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A New Method of Numerical Integration of the Equations of the Laminar Boundary Layer

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A New Method of Numerical Integration of the Equations of the Laminar Boundary Layer

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Summary.—A new method for the numerical solution of the boundary-layer equations is described. This rests in essence on the fact that the equations of steady flow are special cases of the equations of general motion. The velocity profiles are found at successive sections across the boundary layer. Trial values of the velocity are assumed at any section ; from these, space derivatives of the velocity are deduced by using finite differences, and time derivatives by using the equations of motion. The trial values are then adjusted to give zero time derivatives of the velocity at the section. The method in some respects resembles Southwell's relaxation method.

The method has been applied to two problems already discussed numerically by Hartree. It is not suitable for use with a differential analyser, though the development of new calculating machines may bring it within the range of machine integration ; but rather less labour was required to achieve manually with it results rather more accurate than obtained by Hartree with the differential analyser. The results did not, however, differ greatly from Hartree's.

1. *Introduction.*—The work described below was inspired by two reports by Prof. D. R. Hartree[1, 2] giving an account of his method of numerical solution of the laminar boundary-layer equations. In terms of suitable non-dimensional variables, these equations may be written

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad \ldots \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \quad \ldots \quad (2)$$

Here

- x distance along the wall of the boundary layer, divided by standard length L,
- u x-component of velocity, divided by a standard velocity U_0,
- y distance from the wall, divided by $(\nu L/U_0)^{1/2}$, where ν is the kinematic viscosity,
- v y-component of velocity, divided by $(\nu U_0/L)^{1/2}$,
- U value of u just outside the boundary layer.

Hartree's method was, in brief, as follows. He expressed u and v in terms of a stream function, ψ, where

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

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Substituting these values in (1), he obtained a differential equation involving only first-order x-derivatives. These were replaced by finite differences before integrating; the differential equation was then integrated exactly with respect to y for a series of values of x.

This method was not simple. A non-linear equation of third order in y had to be solved, and the solution near $y = 0$ had to be adjusted by trial to secure that a boundary condition was satisfied at $y = \infty$. Actually, an approximation to the solution for any given x could have been inferred from the solutions for earlier values of x, but the method was unable to use this information. Moreover, it did not seem accurate. Replacement of the x-derivatives by finite differences meant a very crude approximation; this was improved by obtaining a second solution in which the x-interval is halved, and using Richardson's h^2-extrapolation formula. Roughly, the original approximation is accurate only if u is linear in x; Richardson's formula corrects for terms quadratic in x, and partially for terms of higher degree. In the author's own work, described later, sixth-order x-differences of u were occasionally needed. While this was partly because a slowly converging backward-difference formula was used, it was not clear that Richardson's formula was adequate. For these reasons, an alternative method was sought.

The first attempt was by integrating with respect to x instead of y. Equations (1) and (2) can be written.

$$-u^2 \frac{\partial}{\partial y}\left(\frac{v}{u}\right) = U \frac{\partial U}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \qquad (1')$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}. \qquad (2')$$

Suppose that, for a given x, we know u at $y = b, 2b, 3b, \ldots$ By constructing successive y-differences, we can find $\partial u/\partial y$ and $\partial^2 u/\partial y^2$; then (1′) gives $\partial(v/u)/\partial y$. Integrating, and using the condition that $v/u = 0$ when $y = 0$, we find v/u and so v; then $\partial v/\partial y$ is given by

$$\frac{\partial v}{\partial y} = u \frac{\partial}{\partial y}\left(\frac{v}{u}\right) + \frac{v}{u}\frac{\partial u}{\partial y}.$$

Hence we can find $\partial u/\partial x$ from (2′); this enables values of u to be determined for a new value of x.

This method is simple, but in practice it proved disappointing. In (1′), $\partial(v/u)/\partial y$ is given as the ratio of two quantities, both of which vanish at $y = 0$; as a consequence, it cannot be determined very accurately near $y = 0$. This affects the values of v/u for all y, and the effect is not small, since $\partial(v/u)\partial y$ is fairly large near $y = 0$. The method was therefore rejected.*

2. *The New Method.*—After other methods had been tried without success, a method was found which combined accuracy with some measure of practicability. This rested on the fact that (1) is the steady-motion form of the general equation of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \qquad (3)$$

where t denotes a non-dimensional variable proportional to the time. Equation (3) indicates not only the state of steady motion, but the way in which u varies when this state is being attained.

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* After the completion of the work described in the present report, it was found that Prandtl (*Zs. f. angew. Math. u. Mech.*, Vol. 18, p.81, 1938) had suggested a method of numerical integration with respect to x. He pointed out the difficulties of the direct method outlined above, and suggested a more refined method whereby these might possibly be obviated. It seems doubtful, however, whether even this more refined method is satisfactory.

Suppose that values of u have already being found for $x = x_0,\ x_0 + a,\ x_0 + 2a, \ldots\ldots,$ $x_0 + na$, with $y = b, 2b, 3b, \ldots\ldots$. By extrapolation or some other method, let approximate values of u be found for $x = x_0 + (n + 1)a$, with the same values of y. These are regarded as the values of u in an unsteady motion which is tending to the steady motion as a limit. From x-differences may be found the values of $\partial u/\partial x$ for $x = x_0 + (n + 1)\, a$, and from y-differences those of $\partial u/dy$ and $\partial^2 u/\partial y^2$; also, by (2)

$$v = -\int_0^y \frac{\partial u}{\partial x}\, dy\,.$$

Thus we find from (3) the values of $\partial u/\partial t$ for $x = x_0 + (n + 1)\, a$ corresponding to the non-steady motion. The values of $\partial u/\partial t$ indicate the corrections to the approximate values required to get the steady-state values of u for $x = x_0 + (n + 1)\, a$. The approximate values are modified accordingly, and the work is repeated with the new values of u. After a few trials, a set of values of u is found which makes $\partial u/\partial t$ negligible for all y. These are taken as the steady-state values for $x = x_0 + (n + 1)\, a$, and the work is then repeated for $x = x_0 + (n + 2)\, a$, etc.

The method is in some respects similar to Southwell's relaxation method, $\partial u/\partial t$ corresponding to the relaxation force. It differs from this method in discussing the successive values of x separately. This is possible because in boundary-layer flow the velocity profile at any x is wholly determinate if the profiles for all *smaller* values of x are known. The present method also corrects all the values of u corresponding to a given value of x simultaneously; Southwell's method corrects them severally.

3. *Howarth's Problem.*—The method has been applied to both of the problems considered by Hartree. The first of these is Howarth's problem of retarded flow along a flat plate[3] in which

$$U = 1 - \frac{1}{8}x\,, \qquad \ldots \ldots \ldots \ldots \ldots \qquad (4)$$

where x is measured from the leading edge. The units U_0 and L, in terms of which U and x are measured are, in Howarth's notation, b_0 and $b_0/8b_1$; thus our x is identical with Howarth's $8x$, or with Hartree's x.

In practice, the method used was slightly different from that described in section (2), because in place of y and v, variables η and V were introduced, where

$$\eta = \tfrac{1}{2} y x^{-1/2}\,, \quad V = 2 v x^{1/2}\,. \qquad \ldots \ldots \ldots \ldots \qquad (5)$$

The variable η is that used by Howarth and Hartree. In terms of these, (2) and (3) become

$$4x\frac{\partial u}{\partial t} - u\frac{\partial V}{\partial \eta} + V\frac{\partial u}{\partial \eta} = 4xU\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial^2 \eta}\,, \qquad \ldots \ldots \qquad (6)$$

$$\frac{\partial V}{\partial \eta} = -\,4x\frac{\partial u}{\partial x} + 2\eta\frac{\partial u}{\partial \eta}\,. \qquad \ldots \ldots \ldots \ldots \qquad (7)$$

The method was applied to these modified equations, using derivatives with respect to η in place of those with respect to y. For any x, values of $2u$ were tabulated at $\eta = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, \ldots$; difference methods gave $\partial u/\partial x$, $\partial u/\partial \eta$ and $\partial^2 u/\partial \eta^2$; then $\partial V/\partial \eta$ was given by (7) and V found by integration; finally (6) gave $\partial u/\partial t$.

Values of $2u$ were tabulated to 5 decimal places. This enabled $4\partial \eta/\partial \eta$ to be found to 4 decimals, and $8\partial^2 u/\partial \eta^2$ to 3; also $2\partial V/\partial \eta$ was found to 3 decimals, $2V$ to 4, and $32x\partial u/\partial t$ to 3. The error introduced by approximations made during calculation is certainly not greater than $0{\cdot}001$ for $32x\partial u/\partial t$ when η is small, though it may be rather greater when η is large.

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Changes in u, made while approximating to the steady motion for a given x, affect $\partial u/\partial t$ mainly through the resultant changes in $\partial u/\partial x$ and $\partial^2 u/\partial \eta^2$, to a less extent through changes in $\partial u/\partial \eta$, and very little as a result of the direct appearance of u in (6). The changes in $\partial u/\partial x$ affect $\partial u/\partial t$ through the terms involving V and $\partial V/\partial \eta$ in (6). The changes in V are relatively unimportant when only a single value of u is altered, but are important when changes of the same sign have to be made in a number of successive values of u. To estimate roughly the corrections to be applied to a trial set of values of $2u$, it is often sufficient to take into account only the effects in (6) of the changes in $\partial V/\partial \eta$ due to changes in $\partial u/\partial x$, and of the changes in $\partial^2 u/\partial \eta^2$. When $\partial u/\partial t$ is consistently of one sign in a given range, however, changes in V become important in (6), and greater corrections have to be applied than would be necessary if such changes were ignored.

When η is small, changes in $\partial^2 u/\partial \eta^2$ are alone important; in this neighbourhood, to annihilate small values of $\partial u/\partial t$ which are consistently of one sign, quite large changes in u may be necessary. Such changes may not even be limited to the region near $\eta = 0$, for changes in $\partial u/\partial x$ near $\eta = 0$ affect the values of V for all η. In consequence, particular care must be taken to ensure that near $\eta = 0$ no non-zero values of $\partial u/\partial t$ are left uncorrected. This was not sufficiently realised at the start of the integration, when the author was content to reduce $\partial u/\partial t$ until $32x\, du/\partial t$ was nowhere greater than $0 \cdot 003$. This is good enough for large values of η, where a change of a unit in the fifth decimal place in $2u$ may produce a change $0 \cdot 006$ in $32x\, \partial u/\partial t$; but much greater care is needed near $\eta = 0$.

Fifth-order η-differences were tabulated for use in calculating $\partial u/\partial \eta$; from these the sixth-order differences could be determined in the few cases when they affected $\partial^2 u/\partial \eta^2$. The η-derivatives were calculated from the central-difference formulae,*

$$hf'(a) = \tfrac{1}{2}\,[\Delta f(a) + \Delta f(a-h)] - \frac{1}{6.2}\,[\Delta^3 f(a-h) + \Delta^3 f(a-2h)]$$
$$+ \frac{1}{30.2}\,[\Delta^5 f(a-2h) + \Delta^5 f(a-3h)] - \ldots\ldots,$$

$$h^2 f''(a) = \Delta^2 f(a-h) - \frac{1}{12}\,\Delta^4 f(a-2h) + \frac{1}{90}\,\Delta^6 f(a-3h) - \ldots\ldots,$$

where h is the spacing (Ref. 4, p. 64). To apply these at $\eta = 0 \cdot 2$ and $0 \cdot 4$, some extrapolation of third and higher order differences was necessary; from the run of the differences, the error so introduced was clearly negligible, and errors would not have been negligible had other formulae been used. On the other hand, in calculating $\partial u/\partial x$ the only differences available were those obtained using earlier values of x, and so $\partial u/\partial x$ had to be found from the slowly converging backward-difference formula

$$hf'(a) = \Delta f(a-h) + \tfrac{1}{2}\Delta^2 f(a-2h) + \frac{1}{3}\,\Delta^3 f(a-3h) + \ldots,$$

(Ref. 4, p. 62, equation (2), using differences sloping up in the differences table instead of those sloping down). Inaccuracies introduced by the use of this formula greatly reduced the advantage in accuracy possessed by the present method over that of Hartree. In integrating $\partial V/\partial \eta$, a central-difference formula was again used, *i.e.*,

$$\frac{1}{h}\int_a^{a+h} f(x)\,dx = \tfrac{1}{2}\,[f(a) + f(a+h)] - \frac{1}{24}\,[\Delta^2 f(a-h) + \Delta^2 f(a)]$$
$$+ \frac{11}{1440}\,[\Delta^4 f(a-2h) + \Delta^4 f(a-h)] - \ldots.$$

* The apparent lack of symmetry of these formulae relative to the value a of the argument is due to the use of forward-differences Δ. It could have been avoided by introducing the central-difference operator δ; but the symbol δ is required for another purpose in section (4).

The repeated use of differences makes the method unsuitable for use with a moment integrator; however, the labour was not excessive with a small calculating machine.

4. *Master and Associated Integrations.*—The method involves a number of trial integrations for each x. In practice, the labour of these was reduced as follows. A first integration was made in the usual way to get a set of values of $\partial u/\partial t$, which were used for a rough correction of the values of u. The corrected values of u, which gave a more regular set of fifth-order η-differences, were used as starting-point for a second integration of the same kind; in this, every step liable to be affected by computing error was most carefully checked. Later integrations used the deviations of the independent variables from their values in this master integration. Let u, V denote the values in the master integration, $u + \delta u$, $V + \delta V$ those in a slightly varied integration. Then δu, δV can be taken to satisfy the equations

$$4x \frac{\partial \delta u}{\partial t} - u \frac{\partial \delta V}{\partial \eta} - \delta u \frac{\partial V}{\partial \eta} + V \frac{\partial \delta u}{\partial \eta} + \delta V \frac{\partial u}{\partial \eta} = \frac{\partial^2 \delta u}{\partial \eta^2}, \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\frac{\partial \delta V}{\partial \eta} = - 4x \frac{\partial \delta u}{\partial x} + 2\eta \frac{\partial \delta u}{\partial \eta}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

These are used to determine values of δu such that $\partial(u + \delta u)/\partial t = 0$.

Slide-rule accuracy was enough in determining the products on the left of (8). The equation used to determine $\partial \delta u/\partial x$ was of the form

$$h\delta f'(a) = \delta \Delta f(a - h) + \tfrac{1}{2}\, \delta \Delta^2 f(a - 2h) + \frac{1}{3}\, \delta \Delta^3 f(a - 3h) + \ldots .$$

In this, $f(a)$, $f(a - h)$, $f(a - 2h)$, ... denote the values of u, for a given η, corresponding to $x = a, a - h, a - 2h, \ldots$; of these only $f(a)$ is affected by the variation δ, and so $\delta \Delta^r f(a - rh) = \delta f(a)$. Hence

$$h\delta f'(a) = \delta f(a) \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots . .\right).$$

The series on the right strictly does not converge; but in practice it was cut off in accordance with the number of x-differences found significant. Thus its value was usually taken to be between $2 \cdot 1$ and $2 \cdot 45$ (corresponding to 4 and 6 differences respectively).

5. *The Separation Point.*—The solution was started by calculating $2u$ from Howarth's tables at $x = 0, 0 \cdot 1, 0 \cdot 2$, and $0 \cdot 3$. Then integration proceeded by steps of $0 \cdot 1$ of x up to $x = 0 \cdot 8$. Here sixth-order x-differences were needed to find $\partial u/\partial x$. Accordingly x-differences for half the interval length were found by the usual subtabulation formulae (Ref. 4, p. 54), and these were used as a basis for integrations at $x = 0 \cdot 85$ and $0 \cdot 9$. The integration was not carried beyond this point, as there is a singularity at separation near $x = 0 \cdot 96$, making numerical integration difficult in its vicinity; it is, in any case, possible to locate the separation point fairly accurately by extrapolation.

Goldstein[1] suggests, in work quoted by Hartree, that near the separation point ($x = X$),

$$\left[\frac{\partial u}{\partial \eta}\right]_{\eta=0} \equiv f(x) = A(X - x)^{1/2} + B(X - x)^{3/4} + C(X - x) + D(X - x)^{5/4} + \quad \ldots \quad (10)$$

Hence $f(x)$ can be represented fairly closely by a polynomial in $(X - x)^{1/2}$, even near separation. This fact supplies two methods of determining X.

(i) Regard x as a regular function of $f(x)$, and use Newton's interpolation formula for unequal intervals to extrapolate the value of x for which $f(x) = 0$.

(ii) Choose a succession of approximate values of X and, regarding $f(x)$ as a regular function of $(X - x)^{1/2}$ for each X, use the same formula to determine $f(X)$; the correct X is that for which $f(X) = 0$.

The separation point has been determined by both these methods; the first gave $X = 0 \cdot 958$, the second $X = 0 \cdot 956$. These estimates are both a little uncertain because the divided differences of third order were rather irregular, perhaps because of the change in interval-length at $x = 0 \cdot 8$. The probable errors of the two estimates due to the irregularity are of order $0 \cdot 005$ and $0 \cdot 001$.

These estimates are very near to those of Howarth ($X = 0 \cdot 96$) and Hartree ($X = 0 \cdot 9589$). Too much weight cannot be attached to the agreement, because our extrapolation has been over a very wide interval of $(X - x)^{1/2}$. Nevertheless, the agreement indicates that the separation point can be determined reasonably well without integrating right up to the awkward singularity. It also suggests that a reasonable approximation to the velocity profile at separation may be got by extrapolation from the known values of $x \leqslant 0 \cdot 9$, using divided differences for each η with $2u$ expressed as a function of $(X - x)^{1/2}$. Values so obtained, taking $X = 0 \cdot 956$, are shown in Table 1; they agree very closely with Hartree's (which are given only for small values of η). Also given in the table are values for $x = 0 \cdot 8$ and $0 \cdot 9$, with values interpolated from Hartree's tables. The values obtained by the new method differ from Hartree's by amounts within the error of the two methods. The writer believes that, had he worked more accurately near $\eta = 0$ for $x = 0 \cdot 5$ and $0 \cdot 6$, the differences would have been slightly greater.

Remembering the form of (10), it might be thought that it would have been better to have extrapolated, taking $f(x)$ and $2u$ as regular functions of $(X - x)^{1/4}$, not of $(X - x)^{1/2}$. This is not so, because $2u$ is known only for values of $(X - x)^{1/4}$ between $0 \cdot 476$ and $0 \cdot 864$, and extrapolation to $X - x = 0$ is therefore very risky. The success of the present extrapolation suggests that the fourth-root terms are not too important in (10).

6. *Schubauer's Pressure Distribution.*—The second problem to which the method was applied was that of boundary-layer flow for Schubauer's observed pressure distribution on an elliptic cylinder[6]. Hartree's smoothed figures for the pressure distribution (Ref. 2, Table 2) were used in most of the work.

The units L, U_0 in terms of which x and u are measured are, in this case, the length of the minor axis of the cylinder and the undisturbed stream velocity; also x is measured from the stagnation point. This makes our x and u identical with those of Schubauer and Hartree; our y is identical with Hartree's, but is $y\sqrt{(LU_0/\nu)}$ in Schubauer's notation. Since the boundary layer expands not less rapidly than $x^{1/2}$ when $X > 0 \cdot 2$, it is convenient, as in section 3, to employ instead of y and v the variables η, V defined by (5); thus the equations integrated were again (6) and (7).

Values of $2u$ were tabulated to three decimal places at $\eta = 0 \cdot 1, 0 \cdot 2, 0 \cdot 3, \ldots$. This meant that $2\partial^2 u/\partial \eta^2$ could be found to only one decimal place. Since the total magnitude of this quantity was found never to exceed $3 \cdot 0$, a less order of accuracy in tabulating $2u$, as used by Hartree at the start of his solution (Ref. 2, Table 7), could clearly give no useful results.

The work was much cruder, and therefore much easier, than that on Howarth's problem, where two more decimals were used. No differences beyond the third order were needed; numerical errors were easily identified, and elaborate cross-checks were unnecessary; only two or three trial integrations were needed for any value of x, and the use of master and associated integrations (section 4) was superfluous. Moreover, numerical accuracy at least equal to that of Hartree's solution was obtained, using no mechanical aids other than an ordinary slide-rule. In consequence, values of $2u$ for a given x could be found in an easy day's work; a week was needed when working on Howarth's problem.

The solution was begun with Hartree's initial data for $x = 0 \cdot 2$, calculated by him from a series expansion. His Table 7[2] gave $2u$ to only two decimals; three decimals were obtained by integrating his $\partial u/\partial y$ values. Values of $\partial u/\partial x$ at $x = 0 \cdot 2$ could in theory also be determined from Hartree's series; but as the details of Hartree's numerical work were not immediately available, in practice the method followed was to estimate $\partial u/\partial x$ from his values of $2u$ at $x = 0 \cdot 2$, $0 \cdot 3$, and $0 \cdot 4$, the estimated values being then checked by a trial integration of the differential equations. The values of $\partial u/\partial x$ so obtained were then used in determining $\partial u/\partial x$ for trial solutions at $x = 0 \cdot 25$, the new values of $\partial u/\partial x$ being connected with assumed values of u at $x = 0 \cdot 25$ by the approximate formula

$$0 \cdot 05\left[\left(\frac{\partial u}{\partial x}\right)_{0 \cdot 2} + \left(\frac{\partial u}{\partial x}\right)_{0 \cdot 25}\right] = 2(u)_{0 \cdot 25} - 2(u)_{0 \cdot 2} .$$

For greater values of x, $\partial u/\partial x$ was found by the difference methods of section 3.

The integration proceeded by steps of $0 \cdot 05$ in x from $x = 0 \cdot 2$ to $x = 0 \cdot 4$, and by steps of $0 \cdot 1$ from $x = 0 \cdot 4$ to $x = 1 \cdot 6$. Thenceforth, since Schubauer's experiments indicated separation near $x = 2$, it was expected (*cf.* section 5) that u would be a nearly regular function of $(2 - x)^{1/2}$, and so $2u$ was found at steps of $0 \cdot 1$ of ξ, where

$$\xi = \sqrt{[1 \cdot 6(2 - x)]} \ ;$$

in this part of the solution $\partial u/\partial x$ was found by first determining $\partial u/\partial \xi$ by difference methods.

As in Hartree's work, separation was not attained by $x = 2$ for the original pressure distribution, and a modified pressure distribution was then substituted beyond $x = 1 \cdot 8$. Hartree found his modified pressure distribution by estimating crudely what pressure at $x = 2$ was required to give separation at $x = 2$; a more exact investigation indicated that this pressure distribution would give separation closer to $x = 1 \cdot 983$. It was not possible to use Hartree[6]s modified distribution in the present work, as it does not vary sufficiently smoothly near $x = 1 \cdot 8$. Instead, values of $(\partial u/\partial \eta)_{\eta=0}$ were assigned beforehand for the different values of ξ, tending to zero like a regular function of ξ as $x \to 2$; the pressures corresponding to the successive values of ξ were adjusted to make the calculated values of $(\partial u/\partial \eta)_{\eta=0}$ agree with these assigned values. The pressure distribution thus found gave $U^2 = 1 \cdot 537$ at $x = 2$, as against $U^2 = 1 \cdot 542$ and $1 \cdot 534$ with Hartree's original and modified distributions (*see* Table 2).

There is no doubt that a modification of Hartree's original pressure distribution is needed to get separation at $x = 2$; the failure to find separation without such a modification cannot be attributed to inaccuracies in the method of integration. Hartree's original distribution appears to slightly over-estimate changes in the pressure gradient between $x = 1 \cdot 4$ and $1 \cdot 8$, and to under-estimate them between $x = 1 \cdot 0$ and $1 \cdot 4$, and between $1 \cdot 8$ and $2 \cdot 0$. The extreme sensitivity of the separation point to slight changes in the pressure gradient near separation has already been noted by earlier workers. However, the failure to obtain the observed separation with Hartree's originally adopted pressure distribution need not be due to actual discrepancies between that distribution and the true one; it may be due to failure of the assumptions of boundary-layer theory near separation due to the thickening of the boundary layer, or to disturbances in the boundary layer near separation.

As in the discussion of Howarth's problem, the present work confirmed the essential accuracy of Hartree's method. The results of the two methods do differ somewhat, the maximum difference between the values of $2u$ being about $0 \cdot 015$, *i.e.*, about 1 per cent. of the total magnitude of $2u$. The order of magnitude of the differences is illustrated by the values of $2u$ for $x = 1$, given in Table 3; by the values of $(\partial u/\partial y)_{y=0}$, given in Table 4; and by the values, given in Table 5, of the displacement and momentum thicknesses, defined by

$$\delta' = \int_0^\infty \left(1 - \frac{u}{U}\right) dy, \quad \vartheta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy . \qquad \qquad (11)$$

The differences are, however, small compared with the probable errors in the values of the respective variables when determined directly from the experimental results.

7. *Solution from the Stagnation Point.*—After the work described in the last section had been completed, a doubt arose, on looking back, as to whether Hartree's starting values for $x = 0 \cdot 2$ might not be more inaccurate than first supposed, and the subsequent integration be thereby affected throughout. Accordingly a further integration was carried out, beginning this time at the stagnation point. Since the boundary-layer thickness is finite at $x = 0$, it was convenient for small values of x to use the variables y, v in place of η, V, and integrate equations (2) and (3) instead of (6) and (7). The pressure distribution used for small values of x is shown in Table 2; it was obtained by interpolation from Hartree's figures for U at $x = 0 \cdot 05$, $0 \cdot 1$, $0 \cdot 15$, $0 \cdot 2$, $0 \cdot 25$ and $0 \cdot 3$ (Ref. 2, Table 2).

To start the solution, the values of u at $x = 0 \cdot 01$ were taken to be those in a flow such that $U = 8x$ (according to Hartree, Ref. 2 equation (14), $U = 7 \cdot 92\, x - 0(x^3)$ when x is small). Flow such that $U = kx$ has been considered by Falkner and Skan[7]; their numerical results have been recalculated by Hartree[8]. As a matter of interest, the present method was used to recalculate the flow for this case correct to three significant figures; it gave results agreeing with Hartree's to within $0 \cdot 2$ per cent.

The values of u thus derived at $x = 0 \cdot 01$ were used, together with the value $u = 0$ at $x = 0$, in determining $\partial u/\partial x$ by difference methods for an integration at $x = 0 \cdot 02$. The integration then proceeded by steps of $0 \cdot 02$ in x from $x = 0 \cdot 02$ to $x = 0 \cdot 2$, and by steps of $0 \cdot 025$ from $x = 0 \cdot 2$ to $x = 0 \cdot 3$. During this part of the integration, $2u$ was tabulated for values of y such that $y\sqrt{2} = 0 \cdot 1, 0 \cdot 2, 0 \cdot 3, \ldots$. After $x = 0 \cdot 3$, the variables η, V were used in place of y, v, and equations (6) and (7) in place of (2) and (3). Integrations were effected at intervals of $0 \cdot 05$ of x from $x = 0 \cdot 3$ to $x = 0 \cdot 5$, and at intervals of $0 \cdot 1$ from $x = 0 \cdot 5$ to $x = 1 \cdot 0$, where the integration was discontinued.

The results showed Hartree's values of $2u$ at $x = 0 \cdot 2$ to be rather high, the maximum error being about $0 \cdot 03$ (rather above 2 per cent.; *see* Table 3). However, the integration of section 6, which used these values as starting values, is not seriously in error save near $x = 0 \cdot 2$. By $x = 0 \cdot 3$, the maximum difference between the values of $2u$ in the present integration and that of section 6 is just greater than 1 per cent.; at $x = 1 \cdot 0$, where the present one was stopped, the difference is less than $\frac{1}{2}$ per cent., *i.e.*, less than the difference of the values in Hartree's integration and in that of section 6. The general effect of the differences, as of those between Hartree's solution and that described in section 6, is that the boundary layer should be thicker than Hartree's work indicates; the increase is above 4 per cent. at $x = 0 \cdot 2$, but not much more than 2 per cent. after $x = 0 \cdot 3$. The change improves the agreement with Schubauer's experimental velocity-profiles, which are compared with the theoretical in Fig. 1.

8. *General Remarks.*—The present method has definite advantages in accuracy, simplicity and directness over Hartree's method. Nevertheless, it is very laborious if the accuracy desired is of the order of that attained in discussing Howarth's problem. It was much more manageable in discussing Schubauer's problem, and it is pointless in most practical problems to attempt to achieve more accuracy by the method than was attained in discussing that problem.

The method does, however, possess certain drawbacks. To determine $\partial u/\partial x$ by difference methods it is necessary to know u for a number of values of x; hence, in starting the solution, more than one set of initial values of u must be known, or, at the very least, values of u and $\partial u/\partial x$ for the initial x. Again, the difference expression for $\partial u/\partial x$ converges slowly near a singularity, and so the accuracy of the method near separation is poor. This drawback is shared by Hartree[6]s method, and we have indicated in section 5 a method of obtaining reasonably accurate results near separation without integrating right up to the singularity.

I should like to thank Professor Hartree for his interest in this work, and for discussions during its progress; also Mr. M. Holt, who read the manuscript and made several useful suggestions.

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TABLE 1

Values of 2u for Howarth's Problem Obtained by the Present Method, with Values Derived from Hartree's Tables for Comparison

η	$x = 0{\cdot}8$		$x = 0{\cdot}9$		Separation	
	Present Method	Hartree	Present Method	Hartree	Present Method	Hartree
0	0	0	0	0	0	0
0·2	0·09368	0·09405	0·05918	0·05945	0·0170	0·0169
0·4	0·21508	0·21560	0·14943	0·14997	0·0674	0·0672
0·6	0·36139	0·36205	0·26853	0·26937	0·1496	0·1493
0·8	0·52784	0·52865	0·41264	0·41370	0·2611	
1·0	0·70780	0·70875	0·57619	0·57742	0·3981	
1·2	0·89308	0·89400	0·75198	0·75337	0·5554	
1·4	1·07460	1·07560	0·93148	0·93295	0·7260	
1·6	1·24341	1·24440	1·10560		0·9017	
1·8	1·39199	1·39295	1·26573		1·0738	
2·0	1·51532	1·51615	1·40496		1·2337	
2·2	1·61155	1·61230	1·51910		1·3742	
2·4	1·68198	1·68260	1·60709		1·4909	
2·6	1·73022	1·73065	1·67074		1·5819	
2·8	1·76114	1·76135	1·71387		1·6492	
3·0	1·77963	1·77975	1·74120		1·6955	
3·2	1·78995	1·79005	1·75738		1·7250	
3·4	1·79534	1·79540	1·76634		1·7424	
3·6	1·79797	1·79800	1·77100		1·7518	
3·8	1·79917	1·79915	1·77328		1·7564	
4·0	1·79968	1·79972	1·77430		1·7587	
4·2	1·79988	1·79995	1·77473		1·7599	
4·4	1·79996	1·8	1·77490		1·7605	
4·6	1·79999	1·8	1·77497		1·7608	
4·8	1·8	1·8	1·77499		1·7609	
5·0	1·8	1·8	1·775		1·7610	

TABLE 2

Values of 2U, U^2 and 4U (dU/dx) in the Schubauer Problem.

(a) when x is small, interpolated from Hartree's Table 2 (Ref. 2).

(b) beyond $x = 1 \cdot 8$, for Hartree's original pressure distribution and the modified distribution of section 6.

(a)

x	$2U$	$2U^2$	$4U \frac{dU}{dx}$	x	$2U$	$2U^2$	$4U \frac{dU}{dx}$
0	0	0	0				
0·02	0·317	0·0502	4·91	0·16	1·750	1·531	9·62
0·04	0·614	0·1884	8·66	0·18	1·849	1·710	8·44
0·06	0·880	0·3872	11·00	0·20	1·932	1·866	7·44
0·08	1·113	0·619	12·08	0·225	2·018	2·036	6·34
0·10	1·314	0·863	12·16	0·25	2·090	2·184	5·45
0·12	1·485	1·102	11·66	0·275	2·150	2·312	4·71
0·14	1·629	1·327	10·76	0·30	2·200	2·420	4·07

(b)

	Original Distribution			Modified Distributions (section 6)			Hartree's Modified Distribution	
x	$2U$	$2U^2$	$4U \frac{dU}{dx}$	$2U$	$2U_2$	$4U \frac{dU}{dx}$	$2U^2$	$4U \frac{dU}{dx}$
1·8	2·530	3·200	−0·59	2·530	3·200	−0·59	3·200	−0·59
1·85	2·518	3·170	−0·60	2·518	3·170	−0·62		
1·9	2·506	3·141	−0·59	2·505	3·139	−0·64	3·137	−0·67
1·95	2·494	3·112	−0·57	2·492	3·107	−0·65		
2·0	2·483	3·084	−0·53	2·479	3·074	−0·66	3·068	−0·71

TABLE 3

Special Velocity Profiles for the Solution of Section 6 (which used Hartree's Initial Values at $x = 0 \cdot 2$) and that of Section 7 (the solution from stagnation) with Hartree's Results for Comparison

$x = 0 \cdot 2$			$x = 1 \cdot 0$			
y	$2u$ (section 7)	$2u$ (Hartree)	y	$2u$ (section 6)	$2u$ (section 7)	$2u$ (Hartree)
0	0	0	0	0	0	0
0·1	0·35	0·35	0·2	0·262	0·261	0·265
0·2	0·65	0·66	0·4	0·519	0·517	0·525
0·3	0·92	0·94	0·6	0·770	0·766	0·777
0·4	1·15	1·17	0·8	1·011	1·007	1·022
0·5	1·34	1·37	1·0	1·242	1·236	1·253
0·6	1·50	1·53	1·2	1·458	1·451	1·470
0·7	1·62	1·655	1·4	1·656	1·648	1·667
0·8	1·72	1·75	1·6	1·834	1·826	1·845
0·9	1·79	1·82	1·8	1·990	1·982	2·001
1·0	1·84	1·87	2·0	2·124	2·116	2·134
1·1	1·87	1·90	2·2	2·235	2·228	2·245
1·2	1·90	1·92	2·4	2·326	2·319	2·335
1·3	1·915	1·925	2·6	2·398	2·391	2·406
1·4	1·925	1·93	2·8	2·453	2·447	2·460
1·5	1·93	1·93	3·0	2·494	2·488	2·500
			3·2	2·523	2·518	2·528
			3·4	2·543	2·540	2·548
			3·6	2·557	2·555	2·561
			3·8	2·566	2·565	2·570
			4·0	2·573	2·572	2·575
			4·2	2·577	2·577	2·578
			4·4	2·579	2·579	2·579
			4·6	2·580	2·580	2·580

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TABLE 4

Values of $4\,(\partial u/\partial y)_{y=0}$ *in the Schubauer Problem for each of the Solutions of Sections* 6 *and* 7, *with Hartree's Values for Comparison*

x	$4\left(\frac{\delta u}{\delta y}\right)_{y=0}$		
	Main Solution (section 6)	Solution from Stagnation (section 7)	Hartree's Solution (Ref. 2, Table 8)
0		0	
0·06		5·48	
0·12		7·56	
0·2	7·37	7·24	7·38
0·3	6·16	6·10	6·17
0·4	5·20	5·14	5·21
0·6	4·02	3·98	4·05
0·8	3·24	3·22	3·28
1·0	2·64	2·63	2·67
1·2	2·23		2·25
1·4	1·87	Solution with	1·89
1·6	1·36	modified pressure	1·37
1·69375	1·09	distribution	
1·775	0·84	(section 6)	
1·84375	0·65	0·64	
1·9	0·50	0·47	0·53
1·94375		0·32	
1·975		0·20	
1·99375		0·09	
2·0		0·0	

TABLE 5

Displacement and Momentum Thicknesses δ', ϑ in the Schubauer Problem, Measured in the Same Units as y

x	Main Solution (section 6)		Hartree's Solution (Ref. 2, Table 9)		Solution from Stagnation (section 7)	
	δ'	ϑ	δ'	ϑ	δ'	ϑ
0					0·228	0·103
0·06					0·253	0·113
0·12					0·301	0·133
0·2	0·372	0·157	0·372	0·157	0·387	0·166
0·3	0·4955	0·208	0·497	0·209	0·506	0·214
0·4	0·6085	0·253	0·610	0·253	0·618	0·258
0·6	0·8125	0·3325	0·808	0·331	0·820	0·336
0·8	0·999	0·404	0·993	0·402	1·007	0·408
1·0	1·190	0·473	1·177	0·468	1·197	0·4765
1·2	1·367	0·537	1·354	0·531	—	—
1·4	1·554	0·599	1·537	0·592	Solution for modified pressure distribution (section 6)	
1·6	1·823	0·677	1·794	0·664		
1·69375	1·981	0·7115				
1·775	2·152	0·7445				
1·84375	2·322	0·7755			2·332	0·7765
1·9	2·480	0·801	2·472	0·799	2·510	0·805
1·94375					2·682	0·8245
1·975					2·850	0·839
1·99375					3·011	0·848
2·0					3·158	0·855

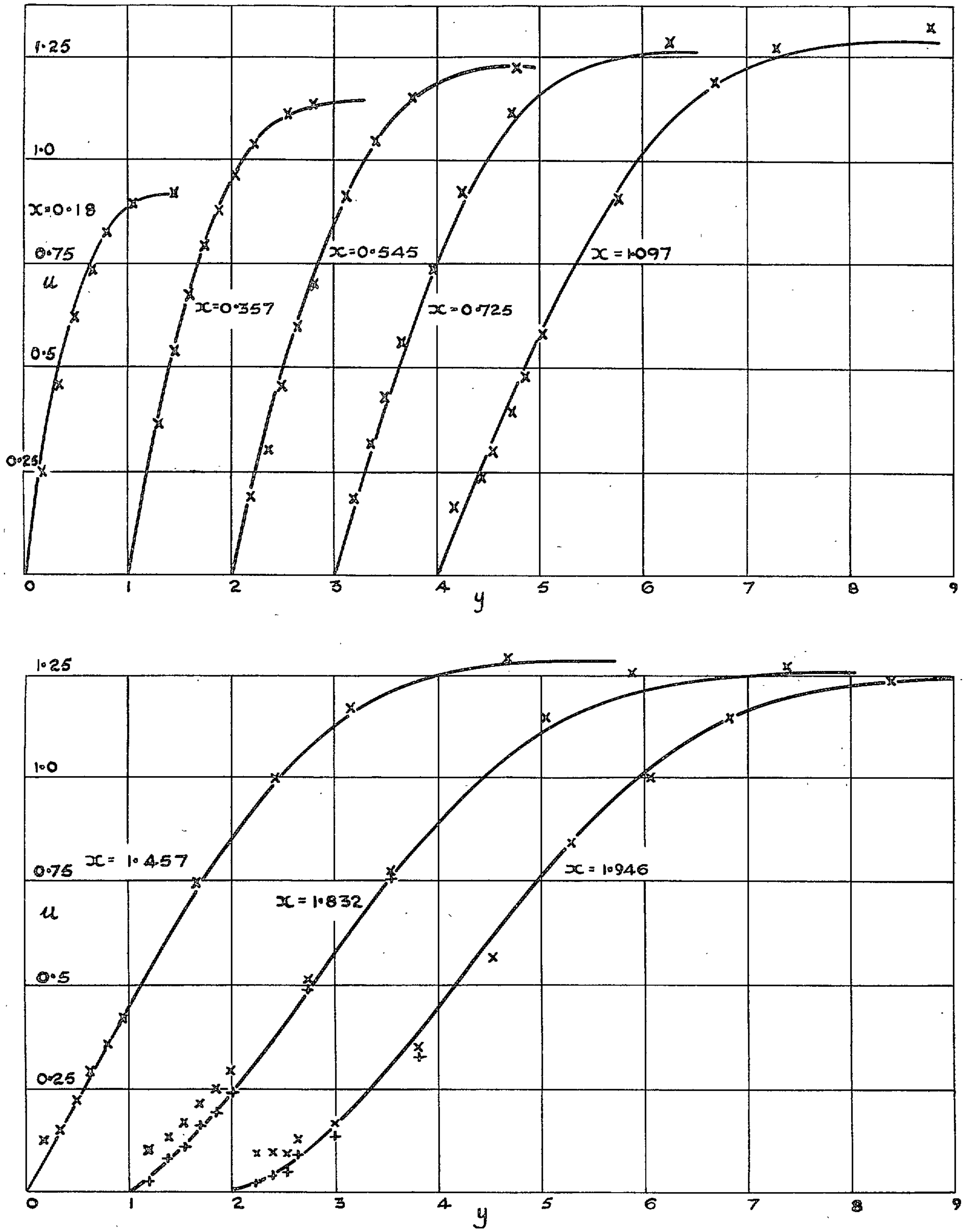

FIG. 1. Calculated Velocity Profiles for the Schubauer Problem for Different Values of x, from the Solutions of 7 (upper curves) and 6 (lower curves).

Schubauer's observed points are marked by crosses,

× uncorrected for heat loss to wall,

+ (over)-corrected for heat loss to wall in still air.

Units of y are shown below the curves ; the origins of successive curves are displaced one unit to the right.

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