outlined is only a guide to the location of the unstable regions of a periodic system but it does provide a systematic approach where otherwise the only method would be to solve the periodic system at a number of arbitrary points.

5. Applications of the Method

5.1. Equations of Motion

The equations of motion for the systems considered in Sections 5.2 and 5.3, namely systems with unsymmetrical rigid rotors in unsymmetrical bearings, have the form,

$$A(t)\ddot{q} + \{D_1 + \omega D_2 + \dot{A}(t)\}\dot{q} + Eq = 0 \tag{27}$$

where

$$A(t) = A_1 + A_2 \sin 2\omega t + A_3 \cos 2\omega t.$$

Also $A(t)$ and E are symmetric matrices, D_1 is a diagonal matrix and D_2 is skew symmetric. Then equation (21) gives,

$$Z = \begin{bmatrix} (D_1 + \omega D_2)A(t)^{-1} & -E \\ A(t)^{-1} & 0 \end{bmatrix}$$

and the system will be reciprocal when $D_1 = 0$. The characteristic roots, λ_i, will occur in complex pairs if the system is stable and the modulus of λ_i will be unity, i.e. $R(\sigma_i) = 0$. Thus the system will be, at the best, neutrally stable.

5.2. An Unsymmetrical Rigid Rotor in Unsymmetrical Bearings

Brosens and Cradall[9] have investigated the stability of an unsymmetrical rigid rotor mounted in unsymmetrical stiffness bearings. This system can be represented by two second-order differential equations with periodic coefficients. The present method will be illustrated using this system and an example from an extension of Brosen and Cradall's work by Gladwell and Stammers[2]. The coefficients for the two cases considered in this section are given in Appendix II.1, it will be noticed that there is no structural damping present, i.e. $D_1 = 0$ and so the system will be, at the best, neutrally stable. The system is of particular interest since the gyroscopic terms, i.e. the elements of the matrix D_2, are large. Shown in Fig. 3a are the principal β_j^* obtained from the constant coefficient solutions and the computed solutions of the periodic system when the rotor has equal mounting stiffness. Shaft whirling instability occurs for a range of ω from 1·203 to 2·118[9] which is an expansion of the predicted $\omega = 1{\cdot}5$, there being a noticeable departure from the constant coefficient solutions in this region. The equal root instability, Type 3, predicted for $\omega = 1{\cdot}05$ becomes a line of limiting stability since $\beta_1 = \beta_2$ at $\omega = 0$ (*see* Section 4). The other predicted instabilities are also lines of limiting stability. For the case of unequal mounting stiffnesses, Fig. 3b, three Type 1 intersections at $\omega = 1{\cdot}6$, 0·55 and 0·44 have been shown to give rise to unstable regions, the range of the unstable region decreasing as ω is reduced. There is an equal root instability, Type 3, in the region of the predicted $\omega = 0{\cdot}95$ but although the lines are distorted in the region of $\omega = 0{\cdot}4$ there was not an instability in the vicinity of this Type 2 intersection.

When the gyroscopic and periodic terms are large, i.e. for a rotor with a large inertia inequality, then the full solutions show big differences from the constant coefficient solutions, Fig. 3c, and the prediction method is poor. However, a few solutions of the periodic system will soon establish the position of the shifted lines and the rules concerning intersections can be applied again, leading quickly to the actual unstable regions. The Type 3 equal root instability in the region of $\omega = 1{\cdot}3$ is predicted despite the large differences in the solutions of the constant coefficient and periodic systems. Also shown is a Type 2 intersection which has given rise to an unstable region. The unstable regions obtained by Gladwell and Stammers[2] are shown for comparison on Fig. 3c, although the solutions are not in exact agreement the comparison is good.

This system does not have an intersection of Type 1 for the constant coefficient solutions emanating from the point $\beta_i = 1{\cdot}2, \omega = 0$, thus the instability of the periodic system at $\omega = 2$ is not predicted. It can also be seen in Fig. 3c that the solutions of the periodic system for values of ω above the critical Type 3 region depart from the principal β_j^* line. This is always a warning and suggests seeking solutions for higher rotor speeds.

5.3. Two Unsymmetrical Rotors on a Flexible Aircraft

The dynamic equations for the system have the form of equation (1) with q defining normal modes of the elastic aircraft with the rotors replaced by non-rotating discs, the moment of inertia of a disc being the mean of the unequal moments of inertia of the rotor. The number of normal modes needed to describe the system was restricted by imposing an arbitrary frequency limit for the normal modes. Solutions are presented for a quaternary and a binary taken from this system of equations. The binary is of interest since there are no gyroscopic coupling terms present, $D_2 = 0$, and the quaternary serves as an example of a larger set of equations. The coefficients for the quaternary are given in Appendix II.2 and the coefficients for the binary are obtained by deleting rows and columns two and four. The form of the equations is such that the periodic terms couple degrees of freedom 1 and 2 with 3 and 4, the gyroscopic terms couple 1 with 2 and 3 with 4 and the system composed of A_1, D_1 and E, i.e. the associated constant coefficient system for $\omega = 0$, is uncoupled.

Shown in Fig. 4a are the principal β_j^* and the full solutions for the quaternary without structural damping and so the system is, at the best, neutrally stable. For $\omega = 0$ the constant coefficient system is uncoupled and we can identify the degree of freedom numbers with the β_j^* solutions, this has been done in Fig. 4a. Since the periodic terms couple degrees of freedom 1 and 2 with 3 and 4 we might expect instabilities, to occur between 1 and 3, 1 and 4, 2 and 3, and 2 and 4. Equal root instabilities, Type 3, do occur and Fig. 4a shows the instabilities occurring at the intersections of the lines associated with the relevant degrees of freedom, i.e. 1 and 3 etc. The eigenvectors associated with these solutions will, of course, contain contributions from the other degrees of freedom. The equal root instabilities are in the regions predicted and are the major unstable regions for this system, Types 1 and 2 becoming line instabilities. The full solutions follow the constant coefficient solutions very closely and departures only occur near an unstable region.

The binary example, Fig. 4b, is degrees of freedom 1 and 3 taken from the quaternary. Since there is no gyroscopic coupling between these degrees of freedom the constant coefficient solutions are lines of constant β and so the principal β_j^* give straight lines. The solutions of the periodic system are essentially the same as the solutions emanating from the degrees of freedom 1 and 3 of the quaternary. The solutions associated with 1 and 3 in the latter are distorted by the gyroscopic coupling present in the quaternary and the instabilities in the regions of $\omega = 0{\cdot}42$ and $\omega = 0{\cdot}43$, Fig. 4a, are not present in the binary since these instabilities arise from the inclusion of degrees of freedom 2 and 4. For the binary, Fig. 4b, there is only one major instability (Type 3) and it is in the region of the predicted $\omega = 0{\cdot}345$. This unstable region is shown in greater detail in Fig. 4c, and shows clearly the bifurcation marking the end of the unstable region. The range of this instability has been extended slightly compared with the same region in the quaternary. The solutions on the $\beta = \omega$ line are line instabilities as are the intersections of Type 2. For a system as shown in Fig. 4b it is a simple matter to locate the likely unstable regions, when the values of β at $\omega = 0$ are known, because all lines are straight.

Large systems of equations will have a multiplicity of intersections and would appear, at first sight, to be unmanageable. The system of this section will give rise to numerous possible instabilities, but examination of binary and quaternary sub-systems suggested that only the Type 3 intersections were important. Also it became apparent, as solutions were obtained, that all the essential information could be gleaned from the binary and quaternary solutions and it was not necessary to solve the large system. Thus by examining sub-systems it should be possible either to avoid solving the large system altogether or to reduce the number of intersections that need be considered.

6. Range of an Instability

The limits of an unstable region for a reciprocal system can be obtained quite simply when three unstable solutions have been found in that region. The limits are calculated from the real part of the characteristic exponent, α_j, assuming the variation with ω to be elliptic when the unstable region is bounded and hyperbolic when there is no upper limit to the instability. These assumptions certainly appear to be true for the unstable regions examined so far.

When the unstable region is bounded an ellipse with an axis of symmetry coincident with the ω-axis is assumed and three points are sufficient to enable the extremities to be calculated. The extremities of the ellipse give the range of ω over which the system is unstable and the maximum value of α_j. Shown in Fig. 5a are the unstable regions computed for the quaternary of Section 5.2. Table 1 lists the calculated extremities of the ellipses for different selections of three unstable solutions and shows how close the variation of α_j with ω is to an ellipse. It should be noted that the accuracy of the quaternary solutions was lower than those of the binary and some of the variation of the limits for the quaternary may be due to this lower accuracy.

When a non-skew symmetric matrix D_1 is included in the system represented by equation (27) (making the system non-reciprocal), the variation of the real part of the characteristic exponent with ω for an unstable region is no longer approximately elliptic.

Figure 5b shows the variation of α_j with ω for 0, 0·5, 1 and 2 per cent critical damping for a binary with widely spaced frequencies at $\omega = 0$ and no gyroscopic coupling. The width of the unstable region can increase with increasing structural damping, but the maximum values of α_j decreases almost linearly with increasing structural damping.

7. Conclusions

A program is now available which can be used to test the stability of large systems of linear second-order differential equations with periodic coefficients. The method described for predicting unstable regions by applying rules to constant coefficient solutions of the system reduces the amount of computation considerably and gives a systematic way of tackling the stability problem. The limits of an unstable region, for a reciprocal system, can be determined quite easily once three unstable solutions have been found within the region. The method has proved very useful in locating the unstable regions of a system describing the dynamic behaviour of a flexible aircraft on which are mounted two unsymmetrical rigid rotors.

LIST OF SYMBOLS

A, D	Periodic matrices defined by equation (2)
$\left.\begin{matrix} A_1, A_2, A_3, E \\ D_1, D_2, D_3, D_4 \end{matrix}\right\}$	$m \times m$ matrices defined by equation (1)
$Z(t)$	$n \times n$ periodic matrix
C, C_1	$n \times n$ constant non-singular matrices
I	Unit matrix
K	Matrix of right hand eigenvectors of $e^{R\tau}$, the Floquet modes
$P(t)$	Periodic part of the solution of $\Phi(t)$
R_1	$n \times n$ matrices defined in equation (11)
m	Order of the square matrices, A_1, A_2, E etc.
n	Order of the square matrices, $A(t), P(t), R$ etc. ($n = 2m$)
p	The column matrix q differentiated with respect to time
q	A column matrix of generalised co-ordinates
r, s	Parameters used in the condensation to principal β_j^*
t	Time
y	A column matrix of the dependent variable of equation (4)
Λ	A diagonal matrix of the characteristic roots λ_j
$\Phi(t)$	A fundamental matrix of equation (4)
Φ_1, Φ_2	Two fundamental matrices of equation (4)
α_j	The real part of the jth characteristic exponent
β_j	The imaginary part of the jth characteristic exponent
β_j^*	The positive value of β_j obtained by solving the constant coefficient system
principal β_j^*	The positive principal value of β_j^*
γ_{ij}	The elementary truncation errors in the matrix Φ arising during the integration
δ	The accuracy required during integration
ε	A factor on the time dependent coefficients
λ_j	The jth characteristic root ($= \mu_j + i\nu_j$)
μ_j	The real part of the jth characteristic root
ν_j	The imaginary part of the jth characteristic root
σ_j	The jth characteristic exponent ($= \alpha_j + i\beta_j$)
τ	The period of the equations ($= \pi/\omega$)
ϕ_i	The ith column of the fundamental matrix

ω	The frequency associated with equation (1)
ω_0	A standard frequency associated with the matrices $\bar{D}_2$, $\bar{D}_3$ and $\bar{D}_4$
$\cdot$	Denotes differentiation with respect to time
$\mathscr{R}, \mathscr{I}$	The real and imaginary parts of a complex number

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APPENDIX I

Details of Stability Program

The method of testing the stability of a set of equations, having the form given in equation (22), has been programmed in ICL 1900 Fortran. The inverse of the matrix $(A_1 + A_2\varepsilon \sin 2\omega t + A_3\varepsilon \cos 2\omega t)$, for any instant of time, is found using Choleski's method and the matrix must be symmetric and positive definite. The equations are integrated using a modified version of the Kutta–Merson variable step length routine[8] and at each selected step the equations are integrated for all initial conditions. The step length during the integration is varied automatically to maintain a specified accuracy, δ, and accuracies varying from 10^{-2} to 10^{-6} have been used, the former being used for preliminary work. The accuracy defined by Merson is as follows:

If $\gamma_{ij}\,(i = 1, 2, \ldots n, j = 1, 2, \ldots n)$ are the elementary truncation errors in Φ_{ij} arising at the end of a particular integration step, and $|\Phi_{ij}|_{max}$ is the maximum modulus of Φ_{ij} that has arisen during the course of the integration, then at the end of each step the following tests are applied:

(a) If for <u>any</u> element, $|\gamma_{ij}| > \delta/2|\Phi_{ij}|_{max}$, return to the beginning of the step and try half the previous interval;

(b) If for <u>all</u> elements, $|\gamma_{ij}| \leqslant \delta/64|\Phi_{ij}|_{max}$, try double the interval for the next step;

(c) Otherwise proceed with the same interval as before.

Form of the Data

(1) An integer m which is the order of the square matrices A_1, A_2 etc. $m \leqslant 5$.

(2) The matrices A_1, A_2 etc. in any order but preceded by the appropriate identifiers i.e. $A1$, $A2$, E etc. Null matrices need not be input and a matrix is only overwritten by a subsequent read with the same identifier. The identifier must be on a record by itself and is read on an A2 format. *See* Note 1.

(3) The character group ** starting a new record.

(4) ω_0, the basic frequency.

(5) δ, the accuracy required in the VINTSTEP integration routine.

(6) An integer k, the number of ω's for which solutions are required.

(7) k pairs of values ω_1, $DJC_1, \ldots, \omega_k$, DJC_k, each pair on a new record, format (E0 . 0, A3). DJC is a data jump character, *see* Note 2.

(8) DJC, another data jump character, format (A3), *see* Note 3.

Notes

(i) All items (1)–(8) must start on a new record. Free format input is used wherever possible.

(ii) The character groups allowed at (7), and the program action, are as follows:

VVV (Where V ≡ space) The eigenvalues are calculated.
VEC Eigenvalues and eigenvectors are calculated.
TDF The time dependent factors, ε, are applied to the periodic terms and the eigenvalues are calculated.
T&V The time dependent factors are applied, eigenvalues and eigenvectors are calculated.

When TDF and T&V are used then j, an integer specifying the number of time dependent factors, and $\varepsilon_1, \ldots, \varepsilon_j$ (where $j \leqslant 10$), the factors, must be supplied. The values, format (I0, 10E 0 . 0), should follow the appropriate ω, DJC pair.

(iii) The character groups allowed at (8), and the program action, are as follows:

NUN Re-enters at (1) to read a new value of m etc.
MTX Re-enters at (2) to read a new matrix etc.
** V Re-enters at (4). (V ≡ space).
ACC Re-enters at (5) to read a new VINTSTEP accuracy etc.
END Execution is stopped.

(iv) If a record starting with the character group CC is read at entry point (2), then the next record is read and output as a title.

Form of the Output

The output to a lineprinter has the following form:

Data titles

The (mxm) matrices $A1, A2, E$ etc. in the order in which they were input and preceded by their identifier. The matrices are printed row by row.

$W0 = \omega_0$ VINTSTEP ACCURACY $= \delta$

$W = \omega$ $EPS = \varepsilon_1$ INTEGRATION STEPS ..., ...

CHARACTERISTIC ROOTS CHARACTERISTIC EXPONENTS

$\{\mu_j\}$ $\{\nu_j\}$ $\{\alpha_j\}$ $\{\beta_j\}$ for $j = 1, 2, \ldots, n$

VECTORS

$\mathscr{R}\{k_1\}$ $\mathscr{I}\{k_1\}$

$\mathscr{R}\{k_2\}$ $\mathscr{I}\{k_2\}$

$\vdots$

$\mathscr{R}\{k_n\}$ $\mathscr{I}\{k_n\}$

If there is more than one value of ω and/or ε then the output will continue,

$W = \ldots$ $EPS = \ldots$

NOTES:

(i) If there are no time dependent factors, then $EPS = \varepsilon_1$ is omitted.

(ii) If the vectors are not required, then VECTORS etc. is omitted.

(iii) When $\omega = 0$ or $\varepsilon = 0$ then INTEGRATION STEPS etc. is omitted and only the characteristic exponents are printed.

(iv) For real characteristic roots the zero imaginary part of the corresponding vector is not printed.

(v) After the caption INTEGRATION STEPS two integers are printed. The first integer is the number of times the integration routine has been used and the second is the number of times the routine has been used usefully. When a step length is halved the calculation has to re-start at the previous step and so one pass through the routine is wasted. Thus the numbers give an indication of the difficulty encountered in the integration of the equations.

APPENDIX II

Coefficients for the Systems of Section 5

II.I. An Unsymmetrical Rigid Rotor in Unsymmetrical Bearings

The coefficients for the system of Brosens and Cradall[9] are given below. The coefficients of the matrix E are dependent on the degree of asymmetry in the stiffness of the bearings. Standard frequency $\omega_0 = 1{\cdot}0$.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0{\cdot}234 \\ 0{\cdot}234 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0{\cdot}234 & 0 \\ 0 & -0{\cdot}234 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{D}_2 = \begin{bmatrix} 0 & 0{\cdot}543 \\ -0{\cdot}543 & 0 \end{bmatrix} \quad \bar{D}_3 = \begin{bmatrix} -0{\cdot}468 & 0 \\ 0 & 0{\cdot}468 \end{bmatrix}$$

Unsymmetrical bearings   Symmetrical bearings

$$\bar{D}_4 = \begin{bmatrix} 0 & 0{\cdot}468 \\ 0{\cdot}468 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1{\cdot}707 & 0 \\ 0 & 0{\cdot}293 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The coefficients for the case considered from the system of Gladwell and Stammers[2] are,

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0{\cdot}6 \\ 0{\cdot}6 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0{\cdot}6 & 0 \\ 0 & -0{\cdot}6 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{D}_2 = \begin{bmatrix} 0 & 1{\cdot}35 \\ -1{\cdot}35 & 0 \end{bmatrix} \quad \bar{D}_3 = \begin{bmatrix} -1{\cdot}2 & 0 \\ 0 & 1{\cdot}2 \end{bmatrix}$$

$$\bar{D}_4 = \begin{bmatrix} 0 & 1{\cdot}2 \\ 1{\cdot}2 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1{\cdot}6 & 0 \\ 0 & 0{\cdot}4 \end{bmatrix}$$

II.2. Two Unsymmetrical Rigid Rotors on a Flexible Aircraft

The coefficients for a quaternary of this system are given below, and for example $2{\cdot}24447, -2 \equiv 2{\cdot}24447 \times 10^{-2}$. Standard frequency $\omega_0 = 0{\cdot}200015$.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 2{\cdot}24447, -2 & -1{\cdot}49399, -1 \\ 0 & 0 & -1{\cdot}61889, -1 & 5{\cdot}33217, -3 \\ 2{\cdot}24447, -2 & -1{\cdot}61889, -1 & 0 & 0 \\ -1{\cdot}49399, -1 & 5{\cdot}33217, -3 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & -2{\cdot}22889, -1 & -2{\cdot}90116, -2 \\ 0 & 0 & 2{\cdot}27265, -2 & 1{\cdot}10932, -1 \\ -2{\cdot}22889, -1 & 2{\cdot}27265, -2 & 0 & 0 \\ -2{\cdot}90116, -2 & 1{\cdot}10932, -1 & 0 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{D}_2 = \begin{bmatrix} 0 & 6{\cdot}26509, -2 & 0 & 0 \\ -6{\cdot}26509, -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6{\cdot}18537, -2 \\ 0 & 0 & -6{\cdot}18537, -2 & 0 \end{bmatrix}$$

$$\bar{D}_3 = \begin{bmatrix} 0 & 0 & 8{\cdot}91622, -2 & 1{\cdot}16055, -2 \\ 0 & 0 & -9{\cdot}09126, -3 & -4{\cdot}43762, -2 \\ 8{\cdot}91622, -2 & -9{\cdot}09126, -3 & 0 & 0 \\ 1{\cdot}16055, -2 & -4{\cdot}43762, -2 & 0 & 0 \end{bmatrix}$$

$$\bar{D}_4 = \begin{bmatrix} 0 & 0 & 8{\cdot}97858, -3 & -5{\cdot}97638, -2 \\ 0 & 0 & -6{\cdot}47602, -2 & 2{\cdot}13304, -3 \\ 8{\cdot}97858, -3 & -6{\cdot}47602, -2 & 0 & 0 \\ -5{\cdot}97638, -2 & 2{\cdot}13304, -3 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1{\cdot}09113, -1 & 0 & 0 & 0 \\ 0 & 2{\cdot}15519, -1 & 0 & 0 \\ 0 & 0 & 1{\cdot}29403, -1 & 0 \\ 0 & 0 & 0 & 3{\cdot}13668, -1 \end{bmatrix}$$

TABLE 1

Calculated Extremities of Unstable Regions

Quaternary of section 5.2	Unstable range	$(\alpha_i)_{max} \times 10^2$
Region 1	$\omega = 0{\cdot}326\ 14 \rightarrow 0{\cdot}348\ 49$	1·222 312
Region 2	$\omega = 0{\cdot}402\ 88 \rightarrow 0{\cdot}446\ 36$	2·158 960
	$0{\cdot}402\ 85 \rightarrow 0{\cdot}446\ 42$	2·159 089
Region 3	$\omega = 0{\cdot}432\ 78 \rightarrow 0{\cdot}464\ 99$	1·662 506
	$0{\cdot}432\ 77 \rightarrow 0{\cdot}465\ 01$	1·662 448
Region 4	$\omega = 0{\cdot}519\ 16 \rightarrow 0{\cdot}620\ 86$	4·296 835
	$0{\cdot}520\ 32 \rightarrow 0{\cdot}620\ 85$	4·303 402
	$0{\cdot}519\ 94 \rightarrow 0{\cdot}620\ 53$	4·296 883
	$0{\cdot}520\ 06 \rightarrow 0{\cdot}620\ 63$	4·296 937

Binary of section 5.2	Unstable range	$(\alpha_i)_{max} \times 10^2$
	$\omega = 0{\cdot}329\ 36 \rightarrow 0{\cdot}369\ 06$	1·984 333
	$0{\cdot}329\ 36 \rightarrow 0{\cdot}369\ 06$	1·984 322

Note: When there is more than one entry for a region the calculated extremities are for different selections of three unstable solutions within the region.

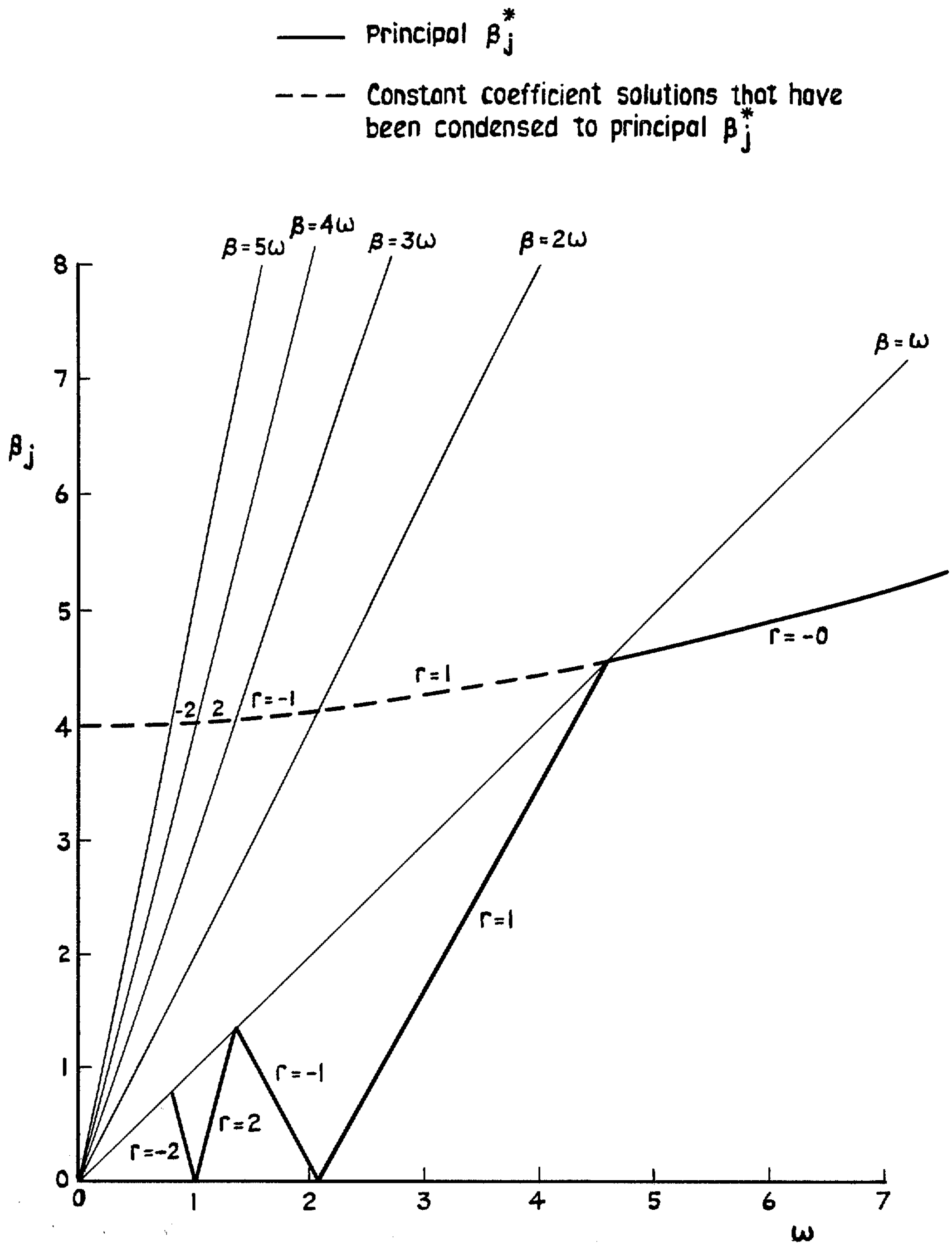

FIG. 1. Condensation of constant coefficient solutions to principal β_j^*.

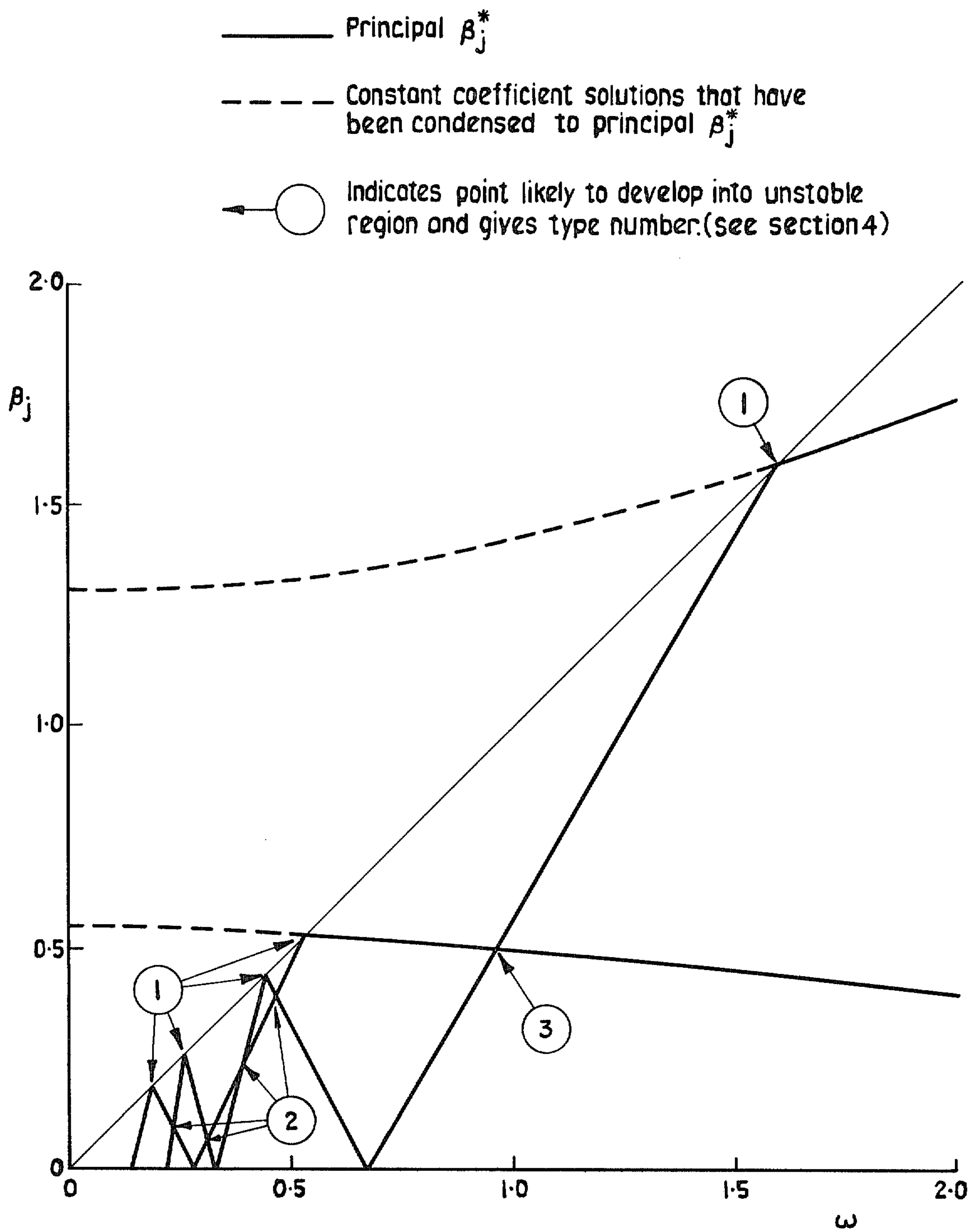

FIG. 2. Predicted unstable regions.

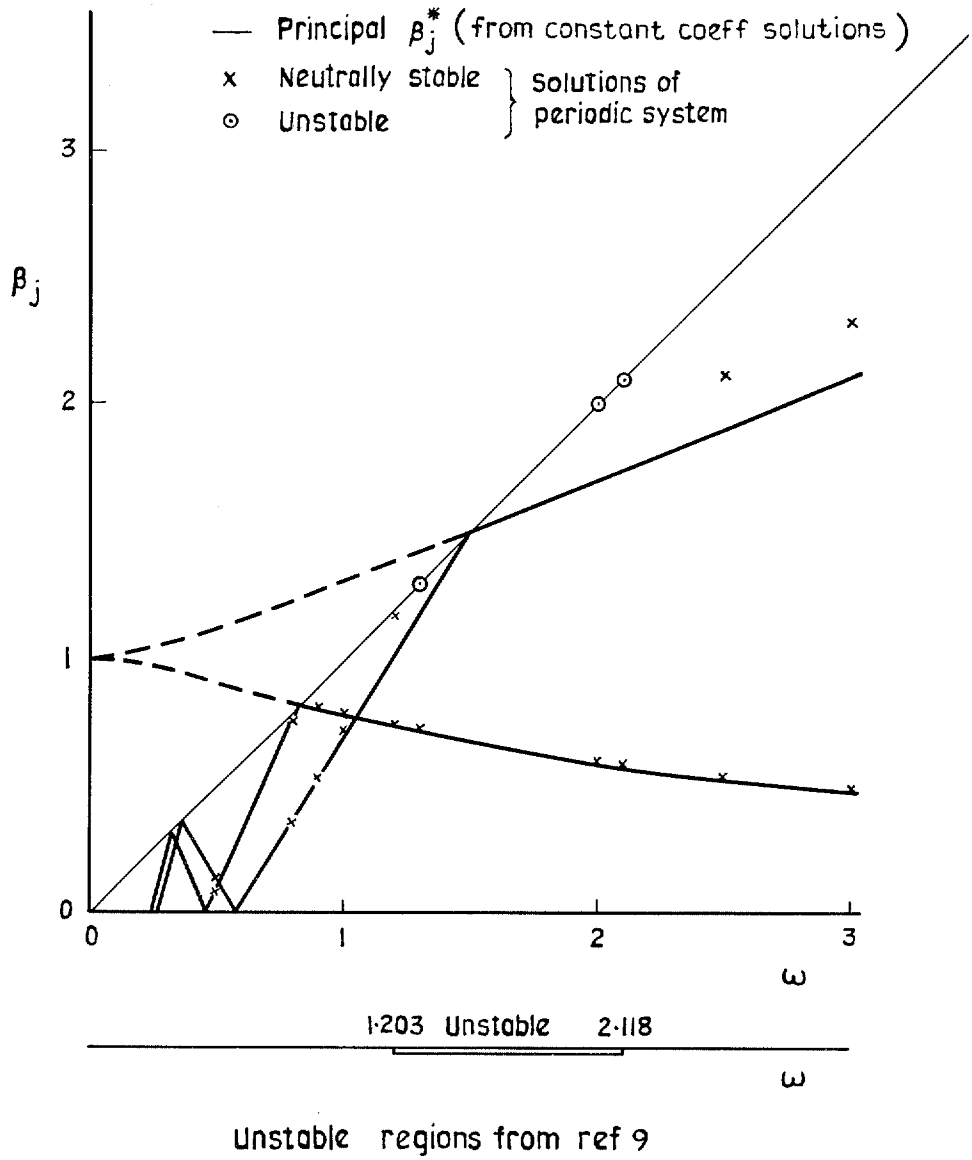

FIG. 3a. Variation of the imaginary part of the characteristic exponent with ω—Unsymmetrical rotor with equal mounting stiffnesses.

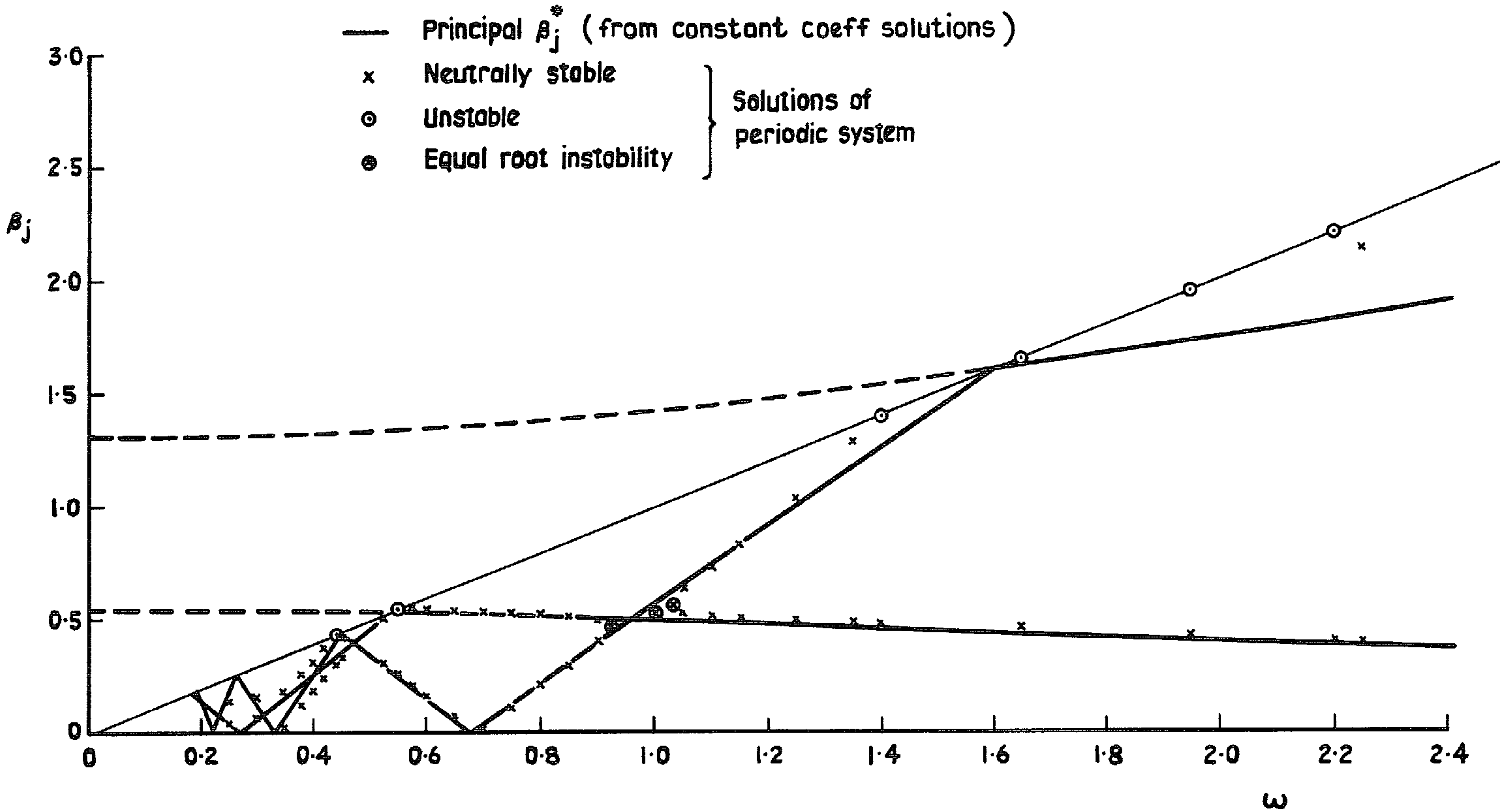

FIG. 3b. Variation of the imaginary part of the characteristic exponent with ω—unsymmetrical rotor with unequal mounting stiffnesses.

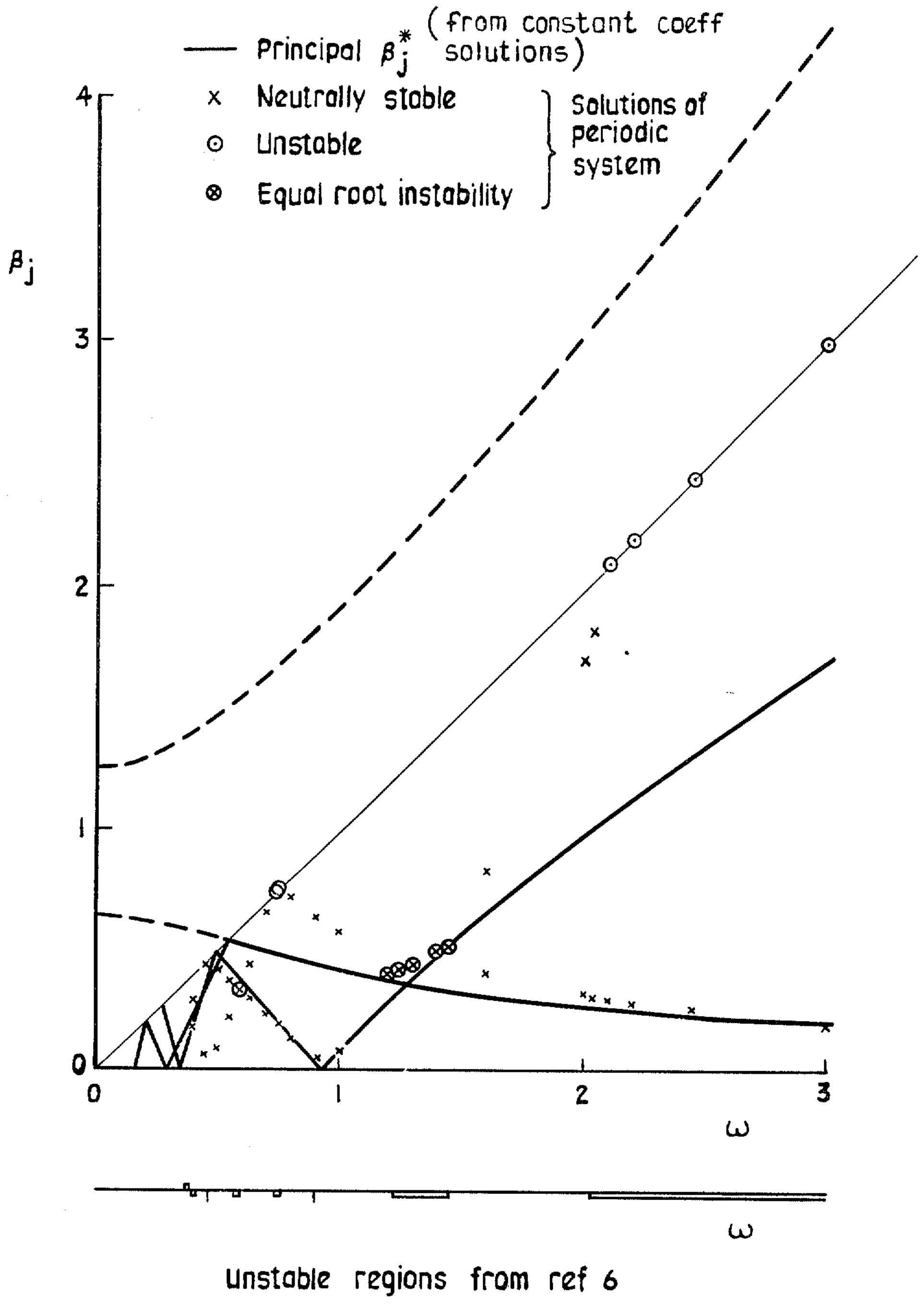

FIG. 3c. Variation of the imaginary part of the characteristic exponent with ω—Rotor with large inertia inequality and unequal mounting stiffnesses.

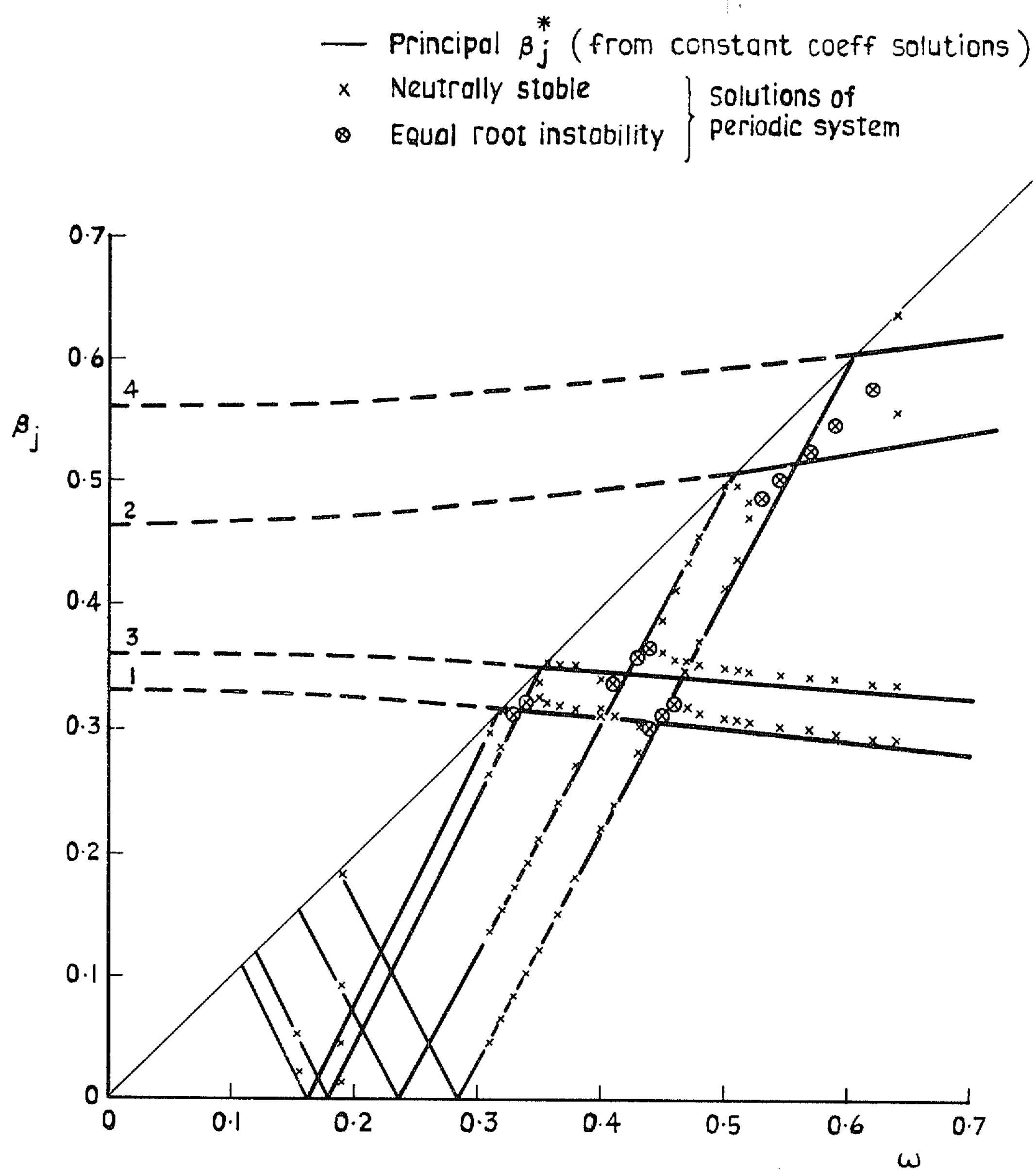

FIG. 4a. Variation of the imaginary part of the characteristic exponent with ω—Solutions for the quaternary of section 5.2.

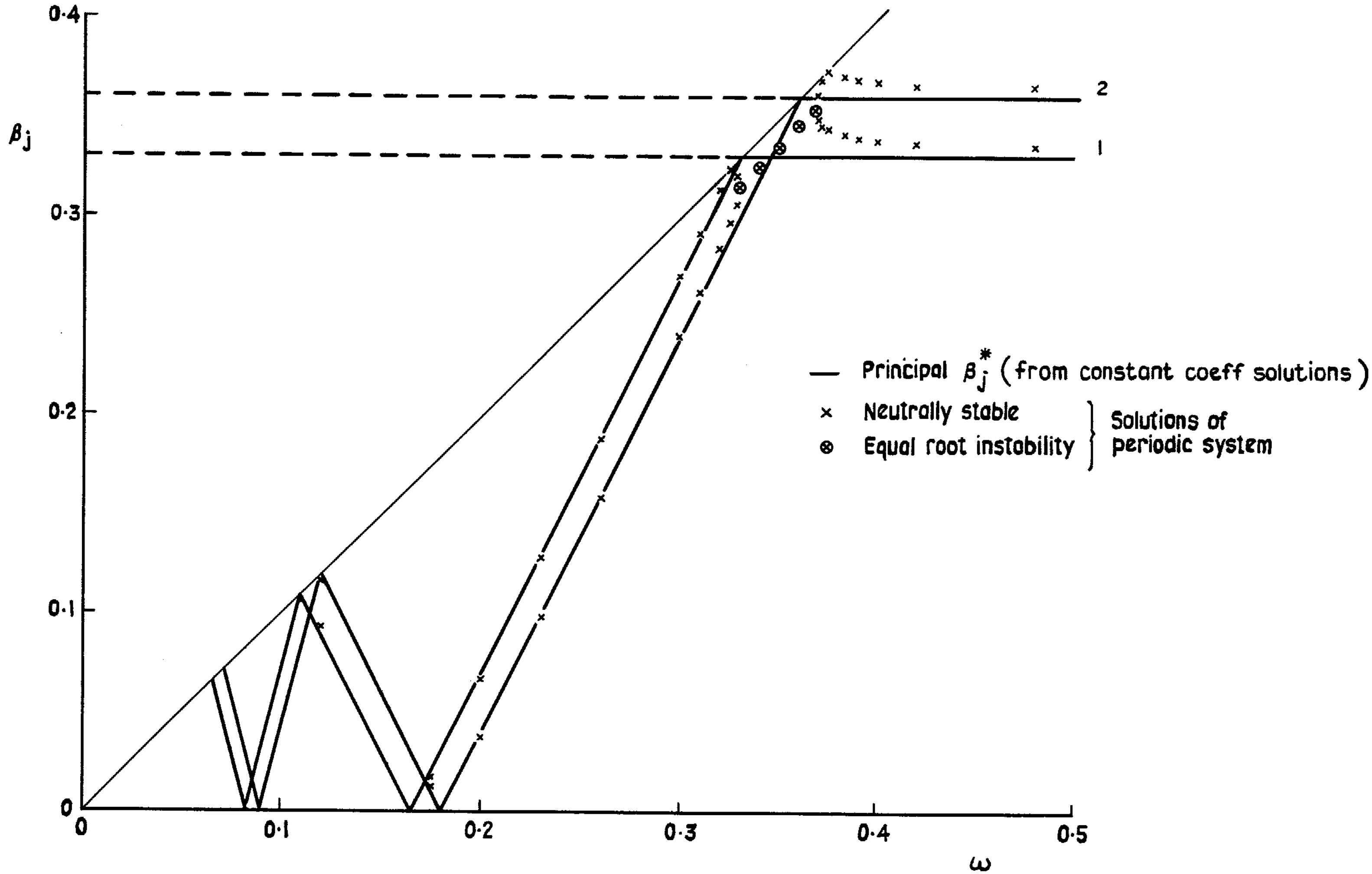

FIG. 4b. Variation of the imaginary part of the characteristic exponent with ω—Solutions for the binary of section 5.2.

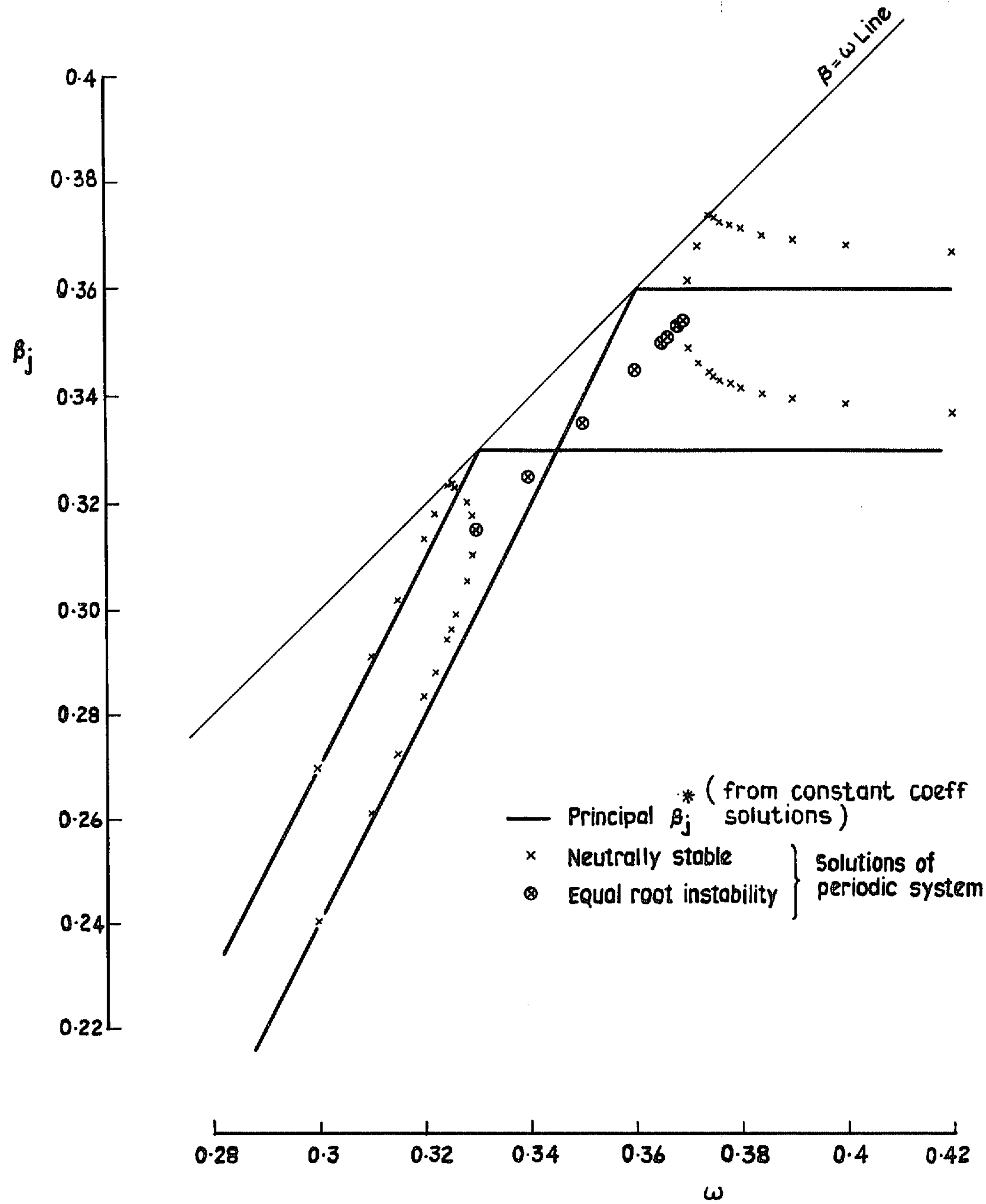

FIG. 4c. Variation of the imaginary part of the characteristic exponent with ω—Solutions in the vicinity of the unstable region of the binary of section 5.2.

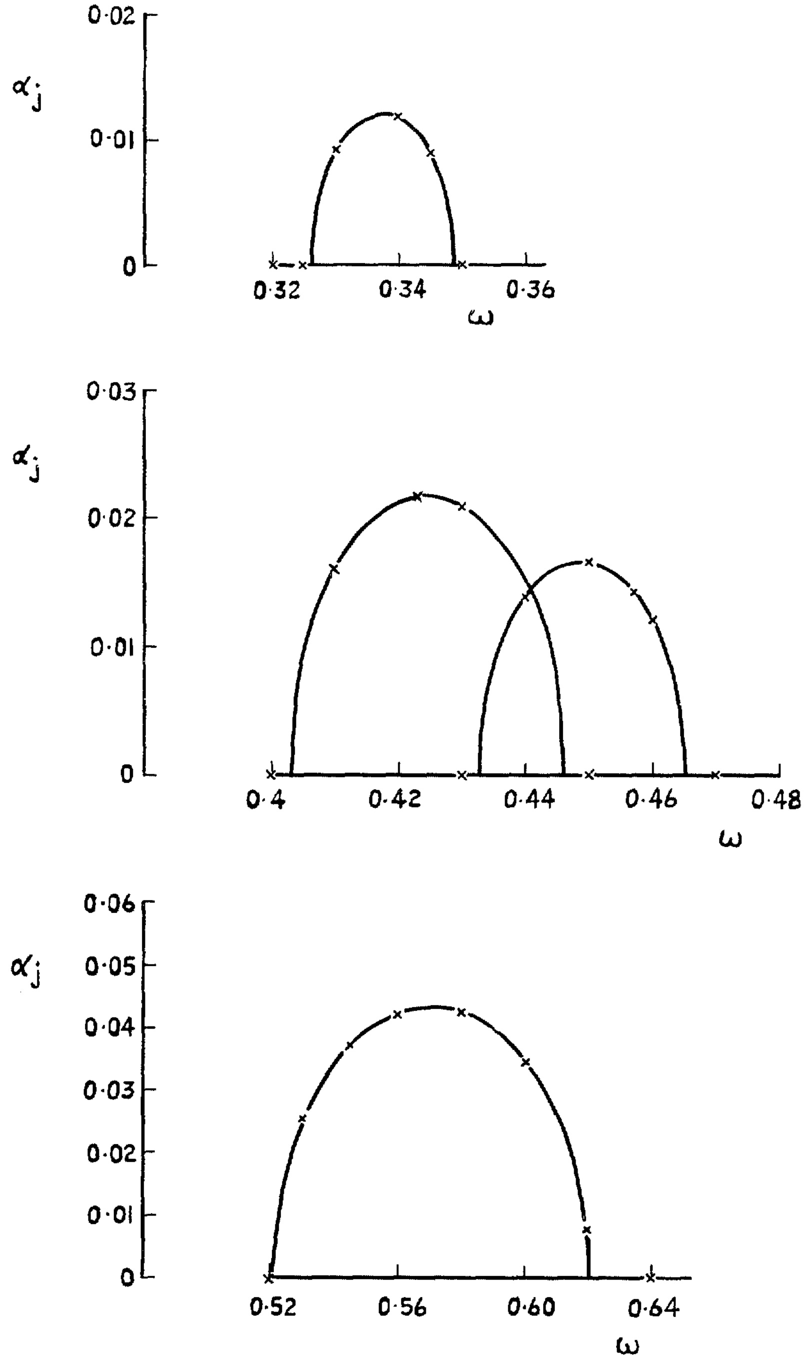

FIG. 5a. Variation of the real part of the characteristic exponent with ω—Quaternary without structural damping.

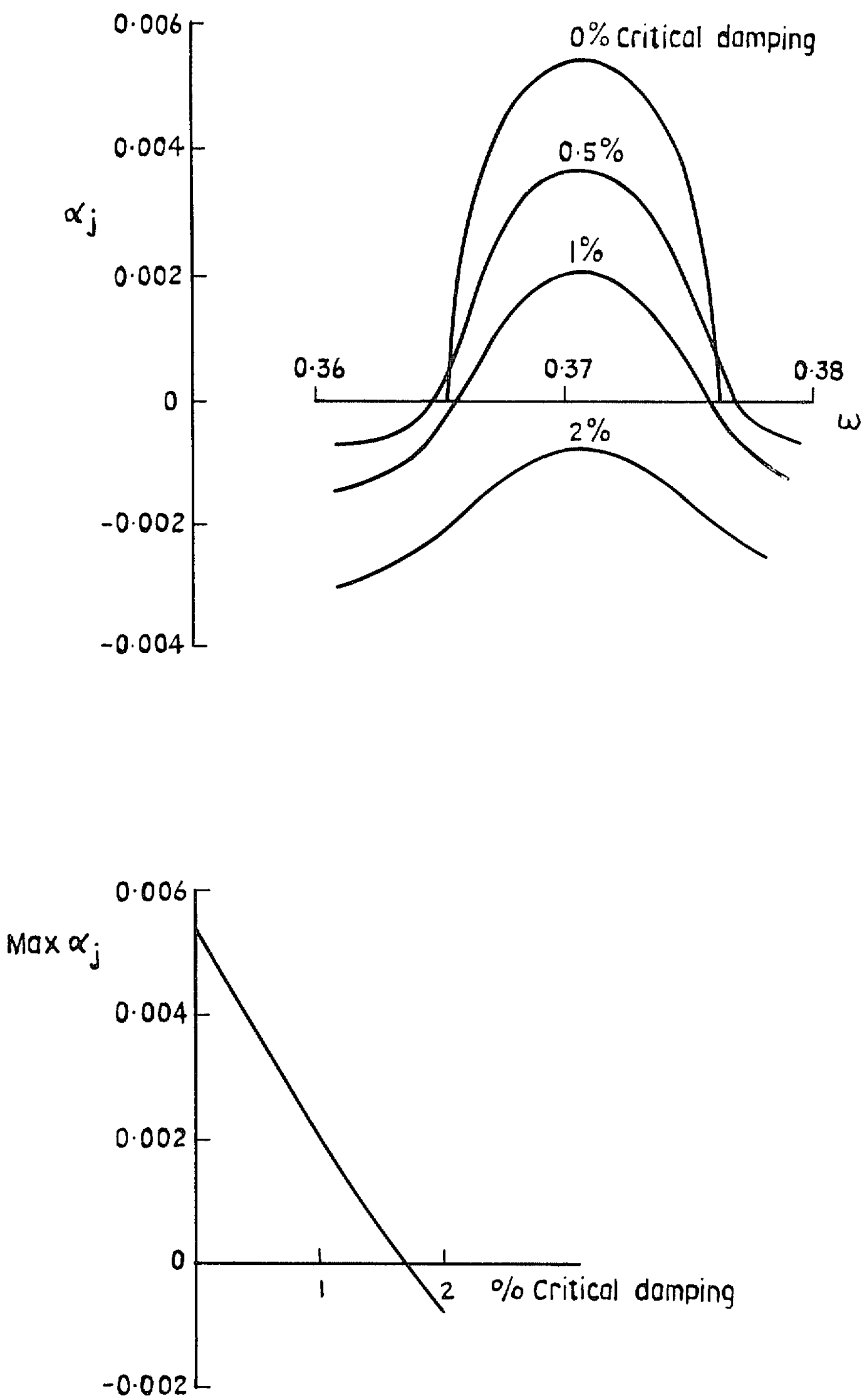

FIG. 5b. Variation of the real part of the characteristic exponent with ω—Binary with viscous structural damping.

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