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Theory of Lifting Surfaces Oscillating at General Frequencies in a Subsonic Stream

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Theory of Lifting Surfaces Oscillating at General Frequencies in a Subsonic Stream

By W. E. A. Acum

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Summary.

This Report gives a description of an extension to wings oscillating at general frequencies of Multhopp's lifting-surface theory for wings in steady subsonic flow.

This particular variant of the 'kernel function' method is published here because it has been used for many years in Aerodynamics Division, N.P.L., and has been the method used to provide results in a number of papers already published.

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*Replaces A.R.C. 17 824, 18 630, 19 229, 20 771.

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1. *Introduction.*

This Report contains a description of a method of calculating the aerodynamic forces on a wing oscillating in a subsonic stream. The method, which has been used for some years in the Aerodynamics Division of the National Physical Laboratory, is of the 'collocation' or 'kernel function' type, involving a numerical solution of the integral equation of linearised theory connecting the lift and upwash distributions. It is in principle the same as the collocation methods of tackling the same problem which have been put forward by other authors both in this country and elsewhere, and have been much used in government establishments and the aircraft industry. (*See*, for example, Refs. 1 to 3.) Nevertheless it was thought worthwhile to publish the details of this particular variation since it has a few distinctive features, and moreover values calculated by it have been quoted elsewhere (Refs. 4 and 5).

The method is essentially an extension to frequency parameters which are not small of that proposed for steady flight by Multhopp[6] and extended to oscillating flight at small frequency parameters by Garner[1]. The distribution of solving points is the same as that in Multhopp's original treatment; although this distribution was developed for steady flow at low Mach numbers it seems to apply satisfactorily to high frequency oscillations in high subsonic flow. The use of an electronic computer is essential for routine application.

In the examples given below, the wings are performing simple rigid oscillations; this is because most of the calculations were performed for comparison with wind-tunnel experiments in which the model had this sort of motion. In fact simple modes of distortion could easily be treated, but modes, such as aileron rotation, which involve discontinuities in the boundary condition are not included.

2. *The Integral Equation.*

This Section, for the sake of completeness, contains a derivation of the integral equation which connects the load distribution on an oscillating wing with the perturbation velocity caused by its oscillatory motion. It is assumed that the thickness and camber of the wing, and the amplitude of its oscillation are all sufficiently small for inviscid linearised flow theory to be applicable.

2.1. *Derivation of the Integral Equation.*

Let (x_1, y_1, z_1) be rectangular co-ordinates arranged so that the undisturbed flow has velocity U in the direction of x_1 increasing. Then Φ, the perturbation velocity potential caused by the small oscillatory motion of the wing, satisfies the equation

$$(1-M^2)\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial y_1^2}+\frac{\partial^2\Phi}{\partial z_1^2}-\frac{2M^2}{U}\frac{\partial^2\Phi}{\partial x_1\,\partial t}-\frac{M^2}{U^2}\frac{\partial^2\Phi}{\partial t^2}=0\,, \tag{1}$$

that is, the equation of sound in moving co-ordinates.

A fundamental solution of equation (1) is the oscillating source of angular frequency ω, whose velocity potential is

$$\Phi_0 = e^{i\omega t}\exp\left(\frac{i\omega}{U}\frac{M^2}{\beta^2}x_1\right)\exp\left(-\frac{i\omega}{U}\frac{M}{\beta^2}R\right)\Big/ R\,, \tag{2}$$

where

$$R = [x_1^2+\beta^2 y_1^2+\beta^2 z_1^2]^{\frac{1}{2}}\,. \tag{3}$$

The flow represented by equation (2) has outgoing waves far from the origin.

The function $\partial\Phi_0/\partial z_1$ then represents an oscillating doublet at the origin. Consider the total potential of a distribution of such doublets over the plane $z_1 = 0$; the resulting expression for Φ is

$$\Phi(x_1, y_1, z_1, t) = e^{i\omega t}\int\!\!\int f(x_1', y_1')\frac{\partial}{\partial z_1}\left[\exp\left\{\frac{i\omega M^2(x_1-x_1')}{U\beta^2}-\frac{i\omega M R}{U\beta^2}\right\}\frac{1}{R}\right]dx_1'\,dy_1'\,, \tag{4}$$

where now

$$R = [(x_1-x_1')^2+\beta^2(y_1-y_1')^2+\beta^2 z_1^2]^{\frac{1}{2}}\,, \tag{5}$$

and f is a function representing the strength of the doublet distribution. The region of integration is the plane $z_1 = 0$.

Now put

$$\left.\begin{aligned} x_1' &= x_1+\beta r_1\cos\theta \\ y_1' &= y_1+r_1\sin\theta \end{aligned}\right\}, \tag{6}$$

then from equation (4)

$$\Phi(x_1, y_1, z_1, t) = e^{i\omega t}\int\limits_{r_1=0}^{\infty}\int\limits_{\theta=0}^{2\pi} f(x_1+\beta r_1\cos\theta, y_1+r_1\sin\theta)\exp\left[-\frac{i\omega M^2}{U\beta}r_1\cos\theta\right]\times$$

$$\times\exp\left[-\frac{i\omega M}{U\beta}\sqrt{r_1^2+z_1^2}\right]\left[\frac{-z_1}{(r_1^2+z_1^2)^{3/2}}-\frac{i\omega M}{U\beta}\frac{z}{(r_1^2+z_1^2)}\right]r_1\,dr_1\,d\theta\,. \tag{7}$$

Consider the limit of Φ as z_1 tends to zero while x_1, y_1 and t remain constant. Divide the plane into two parts by an ellipse $r_1 = \delta$. When $z_1 \to 0$ the contribution from the part outside $r_1 = \delta$ tends to zero. If f is continuous at (x_1, y_1) (and in all the following applications it is continuous) then equation (7) may be rewritten

$$\lim_{z_1 \to 0} \Phi(x_1, y_1, z, t) = -\lim_{z_1 \to 0} e^{i\omega t} \int_{r_1=0}^{\delta} \int_{\theta=0}^{2\pi} [f(x_1, y_1)+\Delta] \exp\left[-\frac{i\omega M}{U\beta}\sqrt{r_1^2+z_1^2}\right] \times$$

$$\times \left[\frac{z_1}{(r_1^2+z_1^2)^{3/2}} + \frac{i\omega M}{U\beta}\frac{z_1}{r_1^2+z_1^2}\right] r_1\, dr_1\, d\theta \tag{8}$$

where

$$\Delta = f(x_1', y_1') \exp\left(-\frac{i\omega M^2}{U\beta^2}(x_1'-x_1)\right) - f(x_1, y_1).$$

Now by integration by parts

$$\lim_{z_1 \to +0} \int_{r_1=0}^{\delta} \exp\left[-\frac{i\omega M}{U\beta}\sqrt{r_1^2+z_1^2}\right]\left[\frac{z_1}{(r_1^2+z_1^2)^{3/2}} + \frac{i\omega M}{U\beta}\frac{z_1}{(r_1^2+z_1^2)}\right] r_1\, dr_1$$

$$= \lim_{z_1 \to +0} \left\{-\exp\left[-\frac{i\omega M}{U\beta}\sqrt{\delta^2+z_1^2}\right]\frac{z_1}{\sqrt{\delta^2+z_1^2}} + \exp\left[-\frac{i\omega M}{U\beta}|z_1|\right]\frac{z_1}{|z_1|}\right\} = 1.$$

Moreover

$$\lim_{z_1 \to +0} \int_{r_1=0}^{\delta} \frac{z_1 r_1\, dr_1}{(r_1^2+z_1^2)^{3/2}} = \lim_{z_1 \to +0} \left\{\frac{z_1}{|z_1|} - \frac{z_1}{\sqrt{\delta^2+z_1^2}}\right\} = 1,$$

and

$$\lim_{z_1 \to +0} \int_{r_1=0}^{\delta} \frac{z_1 r_1\, dr_1}{(r_1^2+z_1^2)} = \lim_{z_1 \to +0} \tfrac{1}{2} z_1 \{\log(\delta^2+z_1^2) - \log(z_1^2)\} = 0.$$

Hence

$$\left|\lim_{z_1 \to +0} \Phi(x_1, y_1, z_1, t) + 2\pi f(x_1, y_1)\, e^{i\omega t}\right| \leqslant 2\pi \max\ \Delta\ ,$$

or, since Δ may be made arbitrarily small by taking δ small enough,

$$\Phi(x, y, +0, t) = -2\pi f(x_1, y_1)\, e^{i\omega t}. \tag{9}$$

Thus equation (4) becomes

$$\Phi(x_1, y_1, z_1, t) = -\frac{1}{2\pi}\int\int \Phi(x_1', y_1', +0, t)\frac{\partial}{\partial z_1}\left[\exp\left\{\frac{i\omega M^2(x_1-x_1')}{U\beta^2}-\frac{i\omega MR}{U\beta^2}\right\}\frac{1}{R}\right]dx_1'\,dy_1' \quad (10)$$

The boundary condition in lifting-surface theory is the fact that the vertical velocity, $w = \partial\Phi/\partial z_1$, is known on the part of the plane $z_1 = 0$ occupied by the wing planform. A possible integral equation is therefore

$$w(x_1, y_1, 0, t) = -\frac{1}{2\pi}\lim_{z_1\to 0}\frac{\partial}{\partial z_1}\int\int_{S+W} \Phi(x_1', y_1', +0, t)\frac{\partial}{\partial z_1}\left[\exp\left\{\frac{i\omega M^2(x_1-x_1')}{U\beta^2}-\frac{i\omega MR}{U\beta^2}\right\}\frac{1}{R}\right.dx_1'\,dy_1', \quad (11)$$

where the region of integration is the wing planform S and the wake W.

For small disturbance flow it is known that

$$p-p_\infty = -\rho_\infty\left(\frac{\partial\Phi}{\partial t}+U\frac{\partial\Phi}{\partial x_1}\right). \quad (12)$$

For an isolated plane wing Φ is an odd function of z_1, and it follows from equation (12) and the fact that there can be no pressure difference across the wake, that in W

$$\Phi(x_1', y_1', \pm 0, t) = \Phi(x_{1_T}', y_1', \pm 0, t)\exp\left[-\frac{i\omega}{U}\{x_1'-x_{1_T}'(y_1')\}\right], \quad (13)$$

where the suffix T indicates the value at the trailing edge. Thus, by equations (11) and (13) w is determined by the distribution of Φ over S.

The integral equation (11) has been much used for both steady and oscillating wings, but in this report another will be used, easily derived from equations (10) and (11), which involves integration over S only.

It may be observed that equation (10) holds not only when Φ is the perturbation potential but also if it is any function, simple harmonic in time, satisfying equation (1) and dying away in outgoing waves at infinity. In particular Φ may be replaced by the acceleration potential ψ, defined by

$$\left.\begin{aligned}\psi &= U\frac{\partial\Phi}{\partial x_1}+\frac{\partial\Phi}{\partial t},\\ &= U\frac{\partial\Phi}{\partial x_1}+i\omega\Phi.\end{aligned}\right\} \quad (14)$$

It follows that

$$\Phi = \frac{e^{-i\omega x_1/U}}{U}\int_{-\infty}^{x_1} e^{i\omega\xi_1/U}\,\psi(\xi_1, y_1, z_1)\,d\xi_1, \quad (15)$$

since both Φ and ψ tend to zero as $x_1 \to -\infty$.

On $z_1 = 0$, ψ is in fact proportional to the wing loading, l, defined by

$$l\,e^{i\omega t} = \frac{\text{pressure difference}}{\frac{1}{2}\rho_\infty U^2} = \frac{4\psi(x_1, y_1, +0, t)}{U^2}. \quad (16)$$

When ψ replaces Φ in equation (10), and equations (15) and (16) are applied, the following expression for Φ is obtained

$$\Phi(x_1, y_1, z_1, t) = -\frac{U}{8\pi} e^{i\omega t} \int\limits_{\xi_1=-\infty}^{x_1} e^{i\omega(\xi_1 - x_1)/U} \iint\limits_S l(x_1', y_1') \times$$

$$\times \frac{\partial}{\partial z_1}\left[\exp\left\{\frac{i\omega M^2(\xi_1 - x_1')}{U\beta^2} - \frac{i\omega MR}{U\beta^2}\right\}\frac{1}{R}\right] dx_1'\, dy_1'\, d\xi_1, \qquad (17)$$

where now

$$R = \{(\xi_1 - x_1')^2 + \beta^2 (y_1 - y_1')^2 + \beta^2 z_1^2\}. \qquad (18)$$

Now let K be any large positive number, and x_{max} the greatest value of x_1 in S, then

$$\left| \int\limits_{\xi_1=-\infty}^{-K} e^{i\omega(\xi_1 - x_1)U} \iint\limits_S l(x_1', y_1') \frac{\partial}{\partial z_1}\left[\exp\left\{\frac{i\omega M^2(\xi_1 - x_1')}{U\beta^2} - \frac{i\omega MR}{U\beta^2}\right\}\frac{1}{R}\right] dx_1'\, dy_1'\, d\xi_1 \right|$$

$$\leqslant \beta^2 |z_1| \int\limits_{\xi_1=-\infty}^{-K} \iint\limits_S |l(x_1', y_1')| \left[\frac{1}{R^3} + \frac{\omega M}{U\beta^2}\frac{1}{R^2}\right] dx_1'\, dy_1'\, d\xi_1$$

$$\leqslant \beta^2 |z_1| \int\limits_{\xi_1=-\infty}^{-K} \iint\limits_S |l(x_1', y_1')|\, dx_1'\, dy_1 \left[\frac{1}{(\xi_1 - x_{max})^3} + \frac{\omega M}{U\beta^2}\frac{1}{(\xi - x_{max})^2}\right] d\xi_1$$

$$= \beta^2 |z_1| \left(\iint\limits_S |l(x_1', y_1')|\, dx_1'\, dy_1'\right)\left[\frac{1}{2(K + x_{max})^2} + \frac{\omega M}{U\beta^2}\frac{1}{(K + x_{max})}\right].$$

Thus the integral from $-\infty$ to $-K$ may be made arbitrarily small for K large enough.
Hence

$$\Phi(x_1, y_1, z_1, t) = -\frac{U e^{i\omega t}}{8\pi} \iint\limits_S l(x_1', y_1')\, e^{-i\omega x_1/U}\, e^{-i\omega M^2 x_1'/U\beta^2} \times$$

$$\times \int\limits_{-\infty}^{x_1} e^{i\omega \xi_1/U\beta^2} \frac{\partial}{\partial z_1}\left[\exp\left\{-\frac{i\omega MR}{U\beta^2}\right\}\frac{1}{R}\right] d\xi_1\, dx_1'\, dy_1',$$

since the integral with respect to ξ_1 from $-\infty$ to $-K$ may again be made arbitrarily small for sufficiently large K.

A more convenient form is obtained by changing the variable ξ_1 to ξ, where $\xi_1 = \xi + x_1'$. Then

$$\Phi(x_1, y_1, z_1, t) = -\frac{U\, e^{i\omega t}}{8\pi} \iint_S l(x_1', y_1')\, e^{-i\omega(x_1 - x_1')/U} \times$$

$$\times \int_{-\infty}^{x_1 - x_1'} e^{i\omega\xi/U\beta^2} \frac{\partial}{\partial z_1}\left[\exp\left(-\frac{i\omega M R}{U\beta^2}\right)\frac{1}{R}\right] d\xi\, dx_1'\, dy_1', \tag{19}$$

where

$$R = \{\xi^2 + \beta^2 (y_1 - y_1')^2 + \beta^2 z_1^2\}^{\frac{1}{2}}. \tag{20}$$

The perturbation velocity caused by l is now obtained as the gradient of Φ as given by equation (19). In particular

$$w(x_1, y_1, 0, t) = -\frac{U\, e^{i\omega t}}{8\pi} \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \iint_S l(x_1', y_1')\, e^{-i\omega(x_1 - x_1')/U} \times$$

$$\times \int_{-\infty}^{x_1 - x_1'} e^{i\omega\xi/U\beta^2} \frac{\partial}{\partial z_1}\left[\exp\left(-\frac{i\omega M R}{U\beta^2}\right)\frac{1}{R}\right] d\xi\, dx_1'\, dy_1' \tag{21}$$

The cross-stream velocity component, v, may be obtained by differentiating with respect to y_1.

Alternatively, provided ω is not zero, equation (21) may be written

$$\frac{w(x_1, y_1, 0, t)}{U} = -\frac{e^{i\omega t}}{8\pi} \iint_S l(x_1', y_1')\, K(x_1 - x_1', y_1 - y_1')\, dx_1'\, dy_1', \tag{22}$$

where the kernel function, K, is given by

$$K(x_1 - x_1', y_1 - y_1') = \lim_{z_1 \to 0} \frac{\partial^2}{\partial z_1^2} \int_{-\infty}^{(x_1 - x_1')} e^{-i\omega(x_1 - x_1')/U}\, e^{i\omega\xi/U\beta^2} \times \exp\left(-\frac{i\omega M R}{U\beta^2}\right)\frac{1}{R}\, d\xi, \tag{23}$$

where R is as in equation (20).

The changes in the positions of the differentiation and the limit operation between equations (21) and (23) may be justified by the uniform convergence of the infinite integral in equation (21) and the convergence of that in equation (23), and by observing that we may suppose that the only singularities in l are along the leading edge and may be removed by a simple change of streamwise variable (e.g. as in equation (61)).

If $\omega = 0$ the infinite integral in equation (21) is easily integrated. (*See*, for example, Ref. 1.)

It may be observed that, unless $\omega = 0$, the two-dimensional forms of equations (10) and (23) may be obtained by spanwise integration with respect to y_1', using the formula

$$H_0^{(2)}(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} e^{-ix\cosh t}\,dt\,.$$

(Ref. 7, p. 80)

2.2. *Transformation of the Integral Equation.*

There appears to be no method of expressing the kernel K, as given by equation (23), as a simple combination of known functions. It is possible to calculate K numerically from equation (23), but the process is expedited by using a transformed equation for K, due to Watkins, Woolston and Cunningham[2], which requires numerical integration over only a finite range. The following method of deriving the altered form of K appears somewhat simpler than that used in Ref. 2.

It is first convenient to change to non-dimensional co-ordinates by division by a typical length, d. Thus

$$\left.\begin{aligned} x_1 = xd\,, \qquad y_1 = yd\,, \qquad z_1 = zd\,, \\ x_1' = x'd\,, \qquad y_1' = y'd\,. \end{aligned}\right\} \tag{24}$$

It is also convenient to define ν, the frequency parameter, by

$$\nu = \omega d/U\,, \tag{25}$$

and to put

$$\left.\begin{aligned} x_0 = x-x'\,, \\ y_0 = y-y'\,. \end{aligned}\right\} \tag{26}$$

Then equations (22) and (23) become

$$\frac{w(x,y,0,t)}{U} = -\frac{e^{i\omega t}}{8\pi}\iint_S l(x',y')\,d^2\,K(x_0,y_0)\,dx'\,dy'$$

where

$$K(x_0,y_0) = \frac{1}{d^2}\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-i\nu x_0}\int_{-\infty}^{x_0}\frac{\exp\left[\frac{i\nu}{\beta^2}(\xi - M\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2})\right]}{\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}}\,d\xi \tag{27}$$

Now consider the part of K involving integration over an infinite range, and change the variable to ϑ by putting

$$\beta\vartheta = \xi - M\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}\,, \tag{28}$$

so that

$$\xi = \frac{1}{\beta}\left[\vartheta + M\sqrt{\vartheta^2+\beta^2 y_0^2+\beta^2 z^2}\right]. \tag{29}$$

Then

$$\int_{-\infty}^{0} \frac{\exp\left\{\frac{iv}{\beta^2}(\xi - M\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2})\right\}}{\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}}\, d\xi$$

$$= \int_{-\infty}^{-M\sqrt{y_0^2+z^2}} \exp\left(\frac{iv\vartheta}{\beta}\right)\frac{1}{\sqrt{\vartheta^2+\beta^2 y_0^2+\beta^2 z^2}}\, d\vartheta$$

$$= \int_0^{\infty} \exp\left(-iv\sqrt{y_0^2+z^2}\,\tau\right)\frac{d\tau}{\sqrt{1+\tau^2}} - \int_0^{M/\beta} \exp\left(-iv\sqrt{y_0^2+z^2}\,\tau\right)\frac{d\tau}{\sqrt{1+\tau^2}}. \tag{30}$$

Now from Ref. 7 (p. 172)

$$\int_0^{\infty} \frac{\cos(v\sqrt{y_0^2+z^2}\,\tau)}{\sqrt{1+\tau^2}}\, d\tau = K_0(v\sqrt{y_0^2+z^2}), \tag{31}$$

and from Ref. 7 (p. 332)

$$\int_0^{\infty} \frac{\sin(v\sqrt{y_0^2+z^2}\,\tau)}{\sqrt{1+\tau^2}}\, d\tau = \frac{\pi}{2}\left[I_0(v\sqrt{y_0^2+z^2}) - L_0(v\sqrt{y_0^2+z^2})\right]$$

when I_0 and K_0 are Bessel functions in the usual notation, and L_0 is the modified Struve function (Ref. 7, p. 329)

$$L_0(x) = \frac{2}{\pi}\left(x + \frac{x^3}{3^2} + \frac{x^5}{3^2\,5^2} + \frac{x^7}{3^2\,5^2\,7^2} + \dots\right), \tag{33}$$

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\,4^2} + \frac{x^6}{2^2\,4^2\,6^2} + \dots\ . \tag{34}$$

A useful transformation may be obtained by considering the integral of the function $e^{i\lambda z}\ \sqrt{1+z^2}$ round a closed contour in the complex z-plane consisting of the positive real axis, the positive imaginary axis indented at $z = i$ and quadrant of an indefinitely large circle. It follows that

$$\int_0^\infty \frac{\cos\lambda x}{\sqrt{1+x^2}}\,dx = \int_1^\infty \frac{e^{-\lambda y}}{\sqrt{y^2-1}}\,dy\,, \tag{35}$$

and

$$\int_0^\infty \frac{\sin\lambda x}{\sqrt{1+x^2}}\,dx = \int_0^1 \frac{e^{-\lambda y}}{\sqrt{1-y^2}}\,dy\,. \tag{36}$$

Equation (35) is merely a well known alternative expression for $K_0(x)$ (cf. equation (31)). Equation (36) leads to the alternative form for equation (32)

$$\frac{\pi}{2}\left[\, I_0\left(v\sqrt{y_0^2+z^2}\right)-L_0\left(v\sqrt{y_0^2+z^2}\right)\right] = \int_0^1 \frac{e^{-v\sqrt{y_0^2+z^2}\,y}}{\sqrt{1-y^2}}\,dy$$

$$= \int_0^{\pi/2} e^{-v\sqrt{y_0^2+z^2}\cos\theta}\,d\theta\,. \tag{37}$$

When the terms are collected together it is found that

$$\int_{-\infty}^{0} \frac{\exp\left[\dfrac{iv}{\beta^2}\left(\xi-M\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}\right)\right]}{\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}}\,d\xi$$

$$= K_0\left(v\sqrt{y_0^2+z^2}\right)-\frac{\pi i}{2}\left\{\, I_0\left(v\sqrt{y_0^2+z^2}\right)-L_0\left(v\sqrt{y_0^2+z^2}\right)\right\}-\int_0^{M/\beta}\exp\left(-iv\sqrt{y_0^2+z^2}\,\tau\right)\frac{d\tau}{\sqrt{1+\tau^2}}\,, \tag{38}$$

and it follows that

$$\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-ivx_0}\int_{-\infty}^{0} \frac{\exp\left[\dfrac{iv}{\beta^2}\left(\xi-M\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}\right)\right]}{\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}}\,d\xi$$

$$= \frac{v\,e^{-ivx_0}}{|y_0|}\left[\,-K_1\left(v\,|y_0|\right)-\frac{\pi i}{2}\left\{I_1\left(v\,|y_0|\right)-L_1\left(v\,|y_0|\right)-\frac{2}{\pi}\right\}\right.$$

$$\left.+\,i\int_0^{M/\beta}\exp\left(-iv\,|y_0|\,\tau\right)\frac{\tau}{\sqrt{1+\tau^2}}\,d\tau\,\right], \tag{39}$$

$$
= \frac{v\, e^{-ivx_0}}{|y_0|}\Bigg[-K_1(v\,|y_0|) - \frac{\pi i}{2}\{I_1(v\,|y_0|) - L_1(v\,|y_0|)\}
$$

$$
+ \frac{i}{\beta}\exp\left(-\frac{ivM}{\beta}|y_0|\right) - v\,|y_0| \int\limits_0^{M/\beta} \exp(-iv\,|y_0|\,\tau)\sqrt{1+\tau^2}\, d\tau \Bigg],
$$

since

$$
K_0'(x) \equiv -K_1(x), \tag{41}
$$

$$
I_0'(x) \equiv I_1(x) \equiv \frac{x}{2} + \frac{x^3}{2^2\,4} + \frac{x^5}{2^2\,4^2\,6} + \dots, \tag{42}
$$

and

$$
L_0'(x) = \frac{2}{\pi} + L_1(x), \tag{43}
$$

where

$$
L_1(x) = \frac{2}{\pi}\left(\frac{x^2}{3} + \frac{x^2}{3^2\,5} + \frac{x^6}{3^2\,5^2\,7} + \dots\right). \tag{44}
$$

The remaining part of the kernel K is

$$
\lim_{z\to 0} e^{-ivx_0} \frac{\partial^2}{\partial z^2} \int\limits_0^{x_0} \frac{\exp\left[\frac{iv}{\beta^2}\left(\xi - M\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}\right)\right]}{\sqrt{\xi^2+\beta^2 y_0^2+\beta^2 z^2}}\, d\xi
$$

$$
= -\beta^2 e^{-ivx_0} \int\limits_0^{x_0} \exp\left[\frac{iv}{\beta^2}\left(\xi - M\sqrt{\xi^2+\beta^2 y_0^2}\right)\right] \left\{\frac{1}{(\xi^2+\beta^2 y_0^2)^{3/2}} + \frac{ivM}{\beta^2}\,\frac{1}{(\xi^2+\beta^2 y_0^2)}\right\} d\xi, \tag{45}
$$

$$= -\frac{e^{-i\nu x_0}}{y_0^2} \frac{x_0}{\sqrt{x_0^2+\beta^2 y_0^2}} \exp\left\{\frac{i\nu}{\beta^2}(x_0 - M\sqrt{x_0^2+\beta^2 y_0^2})\right\}$$

$$+\frac{e^{-i\nu x_0}}{y_0^2}\int_0^{x_0} \exp\left\{\frac{i\nu}{\beta^2}(\xi - M\sqrt{\xi^2+\beta^2 y_0^2})\right\}\left[\frac{i\nu}{\beta^2}\frac{\xi}{\sqrt{\xi^2+\beta^2 y_0^2}} - \frac{i\nu M}{\beta^2}\right] d\xi$$

$$\left(\text{on integrating by parts, observing that } \int \frac{d\xi}{(\xi^2+\beta^2 y_0^2)^{3/2}} = \frac{1}{\beta^2 y_0^2}\frac{\xi}{\sqrt{\xi^2+\beta^2 y_0^2}}\right)$$

$$= -\frac{e^{-i\nu x_0}}{My_0^2}\left\{1+\frac{Mx_0}{\sqrt{x_0^2+\beta^2 y_0^2}}\right\}\exp\left\{\frac{i\nu}{\beta^2}(x_0 - M\sqrt{x_0^2+\beta^2 y_0^2})\right\}$$

$$+\frac{e^{-i\nu x_0}}{My_0^2}\exp\left\{\frac{-i\nu M|y_0|}{\beta}\right\}+\frac{i\nu\, e^{-i\nu x_0}}{My_0^2}\int_0^{x_0}\exp\left\{\frac{i\nu}{\beta^2}(\xi - M\sqrt{\xi^2+\beta^2 y_0^2})\right\}d\xi . \tag{46}$$

K is found by adding equations (40) and (46):

$$d^2 K(x_0, y_0) = \frac{e^{-i\nu x_0}}{y_0^2} \quad -\nu\,|y_0|\, K_1(\nu\,|y_0|) - \frac{\pi i}{2}\nu\,|y_0|\,\{I_1(\nu\,|y_0|) - L_1(\nu\,|y_0|)\}$$

$$+\frac{\beta+i\nu M|y_0|}{M\beta}\exp\left(-\frac{i\nu M|y_0|}{\beta}\right) - \nu^2 y_0^2\int_0^{M/\beta}\exp(-i\nu\,|y_0|\,\tau)\sqrt{1+\tau^2}\,d\tau$$

$$-\frac{1}{M}\left(1+\frac{Mx_0}{\sqrt{x_0^2+\beta^2 y_0^2}}\right)\exp\left\{\frac{i\nu}{\beta^2}(x_0 - M\sqrt{x_0^2+\beta^2 y_0^2})\right\} \;+$$

$$+\frac{iv}{M}\int_0^{x_0} \exp\left\{\frac{iv}{\beta^2}(\xi - M\sqrt{\xi^2+\beta^2 y_0^2})\right\} d\xi \quad . \tag{47}$$

Equation (47) is the expression for K given by Watkins *et al*[2]. An alternative form, suitable for use when M is small or zero may be obtained by adding equations (39) and (45). For steady wings

$$d^2 K(x_0, y_0) = -\frac{1}{y_0^2}\left[1+\frac{x_0}{\sqrt{x_0^2+\beta^2 y_0^2}}\right] ; \tag{48}$$

this follows from equations (47), (42) and (44) and the fact that $xK_1(x)$ tends to unity as x tends to zero, (or, alternatively, directly from the non-dimensionalised form of equation (21) with $\omega = 0$).

The properties of K are discussed in Ref. 2. In particular it is shown that

$$d^2 K(x_0, y_0) = e^{-ivx_0}\left[-\frac{1}{y_0^2}\left\{1+\frac{x_0}{\sqrt{x_0^2+\beta^2 y_0^2}}\right\}\right.$$

$$+\frac{iv}{\sqrt{x_0^2+\beta^2 y_0^2}}-\frac{v^2}{2}\log\frac{v(\sqrt{x_0^2+\beta^2 y_0^2}-x_0)}{2(M-1)}$$

$$\left.+(\text{terms which remain finite when } x_0 \text{ or } y_0 \text{ tend to zero})\right] . \tag{49}$$

The strongly singular nature of K is apparent from the first term in equation (49). Apart from the factor e^{-ivx_0} this singular term is the same as for the steady wing and is treated in the same way. (*See* Section 3.2, below.) The next two terms in equation (49) are also singular but less strongly so, and complicate the integration only through the logarithmic terms in the spanwise integration.

2.3. *Boundary Conditions.*

If the wing surface vibrates according to the equation

$$z_1 = q(x_1, y_1)\, e^{i\omega t} , \tag{50}$$

then the boundary condition on the planform is

$$\frac{w}{U} = \left[\frac{\partial q}{\partial x_1} + \frac{i\omega}{U} q \right] e^{i\omega t},$$

that is

$$\frac{w}{U} = \left[\frac{\partial}{\partial x} \left(\frac{q}{d} \right) + iv \frac{q}{d} \right] e^{i\omega t}. \tag{51}$$

In the particular case of rigid pitching about an axis $x_0 d$ downstream of the origin

$$z = -\alpha_0 (x_1 - x_0 d) e^{i\omega t}, \tag{52}$$

so that

$$\frac{w}{U} = -\alpha_0 e^{i\omega t} [1 + iv (x - x_0)]. \tag{53}$$

If q, in equation (50) is real the wing is performing oscillations of the standing wave type, but travelling waves may also be treated by taking q to be complex.

3. *Extension of Multhopp's Lifting-Surface Theory.*

In Ref. 6 Multhopp described a lifting-surface theory for wings in steady flight at subsonic speeds, in which the kernel of the integral equation was simply that given by equation (48). The distinguishing features of the method are the use of particular chordwise distributions of lift with their corresponding influence functions and the chordwise and spanwise locations of the points at which the upwash is evaluated. Garner[2] (1952) described an extension of this to wings oscillating with low frequency parameter by retaining terms linear in frequency, thus introducing additional influence functions. In this Section a further extension using the complete kernel is described. Multhopp's basic ideas of obtaining influence functions and choice of solving points remain unchanged.

3.1. *Calculation of Influence Functions.*

The integral equation is

$$\frac{w(x, y, 0, t)}{U} = -\frac{e^{i\omega t}}{8\pi} \int \int l(x', y') \{d^2 K(x_0, y_0)\} \, dx' \, dy' \tag{54}$$

where the kernel K may be expressed by equations (27) or (47) (or in other ways, some of which may be found in Ref. 2). In this report equation (47) will be used.

It is convenient to write

$$\bar{w}(x, y, 0) e^{i\omega t} = e^{ivx} (wx, y, 0, t), \tag{55}$$

and

$$\bar{l}(x, y) = e^{ivx} l(x, y), \tag{56}$$

and also to take the origin of co-ordinates so that the wing tips are in the planes $y_1 = \pm s$. (This does not imply that the planform is assumed to be symmetrical.)

Then equation (54) may be re-written

$$\frac{\bar{w}}{U}(x,y,0) = -\frac{1}{8\pi}\int_{y'=-\frac{s}{d}}^{y'=+\frac{s}{d}} \int_{x'=x_L(y')}^{x'=x_T(y')} \frac{\bar{l}(x',y')}{y_0^2}\left[y_0^2\, d^2\, e^{i\nu x_0}\, K(x_0,y_0)\right] dx'\, dy'. \tag{57}$$

It is also convenient to change the spanwise co-ordinates by putting

$$\eta = y_1/s = y\, d/s, \tag{58}$$

$$\eta' = y_1'/s = y'\, d/s, \tag{59}$$

so that

$$y_0 = \frac{s}{d}(\eta - \eta'), \tag{60}$$

and to change the chordwise variable of integration to ϕ by the relation

$$x_1' = x_{1_L}(y_1') + \tfrac{1}{2} c(y_1')(1-\cos\phi), \tag{61}$$

that is

$$x' = x_L(y') + \frac{c(y')}{2d}(1-\cos\phi), \tag{62}$$

so that $\phi = 0$ on the leading edge and $\phi = \pi$ on the trailing edge. Then equation (57) becomes

$$\frac{\bar{w}}{U} = -\frac{1}{8\pi}\frac{d}{s}\int_{\eta'=-1}^{+1} \frac{1}{(\eta-\eta')^2}\int_{\phi=0}^{\pi} \bar{l}(x',y')\left[y_0^2\, d^2\, e^{i\nu x_0}\, K(x_0, y_0)\right]\frac{c(y')}{2d}\sin\phi\, d\phi\, d\eta'. \tag{63}$$

In accordance with Multhopp's approach the loading is now assumed to be of the form

$$\bar{l}(x',y') = \frac{8s}{\pi\, c(y')}\sum_{q=1}^{\infty} \Gamma_q(\eta')\, \Psi_q(\phi) \tag{64}$$

where the function Γ_q are unknown, and

$$\left.\begin{aligned}
\Psi_1(\phi) &= \cot\tfrac{1}{2}\phi, \\
\Psi_2(\phi) &= \cot\tfrac{1}{2}\phi - 2\sin\phi, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\Psi_q(\phi) &= \cot\tfrac{1}{2}\phi - 2\sum_{r=1}^{q-1}\sin r\phi \quad (q \geqslant 2).
\end{aligned}\right\} \tag{65}$$

It follows that

$$\frac{\overline{w}}{U}(x,y,0) = \frac{1}{2\pi} \int_{\eta'=-1}^{+1} \frac{1}{(\eta-\eta')^2} \sum_{q=1}^{\infty} \Gamma_q(\eta') F_q \, d\eta' , \tag{66}$$

where

$$F_q = \frac{1}{\pi} \int_{\phi=0}^{\pi} [-y_0^2 d^2 e^{i\nu x_0} K(x_0, y_0)] \Psi_q(\phi) \sin\phi \, d\phi . \tag{67}$$

The functions F_q are the 'influence functions'; they express the upwash at the point $(x,y,0)$ due to the part of the lift distribution on the elementary strip $\eta' < \eta < \eta' + d\eta'$. The choice of the points (x,y) and the values of η are discussed in Section 3.2. The integral in equation (66) has a strong singularity at $\eta' = \eta$ which has to be accounted for by taking the principal part as in the theory for steady flow.

The expression for $[-y_0^2 d^2 e^{i\nu x_0} K(x_0, y_0)]$, which follows from equation (47) falls naturally, from an algebraic point of view, into four parts.

$$[-y_0^2 d^2 e^{i\nu x_0} K(x_0, y_0)] = G_1 + G_2 + G_3 + G_4 \tag{68}$$

where

$$G_1 = \nu|y_0| K_1(\nu|y_0|) + \frac{\pi i}{2} \nu|y_0| \{I_1(\nu|y_0|) - L_1(\nu|y_0|)\}$$

$$- \frac{iM\nu|y_0| + \beta}{M\beta} \exp\left(-\frac{iM\nu|y_0|}{\beta}\right), \tag{69}$$

$$G_2 = \nu^2 y_0^2 \int_0^{M/\beta} \sqrt{1+\tau^2} \exp(-i\nu|y_0|\tau) \, d\tau , \tag{70}$$

$$G_3 = \frac{1}{M}\left[1 + \frac{Mx_0}{\sqrt{x_0^2 + \beta^2 y_0^2}}\right] \exp\left[\frac{i\nu}{\beta^2}(x_0 - M\sqrt{x_0^2+\beta^2 y_0^2})\right], \tag{71}$$

$$G_4 = -\frac{i\nu}{M} \int_0^{x_0} \exp\left[\frac{i\nu}{\beta^2}(t - M\sqrt{t^2+\beta^2 y_0^2})\right] dt . \tag{72}$$

An alternative form for G_4 may be obtained by changing the variable of integration by putting

$$\beta\vartheta = t - M\sqrt{t^2+\beta^2 y_0^2},$$

After some manipulation it appears that

$$G_4 = \frac{iM\nu|y_0| + \beta}{\beta M} \exp\left(-\frac{i\nu M|y_0|}{\beta}\right) -$$

$$-\left\{\frac{1}{M}+\frac{iv}{\beta^2}\left[\sqrt{x_0^2+\beta^2 y_0^2}-Mx_0\right]\right\}\exp\left[\frac{iv}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2 y_0^2})\right]$$

$$-v^2 y_0^2 \int\limits_{-[x_0-M\sqrt{x_0^2+\beta^2 y_0^2}]/\beta^2|y_0|}^{M/\beta} \exp(-iv|y_0|\tau)\sqrt{1+\tau^2}\,d\tau\,. \tag{73}$$

This form is more suitable for small M since the terms in G_1 and G_3 which become infinite as M tends to zero are cancelled by terms in G_4.

The influence functions F_q depend on the position of the point (x,y) relative to the strip $\eta' < \eta < (\eta'+d\eta')$, $x_L(y') < x < x_T(y')$. The parameters X and Y are therefore defined by

$$X = \frac{x_1-x_{1_L}(y_1')}{c(y_1')} = \frac{d}{c(y')}(x-x_L(y'))\,, \tag{74}$$

$$Y = \frac{\beta\,|y_1-y_1'|}{c(y_1')} = \beta\,|y_0|\frac{d}{c(y')}, \tag{75}$$

so that

$$|y_0| = \frac{c(y')}{d}\,\frac{Y}{\beta}, \tag{76}$$

and

$$x_0 = x-x' = x-x_L(y')-\frac{c(y')}{2d}(1-\cos\phi)$$

$$= \frac{c(y')}{d}\left[X-\tfrac{1}{2}+\tfrac{1}{2}\cos\phi\right]. \tag{77}$$

In terms of these new parameters G_1, G_2, G_3 and G_4 are given by

$$G_1 = \bar{Y}\,K_1(\bar{Y})+\frac{\pi i}{2}\bar{Y}\{I_1(\bar{Y})-L_1(\bar{Y})\}-\frac{iM\,\bar{Y}+\beta}{M\beta}\exp\left(-\frac{iM\,\bar{Y}}{\beta}\right), \tag{78}$$

$$G_2 = \bar{Y}^2\int\limits_0^{M/\beta}\sqrt{1+\tau^2}\exp(-i\bar{Y}\tau)\,d\tau\,, \tag{79}$$

$$G_3 = \frac{1}{M}\left[1+\frac{M(X-\frac{1}{2}+\frac{1}{2}\cos\phi)}{\sqrt{(X-\frac{1}{2}+\frac{1}{2}\cos\phi)^2+Y^2}}\right]\times$$

$$\times\exp\left[\frac{iv}{\beta^2}\frac{c(y')}{d}\left\{(X-\tfrac{1}{2}+\tfrac{1}{2}\cos\phi)-M\sqrt{(X-\tfrac{1}{2}+\tfrac{1}{2}\cos\phi)^2+Y^2}\right\}\right], \tag{80}$$

$$G_4 = -\frac{iv}{M}\int\limits_0^{\frac{c(y')}{d}(X-\frac{1}{2}+\frac{1}{2}\cos\phi)}\exp\left[\frac{iv}{\beta^2}\left(t-M\sqrt{t^2+\left(\frac{c(y')\,Y}{d}\right)^2}\right)\right]dt\,, \tag{81}$$

where

$$\bar{Y} = \frac{vY}{\beta}\frac{c(y')}{d}. \tag{82}$$

The influence functions may now be written

$$F_q = \frac{1}{\pi}\int_0^\pi (G_1+G_2+G_3+G_4)\,\Psi_q \sin\phi\, d\phi\,, \tag{83}$$

or, by equation (65)

$$F_q(X,Y) = \frac{1}{\pi}\int_0^\pi (G_1+G_2+G_3+G_4)\,(\cos(q-1)\,\phi+\cos q\phi)\, d\phi\,. \tag{84}$$

Since G_1 and G_2 do not depend on ϕ

$$F_1 = G_1+G_2+\frac{1}{\pi}\int_0^\pi (G_3+G_4)\,(1+\cos\phi)\, d\phi\,, \tag{85}$$

$$F_q = \frac{1}{\pi}\int_0^\pi (G_3+G_4)\,(\cos(q-1)\,\phi+\cos q\phi)\, d\phi\,, \qquad (q \geqslant 2)\,. \tag{86}$$

The repeated integrals in the terms in G_4 may be avoided by integration by parts. Thus for F_1

$$\frac{1}{\pi}\int_0^\pi G_4\,(1+\cos\phi)\, d\phi$$

$$= \frac{1}{\pi}\Big[(\phi+\sin\phi)\, G_4\Big]_0^\pi - \frac{1}{\pi}\int_0^\pi (\phi+\sin\phi)\frac{\partial G_4}{\partial\phi}\, d\phi$$

$$= -\frac{iv}{M}\int_0^{\frac{c(y')}{d}(X-1)} \exp\left[\frac{iv}{\beta^2}\left(t - M\sqrt{t^2+\left(\frac{c(y')\,Y}{d}\right)^2}\right)\right] dt$$

$$-\frac{iv}{M}\,\frac{1}{\pi}\,\frac{c(y')}{2d}\int_0^\pi (\phi+\sin\phi)\exp\left[\frac{iv}{\beta^2}\,\frac{c(y')}{d}\left\{(X-\tfrac{1}{2}+\tfrac{1}{2}\cos\phi) - M\sqrt{(X-\tfrac{1}{2}+\tfrac{1}{2}\cos\phi)^2+Y^2}\right\}\right]$$

$$\times \sin\phi\, d\phi\,, \tag{87}$$

and similarly for $F_2, F_3, \ldots$.

The calculation of the influence functions is a straightforward task, requiring only standard methods of numerical analysis. The functions K_1, I_1 and L_1, which occur in G_1, have been tabulated, but for use in a computer it may well be more convenient to use expressions for them as infinite series or as integrals. For example $I_1 - L_1$ may be calculated from the series in equation (42) and (44) or from an integral expression such as

$$I_1(x) - L_1(x) = \frac{2}{\pi}\left\{1 - \int_0^{\pi/2} e^{-x\cos\theta}\cos\theta\, d\theta\right\}, \tag{88}$$

which follows from equation (37). Expressions for K_1, may be found, for example, in Ref. 7. The function G_2 requires only simple numerical integration. The only complication arises from the fact that when Y is small G_3 changes rapidly when $\cos\phi$ is near to $(1-2X)$, and a close spacing of values of the variable of integration may be necessary.

When $Y = 0$, and $0 < X < 1$, the influence functions are easily evaluated by observing that

$$\left.\begin{aligned} G_1 + G_2 + G_3 + G_4 &= 2 \text{ for } \phi < \cos^{-1}(1-2X), \\ &= 0 \text{ for } \phi > \cos^{-1}(1-2X). \end{aligned}\right\} \tag{89}$$

Thus, by equation (84)

$$F_q(X,0) = \frac{2}{\pi}\int_0^{\cos^{-1}(1-2X)} (\cos(q-1)\phi + \cos q\phi)\, d\phi . \tag{90}$$

In particular

$$\left.\begin{aligned} F_1(X,0) &= \frac{2}{\pi}\cos^{-1}(1-2X) + \frac{4}{\pi}X^{\frac{1}{2}}(1-X)^{\frac{1}{2}} \\ F_2(X,0) &= \frac{8}{\pi}X^{\frac{1}{2}}(1-X)^{3/2}, \\ F_3(X,0) &= \frac{8}{\pi}X^{\frac{1}{2}}(1-X)^{3/2}\left(1 - \frac{8X}{3}\right), \end{aligned}\right\} \tag{91}$$

and in general, for $q \geqslant 2$

$$F_q(X,0) = \frac{2}{\pi}\left\{\frac{\sin[(q-1)\cos^{-1}(1-2X)]}{q-1} + \frac{\sin[q\cos^{-1}(1-2X)]}{q}\right\}. \tag{92}$$

If $Y = 0$ and $X < 0$ then $(G_1 + G_2 + G_3 + G_4) = 0$, while if $Y = 0$ and $X > 1$ then $(G_1 + G_2 + G_3 + G_4) = 2$, but in practice these cases are not required.

3.2. *The Solving Points and Simultaneous Equations.*

The distribution of solving points was taken to be the same as that used by Multhopp[6] for steady flight, and by Garner[1] for low frequency oscillations. It seemed reasonable to suppose that the satisfactory nature of this distribution would carry over into part of the finite frequency range and in fact no difficulty attributable to the selection of solving points has been encountered.

According to Multhopp's scheme the unknown lift distributions are assumed to be defined by their values at the spanwise stations defined by

$$\frac{y_1'}{s} = \eta_n = \frac{\sin n\pi}{m+1}, \tag{93}$$

where m is an odd integer, and takes the integral values from $-\frac{1}{2}(m-1)$ to $+\frac{1}{2}(m-1)$. In practice the smallest possible value of m seems to be 7. In effect taking this distribution is equivalent to assuming a change of spanwise variable to θ, defined by

$$\eta = \cos\theta \, (0 \leqslant \theta \leqslant \pi), \tag{94}$$

and taking equal intervals in θ.

The unknowns are then the values of Γ_q (equation (64)), and we may define $\Gamma_{q,n}$ as the value of $\Gamma_q(\eta)$ at $\eta = \eta_n$. Since it is assumed that the Γ_q's are zero at the wing tips there are m unknowns for each Γ_q.

The points at which the upwash is to be evaluated are taken to lie in the spanwise positions specified by equation (93). In a calculation it is necessary to include only a finite number of terms, N say, in the lift distribution as postulated in equation (64), and the number of points at each spanwise position is also taken to be N. There are therefore mN solving points and mN unknowns $\Gamma_{q,n}$, $1 \leqslant q \leqslant N$, $-\frac{1}{2}(m-1) \leqslant n \leqslant +\frac{1}{2}(m-1)$. Thus the solving points are taken to lie on the lines

for

$$\left.\begin{aligned} \frac{y_1}{s} &= \eta_\nu = \sin\frac{\nu\pi}{m+1}, \\ -\tfrac{1}{2}(m-1) &\leqslant \nu \leqslant \tfrac{1}{2}(m-1). \end{aligned}\right\} \tag{95}$$

The positions of the N solving points on each line $\eta = \eta_\nu$ are defined as

$$x_1 = x_{1_L}(\eta_\nu) + \tfrac{1}{2}c(\eta_\nu)\left\{1 - \cos\frac{2\pi p}{2N+1}\right\}, \qquad p = 1,2,\ldots N$$

which may also be written

$$x_1 = x_{1_L}(\eta_\nu) + \tfrac{1}{2}c(\eta_\nu) \quad 1 + \cos\frac{(2r-1)\pi}{2N+1} \quad . \qquad r = 1,2,\ldots N \tag{96}$$

In practice N is taken to be small, usually 2, 3 or 4.

For $N = 2$, there are 2 points on each chord

$$x_1 = x_{1_L}(\eta_\nu) + 0{\cdot}9045\, c(\eta_\nu), \tag{97}$$

and

$$x_1 = x_{1_L}(\eta_\nu) + 0{\cdot}3455\, c(\eta_\nu). \tag{98}$$

If $N = 3$ the two numbers 0·9045 and 0·3455 are replaced by the three numbers 0·9505, 0·6113, and 0·1883, and so on.

In calculations non-dimensional co-ordinates are used, and the positions are defined by

$$x = x_\nu^{(r)} = x_L(\eta_\nu)+\tfrac{1}{2}\frac{c(\eta_\nu)}{d}\left\{1+\cos\frac{(2r-1)\pi}{2N+1}\right\}. \qquad r = 1,2,\ldots N \tag{99}$$

The corresponding values of X and Y to be used in calculating the influence functions are thus by equations (74) and (75)

$$X_{\nu,n}^{(r)} = \frac{d}{c(\eta_n)}\left[x_\nu^{(r)}-x_L(\eta_n)\right], \tag{100}$$

and

$$Y_{\nu,n} = \frac{\beta s}{c(\eta_n)}\left|\eta_\nu-\eta_n\right|. \tag{101}$$

The suffix ν denotes the spanwise position at which the upwash is to be evaluated; the suffix n denotes the spanwise position of the lifting elements which cause the upwash.

Now it is shown in Appendix 1 of Ref. 6 that the singular integral in equation (66) has to be evaluated by a principal value by the formula

$$\frac{1}{2\pi}\int_{-1}^{+1}\frac{f(\eta')}{(\eta-\eta')^2}\,d\eta' = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left\{\int_{-1}^{\eta-\varepsilon}\frac{f(\eta')}{(\eta-\eta')^2}\,d\eta'+\int_{\eta+\varepsilon}^{1}\frac{f(\eta)}{(\eta-\eta')^2}\,d\eta'-\frac{2f(\eta)}{\varepsilon}\right\} \tag{102}$$

where $f(\eta)$ represents the numerator in the integrand of equation (66).

If the variable of integration is changed by putting

$$\eta' = \cos\theta', \tag{103}$$

and it is assumed that in equation (102)

$$f(\eta') = \sum_{1}^{\infty} a_p \sin p\theta \tag{104}$$

where the a_p's are constants, then it follows after some analysis that equation (102) may be evaluated numerically according to the formula

$$\frac{1}{2\pi}\int_{-1}^{+1}\frac{f(\eta')}{(\eta_\nu-\eta')^2}\,d\eta' = -b_{\nu\nu}f(\eta_\nu)+\sum_{\substack{n=-\frac{(m-1)}{2}\\ n\neq\nu}}^{+\frac{(m-1)}{2}} b_{\nu n}\, f(\eta_n)$$

where

$$b_{\nu\nu} = \frac{m+1}{4\cos\left(\dfrac{\nu\pi}{m+1}\right)}$$

$$b_{\nu n} = \frac{\cos\left(\frac{n\pi}{m+1}\right)}{(m+1)\left[\sin\left(\frac{n\pi}{m+1}\right) - \sin\left(\frac{\nu\pi}{m+1}\right)\right]^2} \qquad (|\nu-n| \text{ even})$$

$$= 0 \qquad |\nu - n| \text{ odd}. \tag{105}$$

It follows that equation (66) may be replaced by

$$\frac{\bar{w}}{U}(x_\nu^{(r)}, y_\nu, 0) = -b_{\nu\nu} \sum_{q=1}^{N} \Gamma_{q,\nu} F_q(X_{\nu\nu}^{(r)}, Y_{\nu\nu})$$

$$+ \sum_{\substack{n = -\frac{(m-1)}{2} \\ n \neq \nu}}^{n = +\frac{(m-1)}{2}} b_{\nu n} \sum_{q=1}^{N} \Gamma_{q,n} F_q(X_{\nu n}^{(r)}, Y_{\nu n}). \tag{106}$$

By equations (100) and (101),

$$X_{\nu\nu}^{(r)} = \tfrac{1}{2}\left(1 + \cos\frac{(2r-1)\pi}{2N+1}\right), \; Y_{\nu\nu} = 0. \tag{107}$$

In general there are mN equations of the sort (106), one for each pair of the N values of r $(1 \leqslant r \leqslant N)$, and $\nu\left(-\frac{(m-1)}{2} \leqslant \nu \leqslant \frac{(m-1)}{2}\right)$ There are mN unknowns $\Gamma_{q,n}$ since $1 \leqslant q \leqslant N$, and, like ν, $-\frac{(m-1)}{2} \leqslant n \leqslant \frac{(m-1)}{2}$. However, most wing planforms are symmetrical and any mode of vibration may be split into a symmetrical part and an antisymmetrical part. Thus for a symmetrical oscillation of a symmetrical planform

$$\frac{\bar{w}(x_\nu^{(r)}, y_\nu, 0)}{U} = \frac{\bar{w}(x_{-\nu}^{(r)}, y_{-\nu}, 0)}{U}, \tag{108}$$

and

$$\Gamma_{q,n} = \Gamma_{q,-n}. \tag{109}$$

The number of equations and unknowns is then reduced to $\frac{1}{2}N(m+1)$. Similarly for an antisymmetrical oscillation the number of equations is reduced to $\frac{1}{2}N(m-1)$.

Equations (106) are an approximate form of the integral equation (66), but they were deduced on the assumption that the influence functions behave like the right hand side of equation (104) which has continuous derivatives for $|\eta'| < 1$. In fact the numerator in equation (66) contains terms in $(\eta-\eta')^2 \log |\eta-\eta'|$ which require a small (but often significant) alteration in equations (106). This correction is discussed in the next Section (Section 3.3).

3.3. *The Logarithmic Terms in the Influence Functions.*

As noted at the end of Section 3.2 equations (106) have to be adjusted to account for logarithmic terms in the influence functions. This is done by adding to $F_q(X_{\nu\nu}^{(r)}, Y_{\nu\nu})$ a term proportional to the coefficient of $(\eta-\eta')^2 \log|\eta-\eta'|$ in F_q.

In order to evaluate this coefficient it is convenient first to write

$$X_0 = \tfrac{1}{2}(1-\cos\phi). \tag{110}$$

Then by equation (84)

$$F_q = \frac{1}{\pi}\int_0^1 [G_1+G_2+G_3+G_4]\left[\cos\{(q-1)\cos^{-1}(1-2X_0)\} + \cos\{q\cos^{-1}(1-2X_0)\}\right]\left(\frac{d\phi}{dX_0}\right) dX_0. \tag{111}$$

Thus, for example,

$$F_1 = \int_0^1 [G_1+G_2+G_3+G_4]\frac{2}{\pi}\sqrt{\frac{1-X_0}{X_0}}\, dX_0, \tag{112}$$

$$F_2 = \int_0^1 [G_1+G_2+G_3+G_4]\frac{2}{\pi}\sqrt{\frac{1-X_0}{X_0}}(1-4X_0)\, dX_0, \tag{113}$$

and so on.

Consider, therefore the expansion, for small Y of

$$F_q = \int_0^1 [G_1+G_2+G_3+G_4] f_q(X_0)\, dX_0. \tag{114}$$

where f_q is a function of X_0 defined by equations (111) and (110).

(i) By equation (78)

$$\int_0^1 G_1 f_q(X_0)\, dX_0 = G_1 \int_0^1 f_q(X_0)\, dX_0. \tag{115}$$

Now

$$G_1 = \beta\mu Y\, K_1(\beta\mu Y) + (\text{a power series in } Y), \tag{116}$$

where

$$\mu = \frac{\nu}{\beta^2}\,\frac{c(y')}{d}. \tag{117}$$

Then it follows from the expansion of K_1, Ref. 7 (p. 80), that

$$\int_0^1 G_1 f_q(X_0)\,dX_0 = \left\{ \int_0^1 f_q(X_0)\,dX_0 \right\} \quad \tfrac{1}{2}\beta^2\mu^2 Y^2 \log Y$$

$$+ [\text{terms in } Y^4 \log Y,\ Y^6 \log Y \ldots]$$

$$+ [\text{a power series in } Y] \Big\}. \qquad (118)$$

(ii) From equation (79)

$$\int_0^1 G_2 f_q(X_0)\,dX_0 = \text{a power series in } Y. \qquad (119)$$

(iii) By equation (80)

$$\int_0^1 G_3 f_q(X_0)\,dX_0$$

$$= \frac{1}{M}\int_0^1 f_q(X_0)\left[1+\frac{M(X-X_0)}{\sqrt{(X-X_0)^2+Y^2}}\right]\exp\left[i\mu\{(X-X_0)-M\sqrt{(X-X_0)^2+Y^2}\}\right]dX_0 \qquad (120)$$

$$= \frac{1}{M}\int_0^X f_q(X-t)\exp(i\mu t)\left[1+\frac{Mt}{\sqrt{t^2+Y^2}}\right]\exp(-iM\mu\sqrt{t^2+Y^2})\,dt$$

$$+\frac{1}{M}\int_0^{1-X} f(X+t)\exp(-i\mu t)\left[1-\frac{Mt}{\sqrt{t^2+Y^2}}\right]\exp(-iM\mu\sqrt{t^2+Y^2})\,dt. \qquad (121)$$

Now the terms in $Y^2 \log Y$ may be found by using the following integrals

$$\left.\begin{aligned}
\int \sqrt{t^2+Y^2}\,dt &= \frac{t\sqrt{t^2+Y^2}}{2}+\frac{Y^2}{2}\log(t+\sqrt{t^2+Y^2})\\
\int t^{2n}\sqrt{t^2+Y^2}\,dt &= \frac{(-1)^n}{2}\,\frac{1.3.5\ldots(2n-1)}{4.6.8\ldots(2n+2)}\,Y^{2n+2}\log(t+\sqrt{t^2+Y^2})\\
&\quad +\{\text{a polynomial in } t,\ Y \text{ and } \sqrt{t^2+Y^2}\}\\
\int t^{2n+1}\sqrt{t^2+Y^2}\,dt &= \{\text{a polynomial in } t,\ Y \text{ and } \sqrt{t^2+Y^2}\}\\
\int \frac{dt}{\sqrt{t^2+Y^2}} &= \log(t+\sqrt{t^2+Y^2})\\
\int \frac{t^2}{\sqrt{t^2+Y^2}}\,dt &= \frac{t\sqrt{t^2+Y^2}}{2}-\frac{Y^2}{2}\log(t+\sqrt{t^2+Y^2})\\
\int \frac{t^{2n}}{\sqrt{t^2+Y^2}}\,dt &= \frac{(-1)^n}{2}\,\frac{3.5\ldots(2n-1)}{4.6\ldots(2n)}\,Y^{2n}\log(t+\sqrt{t^2+Y^2})\\
&\quad +\{\text{a polynomial in } t,\ Y \text{ and } \sqrt{t^2+Y^2}\}\\
\int \frac{t^{2n+1}}{\sqrt{t^2+Y^2}}\,dt &= \{\text{a polynomial in } t,\ Y \text{ and } \sqrt{t^2+Y^2}\}.
\end{aligned}\right\} \qquad (122)$$

By expanding the integrand of equation (121) it follows that

$$\int_0^1 G_3 f_q(X_0)\, dX_0 = Y^2 \log Y \{2i\mu f_q(X) - f_q'(X)\}$$

$$+\{\text{terms in } Y^4 \log Y,\ Y^6 \log Y \ldots\}$$

$$+\{\text{a power series in } Y\}.$$

(iv) Similarly

$$\int_0^1 G_4 f\,(X_0)\, dX = Y^2 \log Y\, \beta^2 \mu^2 \left\{ -\tfrac{1}{2} \int_0^1 f_q(X_0)\, dX_0 + \int_0^X f_q(X_0)\, dX_0 \right\}$$

$$+\{\text{terms in } Y^4 \log Y,\ Y^6 \log Y \ldots\}$$

$$+\{\text{a power series in } Y\}. \tag{124}$$

When all the terms are collected together it is found that

$$F_q = \int_0^1 (G_1 + G_2 + G_3 + G_4) f_q(X_0)\, dX_0$$

$$= Y^2 \log Y \left\{ \beta^2 \mu^2 \int_0^X f_q(X_0)\, dX_0 + 2i\mu f(X_0) - f'(X) \right\}$$

$$+\{\text{terms in } Y^4 \log Y,\ Y^6 \log Y, \ldots\}$$

$$+\{\text{a power series in } Y\}. \tag{125}$$

Then for small Y;

$$F_q(X,Y) = F_q(X,0) + K_q(X)\, Y^2 \log Y$$

$$+\{\text{terms in } Y^4 \log Y,\ Y^6 \log Y \ldots\}$$

$$+\{\text{a power series in } Y\}, \tag{126}$$

where

$$K_q(X) = \beta^2 \mu^2 \int_0^X f_q(X_0)\, dX_0 + 2i\mu f_q(X) - f_q'(X). \tag{127}$$

In particular

$$K_1(X) = \left(\frac{\nu\, c(y')}{\beta d}\right)^2 \left[\frac{\cos^{-1}(1-2X)}{\pi} + \frac{2}{\pi}\sqrt{X(1-X)}\right]$$

$$+\frac{4}{\pi}\frac{i\nu\, c(y')}{\beta^2 d}\sqrt{\frac{1-X}{X}} + \frac{1}{\pi X^{3/2}(1-X)^{1/2}} \tag{128}$$

$$K_2(X) = \left(\frac{\nu\, c(y')}{\beta d}\right)^2 \frac{4}{\pi} X^{1/2}(1-X)^{3/2}$$

$$+\frac{4}{\pi}\frac{i\nu\, c(y')}{\beta^2 d}\left[\sqrt{\frac{1-X}{X}} - 4\sqrt{X(1-X)}\right] + \frac{1}{\pi}\frac{(1+4X-8X^2)}{X^{3/2}(1-X)^{1/2}} \tag{129}$$

$$K_3(X) = \left(\frac{\nu\, c(y')}{\beta d}\right)^2 \frac{4}{3\pi} X^{1/2}(1-X)^{3/2}(3-8X)$$

$$+\frac{4}{\pi}\frac{i\nu\, c(y')}{\beta^2 d}\sqrt{\frac{1-X}{X}}(1-12X+16X^2)$$

$$+\frac{1}{\pi}\frac{1}{X^{3/2}(1-X)^{1/2}}(1+12X-72X^2+64X^3), \tag{130}$$

and so on.

Then it is shown in Ref. 8 that equations (106) have to be modified by replacing

$$F_q(X_{\nu\nu}^{(r)}, Y_{\nu\nu}) = F_q(X_{\nu\nu}^{(r)}, 0)$$

by

$$\bar{F}_q(X_{\nu\nu}^{(r)}, 0) = F_q(X_{\nu\nu}^{(r)}, 0) + K_q(X_{\nu\nu}^{(r)})\frac{\beta s}{c(\eta_\nu)} G_\nu,$$

where

$$G_\nu = \frac{4}{(m+1)^2}\left[\sum_{\substack{n=-\frac{(m-1)}{2} \\ n\neq\nu}}^{n=\frac{m-1}{2}} \cos^2\frac{n\pi}{m+1}\log\left|\sin\frac{\nu\pi}{m+1} - \sin\frac{n\pi}{m+1}\right| + \frac{m+1}{8}\left(\log 4 + \cos\frac{2\nu\pi}{m+1}\right)\right] \tag{132}$$

Numerical values of the quantities G_ν are given in Ref. 1, for $m = 7$, 11 and 15.

3.4. *Kinked Leading and Trailing Edges.*

Many of the planforms encountered in practice have sudden changes in direction in their leading and trailing edges. The commonest is perhaps the centre section of a swept wing; a more extreme example is treated in Ref. 4. These kinks violate the assumptions in the spanwise integration formula. Those calculations described in Section 4 which related to such planforms have therefore been carried out for 'smoothed' planforms, obtained by the rule suggested by Multhopp for steady flight.

It is assumed that the kink in the leading or trailing edge occurs at a spanwise solving station $\eta = \eta_\nu$. (In practice kinks rarely occur except at the centre section.) Then the equivalent planform is calculated by the rules

$$x_L(\eta_\nu) = \frac{1}{12}x_L(\eta_{\nu-1})+\frac{5}{6}x_L(\eta_\nu)+\frac{1}{12}x_L(\eta_{\nu+1}), \tag{133}$$

$$x_T(\eta_\nu) = \frac{1}{12}x_T(\eta_{\nu-1})+\frac{5}{6}x_T(\eta_\nu)+\frac{1}{12}x_T(\eta_{\nu-1}), \tag{134}$$

where the quantities in the right hand sides are those obtained from the original geometry of the planform.

The value of m should obviously be taken large enough to ensure that the alterations to the geometry are small. The number of solving points may also have to be large if the mode of vibration is one in which the altered areas of the planform are specially significant. The obvious method of deciding whether the number of solving points is big enough is by performing calculations for a sequence of increasing values of m, but the amount of computation required may be prohibitive.

3.5. *Calculation of Lift Distributions and Solving Points.*

The solution of equations (106) gives values of the function $\Gamma_{q,n}$ for $-\frac{1}{2}(m-1)\leqslant n\leqslant\frac{1}{2}(m-1)$. Then according to equations (56) and (64)

$$l(x,y_n) = e^{-i\nu x}\frac{8s}{\pi\, c(\eta_n)}\sum_{q=1}^{N}\Gamma_q(\eta_n)\,\psi_q(\phi), \tag{135}$$

on

$$\eta = \eta_n = \sin\frac{n\pi}{m+1}, \tag{136}$$

that is on

$$y = y_n = \frac{s}{d}\frac{\sin n\pi}{m+1}, \tag{137}$$

where

$$x = x_L(\eta_n)+\tfrac{1}{2}\frac{c(\eta_n)}{d}(1-\cos\phi). \tag{138}$$

Thus l may be calculated on any of the lines $\eta = \eta_n$. Provided m is large enough the lift at any point of the wing may be found by interpolation.

The overall generalised forces may be regarded as weighted integrals of l over the planform. Consider

$$F = \iint_S W(x_1,y_1)\,\Delta p(x_1,y_1)\,dx_1\,dy_1 \tag{139}$$

where W is the weighting function.

Then

$$F = \tfrac{1}{2}\rho_\infty U^2 \int_{y_1=-s}^{s} \int_{x_1 = x_{1_L}(y_1)}^{x_{1_T}(y_1)} W(x_1,y_1)\,l(x_1,y_1)\,dx_1\,dy_1\,, \tag{140}$$

that is

$$F = \tfrac{1}{2}\rho_\infty U^2 \int_{\eta=-1}^{+1} W(x_1,y_1)\,l(x_1,y_1)\tfrac{1}{2}\,s\,c(\eta)\sin\phi\,d\phi\,d\eta\,, \tag{141}$$

where

$$x_{1_L} = x_{1_L}(\eta)+\tfrac{1}{2}c(\eta)\,(1-\cos\phi)\,, \tag{142}$$

$$y_1 = s\eta\,, \tag{143}$$

and

$$l = \frac{8s}{\pi\,c(\eta)}\exp\left[\,-iv\left(x_L(\eta)+\tfrac{1}{2}\frac{c(\eta)}{d}(1-\cos\phi)\right)\right]\sum_{q=1}^{N}\Gamma_q(\eta)\,\Psi_q(\phi)\,. \tag{144}$$

Hence equation (141) may be rewritten

$$F = \tfrac{1}{2}\rho_\infty U^2\frac{4s^2}{\pi}\int_{\eta=-1}^{+1}\int_{\phi=0}^{\pi} W(x_1,y_1)\exp\left[\,-iv\left(x_L(\eta)+\tfrac{1}{2}\frac{c(\eta)}{d}(1-\cos\phi)\right)\right]\times$$

$$\times\sum_{q=1}^{N}\Gamma_q(\eta)\,\{\cos(q-1)\,\phi+\cos q\phi\}\,d\phi\,d\eta\,. \tag{145}$$

If W is a polynomial in (x_1,y_1) it may also be expressed in the form

$$W = \sum_{p=0}^{P} A_p(\eta)\cos p\phi\,,\quad 0\leqslant\phi\leqslant\pi\,, \tag{146}$$

so that the integration with respect to ϕ depends on integrals of the type

$$\int_0^{\pi} \cos n\phi \exp\left\{ \frac{i\nu\, c(\eta)}{2d} \cos\phi \right\} d\phi = \pi i^n J_n \left(\frac{\nu\, c(\eta)}{2d} \right) \tag{147}$$

(Ref. 7, pp. 20, 21).
The spanwise integration may then be carried out by the formula

$$\int_{-1}^{+1} f\, d\eta = \frac{\pi}{m+1} \sum f_n \cos\frac{n\pi}{m+1}. \tag{148}$$

In the examples given in Section 4 the forces evaluated are the overall lift, for which $W = 1$, and the pitching moment, for which $W = -x_1$.

By equation (145)

$$\text{Lift} = \tfrac{1}{2}\rho_\infty U^2 S A \int_{\eta=-1}^{+1} \exp\left[-i\nu \left(x_L(\eta) + \frac{c(\eta)}{2d} \right) \right] \times$$

$$\times \sum_{q=1}^{N} \Gamma_q(\eta) \left\{ i^{q-1} J_{q-1} \left(\frac{\nu\, c(\eta)}{2d} \right) + i^q J_q \left(\frac{\nu\, c(\eta)}{2d} \right) \quad d\eta . \right\} \tag{149}$$

Pitching moment (about $x_1 = 0$, positive nose up)

$$= -\tfrac{1}{2}\rho_\infty U^2 S d A \int_{\eta=-1}^{+1} \exp\left\{ -i\nu \left[x_L(\eta) + \tfrac{1}{2}\frac{c(\eta)}{d} \right] \right\} \times$$

$$\times \left\{ \left[x_L(\eta) + \tfrac{1}{2}\frac{c(\eta)}{d} \right] \sum_{q=1}^{N} \Gamma_q(\eta) \left[i^{q-1} J_{q-1} \left(\frac{\nu\, c(\eta)}{2d} \right) + i^q J_q \left(\frac{\nu\, c(\eta)}{2d} \right) \right] \right.$$

$$-\tfrac{1}{4}\frac{c(\eta)}{d} \sum_{q=1}^{N} \Gamma_q(\eta) \left[i^{q-2} J_{q-2} \left(\frac{\nu\, c(\eta)}{2d} \right) + i^{q-1} J_{q-1} \left(\frac{\nu\, c(\eta)}{2d} \right) \right.$$

$$\left. \left. + i^q J_q \left(\frac{\nu\, c(\eta)}{2d} \right) + i^{q+1} J_{q+1} \left(\frac{\nu\, c(\eta)}{2d} \right) \right] \right\} d\eta . \tag{150}$$

The spanwise integration is now to be carried out by equation (148).

3.6. *Low Frequency Theory.*

When the frequency parameter ν is very small a low-frequency theory may be constructed in which terms of order ν are neglected.

From equations (78) to (81) it follows that when ν is small

$$G_1+G_2+G_3+G_4 = 1+\frac{X-\frac{1}{2}+\frac{1}{2}\cos\phi}{[(X-\frac{1}{2}+\frac{1}{2}\cos\phi)^2+Y^2]^{\frac{1}{2}}}$$

$$-\frac{iv}{\beta^2}\frac{c(y')}{d}\frac{Y^2}{[(X-\frac{1}{2}+\frac{1}{2}\cos\phi)^2+Y^2]^{\frac{3}{2}}}+0(v^2\log v). \qquad (151)$$

The influence functions are then obtained by using equation (151) in equation (85).

When $v = 0$ these influence functions reduce to those of Ref. 6, but for low frequency they are not the same as those of Ref. 1. This arises from the fact that in Ref. 1 the ratio $\frac{w}{\bar{w}} = \frac{l}{\bar{l}}$ is not e^{-ivx} as defined in equations (55) and (56), but $\exp(ivM^2\, x/\beta^2)$. In view of the theory contained in Ref. 1 it appears unnecessary to describe the present low-frequency theory further. There appears to be no analytical method of deciding which is the more accurate.

3.7. *Method of Calculation.*

The following is an outline of the steps which have to be carried out to compute aerodynamic forces by the method described in the preceding Section. The procedure is described as for an asymmetric wing in an asymmetric mode of oscillation, but in most practical examples the calculation is greatly shortened by considerations of symmetry.

Data.

The following information is assumed to have been given:
(i) Geometry of the planform.
(ii) Mode of vibration, $q(x_1,y_1)$ in equation (50).
(iii) Angular frequency, ω.
(iv) Mach number M.
It is also assumed that the following have been chosen:
(v) m the number of spanwise solving stations (m odd).
(vi) N the number of solving points on each chord.
(Some consideration is given to the choice of m and N in Section 5.)
(vii) A representative length d. (Usually either the mean chord, $\bar{c}$, or the root chord c_r.)
(viii) The origin, taken to be midway between the wing tips, and at any convenient streamwise position.

Calculation.

(a) Calculate the following quantities for $n = -\frac{(m-1)}{2}$ to $+\frac{(m-1)}{2}$.

(i) $\eta_n = \sin\frac{n\pi}{m+1}$.

(ii) $\frac{c(\eta_n)}{d}$ (from the wing geometry).

(iii) $x_L(\eta_n)$ (from the wing geometry).

(iv) $x_T(\eta_n)$ (from the wing geometry).

(v) Values of (ii), (iii) and (iv) modified using equations (133) and (134) to account for any kinks in the leading or trailing edges.

(vi) $x_n^{(r)}$, from equation (99) for $r = 1,2,\ldots N$.

(b) Calculate the following quantities for all combinations of

$$v = -\frac{(m-1)}{2} \text{ to } +\frac{(m-1)}{2},$$

$$n = -\frac{(m-1)}{2} \text{ to } +\frac{(m-1)}{2},$$

and

$$r = 1,2, \ldots N,$$

excluding those for which $|\nu - n|$ is an even positive number.

(i) $$X_{\nu,n}^{(r)} = \frac{d}{c(\eta_n)} [x_\nu^{(r)} - x_L(\eta_n)]$$

(ii) $$Y_{\nu,n} = \beta \frac{s}{d} \frac{d}{c(\eta_n)} |\eta_\nu - \eta_n|$$

(c) Calculate the influence function F_q.

(i) $F_q(X_{\nu,n}^{(r)}, Y_{\nu,n})$

for all combinations of

$$q = 1,2, \ldots N$$

$$r = 1,2, \ldots N$$

$$n = -\frac{(m-1)}{2} \text{ to } +\frac{(m-1)}{2}$$

$$\nu = -\frac{(m-1)}{2} \text{ to } +\frac{(m-1)}{2}$$

excluding those for which $|\nu - n|$ is even, using equations (84), and (78) to (82), (or some convenient modified form, such as those obtained using equations (73) or (87)).

(ii) $$F_q(X_{\nu,\nu}^{(r)}, 0)$$

for

$$q = 1, \ldots N$$

$$r = 1, \ldots N$$

$$\nu = -\frac{(m-1)}{2} \text{ to } \frac{(m-1)}{2}$$

using the formulae (90) and (91).

(d) Calculate the logarithmic correction

$$K_q(X_{\nu,\nu}^{(r)})$$

for

$$r = 1, \dots N$$

$$q = 1, \dots N$$

$$\nu = -\frac{(m-1)}{2} \text{ to } +\frac{(m-1)}{2}$$

using equations (127), or (128) to (130), and then calculate

$$\bar{F}_q(X_{\nu,\nu}^{(r)}, 0)$$

for the same values of r, q and ν, using equations (131) and (132).

(e) Calculate

$$\frac{\bar{w}}{U}(x_\nu^{(r)}, y_\nu, 0)$$

for

$$r = 1, \dots N$$

$$\nu = -\frac{(m-1)}{2} \text{ to } +\frac{(m-1)}{2}$$

using equations (55), (50) and (51), and the given mode shape q.

(f) Construct the mN equations (106) in the mN unknowns $\Gamma_{q,\nu}$ using equations (105) and the influence functions calculated in (d), with left hand sides from (e).

(g) Solve the equations (f) for $\Gamma_{q,\nu}$.

(h) Calculate whatever generalised forces are required using equations (145) and (148), (or (149) and (150) if lift and moment are wanted).

The lift distribution on any wing section may be calculated using equations (64), (62) and (56).

4. *Calculated Examples.*

The calculations given below are for wings which are performing rigid pitching and heaving. In all these examples the typical length d is taken to be the geometrical mean chord $\bar{c}$, so the frequency parameter becomes $\bar{\nu} = \omega\bar{c}/U$.

In the earlier examples the influence functions were calculated by a program for the (now obsolete) Deuce computer. Subsequent calculations are to be done by a KDF 9 computer until this in turn is replaced.

In order to define the quantities calculated it is assumed that, when the wing is oscillating in a mode combining heaving and pitching, its surface is defined by the equation

$$z = -\{z_0 + (x - x_0)\,\alpha_0\}\, e^{i\omega t} \tag{152}$$

where $\bar{c}\, z_0$ is the amplitude of the heaving oscillation, and x_0 defines the axis of rotation. Then the lift and pitching moment are expressed by

$$\text{Lift} = \rho V^2 S\, e^{i\omega t}\, \{(l_z + i\bar{\nu}\, l_{\dot{z}})\, z_0 + (l_\alpha + i\bar{\nu}\, l_{\dot{\alpha}})\, \alpha_0\}\,, \tag{153}$$

and

$$\text{Pitching Moment} = \rho V^2 S\bar{c}\, e^{i\omega t}\{(m_z + i\nu\, m_{\dot{z}})\, z_0 + (m_\alpha + i\bar{\nu}\, m_{\dot{\alpha}})\, \alpha_0\}\,, \tag{154}$$

which defines the eight derivatives l_z, $l_{\dot{z}}$, l_α, $l_{\dot{\alpha}}$, m_z, $m_{\dot{z}}$, m_α and $m_{\dot{\alpha}}$. The pitching moment is taken about the axis $x = x_0$, and is positive if it tends to raise the leading edge.

These derivatives depend on the position of the pitching axis and vary with it according to the following equations:

$$\left.\begin{aligned}
l_z(x_0) &= l_z(0) \\
l_{\dot{z}}(x_0) &= l_{\dot{z}}(0) \\
l_\alpha(x_0) &= l_\alpha(0) - x_0\, l_z(0) \\
l_{\dot{\alpha}}(x_0) &= l_{\dot{\alpha}}(0) - x_0\, l_{\dot{z}}(0) \\
m_z(x_0) &= m_z(0) + x_0\, l_z(0) \\
m_{\dot{z}}(x_0) &= m_{\dot{z}}(0) + x_0\, l_{\dot{z}}(0) \\
m_\alpha(x_0) &= m_\alpha(0) + x_0(l_\alpha(0) - m_z(0)) - x_0^2\, l_z(0) \\
m_{\dot{\alpha}}(x_0) &= m_{\dot{\alpha}}(0) + x_0\,(l_{\dot{\alpha}}(0) - m_{\dot{z}}(0)) - x_0^2\, l_{\dot{z}}(0)
\end{aligned}\right\} \tag{155}$$

In all examples the origin is taken to be at the leading edge of the centre section.

4.1. *The Rectangular Wing of Aspect Ratio* 4.

Table 1 gives the derivative coefficients as calculated with $m(N) = 7(2)$ at a high subsonic Mach number, $M = \frac{1}{2}\sqrt{3}$, for a range of frequency parameters. The values for $\bar{\nu} \to 0$ were obtained by the theory of this report for small $\bar{\nu}$ (Section 3.6); the low frequency theory of Ref. 1 gives $l_{\dot{\alpha}} = 0{\cdot}531$, and $-m_{\dot{\alpha}} = 1{\cdot}188$ which differ negligibly from the values in Table 1. All the other 6 derivatives are necessarily the same for both low-frequency theories.

Figs. 1 to 3 show the variation of l_α, $l_{\dot{\alpha}}$, $-m_\alpha$, and $-m_{\dot{\alpha}}$ with pitching axis position x_0. For all derivatives the variation with $\bar{\nu}$ is reasonably systematic, although the value of m is probably too small for high absolute accuracy. Figs. 2 and 3 also show experimental values measured in the 36 in. by 14 in. wind tunnel in Aerodynamics Division N.P.L.,* which has solid side walls and longitudinally slatted roof and flow.

4.2. *The Rectangular Wing of Aspect Ratio* 2.

Table 2 gives values of the pitching and heaving derivatives for a range of frequency parameters. It may be noted that changing m from 7 to 11 makes little difference and $m = 7$ is presumably high enough for this planform in these modes of oscillation. Again the derivatives tabulated for $\bar{\nu} \to 0$ were calculated by the low frequency version of the present theory. The method of Ref. 1 gives for $\bar{\nu} \to 0$, $l_{\dot{\alpha}} = 1{\cdot}633$, and $-m_{\dot{\alpha}} = 1{\cdot}060$, so that the difference is negligible.

Fig. 4 shows the variation of $-m_{\dot{\alpha}}$ with pitching axis position, and a comparison with experiments from the N.P.L. 36 in. by 14 in. wind tunnel. The theory shows reasonably consistent variations with $\bar{\nu}$ and agrees fairly well with experiment.

Figs. 5 and 6 show the variation of $l_{\dot{\alpha}}$, m_α and $m_{\dot{\alpha}}$ with frequency parameter. This variation is reasonably self consistent and also fits in well with the tangents to the curves at $\bar{\nu} = 0$ as predicted by the theory of Ref. 9, in particular by the following formulae for small $\bar{\nu}$:

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*Unpublished communication from K. C. Wight.

$$l_{\dot{\alpha}} = (l_{\dot{\alpha}})_{\bar{\nu}\to 0} + \bar{\nu}\frac{A}{16}(l_{\alpha}^{2})_{\bar{\nu}\to 0} + o(\bar{\nu}), \tag{156}$$

$$-m_{\dot{\alpha}} = (-m_{\dot{\alpha}})_{\bar{\nu}\to 0} + \bar{\nu}\left(\frac{A}{16}\right)(-l_{\alpha}\, m_{\alpha})_{\bar{\nu}\to 0} + o(\bar{\nu}). \tag{157}$$

4.3. *Wings of Non-Rectangular Planform.*

Table 3 gives the derivatives for a swept tapered wing of aspect ratio 2, for three frequencies at $M = 0{\cdot}781$ ($\beta = 5/8$) and one frequency at $M = 0{\cdot}927$ ($\beta = 3/8$). The planform is shown in Fig. 7. The curves in Fig. 7 illustrate the theoretical behaviour of $-m_{\dot{\alpha}}$ as a function of the axis position x_0. The curve for $\bar{\nu} = 0{\cdot}5$ has been omitted to avoid confusion. The comparatively small effects of changes in $\bar{\nu}$ and M may be noted. The experimental points in Fig. 7 have been obtained either by taking mean values from Ref. 10 or from Fig. 20 of Ref. 11. The theory appears to agree best with the experiments of Ref. 11.

Fig. 8 contains some curves of the derivatives for lift and pitching moment, for pitching about the axis $x_0 = 1$. (It should be noted that the scales are larger than in the previous figures.) The curves for non-zero frequency obtained by the present theory fit in well with the values of l_α and m_α for $\bar{\nu} = 0$, and fairly well for $l_{\dot{\alpha}}$ and $m_{\dot{\alpha}}$ with the tangents for small $\bar{\nu}$ obtained by using the method of Ref. 1 and the formulae of Ref. 9 (equations (156) and (157) of this Report).

Fig. 8 also shows curves obtained using the relations between the derivatives in backward flight and forward flight which are predicted by the reverse flow theorem (Ref. 12). The differences between the direct and reverse flow solution is fairly small.

Finally Table 4 contains the derivatives for two other wings whose planforms are given in Fig. 9. The effect of increasing m is fairly small but larger for the delta wing possibly owing to the greater change in leading edge angle at the centre section.

5. *Concluding Remarks.*

The results described in Section 4 indicate that, in common with most kernel function methods, that of this report can be used to provide satisfactory solutions of the linearised problem of three-dimensional theory for rigid modes of oscillation. It seems reasonable to suppose that this will also be true for modes involving smooth distortion provided the mode shape is not so complicated that an excessive number of solving points is required, and the frequency is not excessively large.

Possibly the most important decision which has to be made at the start of a calculation is the choice of the number and position of solving points. Obviously a first consideration is that the mode shape must be adequately defined by its values at the solving points. After this the only general method is to carry out calculations with increasing numbers of integration points until some limiting solution is reached. This is open to the objection that it requires lengthy computation which, even if practicable, is eventually discarded, but there appears to be no alternative. Some guidance as to the minimum number of stations across the span may be obtained from Ref. 13, which deals with the steady case.

Hinged control-surface derivatives, or forces caused by oscillations in other modes whose shape has a discontinuity in shape, are not covered in this report. The usual methods of adapting lifting-surface theory to this problem involve either the use of the reverse-flow theorem or the replacement of the discontinuous mode by a smooth one designed to give the same derivatives (Refs. 12, 14 and 15).

There have been many other applications of lifting-surface theory to oscillating wing derivatives, each with its own method of choosing solving and integration points. All of them should be satisfactory, given sufficiently accurate integration, provided M is low enough to be well within the range for which equation (1) is applicable. The chief interest lies in the derivatives predicted for high subsonic M. A comparison of their behaviour in this respect is given in Ref. 16, in which the present method appears to give reasonably good results.

LIST OF SYMBOLS

A	Aspect ratio
$c(y')$	Wing chord at $y = y'$
$\bar{c}$	Geometric mean chord ($\bar{c} = S/2s$)
c_r	Root chord
c_t	Tip chord for wings with straight tips parallel to the flow
d	Representative length associated with the wing (in the examples in Section 4, $d = \bar{c}$)
f	Strength of doublet distribution (in Section 1)
f	Representative function in integration formula, equation (102) (in Section 3.2)
f_q	Function defined by equations (111) and (114)
F_q	Influence function (equations (66) and (67))
G_1,G_2,G_3,G_4	Parts of kernel function, equations (68) to (72)
G_v	Coefficient in correction for logarithmic term in spanwise integration, equations (131) and (132)
I_0,I_1	Bessel functions (Ref. 7)
$J_0,J_1,J_2,\ldots$	Bessel functions (Ref. 7)
K_0,K_1	Bessel functions (Ref. 7)
K	Kernel of integral equation (equation (27))
K_q	Coefficient of $Y^2 \log Y$ in F_q (equations (125) and (126))
l	Wing loading (equation (16))
$\bar{l}$	Equation (56)
$l_z,l_{\dot{z}},l_\alpha,l_{\dot{\alpha}}$	Derivative coefficients defined in equation (153)
L (suffix)	Value at leading edge of wing
L_0,L_1	Modified Struve functions (equations (33) and (44))
m	Number of spanwise solving stations (equation (93))
$m_z,m_{\dot{z}},m_\alpha,m_{\dot{\alpha}}$	Derivative coefficients defined by equation (154)
M	Mach number of undisturbed stream, $0 \leqslant M < 1$
n	Integer denoting spanwise station (equation (93))
N	Number of chordwise solving points (equation (96))
p	Fluid pressure
p_∞	Fluid pressure far upstream of the wing
r	Integer denoting chordwise position of solving point (equation (96))
r_1	Radial distance in polar co-ordinates (equation (6))
R	Distance modified for compressibility (equation (3))

s	Semi-span of wing
S	Area of wing planform
t	Time
T (suffix)	Value at the trailing edge of the wing
u	Perturbation velocity in the streamwise direction
U	Velocity of undisturbed flow
v	Perturbation velocity in the starboard direction
w	Perturbation velocity in the upward direction
$\bar{w}$	Equation (55)
W (suffix)	Wake
W	Weighting function for generalized force (equation (139))
x_1,y_1,z_1	Rectangular co-ordinates, x_1 increasing in the direction of undisturbed flow, y_1 to starboard, z_1 upwards. (In the numerical examples of Section 4 the origin is taken to be at the leading edge of the centre section.)
x_1',y_1'	Variables of integration
x,y,z	x_1/d, y_1/d, z_1/d Non-dimensional co-ordinates
x',y'	x_1'/d, y_1'/d
x_0,y_0	$(x-x')$, $(y-y')$
x_0 (in Section 4)	Non-dimensional x co-ordinate of the axis of pitching oscillation
z_0	Non-dimensional amplitude of heaving oscillation
X	Defined in equation (74)
X_0	Streamwise variable (equation (110))
Y	Defined in equation (75)
$\bar{Y}$	Defined in equation (82)
α_0	Amplitude of oscillatory incidence of a pitching wing (equation (52))
β	$(1-M^2)^{\frac{1}{2}}$
Γ_q	Functions in lift distribution (equation (64))
$\Gamma_{q,n}$	$\Gamma_q(\eta_n)$
δ	Semi-major axis of ellipse dividing (x_1',y_1') plane (equation (8))
Δ	Increment in f (equation (8))
θ	Angular polar co-ordinate (equation (6))
Λ_L,Λ_T	Angle of sweepback of a straight leading or trailing edge
μ	$\dfrac{\nu}{\beta^2}\dfrac{c(y')}{d}$

ν	Frequency parameter $\nu = \omega d/U$
$\bar{\nu}$	Frequency parameter $\bar{\nu} = \omega\bar{c}/U$
ρ	Fluid density
ρ_∞	Fluid density far upstream of the wing
ϕ	Chordwise variable (equation (62))
Φ	Velocity potential of perturbation flow field
Φ_0	Velocity potential of an oscillating source
ψ	Acceleration potential (equation (14))
Ψ_q	Chordwise lift distribution (equation (65))
ω	Angular frequency of oscillation

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TABLE 1

Pitching and Heaving Derivatives for a Rectangular Wing, $A = 4$.

(Pitching Axis at Leading Edge, $x_0 = 0$)

$m(N)$	7(2)	7(2)	7(2)	7(2)
M	0·866	0·866	0·866	0·866
$\bar{\nu}$	$\to 0$	0·3	0·6	1·2
l_z	0	0·077	0·180	0·209
$l_{\dot{z}}$	2·479	2·310	2·098	1·705
$-m_z$	0	−0·041	−0·121	−0·250
$-m_{\dot{z}}$	0·515	0·546	0·620	0·581
l_α	2·479	2·432	2·413	2·184
$l_{\dot{\alpha}}$	0·547	0·892	0·960	0·936
$-m_\alpha$	0·515	0·544	0·634	0·602
$-m_{\dot{\alpha}}$	1·194	1·217	1·086	0·751

TABLE 2

Pitching and Heaving Derivatives for a Rectangular Wing, $A = 2$.

(Pitching Axis at Leading Edge, $x_0 = 0$)

$m(N)$	7(3)	7(3)	11(3)	7(3)
M	0·866	0·866	0·866	0·866
$\bar{\nu}$	$\to 0$	0·3	0·3	0·6
l_z	0	−0·043	−0·043	−0·167
$l_{\dot{z}}$	1·461	1·478	1·477	1·577
$-m_z$	0	−0·052	−0·051	−0·212
$-m_{\dot{z}}$	0·242	0·258	0·260	0·340
l_α	1·461	1·486	1·486	1·625
$l_{\dot{\alpha}}$	1·634	1·692	1·691	1·699
$-m_\alpha$	0·242	0·235	0·237	0·264
$-m_{\dot{\alpha}}$	1·063	1·101	1·102	1·193

TABLE 3

Pitching and Heaving Derivatives for a Sweptback Wing, $A = 2$.

Pitching axis through leading edge of centre section, $x_0 = 0$
($\Lambda_L = 60°$, $\Lambda_T = 26·57°$, $c_r/s = 1·6160$, $c_t/s = 0·3840$)

$m(N)$	15(3)	15(3)	15(3)	15(3)
M	0·781	0·781	0·781	0·927
$\bar{\nu}$	0·25	0·50	1·00	1·00
l_z	−0·017	−0·081	−0·371	−0·228
$l_{\dot{z}}$	1·268	1·260	1·294	1·333
$-m_z$	−0·028	−0·125	−0·548	−0·388
$-m_{\dot{z}}$	1·368	1·362	1·413	1·532
l_α	1·261	1·211	1·020	1·315
$l_{\dot{\alpha}}$	2·351	2·374	2·428	2·272
$-m_\alpha$	1·344	1·246	0·879	1·333
$-m_{\dot{\alpha}}$	2·959	2·994	3·084	3·031

TABLE 4

Pitching and Heaving Derivatives for a Tapered Wing and a Delta Wing.

(For planform details see Fig. 9)
Pitching axis through leading edge of centre section, $x_0 = 0$

Wing	Tapered $A = 4·33$		Delta $A = 1·5$	
$m(N)$	7(3)	11(3)	7(3)	11(3)
M	0·9	0·9	0·9	0·9
$\bar{\nu}$	0·190	0·190	0·15	0·15
l_z	0·056	0·056	−0·010	−0·009
$l_{\dot{z}}$	2·640	2·636	1·066	1·058
$-m_z$	−0·012	−0·012	−0·017	−0·015
$-m_{\dot{z}}$	1·315	1·324	1·273	1·302
l_α	2·742	2·737	1·058	1·050
$l_{\dot{\alpha}}$	1·281	1·278	2·461	2·405
$-m_\alpha$	1·332	1·341	1·255	1·285
$-m_{\dot{\alpha}}$	2·255	2·251	3·487	3·453

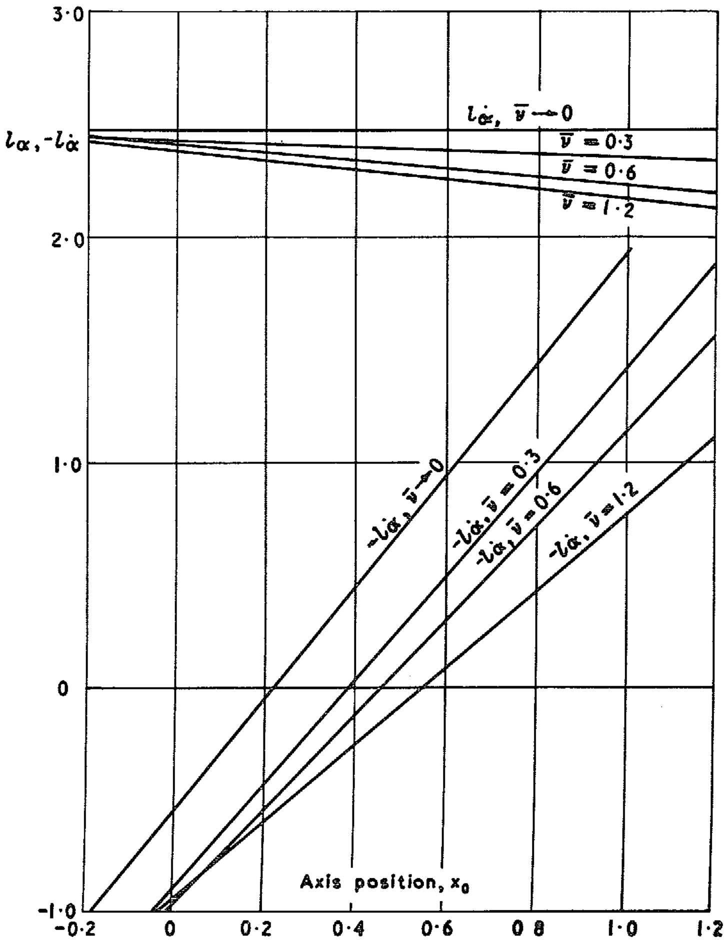


FIG. 1. Values of l_α and $-l_{\dot\alpha}$ (rectangular wing, $A = 4, M = \frac{1}{2}\sqrt{3}$).

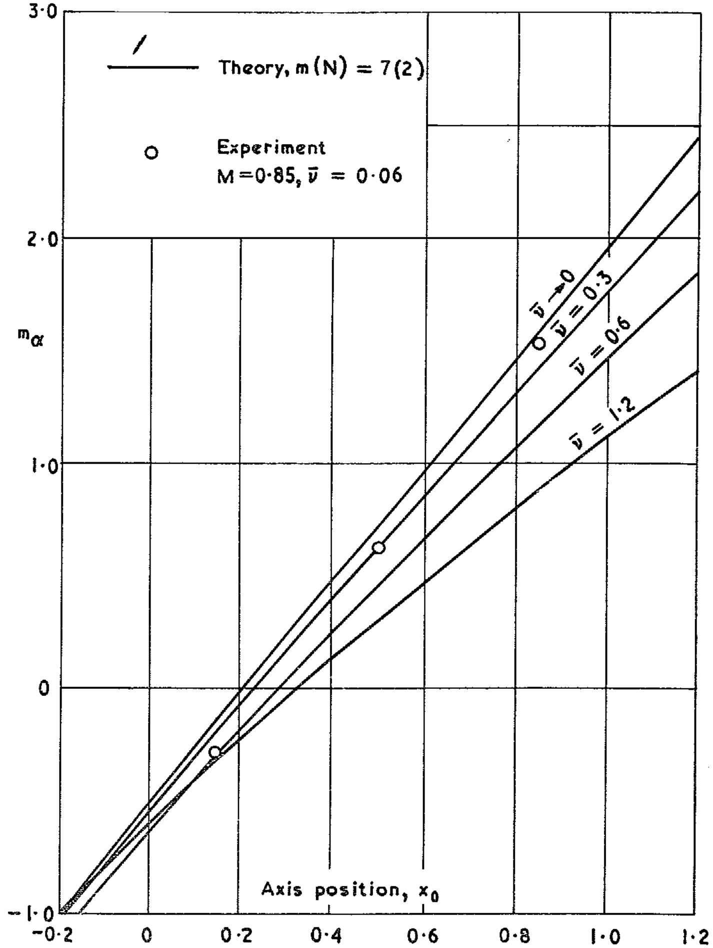
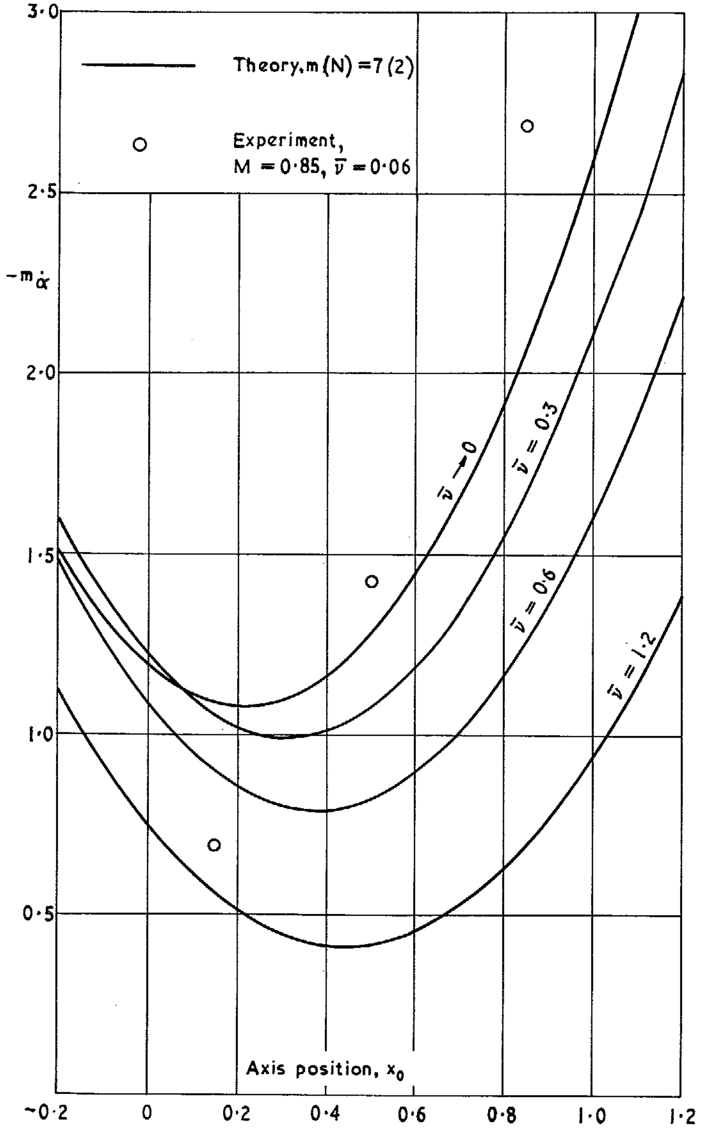


FIG. 2. Values of m_α (Rectangular wing, $A = 4$, $M = \frac{1}{2}\sqrt{3}$).



Values of $-m_{\dot{\alpha}}$ (Rectangular wing, $A = 4$, $M = \frac{1}{2}\sqrt{3}$).

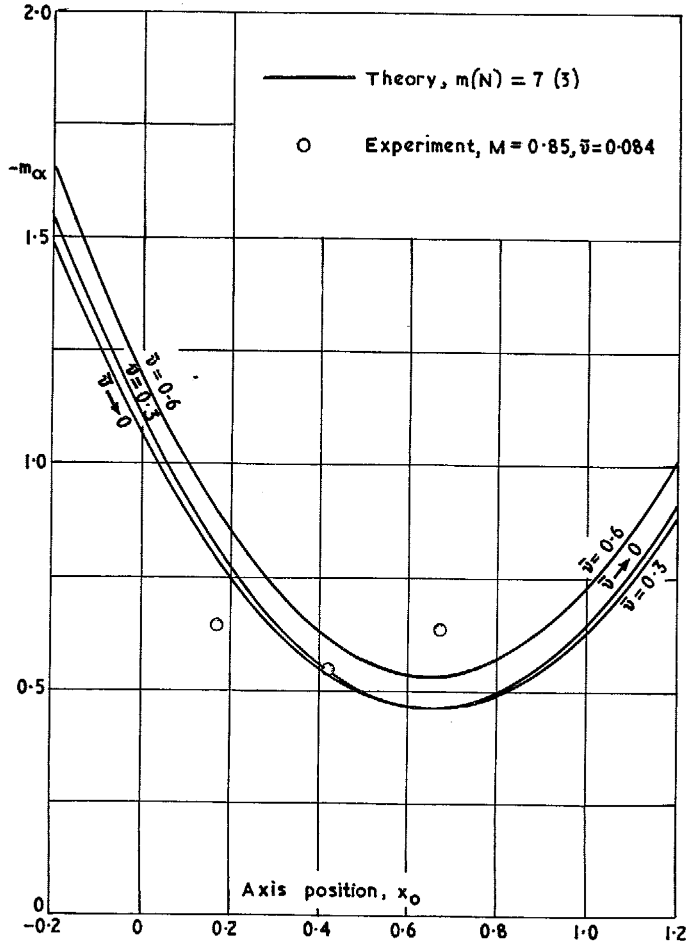


FIG. 4. Values of $-m_{\dot{\alpha}}$ (Rectangular wing, $A = 2$, $M = \frac{1}{2}\sqrt{3}$).

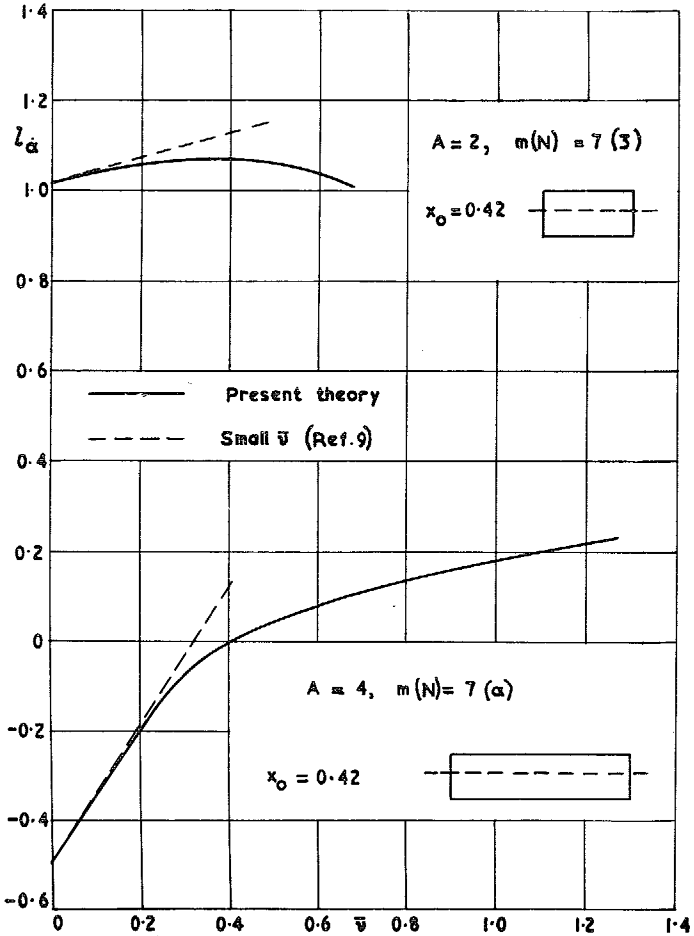


FIG. 5. Lift derivative, $l_{\dot{\alpha}}$, against frequency parameter $\bar{\nu}$ for rectangular wings in pitching motion at $M = 0{\cdot}866$, $x_0 = 0{\cdot}42$.

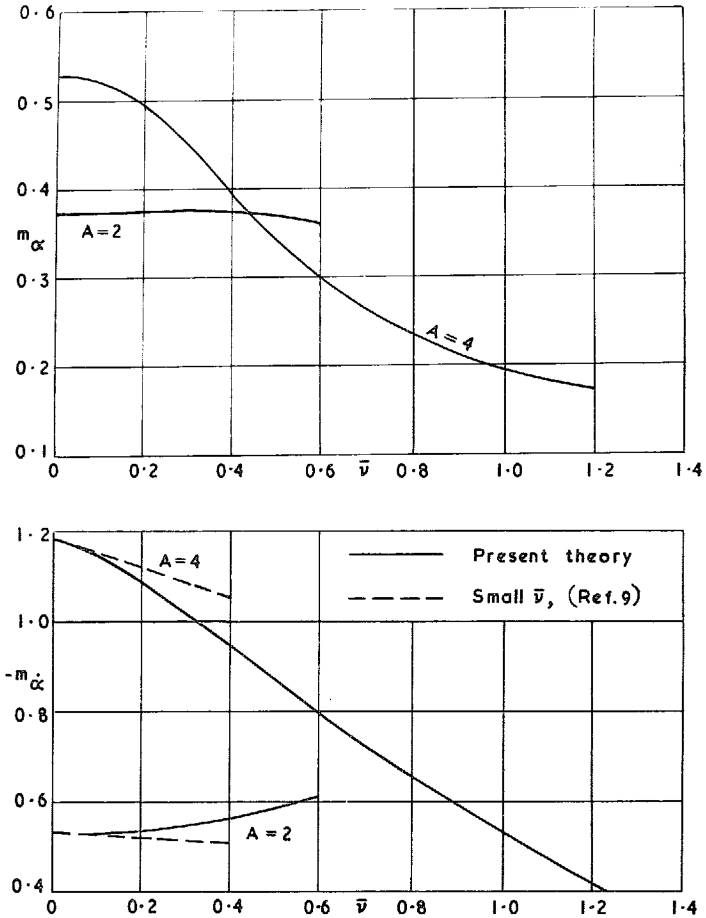


FIG. 6. Pitching moment derivatives, m_α and $-m_{\dot{\alpha}}$ against frequency parameter $\bar{\nu}$ for rectangular wings in pitching motion at $M = 0{\cdot}866$, $x_0 = 0{\cdot}42$.

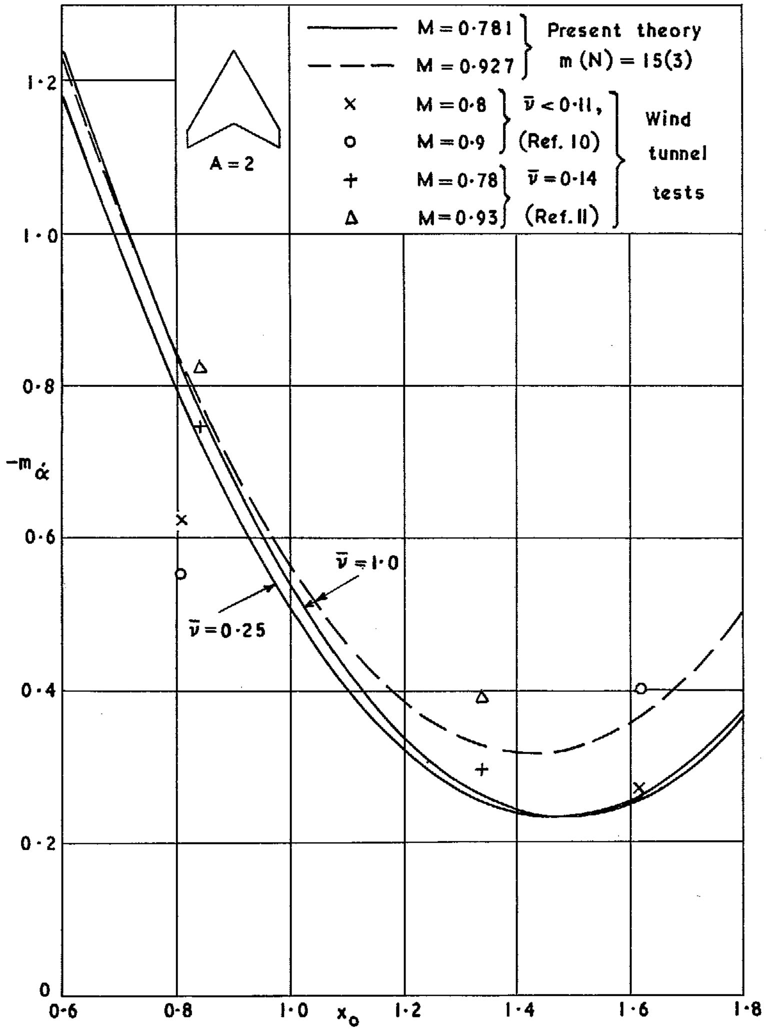


FIG. 7. Pitching damping derivative $-m_{\dot{\alpha}}$ against pitching axis for the swept wing at $M = 0{\cdot}781$ and $M = 0{\cdot}927$.

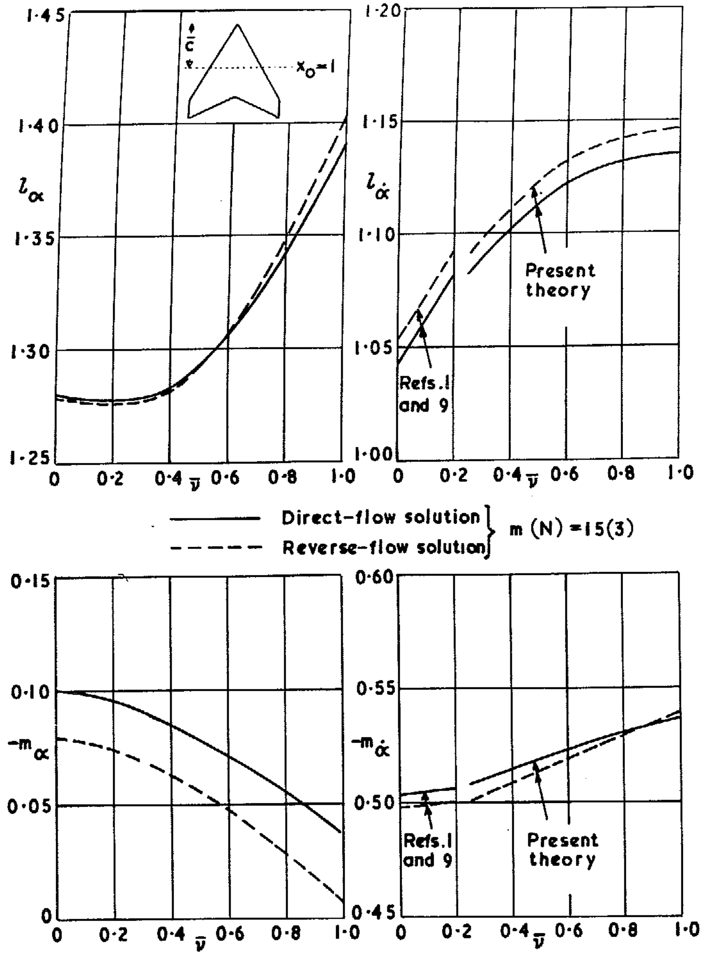
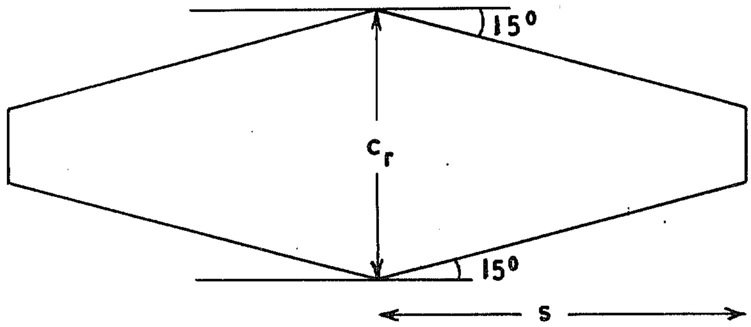
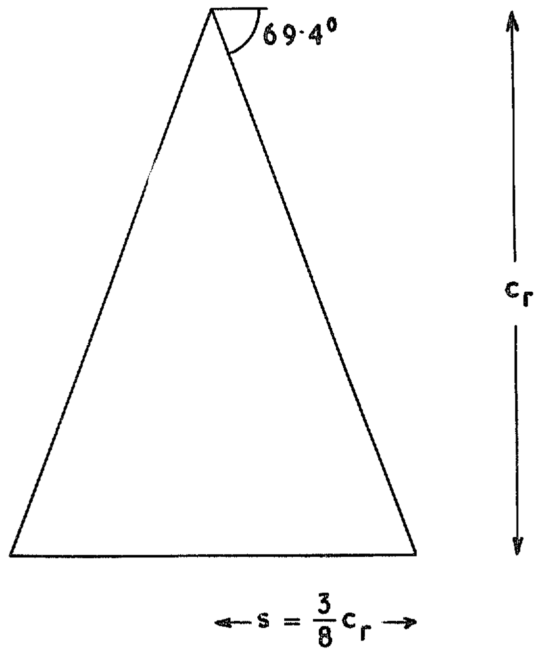


FIG. 8. Effect of frequency parameter on pitching derivatives for the swept wing $A = 2$ at $M = 0{\cdot}781$.



Tapered wing, $A = 4{\cdot}33$, $\Lambda_L = 15^0$, $\Lambda_T = -15^0$

$s = 1{\cdot}37\, c_r$, $c_t = 0{\cdot}266\, c_r$



Delta wing, $A = 1{\cdot}5$, $\Lambda_L = 69{\cdot}4^0$, $\Lambda_T = 0$,

$s = 0{\cdot}375\, c_r$, $c_t = 0$

FIG. 9. Planforms of wings for Table 4.

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